ANISOTROPIC SOBOLEV HOMEOMORPHISMS

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Abstract. Let \( \Omega \subseteq \mathbb{R}^2 \) be a domain. Suppose that \( f \in W_{1,1}^{1,1}(\Omega; \mathbb{R}^2) \) is a homeomorphism. Then the components \( x(w), y(w) \) of the inverse \( f^{-1} = (x, y): \Omega' \rightarrow \Omega \) have total variations given by
\[
|\nabla y| (\Omega') = \int_{\Omega'} |\frac{\partial f}{\partial x}| \, dz, \quad |\nabla x| (\Omega') = \int_{\Omega'} |\frac{\partial f}{\partial y}| \, dz.
\]

1. Introduction

Let \( \Omega \subseteq \mathbb{R}^2 \) and \( \Omega' \subseteq \mathbb{R}^2 \) be domains. Recently, homeomorphisms \( f = (u, v): \Omega \onto \Omega' \) which are a.e. differentiable together with their inverses \( f^{-1} = (x, y): \Omega' \onto \Omega \) have been intensively studied (see [9], [11]).

A homeomorphism \( f: \Omega \onto \Omega' \) which belongs to the Sobolev space \( W_{loc}^{1,1}(\Omega; \mathbb{R}^2) \) is called a \( W^{1,1} \)-homeomorphism. If also \( f^{-1} \) is a \( W^{1,1} \)-homeomorphism, we say that \( f \) is a bi-Sobolev map (see [13]). We recall that a \( W^{1,1} \)-homeomorphism is differentiable a.e. thanks to the well known Gehring–Lehto Theorem (see [6], Theorem 2).

If we adopt the following notations:
\[
f(x, y) = (u(x, y), v(x, y)) \quad \text{for} \quad (x, y) \in \Omega,
\]
\[
f^{-1}(u, v) = (x(u, v), y(u, v)) \quad \text{for} \quad (u, v) \in \Omega',
\]
then the bi-Sobolev condition for \( f \) and \( f^{-1} \) can be precisely expressed by
\[
(u_x, u_y, v_x, v_y) \in L_{loc}^1(\Omega)
\]
and
\[
(x_u, x_v, y_u, y_v) \in L_{loc}^1(\Omega').
\]

The following result derives from [3],[9] and [13].

**Theorem 1.1.** If \( f: \Omega \onto \Omega' \) is a bi-Sobolev map, then
\[
\int_{\Omega} |Df| \, dz = \int_{\Omega'} |Df^{-1}| \, dw.
\]

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If $f$ is a.e. differentiable homeomorphism, then the Jacobian determinant $J_f$ satisfies either the inequality $J_f \geq 0$ or $J_f \leq 0$ a.e. ([2], [12]). For simplicity let us assume $J_f(z) \geq 0$ for a.e. $z \in \Omega$.

Let us point out that if the Jacobians $J_f$ and $J_{f^{-1}}$ of $f^{-1}$ are strictly positive a.e., it is possible to prove (1.3) by mean of the area formula (see Sections 2 and 3). On the other hand, bi-Sobolev mappings do not enjoy such a property; it may happen that their Jacobian vanishes on a set of positive measure ([19], [20], [14]).

The bi-Sobolev assumption rules out the Lipschitz homeomorphism

$$f_0: (0, 2) \times (0, 1) \rightarrow (0, 1) \times (0, 1), \quad f_0(x, y) = (h(x), y),$$

where $h^{-1}(t) = t + c(t)$ and $c: (0, 1) \rightarrow (0, 1)$ is the usual Cantor ternary function because $f_0^{-1}$ does not belong to $W_{1,1}^{1,1}$. On the contrary, our first results deal with $W_{1,1}^{1,1}$-homeomorphisms which include $f_0$ as well (Theorem 1.3). Another interesting property of a bi-Sobolev map $f = (u, v)$ in the plane is that $u$ and $v$ have the same critical points ([13], [17]).

**Theorem 1.2.** Let $f: \Omega \rightarrow \Omega'$ be a bi-Sobolev map $f = (u, v)$. Then $u$ and $v$ have the same critical points:

$$\{z \in \Omega: |\nabla u(z)| = 0\} = \{z \in \Omega: |\nabla v(z)| = 0\} \quad \text{a.e.}$$

The same result holds also for the inverse $f^{-1}$. The analogue of this Theorem is not valid in more than two dimensions (see [13]).

