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EXTENDING HOLOMORPHIC MOTIONS AND MONODROMY

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Abstract. We study some relationships between holomorphic motions, continuous motions, and monodromy. We also study extensions of holomorphic motions over Riemann surfaces and characterize the extendability of holomorphic motions over some planar regions in terms of monodromy.

1. Introduction

Holomorphic motions, which are holomorphic families of injections on a subset of the Riemann sphere, were first introduced in a paper by Mañe, Sad and Sullivan ([10]). Since its inception, it has been a useful tool in complex analysis, in particular, complex dynamics and Teichmüller theory. A fundamental topic in the study of holomorphic motions has been the question of extending holomorphic motions. Namely, the main concern is to understand conditions under which a holomorphic motion of a closed set in the Riemann sphere $\widehat{\mathbf{C}}$ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$.

In the famous paper [15], Slodkowski gave a complete answer to this question for holomorphic motions when the parameter space is the open unit disk in the complex plane $\mathbf{C}$. He showed that in this case, every holomorphic motion can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$. Later, Chirka ([3]) claimed some conditions for extendability of a holomorphic motion, when the parameter space is a Riemann surface.

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For higher dimensional parameter spaces, it is known that there exist holomorphic motions which cannot be extended to $\widehat{\mathbf{C}}$, even if the parameter space is simply connected; see the papers [5], [7], and [9] for some examples. Therefore, a natural problem is to clarify conditions under which a holomorphic motion of a subset of $\widehat{\mathbf{C}}$ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$. This is our main motivation.

In this paper, we obtain some new results on extensions of holomorphic motions. The main idea is to study the relationship between extending holomorphic motions and *monodromy* of holomorphic motions (see §2 for the definitions). As spin-offs of our first theorem, we answer a question of Earle (see Remark 1 in §2.5), strengthen a part of Theorem 1 in Earle's paper [5] (see Theorem 4.2), and prove one direction of a claim in Chirka's paper [3] (see Remark 2 in §2.4). We also study the extendability of holomorphic motions over Riemann surfaces. In particular, we characterize the extendability of holomorphic motions over some planar regions in terms of monodromy (Theorems D and E).

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2. Basic definitions and statements of the main results

2.1. Holomorphic motions.

Definition 2.1. Let V be a connected complex manifold with a basepoint x_0 and let E be a subset of the Riemann sphere $\widehat{\mathbf{C}}$. A *holomorphic motion of* E *over* V is a map $\phi\colon V \times E \to \widehat{\mathbf{C}}$ that has the following three properties:

(1) $\phi(x_0, z) = z$ for all z in E,
(2) the map $\phi(x, \cdot)\colon E \to \widehat{\mathbf{C}}$ is injective for each x in V, and
(3) the map $\phi(\cdot, z)\colon V \to \widehat{\mathbf{C}}$ is holomorphic for each z in E.

We say that V is the *parameter space* of the holomorphic motion ϕ. We will always assume that ϕ is a *normalized* holomorphic motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi(x, \cdot)$ for every x in V. We sometimes write the map $\phi(x, \cdot)$ as $\phi_x(\cdot)$ for x in V.

Definition 2.2. Let V and W be connected complex manifolds with basepoints, and f be a basepoint preserving holomorphic map of W into V. If ϕ is a holomorphic motion of E over V its *pullback* by f is the holomorphic motion

$$f^*(\phi)(x, z) = \phi(f(x), z) \quad \forall (x, z) \in W \times E \tag{2.1}$$

of E over W.

Holomorphic motion of $\widehat{\mathbf{C}}$. Let $M(\mathbf{C})$ denote the open unit ball of the complex Banach space $L^\infty(\mathbf{C})$ and let 0 be its basepoint. For each μ in $M(\mathbf{C})$ let w^μ be the unique quasiconformal homeomorphism of $\widehat{\mathbf{C}}$ onto itself that fixes the points 0, 1, and ∞ and has Beltrami coefficient μ. We can define a map $\Psi_{\widehat{\mathbf{C}}}\colon M(\mathbf{C}) \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ as follows:

$$\Psi_{\widehat{\mathbf{C}}}(\mu, z) = w^\mu(z) \quad \text{for all } z \in \widehat{\mathbf{C}}.$$

By Theorem 11 in [1], this is a holomorphic motion of $\widehat{\mathbf{C}}$ over $M(\mathbf{C})$. This holomorphic motion has the following universal property; see Theorem 4 in [5].

