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UNBOUNDED BILIPSCHITZ HOMOGENEOUS JORDAN CURVES

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Abstract. We prove that an unbounded L-bilipschitz homogeneous Jordan curve in the plane is of B-bounded turning, where B depends only on L. Using this result, we construct a catalogue of snowflake-type curves that includes all unbounded bilipschitz homogeneous Jordan curves, up to bilipschitz self maps of the plane. This catalogue yields characterizations of such curves in terms of certain quasiconformal maps.

1. Introduction

A Jordan curve Γ is *L-bilipschitz homogeneous* provided that for any two points $x, y \in \Gamma$, there exists an L-bilipschitz self homeomorphism of Γ sending x to y. Bishop proved ([Bis01, Theorem 1.1]) that bounded bilipschitz homogeneous Jordan curves in the plane are quasicircles. In this paper we extend this result to unbounded curves. Moreover, in this setting we obtain a quantitative implication, whereas [Bis01, Theorem 1.1] did not yield quantitative control of the bounded turning constant. The proof is carried out in Section 4.

Theorem 1.1. *Suppose $\Gamma \subset \mathsf{R}^2$ is an unbounded L-bilipschitz homogeneous Jordan curve. Then Γ is B-bounded turning, with $B = B(L)$.*

Rohde constructed a catalogue $\mathscr{S}$ of all compact quasicircles in the plane, up to bilipschitz self maps of the plane ([Roh01, Theorem 1.1]). The catalogue $\mathscr{S}$ contains a subcatalogue $\mathscr{HS}$ of all bounded bilipschitz homogeneous Jordan curves. We construct a new catalogue, $\mathscr{HT}$, containing all unbounded bilipschitz homogeneous Jordan curves, up to bilipschitz maps. The following is proved in Section 6.

Theorem 1.2. *Let $\Gamma \subset \mathsf{R}^2$ be an unbounded Jordan curve. The following are quantitatively equivalent:*

(1) *Γ is bilipschitz homogeneous.*
(2) *Γ is bilipschitz equivalent to a curve $T \in \mathscr{HT}$.*
(3) *Γ is bilipschitz equivalent to a curve $T \in \mathscr{HT}$ via a bilipschitz self map of R^2.*
(4) *There exists a quasiconformal homeomorphism $F\colon \mathsf{R}^2 \to \mathsf{R}^2$ with $F(\mathsf{R}) = \Gamma$ and $\alpha \in [0, 1)$ such that for almost all $z, w \in \mathsf{R}^2 \setminus \mathsf{R}$,*

$$\operatorname{dist}(z, \mathsf{R}) \geq \operatorname{dist}(w, \mathsf{R}) > 0 \quad \Longrightarrow \quad 1 \lesssim \frac{JF(w)}{JF(z)} \lesssim \left(\frac{\operatorname{dist}(z, \mathsf{R})}{\operatorname{dist}(w, \mathsf{R})}\right)^{\alpha}.$$

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(5) *There exists a quasiconformal homeomorphism* $F\colon \mathsf{R}^2 \to \mathsf{R}^2$ *with* $F(\mathsf{R}) = \Gamma$ *such that for almost every* $a + ib = z \in \mathsf{R}^2 \setminus \mathsf{R}$, $JF(z) \simeq JF(ib)$

Finally, in Section 7, we provide a characterization of canonical dimension gauges for unbounded bilipschitz homogeneous Jordan curves. This parallels a similar characterization for bounded curves ([Roh01, Corollary 1.5]).

Theorem 1.3. *Suppose that* $\delta\colon \mathsf{R}_+ \to \mathsf{R}_+$ *is a dimension gauge, normalized so that* $\delta(1) = 1$. *Then* δ *is comparable to the canonical dimension gauge for some unbounded bilipschitz homogeneous Jordan curve* $\Gamma \subset \mathsf{R}^2$ *if and only if there exists* $\alpha \in [1, 2)$ *such that for all* $0 < r \leq s < +\infty$,

$$\frac{s}{r} \lesssim \frac{\delta(s)}{\delta(r)} \lesssim \left(\frac{s}{r}\right)^{\alpha}.$$

The constants depend only on one another.

2. Preliminaries

We denote the real line by R, the positive reals by R_+, and the Euclidean plane by R^2. The upper half plane is H, the unit circle is S, and the (open) unit disk is D. Given $x \in \mathsf{R}^2$ and $r > 0$, $D(x; r)$ represents the open disk of radius r centered at x. Then $\overline{D}(x; r)$ indicates the closure of $D(x; r)$ in R^2, and $C(x; r) := \partial D(x; r)$. Given a set E, we write $U(E; r)$ to denote the open set $\bigcup_{x\in E} D(x; r)$. For an open set $\Omega \subset \mathsf{R}^2$ and $z \in \Omega$, define $d_\Omega(z) := \operatorname{dist}(z, \partial\Omega)$.

For a point $z \in \mathsf{H}$, write I_z to denote the subarc of R cut off by the circle orthogonal to R whose intersection with H has Euclidean midpoint z.

Given two positive numbers A and B, we write $A \simeq B$ to indicate the existence of some $C \in [1, +\infty)$ such that $C^{-1}B \leq A \leq C\,B$. Here we require that C is independent of A and B. We write $A \lesssim B$ to indicate $A \leq C\,B$. When C is determined by numbers $N, M, \dots$, we write $C = C(N, M, \dots)$. We say that two conditions are *quantitatively equivalent* if the constants for each condition are determined solely by the constants for the other.

Given a set $E \subset \mathsf{R}^2$ and a scale $r > 0$, we define a *covering number* for E as

$$N(r; E) := \inf\{n \in \mathsf{N}\colon \exists\, \{x_i\}_{i=1}^k \subset \mathsf{R}^n \text{ such that } E \subset \cup_{i=1}^k D(x_i; r)\}.$$

Note that R^2 is *D-doubling*: there exists a constant $D \in [1, +\infty)$ such that, given any $x \in \mathsf{R}^2$ and $r > 0$, we have $N(r; D(x; 2r)) \leq D$. Using this doubling condition, when E is compact and $A \in [1, +\infty)$, one can verify that

$$N(Ar; E) \leq N(r; E) \leq DA^{\log_2(D)} N(Ar; E).$$

For $r > 0$, a set S is *r-separated* if for every pair of distinct points x, y in S, $|x - y| \geq r$. Given a set $E \subset \mathsf{R}^2$ and $r > 0$, we define the *packing number* $P(r; E)$ as the supremal cardinality of r-separated sets in E. We say that a set E is *(H, α)-homogeneous* provided that for every $x \in E$ and all $0 < r \leq s < \operatorname{diam}(E)$, we have $P(r; D(x; s)) \leq H(s/r)^\alpha$. It is well known that this condition is equivalent to the doubling condition defined above. We say that a set $E \subset \mathsf{R}^2$ is *P-porous* provided that for every $x \in \mathsf{R}^2$ and $r > 0$, there exists a point $y \in D(x; r)$ such that $\operatorname{dist}(y, E) \geq r/P$. In R^2, (H, α)-homogeneity (for $\alpha < 2$) is quantitatively equivalent to P-porosity ([Luu98, Theorem 5.2]).

A *Jordan curve* is a proper homeomorphic image of either the unit circle or real line. By the term *proper* we mean that closed and bounded sets are compact. This definition rules out non-complete homeomorphic images of the real line such as the open unit interval (such curves are not bilipschitz homogeneous).

Let Γ denote a Jordan curve. Given two points $x, y \in \Gamma$, we write $\Gamma(x, y)$ to denote the smallest (with respect to diameter) component of $\Gamma \setminus \{x, y\}$. If both components have the same diameter (as may occur for bounded curves), choose one. If we desire to include the points x and y, we write $\Gamma[x, y]$. We say that Γ is of *B-bounded turning* provided that for all $x, y \in \Gamma$ we have $\operatorname{diam}(\Gamma[x, y]) \le B|x - y|$. The curve Γ is of *ε-local B-bounded turning* provided that for all $x, y \in \Gamma$, $|x - y| \le \varepsilon \Rightarrow \operatorname{diam}(\Gamma[x, y]) \le B|x - y|$.

When studying bilipschitz homogeneous (thus bounded turning) Jordan curves $\Gamma \subset \mathsf{R}^2$, a canonical class of dimension gauges presents itself. Suppose first that Γ is bounded. Then for $t > 0$ define $\delta_\Gamma(t) := N(t; \Gamma)^{-1}$. Suppose now that Γ is unbounded. Choose a basepoint x_0 and an orientation on Γ. Given $t > 0$, we move in the positive direction along Γ until we reach the first point x_t such that $|x_t - x_0| = t$. Writing $\Gamma_t := \Gamma[x_0, x_t]$, we define

$$\delta_\Gamma(t) := \begin{cases} N(t; \Gamma_1)^{-1} & \text{if } t \le 1, \\ N(1; \Gamma_t) & \text{if } t \ge 1. \end{cases}$$

Clearly, this definition depends on the choice of x_0. However, due to [HM99, Fact 3.2(a)], different choices of x_0 change δ_Γ only up to a multiplicative constant. Moreover, this constant depends only on the bilipschitz homogeneity and bounded turning constants. We have the following, which is the necessity of Theorem 1.3 (cf. [FH, Proposition 3.20]).