Let us point out that we only assume that $f$ and $f^{-1}$ are in $W_{1,1}^{1,1}$. In the category of $W_{1,1}^{1,p}$-bi-Sobolev maps, that is, $f$ belongs to $W_{1,1}^{1,p}(\Omega; \mathbb{R}^2)$ and $f^{-1}$ belongs to $W_{1,1}^{1,p}(\Omega'; \mathbb{R}^2)$, the case $1 \leq p < 2$ (see [20]) is critical with respect to the so-called N property of Lusin, i.e., that a function maps every set of measure zero to a set of measure zero. Let us mention that for $W_{1,2}^{1,2}$-bi-Sobolev mappings the statement of Theorem 1.2 is obviously satisfied. In fact (see [16], p. 150), for homeomorphisms in $W_{1,2}^{1,2}$ we have the N property. Clearly,

$$\{z \in \Omega: |\nabla u(z)| = 0\} \subset \{z \in \Omega: J_f(z) = 0\} \quad \text{a.e.}$$

We can decompose the set $\{J_f = 0\}$ into a null set $Z$ and countably many sets on which we can use the Sard’s Lemma (see [4], Theorem 3.1.8). It follows that

$$|f(\{J_f = 0\} \setminus Z)| = 0$$

and hence

$$|f(\{\nabla u = 0\} \setminus Z)| = 0.$$

Since $f^{-1}$ satisfies the N property, we obtain $|\{\nabla u = 0\}| = 0$ and analogously $|\{\nabla v = 0\}| = 0$ as well.

We observe that the following identity

$$\left\{z \in \Omega: \frac{\partial f}{\partial x}(z) = 0\right\} = \left\{z \in \Omega: \frac{\partial f}{\partial y}(z) = 0\right\} \quad \text{a.e.}$$

where $|\frac{\partial f}{\partial x}(z)|^2 = u_x^2(z) + v_x^2(z)$ and $|\frac{\partial f}{\partial y}(z)|^2 = u_y^2(z) + v_y^2(z)$, is true for bi-Sobolev maps and parallels (1.5). This is a consequence of the following characteristic property of a bi-Sobolev map which was proved in [3], [13], [9]:

$$J_f(z) = 0 \implies |Df(z)| = 0 \quad \text{a.e.}$$

Our first result is the following, in which we give some identities for $W_{1,1}^{1,1}$-homeomorphism. Notice that the symbol $|\nabla \varphi|(\Omega')$ denotes the total variation of
the real function $\varphi$ belonging to the space $\mathrm{BV}(\Omega')$ of functions of bounded variation on $\Omega'$ (see Section 2).

**Theorem 1.3.** Let $f = (u, v): \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ be a homeomorphism whose inverse is $f^{-1} = (x, y)$. If we assume $u, v \in W^{1,1}_{\text{loc}}(\Omega)$, then $x, y \in \mathrm{BV}_{\text{loc}}(\Omega')$ and

\begin{align}
|\nabla y|(\Omega') &= \int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz, \\
|\nabla x|(\Omega') &= \int_{\Omega} \left| \frac{\partial f}{\partial y}(z) \right| \, dz.
\end{align}

In [11] it was proved that if $f: \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ has bounded variation, $f \in \mathrm{BV}_{\text{loc}}(\Omega; \mathbb{R}^2)$, then $f^{-1} \in \mathrm{BV}_{\text{loc}}(\Omega'; \mathbb{R}^2)$ and both $f$ and $f^{-1}$ are differentiable a.e. We notice that our identities (1.7) and (1.8) represent an improvement of such a result when $f$ is $W^{1,1}$-homeomorphism; in particular the following estimate

\[ |Df^{-1}|(\Omega') \leq 2 \int_{\Omega} |Df| \, dz \]

holds (Corollary 3.4). A $W^{1,p}_{\text{loc}}$-homeomorphism in the plane, $1 \leq p < 2$ whose Jacobian vanishes a.e., has been recently constructed by Hencl [8]; such a mapping satisfies the assumptions of Theorem 1.3. If in Theorem 1.3 we add the hypothesis $J_f > 0$ a.e., we obtain the identities (1.7) and (1.8) using the area formula (see Sections 2 and 3).