Theorem 2.3. *Let $\phi\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a holomorphic motion where V is a connected complex manifold with a basepoint. Then there exists a (unique) basepoint preserving holomorphic map $f\colon V \to M(\mathbf{C})$ such that $f^*(\Psi_{\widehat{\mathbf{C}}}) = \phi$.*

If E is a proper subset of $\widetilde{E}$, and $\phi\colon V \times E \to \widehat{\mathbf{C}}$ and $\widetilde{\phi}\colon V \times \widetilde{E} \to \widehat{\mathbf{C}}$ are two maps, we say that $\widetilde{\phi}$ *extends* ϕ if $\widetilde{\phi}(x,z) = \phi(x,z)$ for all (x,z) in $V \times E$.

2.2. Quasiconformal motions. In their paper [16], Sullivan and Thurston introduced the idea of quasiconformal motions. In what follows, ρ denotes the Poincaré metric on $\widehat{\mathbf{C}} \setminus \{0,1,\infty\}$.

Let V be a connected Hausdorff space with a basepoint x_0. For any map $\phi\colon V \times E \to \widehat{\mathbf{C}}$, x in V, and any quadruplet a, b, c, d of points in E, let $\phi_x(a,b,c,d)$ denote the cross-ratio of the values $\phi(x,a)$, $\phi(x,b)$, $\phi(x,c)$, and $\phi(x,d)$, for x in V. As in §2.1, we will write $\phi(x,z)$ as $\phi_x(z)$ for x in V and z in E. So we have:

$$\phi_x(a,b,c,d) = \frac{(\phi_x(a)-\phi_x(c))(\phi_x(b)-\phi_x(d))}{(\phi_x(a)-\phi_x(d))(\phi_x(b)-\phi_x(c))} \tag{2.2}$$

for each x in V.

Remark 2.4. For each $x \in V$, $\phi_x(a,b,c,d)$ takes values in $\widehat{\mathbf{C}} \setminus \{0,1,\infty\}$ if and only if the map $\phi_x\colon E \to \widehat{\mathbf{C}}$ is injective.

Definition 2.5. A *quasiconformal motion* is a map $\phi\colon V \times E \to \widehat{\mathbf{C}}$ of E over V such that

(1) $\phi(x_0,z) = z$ for all z in E,
(2) for each $x \in V$, the map $\phi_x\colon E \to \widehat{\mathbf{C}}$ is injective, and
(3) given any x in V and any $\epsilon > 0$, there exists a neighborhood U_x of x such that for any quadruplet of distinct points a,b,c,d in E, we have
$$\rho\big(\phi_y(a,b,c,d), \phi_{y'}(a,b,c,d)\big) < \epsilon \quad \text{for all } y \text{ and } y' \text{ in } U_x.$$

As usual, we will always assume that ϕ is a normalized quasiconformal motion; i.e. 0, 1, and ∞ belong to E and are fixed points of the map $\phi_x(\cdot)$ for every x in V.

We will need the following property of quasiconformal motions of $\widehat{\mathbf{C}}$. See [12] for a complete proof.

Theorem 2.6. *Let $\phi\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a map such that $\phi(x_0,z) = z$ for all z in $\widehat{\mathbf{C}}$, and for each x in V, ϕ_x fixes the points $0,1$, and ∞. Then, ϕ is a quasiconformal motion of $\widehat{\mathbf{C}}$ if and only if it satisfies:*

(1) *the map $\phi_x\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is quasiconformal for each x in V, and*
(2) *the map that sends x in V to the Beltrami coefficient of ϕ_x, for each x in V, is continuous.*

2.3. Continuous motions.

Definition 2.7. Let V be a path-connected Hausdorff space with a basepoint x_0. A *normalized continuous motion* of $\widehat{\mathbf{C}}$ over V is a continuous map $\phi\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that:

(1) $\phi(x_0,z) = z$ for all z in $\widehat{\mathbf{C}}$, and
(2) for each x in V, the map $\phi_x\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a homeomorphism, that fixes the points 0, 1, and ∞.