Fact 2.1. *Let $\Gamma \subset \mathsf{R}^2$ be an L-bilipschitz homogeneous B-bounded turning Jordan curve with canonical dimension gauge $\delta := \delta_\Gamma$. Then there exist constants $D \in [1, +\infty)$ and $\alpha \in [1, 2)$ depending only on B and L, such that for all $0 < r \le s < \operatorname{diam}(\Gamma)$ we have*

$$D^{-1}\frac{s}{r} \le \frac{\delta(s)}{\delta(r)} \le D\left(\frac{s}{r}\right)^{\alpha}. \tag{2.1}$$

It is well known (cf. [Ric66]) that bounded turning Jordan curves in R^2 are precisely the images of R or S under quasisymmetric maps. Given a homeomorphism $\eta\colon \mathsf{R}_+ \to \mathsf{R}_+$, recall that an embedding $f\colon X \to Y$ is *η-quasisymmetric* if, for all $x, y, z \in X$, we have

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x - y|}{|x - z|}\right).$$

When studying bilipschitz homogeneous Jordan curves, it is useful to consider *quasihomogeneous maps.* Again we let $\eta\colon \mathsf{R}_+ \to \mathsf{R}_+$ be a homeomorphism. An embedding $f\colon X \to Y$ is η-quasihomogeneous if, for all $x, y, z, w \in X$, we have

$$\frac{|f(x) - f(y)|}{|f(z) - f(w)|} \le \eta\left(\frac{|x - y|}{|z - w|}\right).$$

In some cases, this condition is equivalent to *weak* or *very weak quasihomogeneity* (see [HM99, p. 775]). Weak quasihomogeneity is defined by the existance of a constant

$H \in [1, +\infty)$ such that $|x - y| \le |z - w| \Rightarrow |f(x) - f(y)| \le H|f(z) - f(w)|$. Very weak quasihomogeneity is defined similarly, for $|x - y| = |z - w|$.

3. A qualitative result

Here we prove that unbounded bilipschitz homogeneous Jordan curves are of bounded turning, qualitatively. In Section 4 we use this result to 'bootstrap' our way towards a quantitative implication. See [FH, Lemma 2.2 and Proposition 3.2] for proofs of the following.

Fact 3.1. *Suppose* Γ *is a Jordan curve with homeomorphic parametrization* $\gamma\colon \mathsf{R} \to \Gamma$. *Then* $\lim_{t\to\pm\infty} |\gamma(t) - \gamma(0)| = +\infty$.

Fact 3.2. *Suppose* Γ *is a Jordan curve that is* L*-bilipschitz homogeneous with respect to orientation preserving maps, but is not of bounded turning. Then there exists some* $C = C(L)$ *such that for any* $R > 30L^2$, *there exists* $x \in \Gamma$, $s \in (0, 1/R)$, *and* $r > 0$ *such that* $D(x; r) \cap \Gamma$ *contains an* rs/C *separated set of cardinality greater than* $1/(Cs^2)$. *We can take* $C = 60L^3$.

The requirement of orientation preserving maps in Fact 3.2 is not significant (see [GH99, Lemma 2.5]). The next two lemmas provide useful technical information.

Lemma 3.3. *Let* $\Gamma \subset \mathsf{R}^2$ *be a Jordan curve. Suppose* $x, y \in \Gamma$ *are distinct points satisfying*

$$\frac{\operatorname{diam}(\Gamma[x, y])}{|x - y|} \ge C.$$

Then there exist points $u, v \in \Gamma \cap [x, y]$ *such that* $\Gamma \cap [u, v] = \{u, v\}$ *and*

$$\operatorname{diam}(\Gamma[u, v]) \ge \left(\frac{1}{2} - \frac{1}{C}\right) \operatorname{diam}(\Gamma[x, y]).$$

Proof. First we assume that Γ is unbounded. Let Γ_x denote the component of $\Gamma \setminus \Gamma(x, y)$ with endpoint x, and Γ_y denote the component of $\Gamma \setminus \Gamma(x, y)$ with endpoint y. We order points on $[x, y]$ from x to y, and let z denote the last point in $[x, y]$ that lies in Γ_x. Then define w to be the first point in $[x, y]$ such that $w > z$ and $w \in \Gamma_y$. Note that we may have $z = x$ and $w = y$. By our choice of z and w, $\Gamma_x \cap (z, w) = \varnothing = \Gamma_y \cap (z, w)$. Define Γ_z to be the component of $\Gamma \setminus \Gamma(z, w)$ with endpoint z. Since $\Gamma_z \subset \Gamma_x$, we have $\Gamma_z \cap (z, w) = \varnothing$. Similarly, $\Gamma_w \cap (z, w) = \varnothing$.

Let a denote a point in $\Gamma[x, y]$ of maximal distance from x. Let u denote the last point in $[z, w]$ for which $u \in \Gamma[z, a]$. We may have $u = z$. Let v denote the first point in $[z, w]$ for which $v > u$ and $v \in \Gamma[a, w]$. We may have $v = w$. Then $\Gamma[z, a] \cap [z, w] \subset [z, u]$ and $\Gamma[a, w] \cap [z, w] \subset [z, u] \cup [v, w]$. Since $\Gamma[z, w] = \Gamma[z, a] \cup \Gamma[a, w]$, we have $\Gamma[z, w] \cap (u, v) = \varnothing$. Because $\Gamma = \Gamma_z \cup \Gamma[z, w] \cup \Gamma_w$ and each of these three subarcs is disjoint from (u, v), we have $\Gamma \cap (u, v) = \varnothing$.

Since $u \in \Gamma[z, a]$ and $v \in \Gamma[a, w]$, we have $a \in \Gamma[u, v]$. Therefore,

$$\begin{aligned}\operatorname{diam}(\Gamma[u, v]) &\ge |a - u| \ge |a - x| - |x - u| \\ &\ge \frac{\operatorname{diam}(\Gamma[x, y])}{2} - |x - y| \ge \left(\frac{1}{2} - \frac{1}{C}\right) \operatorname{diam}(\Gamma[x, y]).\end{aligned}$$

Now suppose that Γ is bounded. Choose $b \in \Gamma \setminus \Gamma[x, y]$ of maximal distance from x. We follow the strategy used above, with b playing the role of ∞. $\square$

Lemma 3.4. *Suppose $\Gamma \subset \mathsf{R}^2$ is an L-bilipschitz homogeneous Jordan curve of ε-local bounded turning for some $\varepsilon > 0$. Then there exist arbitrarily large disks in $\mathsf{R}^2 \setminus \Gamma$.*

Proof. If Γ is bounded or of bounded turning then the conclusion is trivial, so we assume Γ is unbounded and not of bounded turning. Then for any $n \in \mathsf{N}$, there exist points $x_n, y_n \in \Gamma$ such that $\operatorname{diam}(\Gamma[x_n, y_n]) \geq n|x_n - y_n|$. Since Γ is of ε-local bounded turning, for large enough n we have $|x_n - y_n| \geq \varepsilon$, and so $\operatorname{diam}(\Gamma[x_n, y_n]) \to +\infty$ as $n \nearrow +\infty$.

There are three main steps to the proof.

Step 1. We show that $\operatorname{diam}(\Gamma[x_n, y_n])/|x_n - y_n| \to +\infty$ implies $|x_n - y_n| \to +\infty$. By way of contradiction, suppose the sequence $(|x_n - y_n|)$ contains a bounded subsequence. For simplicity of notation, we assume that the sequence $(|x_n - y_n|)$ itself is bounded; so there exists $M \in [1, +\infty)$ such that for all n we have $|x_n - y_n| \leq M$. Map each x_n to x_0 via an L-bilipschitz self homeomorphism of Γ. This sends each y_n to some point z_n with $|x_0 - z_n| \leq L|x_n - y_n| \leq LM$. However, $\operatorname{diam}(\Gamma[x_0, z_n]) \geq \operatorname{diam}(\Gamma[x_n, y_n])/L \to +\infty$. Up to a subsequence, for every n, $\Gamma[x_0, z_n] \subset \Gamma[x_0, z_{n+1}]$. Taking another subsequence, we assume that $2\operatorname{diam}(\Gamma[x_0, z_n]) < \operatorname{diam}(\Gamma[x_0, z_{n+1}])$. Therefore,

$$2\operatorname{diam}(\Gamma[x_0, z_n]) < \operatorname{diam}(\Gamma[x_0, z_{n+1}]) \leq \operatorname{diam}(\Gamma[x_0, z_n]) + \operatorname{diam}(\Gamma[z_n, z_{n+1}]),$$

and so

$$\operatorname{diam}(\Gamma[z_n, z_{n+1}]) > \operatorname{diam}(\Gamma[x_0, z_n]) \nearrow +\infty.$$

Let $\gamma\colon \mathsf{R} \to \Gamma$ be a homeomorphism. Since (z_n) is a bounded sequence in Γ, Fact 3.1 tells us that $(\gamma^{-1}(z_n)) = (s_n)$ is a bounded sequence in R. Up to a choice of orientation for Γ, for each n we have $z_n < z_{n+1}$ along Γ. Therefore, we may assume that (s_n) is strictly increasing to some finite number s_∞. By the continuity of γ at s_∞, the subarcs $\Gamma[z_n, z_{n+1}] = \gamma([s_n, s_{n+1}])$ must have diameters tending to zero. Since $\operatorname{diam}(\Gamma[z_n, z_{n+1}])$ cannot simultaneously tend towards both 0 and $+\infty$, our supposition that $(|x_n - y_n|)$ contains a bounded subsequence has led to a contradiction. So $\operatorname{diam}(\Gamma[x_n, y_n])/|x_n - y_n| \to +\infty \Rightarrow |x_n - y_n| \to +\infty$.

Step 2. In this step, for each $n \in \mathsf{N}$, we choose a certain pair of points in Γ. For $n \in \mathsf{N}$,

$$B_n := \sup\left\{\frac{\operatorname{diam}(\Gamma[x, y])}{|x - y|} \colon |x - y| \leq n\right\},$$
$$R_n := \inf\left\{|x - y| \colon \frac{\operatorname{diam}(\Gamma[x, y])}{|x - y|} > \frac{B_n}{2}\right\}.$$

Since $\operatorname{diam}(\Gamma[x, y])/|x - y| \to +\infty$ implies that $|x - y| \to +\infty$, $B_n < +\infty$. We also note that $R_n \leq n$ (else $B_n \leq B_n/2$). Since Γ is not of bounded turning, $B_n \to +\infty$ as $n \nearrow +\infty$. This in turn implies that $R_n \to +\infty$ as $n \nearrow +\infty$, since $|x - y| \to +\infty$ if $\operatorname{diam}(\Gamma[x, y])/|x - y| \to +\infty$.