Condition (1.6) makes it possible, for a given bi-Sobolev mapping $f$, to consider the distortion quotient

\[ \frac{|Df(z)|^2}{J_f(z)} \quad \text{for a.e. } z \in \Omega. \]

Hereafter the undetermined ratio $0/0$ is understood to be equal to 1 for $z$ in the zero set of the Jacobian. The Borel function

\[ K_f(z) := \begin{cases} 
\frac{|Df(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\
1 & \text{otherwise,}
\end{cases} \]

is the distortion function of $f$ and has relevant properties: it is the smallest function $K(z)$ greater or equal to 1 for which the distortion inequality:

\[ |Df(z)|^2 \leq K(z)J_f(z) \quad \text{a.e. } z \in \Omega \]

holds true. Moreover, there are interesting interplay between the integrability of the distortions $K_f$ and $K_{f^{-1}}$, and the regularity of $f$ and $f^{-1}$ (see [13], Theorem 5).

In our general context of $W^{1,1}$-homeomorphisms there are different distortion functions which play a significant role (see Section 4). We obtain conditions under which one of these functions is finite a.e. or integrable.

2. Preliminaries

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^2$. We say that two sets $A, B \subset \mathbb{R}^2$ satisfy $A = B$ a.e. if their symmetrical difference has measure zero, i.e.,

\[ |(A \setminus B) \cup (B \setminus A)| = 0. \]
A homeomorphic mapping $f : \Omega \subset \mathbb{R}^2 \rightarrow \Omega' \subset \mathbb{R}^2$ is said to satisfy the N property of Lusin on the domain $\Omega$ if for every $A \subset \Omega$ such that $|A| = 0$ we have $|f(A)| = 0$.

A function $u \in \mathscr{L}^1(\Omega)$ is of bounded variation, $u \in BV(\Omega)$ if the distributional partial derivatives of $u$ are measures with finite total variation in $\Omega$: there exist Radon signed measures $D_1 u$, $D_2 u$ in $\Omega$ such that for $i = 1, 2$, $|D_i u|_2(\Omega) < \infty$ and

$$\int_{\Omega} u D_i \phi(z) \, dz = - \int_{\Omega} \phi(z) \, dD_i u(z) \quad \forall \phi \in C^0_0(\Omega).$$

The gradient of $u$ is then a vector-valued measure with finite total variation

$$|\nabla u|(\Omega) = \sup \left\{ \int_{\Omega} u \, \text{div} \phi(z) \, dz : \phi \in C^0_0(\Omega, \mathbb{R}^2), \|\phi\|_\infty \leq 1 \right\} < \infty.$$

By $|\nabla u|$ we denote the total variation of the signed measure $Du$.

The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$; indeed for any $u \in W^{1,1}(\Omega)$ the total variation is given by $\int_{\Omega} |\nabla u| = |\nabla u|(\Omega)$. We say that $f = (u, v) \in \mathscr{L}^1(\Omega; \mathbb{R}^2)$ belongs to $BV(\Omega; \mathbb{R}^2)$ if $u, v \in BV(\Omega)$. Finally we say that $f \in BV_{loc}(\Omega; \mathbb{R}^2)$ if $f \in BV(A; \mathbb{R}^2)$ for every open $A \subset \subset \Omega$. In the following, for $f \in BV_{loc}(\Omega; \mathbb{R}^2)$ we will denote the total variation of $f$ by:

$$|Df|(\Omega) = \sup \left\{ \int_{\Omega} u \, \text{div} \phi_1(z) \, dz + \int_{\Omega} v \, \text{div} \phi_2(z) \, dz : \phi_i \in C^0_0(\Omega, \mathbb{R}^2), \|\phi_i\|_\infty \leq 1, i = 1, 2 \right\}.$$

We will need the definition of sets of finite perimeter (see [1]).

**Definition 2.1.** Let $E$ be a Lebesgue measurable subset of $\mathbb{R}^2$. For any open set $\Omega \subset \mathbb{R}^2$ the perimeter of $E$ in $\Omega$, denoted by $P(E, \Omega)$, is the total variation of $\chi_E$ in $\Omega$, i.e.,

$$P(E, \Omega) = \sup \left\{ \int_E \text{div} \varphi \, dz : \varphi \in C^0_0(\Omega, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \right\}.$$

We say that $E$ is a set of finite perimeter in $\Omega$ if $P(E, \Omega) < \infty$.