The following facts were proved in [12].

Proposition 2.8. *A quasiconformal motion* $\phi\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *is also a continuous motion.*

Theorem 2.9. *Let* $\phi\colon V \times E \to \widehat{\mathbf{C}}$ *be a holomorphic motion where* V *is a connected complex Banach manifold with a basepoint, and* E *is a closed subset of* $\widehat{\mathbf{C}}$ *(that contains the points 0, 1, and* ∞*). Then the following are equivalent:*

(1) *There exists a continuous motion* $\widehat{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *that extends* ϕ.
(2) *There exists a quasiconformal motion* $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *that extends* ϕ.

We also note the following result. See Corollary 6.1 in [12].

Proposition 2.10. *Let* $\phi\colon V \times E \to \widehat{\mathbf{C}}$ *be a holomorphic motion where* V *is a simply connected complex Banach manifold with a basepoint. Then there exists a quasiconformal motion* $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ *such that* $\widetilde{\phi}$ *extends* ϕ.

Remark 2.11. In the above proposition, for each x in V, $\widetilde{\phi}_x\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a (normalized) quasiconformal map; this follows from Theorem 2.6. Furthermore, there exists a basepoint preserving continuous map $f\colon V \to M(\mathbf{C})$, where, for each x in V, $f(x)$ is the Beltrami coefficient of the quasiconformal map $\widetilde{\phi}_x$. Thus, for each z in E and x in V, we have $\phi(x,z) = \widetilde{\phi}(x,z) = w^{f(x)}(z)$.

2.4. Monodromy. Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion of E over a connected complex Banach manifold V with basepoint x_0. For each $z \in E \setminus \{0,1,\infty\}$, we have a holomorphic map $\phi^z(\cdot) := \phi(\cdot, z)$ on V. Being a continuous map, it induces a homomorphism $(\phi^z)_*\colon \pi_1(V) \to \pi_1(\widehat{\mathbf{C}} \setminus \{0,1,\infty\})$. We call $(\phi^z)_*$ the *trace monodromy* of ϕ for $z \in E$. The trace monodromy is called *trivial* if it maps every element of $\pi_1(V)$ to the identity of $\pi_1(\widehat{\mathbf{C}} \setminus \{0,1,\infty\})$.

Let $\pi\colon \widetilde{V} \to V$ be a holomorphic universal covering, with the cover transformation group G_V. We take a point $\widetilde{x}_0 \in \widetilde{V}$ so that $\pi(\widetilde{x}_0) = x_0$.

Let $\Phi = \pi^*(\phi)$. Then, $\Phi\colon \widetilde{V} \times E \to \widehat{\mathbf{C}}$ is a holomorphic motion of E over $\widetilde{V}$ with $\widetilde{x}_0$ as the basepoint. By Remark 2.11, there exists a basepoint preserving continuous map $f\colon \widetilde{V} \to M(\mathbf{C})$ such that

$$\Phi(x,z) = w^{f(x)}(z) \quad \text{for each } x \in \widetilde{V} \text{ and each } z \in E. \tag{2.3}$$

For each $z \in E$ and for each $g \in G_V$, we have

$$w^{f\circ g(\widetilde{x}_0)}(z) = \Phi(g(\widetilde{x}_0), z) = \phi(\pi \circ g(\widetilde{x}_0), z) = \phi(x_0, z) = z.$$

Therefore, $w^{f\circ g(\widetilde{x}_0)}$ keeps every point of E fixed.

We claim that the homotopy class for $w^{f\circ g(\widetilde{x}_0)}$ relative to E is well-defined.

Lemma 2.12. *The homotopy class for* $w^{f\circ g(\widetilde{x}_0)}$ *relative to* E *does not depend on the choice of continuous mappings* f.