Suppose that $R_n = n$. Then for every $|x - y| < n$, we have

$$\frac{\operatorname{diam}(\Gamma[x, y])}{|x - y|} \leq \frac{B_n}{2} < B_n.$$

Therefore, $B_n = \sup\{\operatorname{diam}(\Gamma[x,y])/|x-y| \colon |x-y| = n\}$, and there exist $x, y \in \Gamma$ such that

$$\frac{\operatorname{diam}(\Gamma[x,y])}{|x-y|} > \frac{B_n}{2} \quad \text{and} \quad |x-y| = R_n = n.$$

Suppose now that $R_n < n$. Then there exists $\delta_n \in (0,1]$ and points $x, y \in \Gamma$ such that

$$\frac{\operatorname{diam}(\Gamma[x,y])}{|x-y|} > \frac{B_n}{2} \quad \text{and} \quad |x-y| \le (1+\delta_n)R_n < n.$$

In conclusion, whether $R_n = n$ or $R_n < n$ we choose $x := x_n$ and $y := y_n$ in Γ such that

$$\frac{\operatorname{diam}(\Gamma[x,y])}{|x-y|} > \frac{B_n}{2} \quad \text{and} \quad |x-y| \le \min\{2R_n, n\}.$$

By Lemma 3.3, there exist points $u, v \in \Gamma \cap [x,y]$ for which $\Gamma \cap (u,v) = \varnothing$ and

$$\operatorname{diam}(\Gamma[u,v]) \ge \left(\frac{1}{2} - \frac{2}{B_n}\right) \operatorname{diam}(\Gamma[x,y]).$$

For large enough n we have $\operatorname{diam}(\Gamma[u,v]) \ge 7\operatorname{diam}(\Gamma[x,y])/16$. Since $|u-v| \le |x-y| \le n$, we then have

$$|u-v| \ge \frac{\operatorname{diam}(\Gamma[u,v])}{B_n} \ge \frac{7\operatorname{diam}(\Gamma[x,y])}{16B_n} > \frac{7}{16B_n}\frac{B_n|x-y|}{2} = \frac{7|x-y|}{32}.$$

Step 3. In this step we find disks in the compliment of Γ. Let Γ_u denote the component of $\Gamma \setminus \Gamma(u,v)$ with endpoint u. Define Γ_v analogously. Let w denote the midpoint of $[u,v]$. Then define

$$\begin{aligned} U &:= D(u; 3|u-v|/8) \cap D(w; 3|u-v|/16), \\ V &:= D(v; 3|u-v|/8) \cap D(w; 3|u-v|/16). \end{aligned}$$

We assert that a 'large portion' of either U or V lies in the complement of Γ. The remainder of the proof is dedicated to verifying this assertion.

We first show that

$$\Gamma_u \cap V = \varnothing = \Gamma_v \cap U. \tag{3.1}$$

To see this, suppose that there exists a point $b \in \Gamma_u \cap V$. Then we would have $\Gamma[u,v] \subset \Gamma[b,v]$ and $|b-v| < 3|u-v|/8$, so

$$\frac{\operatorname{diam}(\Gamma[b,v])}{|b-v|} > \frac{8\operatorname{diam}(\Gamma[u,v])}{3|u-v|} \ge \frac{7\operatorname{diam}(\Gamma[x,y])}{6|x-y|} > \frac{B_n}{2}.$$

But we would also have

$$|b-v| < \frac{3|u-v|}{8} \le \frac{6R_n}{8} < R_n.$$

This would contradict the definitions of B_n and R_n. Similarly, $\Gamma_v \cap U = \varnothing$.

Next, we demonstrate that if $\Gamma_u \cap U \ne \varnothing$, then $\Gamma_v \cap V = \varnothing$. Similarly, if $\Gamma_v \cap V \ne \varnothing$, then $\Gamma_u \cap U = \varnothing$. To see this, suppose there exist points $a \in \Gamma_u \cap U$ and $b \in \Gamma_v \cap V$. Then

$$|a-b| \le |a-w| + |b-w| < \frac{3}{16}|u-v| + \frac{3}{16}|u-v| = 3|u-v|/8 < R_n$$

and

$$\frac{\operatorname{diam}(\Gamma[a,b])}{|a-b|} > \frac{8\operatorname{diam}(\Gamma[u,v])}{3|u-v|} \geq \frac{7\operatorname{diam}(\Gamma[x,y])}{6|x-y|} > \frac{B_n}{2}.$$

This contradicts the definitions of B_n and R_n. Thus either $\Gamma_u \cap U = \varnothing$ or $\Gamma_v \cap V = \varnothing$.

Now we note that $\Gamma_u \cup [u,v] \cup \Gamma_v$ is an unbounded Jordan curve dividing the plane into two unbounded domains, Ω_1 and Ω_2. Without loss of generality, the arc $\Gamma(u,v) \subset \Omega_1$. We also note that U is symmetric with respect to $[u,v]$, and $U \setminus [u,v]$ consists of two components U_1 and U_2. Similarly, $V \setminus [u,v] = V_1 \cup V_2$. Let U_2 and V_2 denote the components such that points of U_2 and V_2 near $[u,v]$ lie in Ω_2. Note that this does not imply that U_2 or V_2 is contained in Ω_2.

Suppose $\Gamma_u \cap U_2 = \varnothing$. By (3.1) we know that $\Gamma_v \cap U_2 = \varnothing$, which implies that $\partial\Omega_2 \cap U_2 = \varnothing$. Since $U_2 \cap \Omega_2 \neq \varnothing$ (by definition), we must have $U_2 \subset \Omega_2$. Because $\Gamma(u,v) \subset \Omega_1$, we have $\Gamma[u,v] \cap U_2 = \varnothing$. Thus we conclude that $\Gamma = \Gamma_u \cup \Gamma[u,v] \cup \Gamma_v$ does not meet U_2.

Suppose now that $\Gamma_u \cap U_2 \neq \varnothing$. Then $\Gamma_v \cap V = \varnothing$, so $\Gamma_v \cap V_2 = \varnothing$. By (3.1), $\Gamma_u \cap V_2 = \varnothing$, which implies that $\partial\Omega_2 \cap V_2 = \varnothing$. Since $V_2 \cap \Omega_2 \neq \varnothing$ (by definition), we must have $V_2 \subset \Omega_2$. Because $\Gamma[u,v] \subset \Omega_1$, we have $\Gamma[u,v] \cap V_2 = \varnothing$. Thus we conclude that $\Gamma = \Gamma_u \cup \Gamma[u,v] \cup \Gamma_v$ does not meet V_2.

In conclusion, either U_2 or V_2 lies in the complement of Γ. Note that each of U_2 and V_2 contain a disk of diameter $|u-v|/32 \geq 7|x-y|/1024$. Since $|x-y| \to +\infty$ as $n \nearrow +\infty$, this confirms that the complement of Γ contains arbitrarily large disks. $\square$

The following fact ([Bis01, Lemma 2.4]) is crucial to the proof of Theorem 3.6 below.

Fact 3.5. *Suppose $E \subsetneq \mathsf{R}^2$ is a closed bilipschitz homogeneous set. Then $\mathscr{H}^2(E) = 0$.*

We are now ready to proceed in a manner similar to that of Bishop. This proof utilizes a notion of *local Hausdorff convergence*. We say that a sequence of closed subsets $E_i \subset \mathsf{R}^2$ is locally convergent to $E \subset \mathsf{R}^2$ if, for all $r \in \mathsf{R}_+$,

$$d_H(\overline{D}(0;r) \cap E_i, \overline{D}(0;r) \cap E) \to 0$$

as $i \nearrow +\infty$. Here d_H denotes the usual Hausdorff distance.

Theorem 3.6. *Suppose $\Gamma \subset \mathsf{R}^2$ is an unbounded L-bilipschitz homogeneous Jordan curve. Then Γ is of bounded turning.*

Proof. We prove this theorem by way of contradiction. Choose $\varepsilon_n \searrow 0$. By Fact 3.2, there exists a constant $C \in [1,+\infty)$ such that for each n there exists a point $x_n \in \Gamma$, a constant $s_n < \varepsilon_n$, and a constant $r_n > 0$ satisfying the following: each disk $D(x_n; r_n)$ contains an $(r_n s_n/C)$-separated set $S_n = \{x_{ni}\}$ of cardinality greater than $1/(Cs_n^2)$.

Suppose first that Γ is not of ε-local bounded turning for any $\varepsilon > 0$. In the proof of Fact 3.2 each r_n is defined to be the diameter of a subarc $\Gamma[a_n, b_n]$ for which the ratio $\operatorname{diam}(\Gamma[a_n,b_n])/|a_n - b_n|$ is sufficiently large. Since Γ is not of local bounded turning, there exist arbitrarily small subarcs for which this ratio is arbitrarily large. Therefore, we may assume that $r_n \searrow 0$. For large enough n, there exists a disk D_n of radius Lr_n such that $D_n \cap \Gamma = \varnothing$ and such that there exists a point $z_n \in \partial D_n \cap \Gamma$. Using the bilipschitz homogeneity of Γ we may assume the following: Each S_n is contained in D_n, is $(r_n s_n/(CL))$-separated, and has cardinality greater than $1/(Cs_n)^2$.