We say that $f = (u, v) \in W^{1,p}_{loc}(\Omega; \mathbb{R}^2)$, $1 \leq p \leq \infty$, if for each open $A \subset \subset \Omega$, $f$ belongs to the Sobolev space $W^{1,p}(A; \mathbb{R}^2)$, i.e., if $u \in L^p(A)$ and $v \in L^p(A)$ have distributional derivatives in $L^p(A)$.

We are interested in the area formula for a homeomorphism $f \in W^{1,1}_{loc}(\Omega; \mathbb{R}^2)$ with $\Omega \subset \mathbb{R}^2$. In this case we have

$$\int_{\Omega} \eta(f(z)) J_f(z) \, dz \leq \int_{\mathbb{R}^2} \eta(w) \, dw$$

for any non negative Borel function $\eta$ on $\mathbb{R}^2$. This follows from the area formula for Lipschitz mappings (see [4], Theorem 3.2.3), and from a general property of a.e. differentiable functions (see [4], Theorem 3.1.8), namely that $\Omega$ can be exhausted up to a set of measure zero by sets the restriction to which of $f$ is Lipschitz continuous.

Moreover, the area formula

$$\int_E \eta(f(z)) J_f(z) \, dz = \int_{\mathbb{R}^2} \eta(w) \, dw$$

holds on each set $E \subset \Omega$ on which the N property of Lusin is satisfied.
3. The identities for $W^{1,1}$-homeomorphisms

Before proving Theorem 1.3 in its full generality we give now a partial proof under the following additional assumptions:

\begin{align}
(3.1) \quad \{w: J_{f^{-1}}(w) = 0\} = \{w: |\nabla y(w)| = 0\} \quad \text{a.e.,}
(3.2) \quad \{z: J_f(z) = 0\} = \{z: |\partial f(\hat{z})| = 0\} \quad \text{a.e.,}
\end{align}

where $J_{f^{-1}}$ denotes the determinant of the absolutely continuous part of $Df^{-1}$; moreover, we suppose $f^{-1}$ differentiable a.e. in the classical sense. Therefore, we have

$$
\int_{\Omega'} |\nabla y(w)| \, dw = \int_{A'} |\nabla y(w)| \, dw,
$$

where $A'$ is a Borel subset of the set $E'$ where $f^{-1}$ is differentiable with $J_{f^{-1}} > 0$ such that $|A'| = |E'|$.

Applying (2.1), (3.1) and basic linear algebra, we arrive at:

$$
\int_{A'} |\nabla y(w)| \, dw = \int_{A'} \frac{|\nabla y(w)|}{J_{f^{-1}}(w)} J_{f^{-1}}(w) \, dw \leq \int_{f^{-1}(A')} \frac{|\nabla y(f(z))|}{J_{f^{-1}}(f(z))} \, dz
\leq \int_{f^{-1}(A')} \left| \frac{\partial f}{\partial x}(z) \right| \, dz \leq \int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz.
$$

Here we are using the identity $D \text{adj} D = I \det D$ and the fact that $J_f(z)J_{f^{-1}}(f(z)) = 1$ at the points of differentiability with nonzero Jacobian. We have used as well the expression of the inverse matrix to the differential $2 \times 2$ matrix $Df = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ in terms of $Df^{-1} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$, namely

$$
y_u(f(z)) = -v_x(z)J_{f^{-1}}(f(z)), \quad y_v(f(z)) = u_x(z)J_{f^{-1}}(f(z)) \quad \forall z \in f^{-1}(A'),
$$

and the identity

$$
|\nabla y(f(z))|^2 = [v_x(z)^2 + u_x(z)^2] J_{f^{-1}}(f(z))^2 \quad \forall z \in f^{-1}(A').
$$

The opposite inequality follows by a symmetric procedure which relies on (3.2).

Notice that (3.1) and (3.2) are certainly satisfied if $J_f > 0$ a.e. and $J_{f^{-1}} > 0$ a.e. We observe that Theorem 1.1 can be proved using the same technique under the additional assumptions that $J_f > 0$ and $J_{f^{-1}} > 0$ a.e. In the general case the proof of Theorem 1.3 is completely different; to prove the Theorem we need some preliminary results. The next Lemma is known as Coarea Formula (see [1], Theorem 3.40):

**Lemma 3.1.** For any open set $\Omega' \subset \mathbb{R}^2$ and $y \in L^1_{\text{loc}}(\Omega')$ we have

\begin{align}
(3.3) \quad |\nabla y|(\Omega') = \int_{-\infty}^{+\infty} P\left(\{w \in \Omega': y(w) > t\}, \Omega'\right) dt.
\end{align}

We understand the left-hand side of (3.3) to be infinity if $y \notin \text{BV}$.