Proof. Let $f_1, f_2\colon \widetilde{V} \to M(\mathbf{C})$ be basepoint preserving continuous maps which are obtained from the given holomorphic motion $\phi\colon V \times E \to \widehat{\mathbf{C}}$. For each $g \in G_V$, take a path $c_g\colon [0,1] \to \widetilde{V}$ which connects $\widetilde{x}_0$ and $g(\widetilde{x}_0)$ and write

$$H(z,t) := w^{f_1\circ g(\widetilde{x}_0)} \circ \{w^{f_1\circ c_g(t)}\}^{-1} \circ w^{f_2\circ c_g(t)}(z)$$

for $(z,t) \in \widehat{\mathbf{C}} \times [0,1]$. Then, we see that $H(\cdot,\cdot)$ gives a homotopy from $w^{f_1\circ g(\widetilde{x}_0)}$ to $w^{f_2\circ g(\widetilde{x}_0)}$ relative to E. Hence, we conclude that $w^{f_1\circ g(\widetilde{x}_0)}$ and $w^{f_2\circ g(\widetilde{x}_0)}$ belong to the same homotopy class relative to E, as claimed. □

We now assume that E is a finite set and it contains $n(> 3)$ points including $0, 1$ and ∞. Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. The map $w^{f\circ g(\widetilde{x}_0)}$ is a quasiconformal selfmap of the Riemann surface $X_E := \widehat{\mathbf{C}} \setminus E$. Therefore, it represents a mapping class of X_E, and by Lemma 2.12, we have a homomorphism $\rho_\phi\colon \pi_1(V, x_0) \to \mathrm{Mod}(0, n)$ given by

$$\rho_\phi(c) = [w^{f\circ g_c(\widetilde{x}_0)}] \tag{2.4}$$

where $\mathrm{Mod}(0, n)$ is the mapping class group of the n-times punctured sphere, $g_c \in G_V$ is an element corresponding to $c \in \pi_1(V, x_0)$, and $[w]$ denotes the mapping class of X_E for w. We call the homomorphism ρ_ϕ the *monodromy* of the holomorphic motion ϕ of the finite set E. The monodromy is called *trivial* if it maps every element of $\pi_1(V, x_0)$ to the identity of $\mathrm{Mod}(0, n)$.

If ϕ extends to a holomorphic motion of $\widehat{\mathbf{C}}$, then it represents a holomorphic family of quasiconformal mappings with the parameter space V. In fact, by Theorem 2.3, there exists a basepoint preserving holomorphic map $f\colon V \to M(\mathbf{C})$ such that for all $(x, z) \in V \times E$, we have $\phi(x, z) = w^{f(x)}(z)$. Hence, we have the following

Proposition 2.13. *Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion of a finite set E. If ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over V, then the monodromy ρ_ϕ is trivial.*

2.5. Statements of the main results. In what follows, we will always assume that E is a closed subset of $\widehat{\mathbf{C}}$, such that 0, 1, and ∞ belong to E. As usual, let $E^n = E \times \cdots \times E$ (n times). Let V be a connected complex Banach manifold with a basepoint x_0. Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. Let X be a topological space. Let $Y_1 := \mathbf{C} \setminus \{0, 1\}$, and, for $n \geq 2$,

$$Y_n = \{z = (z_1, \cdots, z_n)\colon z_i \neq z_j \text{ for } 1 \leq i \neq j \leq n \text{ and } z_i \neq 0, 1 \text{ for all } i\}.$$

Let $F_n\colon Y_n \to X$ be a continuous map. Define $\phi_n\colon V \times (E^n \cap Y_n) \to \widehat{\mathbf{C}}$ as follows:

$$\phi_n(x, z) = (\phi(x, z_1), \cdots, \phi(x, z_n))$$

where $z = (z_1, \cdots, z_n) \in E^n \cap Y_n$.

Next, we define $G_n\colon V \times (E^n \cap Y_n) \to X$ as follows: $G_n = F_n \circ \phi_n$.

Let γ be a closed curve in V. For any z in $E^n \cap Y_n$, we define

$$G_n(\gamma, z) := \{G_n(x, z)\colon x \in \gamma\}.$$

We now state our first theorem.

Theorem A. If $\phi\colon V \times E \to \widehat{\mathbf{C}}$ can be extended to a continuous motion $\widehat{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, then $G_n(\gamma, z)$ is homotopic to $G_n(x_0, z)$ in X.

The following special cases are important and have independent interests.