Now, for each n, we translate z_n to the origin and scale by a factor of $1/(Lr_n)$ to obtain Γ_n. Up to a rotation, we may assume each Γ_n omits the disk $D(1;1)$. In addition, we define

$$E_n := \Gamma_n \cup \bigcup_i D\left(\frac{x_{ni}-z_n}{Lr_n}; \frac{s_n}{2CL^2}\right).$$

Up to a subsequence, there exists a closed set $E \subset \mathsf{R}^2$ to which E_n is locally convergent in the Hausdorff distance. Since $s_n \to 0$, we have Γ_n and E_n converging to the same set E. Note that each Γ_n remains L-bilipschitz homogeneous. By taking limits of maps via Ascoli's theorem for variable domains ([Väi88, Theorem 2.9]), it follows that E is also L-bilipschitz homogeneous.

For every n, we have

$$\mathscr{H}^2(E_n) \geq \pi \frac{1}{(Cs_n)^2}\left(\frac{s_n}{2CL^2}\right)^2 = \frac{\pi}{4C^4L^4} > 0.$$

By [Bis01, Lemma 2.3], we conclude that $\mathscr{H}^2(E) > 0$. To see that $E \neq \mathsf{R}^2$, we recall that each Γ_n omits the disk $D(1;1)$. Thus E omits the same disk. In this way the assumption that Γ is not of local bounded turning would allow us to construct a closed bilipschitz homogeneous set $E \subsetneq \mathsf{R}^2$ with $\mathscr{H}^2(E) > 0$. By Fact 3.5, such a set cannot exist.

The above paragraph tells us that Γ is of ε-local bounded turning for some $\varepsilon > 0$. We now show that in fact Γ is of bounded turning in the global sense. Indeed, suppose the contrary. In the second paragraph of the proof of Lemma 3.4, we proved that, under our assumptions on Γ, $\operatorname{diam}(\Gamma[a_n,b_n])/|a_n-b_n| \to +\infty \Rightarrow \operatorname{diam}(\Gamma[a_n,b_n]) \to +\infty$. Since r_n is defined to be the diameter of some $\Gamma[a_n,b_n]$ for which $\operatorname{diam}(\Gamma[a_n,b_n])/|a_n-b_n| \to +\infty$, we conclude that $r_n \to +\infty$. By Lemma 3.4 there exist arbitrarily large disks in the complement of Γ, so let D_n denote a disk of radius r_n such that $D_n \cap \Gamma = \varnothing$ and such that there exists a point $z_n \in \partial D_n \cap \Gamma$. From here we follow the same strategy as in the above paragraph, reaching the same contradiction. Therefore, we conclude that Γ is of bounded turning. $\square$

4. A quantitative result

Here we use the results of the previous section to obtain a quantitative result. Our basic strategy is to find both lower and upper bounds on the Assouad dimension of an unbounded bilipschitz homogeneous Jordan curve in the plane. The key is to obtain such bounds in terms of the bilipschitz homogeneity and bounded turning constants. We begin by recording the following fact (see [Väi87, Theorem 3.17]).

Fact 4.1. *Suppose $f\colon A \to \mathsf{R}^2$ is an L-bilipschitz map, where $A \subset \overline{D}(x;r)$ and $x \in A$. Suppose also that $0 < \varepsilon \leq 1/(16L^3(L+1))$ and that $\overline{D}(x;r) \subset U(A;r\varepsilon)$. Then $\overline{D}(f(x);r/(2L)) \subset U(f(A);rL\varepsilon)$.*

Proposition 4.2. *Suppose Γ is an L-bilipschitz homogeneous Jordan curve in R^2. Then Γ is P-porous, with $P = P(L) = 160L^4$.*

Proof. By Theorem 3.6 we know that Γ is of bounded turning, and so Γ is porous for some finite constant. Suppose Γ fails to satisfy the P-porosity condition for $P := 160L^4$.

By the failure of P-porosity, there exists $x_0 \in \Gamma$ and $r > 0$ such that for every $z \in D(x_0;r)$ we have $\operatorname{dist}(z,\Gamma) < r/P$. Since Γ is porous for some finite constant, there exist arbitrarily large disks in the complement of Γ. In particular, there exists a disk D of radius r such that $D\cap\Gamma=\varnothing$ and such that there exists a point $y_0\in\partial D\cap\Gamma$. Let $f\colon\Gamma\to\Gamma$ be an L-bilipschitz homeomorphism with $f(x_0)=y_0$.

Now let S denote a maximal (r/P)-separated set in $\overline{D}(x_0;r)\cap\Gamma$ such that $x_0\in S$. By the choice of x_0, for every $z\in\overline{D}(x_0;r)$ we have $\overline{D}(z;r/P)\cap\Gamma\neq\varnothing$. It follows that $\overline{D}(x_0;r)\subset U(S;5r/P)$. Since $5/P=1/(32L^4)\leq 1/(16L^3(L+1))$, by Fact 4.1 we know that $\overline{D}(y_0;r/(2L))\subset U(f(S);5rL/P)$.

At this point we recall that y_0 was contained in the boundary of a disk D of radius r with $D\cap\Gamma=\varnothing$; so $f(S)\cap D=\varnothing$. Writing w to denote the center of D, let z denote the point at which the line segment $[y_0,w]$ intersects $\partial D(y_0;2r/L)$. Since $D(z;r/(2L))\subset D$, $\operatorname{dist}(z;f(S))\geq r/(2L)>5rL/P$. Therefore, $\overline{D}(y_0;r/(2L))\not\subset U(f(S);5rL/P)$. This contradicts the conclusion of the preceding paragraph. Since our assumption on P leads to a contradition, we conclude that Γ is indeed P-porous for $P=160L^4$. □

The above proposition in conjunction with [Luu98, Theorem 5.2] yields a quantitative upper bound on the Assouad dimension of Γ that is independent of the bounded turning constant. The following lemma provides a quantitative lower bound that increases with the bounded turning constant.

Lemma 4.3. *Let $\Gamma\subset\mathsf{R}^2$ be an L-bilipschitz homogeneous unbounded Jordan curve that is B-bounded turning. Suppose there exist distinct points $x_0,x_1\in\Gamma$ such that*

$$2\operatorname{diam}(\Gamma[x_0,x_1])\geq B|x_0-x_1|.$$

Writing $r:=\operatorname{diam}(\Gamma[x_0,x_1])$, there exists a $(r/4BL^2)$-separated set in Γ whose cardinality is at least $B^2/64$ and whose diameter is no greater than $10L^2r$.

Proof. We may assume that $B\geq 8$, for otherwise the claim is satisfied by $\{x_0,x_1\}$. Define an orientation on Γ by moving from x_0 to x_1 along $\Gamma[x_0,x_1]$. By [GH99, Lemma 2.5] there exists an orientation preserving L^2-bilipschitz map $f_1\colon\Gamma\to\Gamma$ with $f_1(x_0)=x_1$. Define $x_2:=f_1(x_1)$. In the same way we map x_0 to x_2 via $f_2\colon\Gamma\to\Gamma$ and define $x_3:=f_2(x_1)$. We iterate this process until we obtain x_M with $B+1\leq M\leq B+2$. Now we examine the circles $C(x_0;nr/B)$, for $n\in\mathsf{N}_0$. By the choice of r, we find that when $n\leq B/4$, we have $C(x_0;nr/B)\cap\Gamma[x_0,x_1]\neq\varnothing$. In particular, we have $C(x_0;nr/B)\cap\Gamma[x_0,y_0]\neq\varnothing$, where y_0 is the first point of $\Gamma[x_0,x_1]\cap C(x_0;r/4)$. For every such n, choose a point $z_{0n}\in C(x_0;nr/B)\cap\Gamma[x_0,y_0]$. Thus we obtain a collection $\{z_{0n}\}\subset\Gamma[x_0,y_0]$ consisting of at least $B/8$ points. Note that $z_{00}=x_0$. For every $m\leq M$, define $z_{mn}:=f_m(z_{0n})\in\Gamma[x_m,y_m]$, where $y_m:=f_m(y_0)$. Thus we obtain a collection $\{z_{mn}\}$ of cardinality at least $B(B+1)/8$.

We claim that $\{z_{mn}\}$ is $r/4BL^2$-separated. Let z_{mi} and z_{nj} be given. If $m=n$,

$$|z_{mi}-z_{mj}|=|f_m(z_{0i})-f_m(z_{0j})|\geq|z_{0i}-z_{0j}|/L^2\geq r/(BL^2).$$

Suppose now that $m<n$. Since $z_{mi}\leq y_m<x_{m+1}\leq z_{nj}$, we have

$$|z_{mi}-z_{nj}|\geq\operatorname{diam}(\Gamma[z_{mi},z_{nj}])/B\geq\operatorname{diam}(\Gamma[y_m,x_{m+1}])/B\geq r/(4BL^2).$$

The final inequality follows from the fact that $|x_0-y_0|=r/4$ while $\Gamma[y_0,x_1]$ contains a point on the circle $C(x_0;r/2)$.

Finally, we claim that $\operatorname{diam}(\{z_{mn}\}) \le 10L^2 r$. Let z_{mn} be given. We have

$$|z_{mn} - x_0| \le |x_0 - x_m| + |x_m - z_{mn}| \le \sum_{i=0}^{M-1} |x_i - x_{i+1}| + L^2 r/4$$
$$\le (B+2)L^2|x_0 - x_1| + L^2 r/4 \le L^2(4r + r/4) \le 5L^2 r. \qquad \square$$

Proof of Theorem 1.1. Let $B \in [1, +\infty)$ denote the optimal bounded turning constant for Γ (which exists by Theorem 3.6). Then by Lemma 4.3, there exists a scale $r > 0$ and a finite point set $S_0 \subset \Gamma$ such that $\operatorname{card}(S_0) \ge B^2/64$, $\operatorname{sep}(S_0) \ge r/(4BL^2)$, and $\operatorname{diam}(S_0) \le 10L^2 r$. Here $\operatorname{card}(S)$ denotes the cardinality of a given set S, and (when $\operatorname{card}(S) > 1$) $\operatorname{sep}(S) := \inf\{r \colon S \text{ is } r\text{-separated}\}$.