The following Lemma is the main step towards the equality in the area formula (see Theorem 1.3 of [3] and also [13], where the case $f$ ACL, i.e., absolutely continuous on lines, is treated).
Lemma 3.2. Let \( f \in \mathcal{W}^{1,1}(\Omega \subset (-1,1)^2; \mathbb{R}^2) \) be a homeomorphism. Then for almost every \( t \in (-1,1) \) the mapping \( f_t \) satisfies the N property of Lusin, i.e., for every \( A \subset (-1,1) \times \{t\} \), \( \mathcal{H}^1(A) = 0 \) implies \( \mathcal{H}^1(f(A)) = 0 \).

Proof of Theorem 1.3. Without loss of generality we take \( \Omega = (-1,1) \times (-1,1) \). Let us apply Lemma 3.2 to the homeomorphism \( f \). Then, the mapping

\[
 f (\cdot, t) : x \in (-1,1) \mapsto (u(x,t), v(x,t)) \in \Omega'
\]

belongs to \( \mathcal{W}^{1,1}((-1,1), \mathbb{R}^2) \) for a.e. \( t \) and satisfies the N property. In particular, the area formula holds for \( f (\cdot, t) \) on \((-1,1)\):

\[
\int_{-1}^{1} \left| \frac{\partial f}{\partial x}(x,t) \right| \, dx = \mathcal{H}^1(f ((-1,1) \times \{t\})) .
\]

Integrating with respect to \( t \) we obtain:

\[
\int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz = \int_{-1}^{1} \mathcal{H}^1(f ((-1,1) \times \{t\})) \, dt .
\]

Since it is clear that

\[
f ((-1,1) \times \{t\}) = \{ w \in \Omega' : y(w) = t \},
\]

then

\[
\int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz = \int_{-1}^{1} \mathcal{H}^1(\{ w \in \Omega' : y(w) = t \}) \, dt .
\]

As \( y \) is continuous, then the set \( \{ w \in \Omega' : y(w) = t \} \) is the boundary of the level set \( \{ w \in \Omega' : y(w) > t \} \). By assumptions we know that for a.e. \( t \), \( \mathcal{H}^1(\{ w \in \Omega' : y(w) = t \}) < \infty \) and from [1] (p. 209) we have

\[
\mathcal{H}^1(\{ w \in \Omega' : y(w) = t \}) = P(\{ w \in \Omega' : y(w) > t \}, \Omega') \quad \text{a.e.} \ t \in (-1,1).
\]

Using Coarea Formula from Lemma 3.1, we obtain

\[
|\nabla y|(\Omega') = \int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz
\]

and we deduce that \( y \in \text{BV}_{\text{loc}}(\Omega') \).

The equality (1.8) is proved using the same technique. \( \square \)

Remark 3.3. From the above proof it is clear that if \( f \) is a homeomorphism in \( \text{BV}_{\text{loc}}(\Omega; \mathbb{R}^2) \) such that \( \frac{\partial f}{\partial x} \in L^1(\Omega; \mathbb{R}^2) \), then (1.7) holds true.

Since the total variation of a map is less or equal than the sum of total variation of the components, by Theorem 1.3 we immediately get

Corollary 3.4. Let \( f = (u,v) : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2 \) be a homeomorphism whose inverse is \( f^{-1} = (x,y) \). If we assume \( u, v \in \mathcal{W}^{1,1}_{\text{loc}}(\Omega) \), then

\[
|Df^{-1}|(\Omega') \leq 2 \int_{\Omega} |Df| .
\]
4. The distortions of anisotropic Sobolev maps

In Section 1 we have already mentioned the known fact that, if $f: \Omega \subset \mathbb{R}^2 \rightarrow \Omega' \subset \mathbb{R}^2$ is bi-Sobolev, then we have

$$\{z : J_f(z) = 0\} = \{z : |Df(z)| = 0\} \quad \text{a.e.}$$

and this makes it possible to consider the distortion function

$$K_f(z) := \begin{cases} \frac{|Df(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, the distortion inequality

$$|Df(z)|^2 \leq K_f(z)J_f(z)$$

holds for a.e. $z \in \Omega$. According to a well established terminology, we say that $f$ has finite distortion $K_f$.