By Proposition 4.2 we know that Γ is P-porous with $P = P(L)$. Therefore, by [Luu98, Theorem 5.2] we know that Γ is (H,α)-homogeneous for some $H \in [1, +\infty)$ and $\alpha \in [1, 2)$, each depending only on L. Indeed, let k be any integer satisfying $k > 4\sqrt{2}P$. We may set $\alpha = \log(k^2 - 1)/\log(k)$ and $H = (k\sqrt{2})^\alpha$ (see the proof of [Luu98, Theorem 5.2]).

It follows from the definition of (H,α)-homogeneity that for any non-trivial finite point set $S \subset \Gamma$, we have $\operatorname{card}(S) \le 2^\alpha H(\operatorname{diam}(S)/\operatorname{sep}(S))^\alpha$. Using the set S_0 obtained above, this inequality yields $B \le (64H(80L^4)^\alpha)^{1/(2-\alpha)}$. $\square$

5. Constructing unbounded snowflakes

Recall the construction of the catalogue $\mathscr{S}$ in [Roh01], which lists all bounded quasicircles in R^2 up to bilipschitz equivalence. The subcatalogue $\mathscr{HS}$ lists all bounded bilipschitz homogeneous Jordan curves. We point out that $\mathscr{HS}$ does not account for unbounded bilipschitz homogeneous curves. Moreover, this cannot be remedied by the use of auxiliary Möbius maps, for such maps need not preserve bilipschitz homogeneity (see [Fre, Theorem 1.1]). Therefore, we construct a new catalogue $\mathscr{HT}$ that lists all unbounded bilipschitz homogeneous Jordan curves in R^2, up to bilipschitz equivalence.

The construction of $\mathscr{HT}$ resembles that of $\mathscr{HS}$. Let $p \in [1/4, 1/2)$. We start with the unit interval $I = [0,1] \subset \mathsf{R} \subset \mathsf{R}^2$, and carry out a $\mathscr{HS}_p$-type construction on I (see [Roh01] for details). This yields a sequence of piecewise linear arcs $(I_k)_{k=0}^\infty$ converging to a *snowflake arc* J_0. In particular, we refer to J_0 as a 0-*arc*. Then we take three isometric copies of J_0 and arrange them to the right of J_0 so that their endpoints coincide with the vertices of a similarity (scaled, rotated, translated) copy of either I or J_p (see Figure 1). We leave J_0 unaltered in this construction (see Figure 2).

This forms a new arc J_1, which we refer to as a 1-*arc*. Note that J_1 consists of four 0-arcs. Next, we take three copies of J_1, and arrange them to the right of J_1 so that their endpoints coincide with the vertices of a similarity copy of either I or J_p. We leave J_1 unaltered in this construction. Thus we form a new 2-arc J_2 consisting of 4^2 0-arcs and 4^1 1-arcs. We continue inductively to form an increasing sequence of arcs, (J_n). The union $T^+ := \bigcup_{n=0}^\infty J_n$ is contained in the first quadrant of the plane. We then form the unbounded Jordan curve $T := T^+ \cup T^-$, where T^- is the reflection of T^+ through the origin. The collection of all curves constructed in this way forms $\mathscr{HT}_p$; then $\mathscr{HT} := \cup_{p\in[1/4,1/2)} \mathscr{HT}_p$.

Figure 1. The arcs I and J_p.

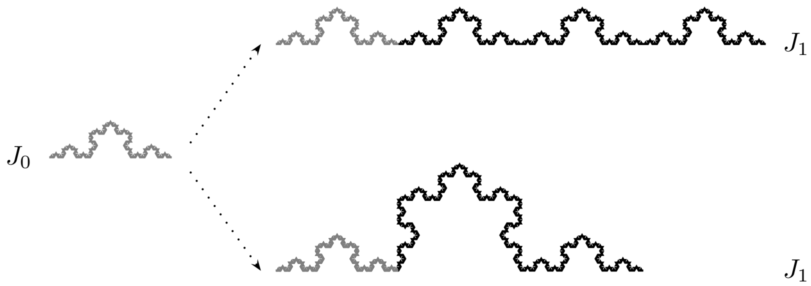

Figure 2. Building J_1 from J_0.

To obtain a closer parallel with the construction of curves $S \in \mathscr{HS}$ (which are the limits of polygonal arcs S_n), we define piecewise linear curves T_n converging to a given curve $T \in \mathscr{HT}$. To do this, we note that such a curve T consists entirely of copies of the 0-arc J_0 (as in the above paragraph). Denote these copies by $\{J_0^i\}_{i\in\mathsf{Z}}$. Each J_0^i is the limit of the sequence of piecewise linear arcs I_n^i. We refer to the linear segments comprising each I_n^i as *n-edges*; so each I_n^i consists of 4^n n-edges. We form the piecewise linear arc $T_n := \bigcup_{i\in\mathsf{Z}} I_n^i$, composed entirely of n-edges. Moreover, $T_n \to T$ as $n \nearrow +\infty$. We also emphasize the distinction between n-edges and n-arcs. Indeed, n-edges are linear segments in T_n whose lengths decrease as n increases. On the other hand, n-arcs are snowflake arcs in T whose diameters increase as n increases.

We need to analyze curves $T \in \mathscr{HT}$ on scales smaller than $\operatorname{diam}(J_0)$. To do this, we extend the notion of n-arcs to include negative indices. Indeed, for $n \in \mathsf{N}$, we use the term *$(-n)$-arc* in T to describe a subarc whose endpoints coincide with the endpoints of a copy of an n-edge in T_n.

The following can be proved as in [Roh01, Lemma 3.1].

Lemma 5.1. *If $T \in \mathscr{T}_p$, then T is of B-bounded turning, with $B = B(p)$.*

Now define a set of points $W \subset \mathsf{H}$ as follows: For every $n, k \in \mathsf{Z}$, set

$$v_{n,k} := 4^n(k + (1+i)/2).$$

For a given $v \in W$, we have $v = v_{n,k}$ for some n, k, and we write $d(v) = n$. We say that a given $v_{n,k}$ has four *children*, namely $v_{n-1,4k}, v_{n-1,4k+1}, v_{n-1,4k+2}, v_{n-1,4k+3}$ (when $k \geq 0$) or $v_{n-1,4k}, v_{n-1,4k-1}, v_{n-1,4k-2}, v_{n-1,4k-3}$ (when $k \leq 0$). Given $v \in W$, let I_v denote the subarc of R defined in Section 2. When $d(v) = n$, we again use the term n-arc to describe to I_v. See Figure 3.

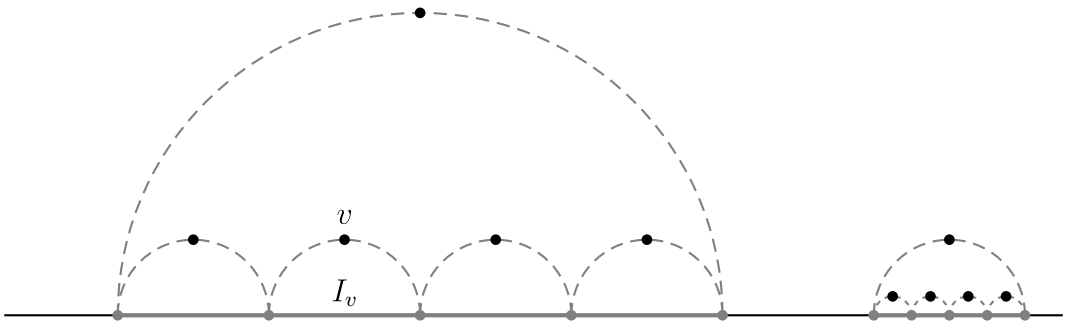

Figure 3. Points from W and associated subarcs in R.

Given a curve $T \in \mathscr{HT}$, we may use the set W to define a canonical parametrization $\varphi_T \colon \mathsf{R} \to T$. We begin by recalling the construction of T. We start with a compact interval $I \subset \mathsf{R} \subset \mathsf{R}^2$. We then obtain a sequence of piecewise linear curves T_n converging to T. Recall that T_n is composed of n-edges. We first define $\varphi_0 \colon \mathsf{R} \to T_0$ by sending the 0-arcs in R onto the corresponding 0-edges in T_0 in a linear manner. As a point of reference, we map the 0-arc $[0,1]$ onto the 0-edge whose endpoints coincide with the snowflake arc $J_0 \subset T$ (here we use the notation from the construction of $\mathscr{HT}$). We proceed inductively: for $n \in \mathsf{N}$, we map $(-n)$-arcs of R onto corresponding n-edges of T_n by a piecewise linear map φ_n. Since $T_n \to T$ in the Hausdorff distance, (φ_n) converges (uniformly) to a map $\varphi_T \colon \mathsf{R} \to T$.

Lemma 5.2. *Given a curve $T \in \mathscr{HT}_p$, the canonical parametrization $\varphi_T \colon \mathsf{R} \to T$ is η-quasihomogeneous, with η depending only on p.*

Proof. By [HM99, Fact 2.3], we only need to show that φ_R is weakly quasihomogeneous. That is, we need to find some constant H such that

$$|\varphi_T(x) - \varphi_T(y)| \le H|\varphi_T(z) - \varphi_T(w)|$$

when $|x-y| \le |z-w|$. To this end, let $|x-y| \le |z-w|$ in R, and let $n \in \mathsf{Z}$ be the largest index for which z and w are contained in non-adjacent n-arcs of R. Thus z and w are contained in the union of 8 consecutive n-arcs in R; so x and y are contained in 9 consecutive n-arcs. In particular, there exists an n-arc J that is mapped inside the subarc $T[\varphi_T(z), \varphi_T(w)]$ by φ_T. Since every n-edge in T has the same diameter, we have

$$\begin{aligned}|\varphi_T(x) - \varphi_T(y)| &\le \operatorname{diam}(\varphi_T((x,y))) \le 9 \operatorname{diam}(\varphi_T(J)) \\ &\le 9 \operatorname{diam}(T[\varphi_T(z), \varphi_T(w)]) \le 9B|\varphi_T(z) - \varphi_T(w)|.\end{aligned}$$

Here $B = B(p)$ denotes the bounded turning constant for T (Lemma 5.1). Thus φ_T is $9B$-weakly quasihomogeneous, and thus η-quasihomogeneous with η depending only on p. $\square$

6. Characterizing unbounded curves

We will make frequent use of the following fact (see [Leh87, pp. 35–36] and [Tuk81]).

Fact 6.1. *Suppose $f\colon \mathsf{R} \to \mathsf{R}^2$ is an η-quasisymmetric map. Then f has a K-quasiconformal extension $F\colon \mathsf{R}^2 \to \mathsf{R}^2$ with the following property: for all $x \in \mathsf{R}^2$ we have*

$$\operatorname{diam}(F(I_x)) \simeq |DF(x)| \operatorname{diam}(I_x).$$

The constants depend only on η.

We will also require the following two items ([Roh01, Lemma 2.1], [Fre09]).

Fact 6.2. *Suppose $F\colon \mathsf{R}^2 \to \mathsf{R}^2$ is a quasiconformal homeomorphism. Then $JF \simeq 1$ almost everywhere if and only if F is bilipschitz. The bilipschitz constant depends only on the dilatation of F and the constant involved in $JF \simeq 1$, and conversely.*

Fact 6.3. *An unbounded Jordan curve $\Gamma \subset \mathsf{R}^2$ is L-bilipschitz homogeneous and of B-bounded turning if any only if there exists a parametrization $h\colon \mathsf{R} \to \Gamma$, a dimension gauge δ satisfying (2.1), and a constant $C \in [1, +\infty)$ such that for every $x, y \in \mathsf{R}$,*

$$C^{-1}|x-y| \le \delta(|h(x)-h(y)|) \le C|x-y|. \tag{6.1}$$

The constants depend only on each other.

Proof of Theorem 1.2. We prove the following implications:

$$(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1),$$

noting that the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) are immediate.

(1) $\Rightarrow$ (3). Let $\Gamma \subset \mathsf{R}^2$ be an unbounded L-bilipschitz homogeneous Jordan curve with canonical dimension gauge δ. Let $D \in [1, +\infty)$ and $\alpha \in [1, 2)$ denote the constants from (2.1). We increase D by a factor of 2^α so that $\delta(2r) \le D\delta(r)$. Let $h\colon \mathsf{R} \to \Gamma$ be the parametrization described in Fact 6.3, satisfying (6.1) with constant C. By Theorem 1.1, the constants D, α and C depend only on L. Define

$$\rho(r) := \sup\{t\colon \delta(t) \le r\}.$$

It follows from (2.1) and (6.1) that for every $x, y \in \mathsf{R}$ we have $\rho(|x-y|) \simeq |h(x)-h(y)|$ up to the constant DC.

Recalling the collection of points $\{v_{n,k}\} = W \subset \mathsf{H}$, we define a labeling $\ell\colon W \to \mathsf{R}_+$. Given $v \in W$, set $\ell(v) := \rho(4^{d(v)})$. Using the fact that ρ is non-decreasing along with (2.1), for $m \le n$ we have

$$\frac{1}{D}\frac{\rho(4^n)}{\rho(4^m)} \le \frac{\delta(\rho(4^n))}{\delta(\rho(4^m))} \le D\left(\frac{\rho(4^n)}{\rho(4^m)}\right)^\alpha.$$

At the same time,

$$D^{-2}4^{n-m} \le \frac{\delta(\rho(4^n))}{\delta(\rho(4^m))} \le D^2 4^{n-m}.$$

Therefore, for $v, w \in W$ with $d(v) = m \le n = d(w)$,

$$D^{-3}A^{n-m} \le \frac{\ell(w)}{\ell(v)} \le D^3 4^{n-m},$$

where $A := 4^{1/\alpha} \in (2, 4]$.

Now we define another labeling $\ell'\colon W \to \mathsf{R}_+$ such that, for $d(v) = d(w) - 1$,

$$\frac{\ell'(w)}{\ell'(v)} \in \{4, A\}.$$

Moreover, our construction also yields $\ell' \simeq \ell$, up to the constant D^3. For ease of notation, let v_n denote any element of W with $d(v) = n$. Define $\ell'(v_0) := \rho(4^0) = \rho(1)$. Then for $n \geq 1$, define

$$\ell'(v_{n+1}) := \begin{cases} A\ell'(v_n) & \text{if } \ell(v_n) \leq \ell'(v_n), \\ 4\ell'(v_n) & \text{if } \ell(v_n) > \ell'(v_n). \end{cases}$$

For $n \leq -1$, define

$$\ell'(v_{n-1}) := \begin{cases} \ell'(v_n)/4 & \text{if } \ell(v_n) \leq \ell'(v_n), \\ \ell'(v_n)/A & \text{if } \ell(v_n) > \ell'(v_n). \end{cases}$$

To see that $\ell \simeq \ell'$ on W, let v_n be given, with $n \geq 1$. Suppose that $\ell(v_n) > \ell'(v_n)$. Let $0 \leq m < n$ be the smallest non-negative integer such that for every $m+1 \leq l \leq n$ we have $\ell(v_l) > \ell'(v_l)$; so $\ell(v_m) \leq \ell'(v_m)$. By the definition of ℓ', the properties of ℓ, and the choice of m, we have

$$\ell(v_n) > \ell'(v_n) = 4^{n-m}\ell'(v_m) \geq 4^{n-m}\ell(v_m) \geq D^{-3}\ell(v_n).$$

Now suppose that $\ell(v_n) \leq \ell'(v_n)$. Choose $0 \leq m < n$ to be the smallest non-negative integer such that for every $m + 1 \leq l \leq n$ we have $\ell(v_l) \leq \ell'(v_l)$; so $\ell(v_m) \geq \ell'(v_m)$. Similar to our above calculations, we have

$$\ell(v_n) \leq \ell'(v_n) = A^{n-m}\ell'(v_m) \leq A^{n-m}\ell(v_m) \leq D^3\ell(v_n).$$

Therefore, $\ell(v_n) \simeq \ell'(v_n)$ for $n \geq 1$, with constant D^3. A parallel strategy can be used to verify that the same comparability holds for $n \leq -1$.

We use ℓ' to construct a curve $T \in \mathscr{HT}$. Define $I = [0, \ell'(v_0)] \subset \mathsf{R}$. We begin our construction on decreasing scales. If $\ell'(v_{-1}) = \ell'(v_0)/4$, then we replace I by a similarity copy of I. If $\ell'(v_{-1}) = \ell'(v_0)/A$, then we replace I by a similarity copy of J_p. Here $p := A^{-1} \in [1/4, 1/2)$. Thus we form an arc we call I_1, and which consists of four 1-edges. Inductively, for $n \geq 2$, if $\ell'(v_{-n}) = \ell'(v_{-n+1})/4$ then we replace each $(n-1)$-edge in I_{n-1} by a similarity copy of I. If $\ell'(v_{-n}) = \ell'(v_{-n+1})/A$ then we replace each $(n-1)$-edge in I_{n-1} by a similarity copy of J_p. Thus we obtain I_n, consisting of 4^n n-edges. Continuing in this manner, we obtain a limit arc, which we denote by J_0. We refer to J_0 as a 0-arc and for $n \geq 0$ we use the term $(-n)$-arc to refer to subarcs of J_0 whose endpoints coincide with the n-edges in I_n.

Now we perform our construction on increasing scales. If $\ell'(v_1) = 4\ell'(v_0)$, then we place three copies of J_0 to the right of J_0 so that the endpoints of the four arcs coincide with the vertices of a similarity copy of I. We leave J_0 unaltered in this process, and so we form J_1. If $\ell'(v_1) = A\ell'(v_0)$, then we place three copies of J_0 to the right of J_0 so that the endpoints of the four arcs coincide with the vertices of a similarity copy of J_p. We leave J_0 unaltered in this process, and so we form J_1. In either case, J_1 is a 1-arc consisting of four 0-arcs. We continue inductively. Let J_n be given (for $n \geq 2$). If $\ell'(v_{n+1}) = 4\ell'(v_n)$, then we place three copies of J_n to the right of J_n so that the endpoints of the four arcs coincide with the vertices of a similarity copy of I. If $\ell'(v_{n+1}) = A\ell'(v_n)$, then we place three copies of J_n to the right of J_n so that the endpoints of the four arcs coincide with the vertices of a similarity copy of

J_p. We continue in this manner, obtaining a limit arc. Taking the union of this arc with it's reflection through the origin, we obtain a Jordan curve $T \in \mathscr{HT}$.