For a Sobolev homeomorphism, under suitable assumptions, it is possible to introduce different distortion functions (see [21]). Namely, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \{z : |\nabla u(z)| = 0\} \quad \text{a.e.,}$$

then we are allowed to define the Borel function

$$K_f^{(1)}(z) := \begin{cases} \frac{|\nabla u(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \{z : |\nabla v(z)| = 0\} \quad \text{a.e.,}$$

then the Borel function

$$K_f^{(2)}(z) := \begin{cases} \frac{|\nabla v(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise,} \end{cases}$$

is well defined. On the other hand, if $f = (u, v)$ satisfies the condition

$$\{z : J_f(z) = 0\} = \left\{z : \left|\frac{\partial f}{\partial x}(z)\right| = 0\right\} \quad \text{a.e.,}$$

then we can define the Borel function

$$H_f^{(1)}(z) := \begin{cases} \frac{|\frac{\partial f}{\partial x}(z)|^2}{J_f(z)} & \text{if } J_f(z) > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, for $f$ satisfying

$$\{z : J_f(z) = 0\} = \left\{z : \left|\frac{\partial f}{\partial y}(z)\right| = 0\right\} \quad \text{a.e.}$$
we define

\[
H_f^2(z) := \begin{cases} 
\left| \frac{\partial f}{\partial y}(z) \right|^2 / J_f(z) & \text{if } J_f(z) > 0, \\
1 & \text{otherwise}.
\end{cases}
\]  

In the following, given a \( W^{1,1} \)-homeomorphism \( f \), we establish conditions which guarantee that one of its distortions is finite a.e. or \( L^1 \). Let us begin with the following

**Theorem 4.1.** Let \( f = (u, v) : \Omega \subset \mathbb{R}^2 \rightarrow \Omega' \subset \mathbb{R}^2 \) be a \( W^{1,1} \)-homeomorphism whose inverse is \( f^{-1} = (x, y) \). If \( x \in W^{1,1}_{\text{loc}}(\Omega') \) and \( v_y \neq 0 \) on a positive measure set \( P \subset \Omega \), then

\[
\{ z \in P : J_f(z) = 0 \} = \{ z \in P : \left| \frac{\partial f}{\partial y}(z) \right| = 0 \} \quad \text{a.e.}
\]

and the distortion \( H_f^2(z) \) is finite a.e. Moreover, we have the following identities

\[
\int_{\Omega'} |\nabla x(w)| \, dw = \int_{\Omega} \left| \frac{\partial f}{\partial y}(z) \right| \, dz \\
\int_{\Omega'} |\nabla y(w)| \, dz = \int_{\Omega} \left| \frac{\partial f}{\partial x}(z) \right| \, dz.
\]  

**Proof.** By contradiction we suppose that there exists a set \( A \subset P \) with positive Lebesgue measure such that \( f \) is differentiable in \( A \) and \( J_f(z) = 0 \) and \( \left| \frac{\partial f}{\partial y}(z) \right| > 0 \ \forall z \in A \). We can assume that \( f \) is Lipschitz on \( A \) and use the area formula (2.2) to get

\[
|f(A)| = 0 \quad \text{since} \quad \int_A J_f(z) \, dz = 0.
\]

We denote by

\[
p_2 : (x_1, x_2) \in \mathbb{R}^2 \rightarrow H_2 = \{ x \in \mathbb{R}^2 : x_2 = 0 \}
\]

the orthogonal projection and by

\[
p_2^{(2)} : (x_1, x_2) \in \mathbb{R}^2 \rightarrow x_2 \in \mathbb{R}
\]

the second coordinate function.

We observe that

\[
\{ \omega \in \Omega' : x(\omega) = t \} = (p_2 \circ f^{-1})^{-1} \{ (t, 0) \} \quad \forall t \in \mathbb{R}.
\]

By assumptions we know that

\[
\mathcal{H}^1 (\{ w \in f(A) : x(w) = t \}) < \infty
\]

and from [1] (p. 209)

\[
\mathcal{H}^1 (\{ w \in f(A) : x(\omega) = t \}) = P (\{ w \in f(A) : x(w) > t \}, \Omega').
\]