Let $\varphi_T \colon \mathsf{R} \to T$ denote the canonical parametrization of T. Given $v \in W$, we write $T(v)$ to denote $\varphi_T(I_v)$. We observe that for any $n \in \mathsf{Z}$ and any $v_n \in W$, we have

$$\ell'(v_n) = \operatorname{diam}(T(v_n)).$$

Since φ_T is quasihomogeneous (Lemma 5.2), it is quasisymmetric. Let $\Phi \colon \mathsf{R}^2 \to \mathsf{R}^2$ denote the quasiconformal extenstion of φ_T given by Fact 6.1. We note that $h \colon \mathsf{R} \to \Gamma$ is also quasisymmetric (recall that h is the parametrization obtained from Fact 6.3), and so h also has a quasiconformal extension $H \colon \mathsf{R}^2 \to \mathsf{R}^2$ given by Fact 6.1. For $x \in \mathsf{R}^2$, let $n \in \mathsf{Z}$ be such that I_x has endpoints a, b for which $|a - b| \simeq 4^n$, up to a factor of 2. Then

$$\begin{aligned} |DH(x)| \operatorname{diam}(I_x) &\simeq \operatorname{diam}(H(I_x)) \simeq |h(a) - h(b)| \simeq \rho(|a - b|) \simeq \ell(v_n) \\ &\simeq \ell'(v_n) = \operatorname{diam}(T(v_n)) \simeq \operatorname{diam}(\Phi(I_x)) \simeq |D\Phi(x)| \operatorname{diam}(I_x). \end{aligned}$$

Here all the constants are absolute or depend only on L. By Fact 6.2 (and the chain rule) we know that $H \circ \Phi^{-1} \colon \mathsf{R}^2 \to \mathsf{R}^2$ is a bilipschitz map for which $H \circ \Phi^{-1}(T) = \Gamma$. Moreover, the bilipschitz constant depends only on L.

(1) $\Rightarrow$ (4). Since (1) $\Rightarrow$ (3) has been established, we may assume that $T \in \mathscr{HT}_p$ is L-bilipschitz equivalent to Γ via some map $f \colon \mathsf{R}^2 \to \mathsf{R}^2$. Here $p = p(L)$. Let $\Phi \colon \mathsf{R}^2 \to \mathsf{R}^2$ be the quasiconformal extension of $\varphi_T \colon \mathsf{R} \to T$ given by Fact 6.1. For $z, w \in \mathsf{R}^2 \setminus \mathsf{R}$ with $0 < \operatorname{dist}(w, \mathsf{R}) \le \operatorname{dist}(z, \mathsf{R})$, let I_m and I_n be 4-adic intervals in R such that

$$\begin{aligned} \operatorname{diam}(I_m) &= 4^m \simeq \operatorname{diam}(I_w) = 2 \operatorname{dist}(w, \mathsf{R}), \\ \operatorname{diam}(I_n) &= 4^n \simeq \operatorname{diam}(I_z) = 2 \operatorname{dist}(z, \mathsf{R}), \end{aligned}$$

where the comparability is up to a factor of 2. Then

$$\frac{J\Phi(w)}{J\Phi(z)} \simeq \left(\frac{|D\Phi(w)|}{|D\Phi(z)|} \right)^2 \simeq \left(\frac{\operatorname{diam}(\varphi_T(I_m))}{\operatorname{diam}(\varphi_T(I_n))} \right)^2 4^{2(n-m)}.$$

At the same time, by the construction of T we have

$$4^{m-n} \le \frac{\operatorname{diam}(\varphi_T(I_m))}{\operatorname{diam}(\varphi_T(I_n))} \le (1/p)^{m-n}.$$

Therefore,

$$1 \lesssim \frac{J\Phi(w)}{J\Phi(z)} \lesssim (4p)^{2(n-m)} \simeq \left(\frac{\operatorname{dist}(z, \mathsf{R})}{\operatorname{dist}(w, \mathsf{R})} \right)^\beta .$$

Here $\beta := \log_2(4p) \in [0, 1)$, and all the comparability constants depend only on L.

Note that $\Psi := f \circ \Phi$ is a quasiconformal map of R^2 with $\Psi(\mathsf{R}) = \Gamma$. By the chain rule, $J\Psi \simeq J\Phi$ almost everywhere, up to a constant depending only on L. It follows that Ψ is our desired map.

(5) $\Rightarrow$ (1). Let F be the K-quasiconformal map given by (5); so there exists $C \in [1, +\infty)$ such that for almost every $a + ib = z \in \mathsf{R}^2$ we have

$$C^{-1} JF(z) \le JF(ib) \le C JF(z). \tag{6.2}$$

Choose points $x, y \in \Gamma$ and define the horizontal translation $G(z) = z + (F^{-1}(y) - F^{-1}(x))$. Then $G(F^{-1}(x)) = F^{-1}(y)$, and so $F \circ G \circ F^{-1}$ is a self homeomorphism of Γ that sends x to y. Moreover, this map is bilipschitz. Indeed, the chain rule and (6.2) tell us that for almost every $a+ib=z\in\mathsf{R}^2$, $J(F\circ G)(z)=JF(G(z))JG(z)\simeq JF(ib)\simeq JF(z)$. Thus $J(F\circ G\circ F^{-1})\simeq 1$ almost everywhere in R^2, up to the constant C^2. By Fact 6.2 $F\circ G\circ F^{-1}$ is bilipschitz, so Γ is bilipschitz homogeneous with constant depending only on C and K. □

7. A characterization of canonical dimension gauges

The techniques involved in proving Theorem 1.2 allow us to further explore the behavior of certain quasiconformal Jacobians and to characterize dimension gauges satisfying (2.1). This section closely follows ideas behind analogous results in [Roh01]. Indeed, the following proposition is a direct analogue to [Roh01, Corollary 1.4].

Proposition 7.1. *Let $\rho\colon(0,+\infty)\to(0,+\infty)$ be a function. Suppose there exists $\alpha\in[0,1/2)$ and $D\in[1,+\infty)$ such that for all $0\le r\le s$ we have*

$$D^{-1}\left(\frac{r}{s}\right)^{\alpha}\le\frac{\rho(s)}{\rho(r)}\le D.$$

Then there exists a constant $C\in[1,+\infty)$ and a K-quasiconformal map $F\colon\mathsf{R}^2\to\mathsf{R}^2$ such that for $z\in\mathsf{R}^2\setminus\mathsf{R}$,

$$C^{-1}\rho(\operatorname{dist}(z,\mathsf{R}))\le JF(z)^{1/2}\le C\rho(\operatorname{dist}(z,\mathsf{R})).$$

Here the constants C and K depend only on D and α.

Proof. We make use of notation from the proof of Theorem 1.2. Let $W\subset\mathsf{H}$ denote the collection $\{v_{n,k}\}$, where v_n denotes any element of W such that $d(v)=n$. Define the labeling $\ell\colon W\to\mathsf{R}_+$ by $\ell(v_n):=4^n\rho(4^n)=2d_{\mathsf{H}}(v_n)\rho(2d_{\mathsf{H}}(v_n))$. For $m\le n$,

$$\frac{\ell(v_n)}{\ell(v_m)}=\frac{4^n\rho(4^n)}{4^m\rho(4^m)}\le D4^{n-m}.$$

As well,

$$\frac{\ell(v_n)}{\ell(v_m)}=\frac{4^n\rho(4^n)}{4^m\rho(4^m)}\ge D^{-1}4^{n-m}(4^{m-n})^{\alpha}=D^{-1}A^{n-m},$$

where $A=4^{1-\alpha}\in(2,4]$. Thus we have $D^{-1}A^{n-m}\le\ell(v_n)/\ell(v_m)\le D4^{n-m}$. As in the proof of Theorem 1.2, using the above inequalities we construct a labeling $l'\simeq l$ such that for $d(v)=d(w)-1$ we have $\ell'(w)/\ell'(v)\in\{4,A\}$. From ℓ' we obtain $T\in\mathscr{H}\mathscr{T}$ such that $\ell'(v)=\operatorname{diam}(T(v))$. Via Lemma 5.2, we know that the canonical parametrization $\varphi_T\colon\mathsf{R}\to T$ is η-quasihomogeneous, with η depending only on D and α. Fact 6.1 yields a quasiconformal extension $\Phi\colon\mathsf{R}^2\to\mathsf{R}^2$.

For $z\in\mathsf{R}^2\setminus\mathsf{R}$, there exists $n\in\mathsf{Z}$ such that $\operatorname{dist}(z,\mathsf{R})\simeq 4^n$ up to a factor of 2. We have

$$\begin{aligned}J\Phi(z)^{1/2}&\simeq|D\Phi(z)|\simeq\frac{\operatorname{diam}(\varphi_T(I_z))}{\operatorname{diam}(I_z)}\simeq\frac{\operatorname{diam}(\varphi_T(I_{v_n}))}{\operatorname{diam}(I_z)}\simeq\frac{\ell'(v_n)}{4^n}\\&\simeq\frac{\ell(v_n)}{4^n}=\rho(4^n)\simeq\rho(\operatorname{dist}(z,\mathsf{R})).\end{aligned}$$

Thus $J\Phi(z)^{1/2}\simeq\rho(\operatorname{dist}(z,\mathsf{R}))$ for $z\in\mathsf{R}^2\setminus\mathsf{R}$, up to a constant depending only on D,α. □

We supply the following technical lemma.

Lemma 7.2. *Let $f\colon \mathsf{R} \to \mathsf{R}^2$ be an η-quasihomogeneous map and let $I \subset \mathsf{R}$ be a compact interval. Then for $z \in \mathsf{H}$,*

$$N(JF(z)^{1/2} d_{\mathsf{H}}(z); F(I)) \simeq N(d_{\mathsf{H}}(z); I).$$

Here F denotes the quasiconformal extension of f given by Fact 6.1, and the comparability constant depends only on η.

Proof. We begin by noting that f is η-quasisymmetric, so $\Gamma = f(\mathsf{R})$ is of B-bounded turning for some B depending only on η. Next, the properties of F described in Fact 6.1 tell us that for $a + ib = z \in \mathsf{H}$, $JF(z) \simeq JF(ib)$, where the comparability depends only on η.

Let $\{J_i\}_{i=1}^n$ be a finite cover of I by consecutive, non-overlapping intervals, each of which has the same diameter $2d_{\mathsf{H}}(z)$. Note that n may equal 1. Again using quasihomogeneity, for our calculations we may assume that $J_1 = I_z$. Furthermore, for $i, j \in \{1, 2, \dots, n\}$ we have $\operatorname{diam}(F(J_i)) \simeq \operatorname{diam}(F(J_j))$ up to the constant $\eta(1)$. Therefore,

$$N(2\eta(1) \operatorname{diam}(F(I_z)); F(I)) \le N(2d_{\mathsf{H}}(z); I).$$

Let x_i denote the left endpoint of each J_i. Since $F(I)$ is of B-bounded turning and F is quasihomogeneous, the points $F(x_i)$ form a $(\operatorname{diam}(F(I_z))/(\eta(1)B))$-separated collection in $F(I)$. Therefore,

$$P(2d_{\mathsf{H}}(z); I) \le P\left(\frac{\operatorname{diam}(F(I_z))}{\eta(1)B}; F(I)\right).$$

Using the comparability of covering and packing numbers along with the above estimates, we conclude that $N(d_{\mathsf{H}}(z); I) \simeq N(\operatorname{diam}(F(I_z)); F(I))$, up to a constant depending only on η. Finally, using the properties of F and the metric doubling property, we have

$$N(JF(z)^{1/2} d_{\mathsf{H}}(z); F(I)) \simeq N(\operatorname{diam}(F(I_z)); F(I)) \simeq N(d_{\mathsf{H}}(z); I).$$

Again the constants depend only on η. □

Proof of Theorem 1.3. As already mentioned, the necessity is Fact 2.1. To prove the sufficiency, we define $\rho\colon \mathsf{R}_+ \to \mathsf{R}_+$ as $\rho(s) := \sup\{t \in \mathsf{R}_+ \colon \delta(ts) \le s\}$.

Claim 1. We first claim that ρ satisfies the hypotheses of Proposition 7.1. By our assumptions on δ, it is clear that $\rho(s) > 0$ on R_+.

Subclaim 1a. For any $s > 0$, $\delta(\rho(s)s) \simeq s$, up to the constant $2^\alpha D$. To verify this subclaim, suppose first that $\delta(\rho(s)s) > s$. Choose $r < \rho(s)$ such that $\rho(s) \le 2r$. By the definition of ρ we must have $\delta(rs) \le s$. Thus we have

$$s < \delta(\rho(s)s) \le \delta(2rs) \le 2^\alpha D \delta(rs) \le 2^\alpha D s.$$

When $\delta(\rho(s)s) \le s$ we proceed similarly. Thus our first subclaim is verified.

Subclaim 1b. Given $s, t \in \mathsf{R}_+$ for which $s < t$, we have $\rho(s)s \le 4^\alpha D^3 \rho(t)t$. To see this, suppose that $\rho(s)s > \rho(t)t$. By the properties of δ and Subclaim 1a,

$$\frac{\rho(s)s}{\rho(t)t} \le D \frac{\delta(\rho(s)s)}{\delta(\rho(t)t)} \le 4^\alpha D^3 \left(\frac{s}{t}\right) \le 4^\alpha D^3.$$

Thus our second subclaim is verified.

Using these two subclaims, when $0 < s < t$ it is straightforward to check that

$$C^{-1}\left(\frac{s}{t}\right)^{\beta} \leq \frac{\rho(t)}{\rho(s)} \leq C.$$

where $C := D(4^{\alpha}D^3)^{2+\log_2(D)}$ and $\beta := (\alpha-1)/\alpha \in [0, 1/2)$. Therefore, Claim 1 is verified.

By Proposition 7.1, there exists a quasiconformal map $F\colon \mathsf{R}^2 \to \mathsf{R}^2$ such that $JF(z)^{1/2} \simeq \rho(\operatorname{dist}(z, \mathsf{R}))$ in $\mathsf{R}^2 \setminus \mathsf{R}$, where the dilatation and comparability depend only on D, α. By Subclaim 1a and the properties of δ, for $z \in \mathsf{H}$ we conclude that

$$\delta(JF(z)^{1/2} d_{\mathsf{H}}(z)) \simeq d_{\mathsf{H}}(z), \tag{7.1}$$

up to a constant depending only on D, α.

By construction of ρ we find that F satisfies Theorem 1.2 (5), with constants depending only on D and α. Therefore, we know that $\Gamma := F(\mathsf{R})$ is a B-bounded turning, L-bilipschitz homogeneous unbounded Jordan curve, with B and L depending only on D and α.

Claim 2. Let δ_Γ denote the canonical dimension guage for Γ. We claim that for $z \in \mathsf{H}$ we have $\delta_\Gamma(JF(z)^{1/2} d_{\mathsf{H}}(z)) \simeq d_{\mathsf{H}}(z)$, up to a constant depending only on D and α. To verify this claim, we start by observing that F is η-quasihomogeneous on R, with $\eta\colon \mathsf{R}_+ \to \mathsf{R}_+$ depending only on D, α. By [HM99, Lemma 2.2 and Fact 2.3] it suffices to check that F is very weakly quasihomogeneous, which is straightforward.

Let $\Omega := F(\mathsf{H})$; thus Ω is a quasidisk. Given $z \in \mathsf{H}$, by [Hei89, Theorem 3.1] (here we use implication III $\Rightarrow$ IV, which does not require the boundedness of Ω), $\operatorname{diam}(F(I_z)) \simeq d_\Omega(F(z))$. Then by [AG85, Theorem 1.8], $\operatorname{diam}(F(I_z)) \simeq JF(z)^{1/2} d_{\mathsf{H}}(z)$. The comparability constants depend only on D, α.

Choose a point $w_1 \in \mathsf{H}$ such $d_{\mathsf{H}}(w_1) = \delta(1) = 1$; so $\operatorname{diam}(F(I_{w_1})) \simeq JF(w_1)^{1/2}$. We have

$$\delta(\rho(1)) = \delta(\rho(d_{\mathsf{H}}(w_1)) d_{\mathsf{H}}(w_1)) \simeq d_{\mathsf{H}}(w_1) = 1 = \delta(1);$$

so property (2.1) yields $\rho(1) \simeq 1$. Using the relationship between JF and ρ, we obtain

$$\operatorname{diam}(F(I_{w_1})) \simeq JF(w_1)^{1/2} d_{\mathsf{H}}(w_1) \simeq \rho(d_{\mathsf{H}}(w_1)) = \rho(1) \simeq 1.$$

Using [Fre, Lemma 2.2 and Fact 3.1(b)], the above estimate tells us that, for any $r \leq 1$,

$$\delta_\Gamma(r) = N(r; \Gamma_1)^{-1} \simeq N(r; F(I_{w_1}))^{-1}, \tag{7.2}$$

where comparability depends only on D, α.

We are now ready to verify Claim 2 regarding δ_Γ. Suppose first that for $z \in \mathsf{H}$ we have $d_{\mathsf{H}}(z) \leq 1$. Then by Subclaim 1b and the construction of F and w_1, for such z we have $JF(z)^{1/2} d_{\mathsf{H}}(z) \lesssim JF(w_1) d_{\mathsf{H}}(w_1) \simeq 1$. We write $\omega(z) := JF(z)^{1/2} d_{\mathsf{H}}(z)$. By the doubling property, the definition of δ_Γ, and Lemma 7.2,

$$\delta_\Gamma(\omega(z)) \simeq N(\omega(z); F(I_{w_1}))^{-1} \simeq N(d_{\mathsf{H}}(z); I_{w_1})^{-1} \simeq \frac{d_{\mathsf{H}}(z)}{\operatorname{diam}(I_{w_1})} \simeq d_{\mathsf{H}}(z).$$

All comparability constants depend only on D, α.

Suppose now that $d_{\mathsf{H}}(z) \geq 1$. Again by Subclaim 1b, $\omega(z) \gtrsim 1$. Note also that $\operatorname{diam}(\Gamma_{\omega(z)}) = \omega(z) \simeq \operatorname{diam}(F(I_z))$, so as in (7.2), $\delta_\Gamma(\omega(z)) \simeq N(1; \Gamma_{v(z)}) \simeq$

$N(1; F(I_z))$. Since $\omega(w_1) \simeq 1$, by Lemma 7.2, we have

$$N(1; F(I_z)) \simeq N(\omega(w_1); F(I_z)) \simeq N(d_{\mathsf{H}}(w_1); I_z) \simeq \frac{\operatorname{diam}(I_z)}{d_{\mathsf{H}}(w_1)} \simeq d_{\mathsf{H}}(z).$$

Again, all comparability constants depend only on D, α, and Claim 2 is verified.

By Subclaim 1a, $\lim_{s\to 0} \rho(s)s = 0$ while $\lim_{s\to+\infty} \rho(s)s = +\infty$. Using this observation along with Claim 1, the hypotheses of Proposition 7.1 allow us to conclude that for all $r > 0$, there exists $s > 0$ such that $\rho(s)s \simeq r$ up to the constant $2C$ (recall, $C = D(4^\alpha D^3)^{2+\log_2(D)}$). Then there exists $z \in \mathsf{H}$ such that $d_{\mathsf{H}}(z) = s$ and $JF(z)^{1/2} \simeq \rho(d_{\mathsf{H}}(z))$. Therefore, $\rho(s)s \simeq JF(z)^{1/2} d_{\mathsf{H}}(z)$. Using (7.1), it follows that

$$\delta(r) \simeq \delta(JF(z)^{1/2} d_{\mathsf{H}}(z)) \simeq d_{\mathsf{H}}(z) \simeq \delta_\Gamma(JF(z)^{1/2} d_{\mathsf{H}}(z)) \simeq \delta_\Gamma(r).$$

The comparability constants depend only on D, α. □

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