By Lemma 3.1 and the assumption that \( x \) belongs to \( W^{1,1}_{\text{loc}}(\Omega') \), we have

\[
\int_\mathbb{R} \mathcal{H}^1 (\{ w \in f(A) : x(w) = t \}) = \int_{f(A)} |\nabla x(\omega)| \, dw = 0.
\]
Thus the curve \( \{ w \in f(A) : x(w) = t \} \) has zero one dimensional measure for a.e. \( t \in \mathbb{R} \) and in particular its second projection to the axis have zero one-dimensional measure as well:

\[
\mathcal{H}^1 \left( \rho^{(2)} \left( \{ w \in f(A) : x(w) = t \} \right) \right) = 0 \quad \text{a.e. } t \in \mathbb{R}.
\]

On the other hand, using Fubini Theorem, we have

\[
|A| = \int_{\mathbb{R}} |A \cap p_2^{-1}\{ (t, 0) \}| dt > 0.
\]

Hence, there exists \( t_0 \in \mathbb{R} \) such that

\[
\mathcal{H}^1 \left( A \cap p_2^{-1}\{ (t_0, 0) \} \right) > 0.
\]

Applying the area formula to the differentiable function \( v(t_0, \cdot) : \tau \in \rho^{(2)}(A) \rightarrow v(t_0, \tau) \), we have

\[
0 < \int_{A \cap p_2^{-1}(t_0)} |v_y(t_0, \tau)| d\mathcal{H}^1(\tau) \leq \int_{\mathbb{R}} N(v, A \cap p_2^{-1}(t_0), \sigma) d\sigma
\]

\[
= \int_{\rho^{(2)}(f(A) \cap p_2^{-1}(t_0))} N(v, A \cap p_2^{-1}(t_0), \sigma) d\sigma,
\]

where \( N(v, A \cap p_2^{-1}(t_0), \sigma) \) is the number of preimages of \( \sigma \) under \( v \) in \( A \cap p_2^{-1}(t_0) \). The last integral is zero by (4.9) and this is a contradiction. \( \Box \)

The following result shows that if the distortion \( K_f^{(2)} \) is \( \mathcal{L}^1 \), then \( f^{-1} \) has better Sobolev regularity.

**Theorem 4.2.** Let \( f = (u, v) : \Omega \subset \mathbb{R}^2 \rightarrow \Omega' \subset \mathbb{R}^2 \) be a \( \mathcal{W}^{1,1} \)-homeomorphism and denote by \( f^{-1} = (x, y) \) its inverse. If we assume

\[
\{ w \in \Omega' : J_{f^{-1}}(w) = 0 \} = \left\{ w \in \Omega' : \left| \frac{\partial f^{-1}}{\partial u} (w) \right| = 0 \right\},
\]

\[
\{ z \in \Omega : J_f(z) = 0 \} = \{ z \in \Omega : |\nabla v(z)| = 0 \}
\]

and \( K_f^{(2)} \in \mathcal{L}^1 \), then

\[
\left| \frac{\partial f^{-1}}{\partial u} \right| \in \mathcal{L}^2(\Omega)
\]

and

\[
\int_{\Omega'} \left| \frac{\partial f^{-1}}{\partial u} (w) \right|^2 dw \leq \int_{\Omega'} K_f^{(2)}(z) dz.
\]

**Proof.** Let \( A' \) be the Borel subset of the set \( E' \) where \( f^{-1} \) is differentiable with \( J_{f^{-1}} > 0 \), such that \( |A'| = |E'| \). Applying the area formula, we obtain

\[
\int_{\Omega'} \left| \frac{\partial f^{-1}}{\partial u} (w) \right|^2 dw = \int_{A'} \left| \frac{\partial f^{-1}}{\partial u} (w) \right|^2 dw = \int_{A'} \frac{\left| \frac{\partial f^{-1}}{\partial u} (w) \right|^2}{J_{f^{-1}}(w)} J_{f^{-1}}(w) dw

\leq \int_{f^{-1}(A')} \frac{1}{J_{f^{-1}}(f(z))} \left| \frac{\partial f^{-1}}{\partial u} (f(z)) \right|^2 dz = \int_{f^{-1}(A')} \frac{1}{J_{f^{-1}}(z)} |\nabla v(z)|^2 dz
\]
\[ \int_{f^{-1}(A')} \frac{\sqrt{\nabla v(z)^2}}{f'(z)} \, dz \leq \int_{\Omega'} K_f^{(2)}(z) \, dz. \]

\[ \square \]

References


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