Case 1. When $n = 4$, $X = \mathbf{C} \setminus \{0, 1\}$, and let F_4 be the cross-ratio map

$$F_4(a, b, c, d) = \frac{(a - c)(b - d)}{(a - d)(b - c)},$$

where a, b, c, d are distinct points in E.

Corollary 1. *Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. For each z in $E^4 \cap Y_4$, define $H_z(x) := G_4(x,z) = F_4 \circ \phi_4$. If ϕ extends to a continuous motion $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, then the map H_z is null-homotopic, which means the induced homomorphism*

$$H_{z*}\colon \pi_1(V) \to \pi_1(\mathbf{C} \setminus \{0,1\})$$

is trivial.

Remark 1. Corollary 1 gives a positive answer to the following question asked by Earle (e-mail communication). We thank him for this question. Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. If ϕ extends to a continuous motion of $\widehat{\mathbf{C}}$, then for any distinct points a, b, c, d in E, and any closed curve γ in V, is it true that the closed curve $t \mapsto \phi_{\gamma(t)}(a,b,c,d)$ is null-homotopic, where $\phi_{\gamma(t)}$ is the cross-ratio of the points $\phi_{\gamma(t)}(a)$, $\phi_{\gamma(t)}(b)$, $\phi_{\gamma(t)}(c)$, $\phi_{\gamma(t)}(d)$.

Case 2. When $n = 1$; $X = \mathbf{C} \setminus \{0,1\}$, $F(z) = z$. For each $z \in E \setminus \{0,1,\infty\}$, we consider the the trace monodromy $(\phi^z)_*$ of ϕ for $z \in E$.

Corollary 2. *Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion that extends to a continuous motion $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$. Then, the trace monodromy $(\phi^z)_*$ is trivial for every point z in E.*

Case 3. When $n = 2$; X is the set of all integers. For any two smooth closed curves $\delta_1, \delta_2\colon [0,1] \to \mathbf{C}$ in $\mathbf{C}$, we still continue with the notation F_2, which is defined as follows:

$$F_2(\delta_1,\delta_2) = \frac{1}{2\pi}\int_0^1 d\left(\arg(\delta_1(t)) - (\delta_2(t))\right),$$

if $\delta_1(t) \neq \delta_2(t)$ for any $t \in [0,1]$. Note that F_2 takes values in X and it is the winding number of a closed curve $\delta_1 - \delta_2$ defined by

$$(\delta_1 - \delta_2)(t) := \delta_1(t) - \delta_2(t)$$

for $t \in [0,1]$.

In fact, F_2 is not exactly defined as a map on Y_2. It is defined on the product of the spaces of closed curves in $\mathbf{C}$. However, as we will see in the proof of Corollary 3, this slight abuse of notations will not be relevant.

Corollary 3. *Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. For any z_1, z_2 in E ($z_1 \neq z_2$), and a closed curve γ in V, we define*

$$H(\gamma) := F_2\left(\phi(\gamma(t),z_1), \phi(\gamma(t),z_2)\right). \tag{2.5}$$

If ϕ extends to a continuous motion $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, then $H(\gamma)$ is zero. In particular, the winding number of the closed curve $t \mapsto \phi(\gamma(t),z_1) - \phi(\gamma(t),z_2)$ about the origin is zero.

Remark 2. This proves one direction of Chirka's claim (see [3]).

Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. The statements of Proposition 2.13 and Corollary 2 lead to the following question.

Question. If the trace monodromy $(\phi^z)_*$ is trivial for every z in E or if the monodromy is trivial for holomorphic motion of any finite subset of E, does ϕ have an extension as a holomorphic motion (or a continuous motion) to the whole of $\widehat{\mathbf{C}}$ over V?

If the set E consists of four points, it is a special case; we have an affirmative answer.

Theorem B. Let $\phi\colon V \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion, where $E = \{0, 1, a, \infty\}$ for some $a \neq 0, 1, \infty$. Then, the following are equivalent.

(1) The holomorphic motion ϕ can be extended to a holomorphic motion $\widehat{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$;
(2) the trace monodromy $(\phi^a)_*\colon \pi_1(V) \to \pi_1(\mathbf{C} \setminus \{0, 1\})$ is trivial;
(3) the monodromy $\rho_\phi\colon \pi_1(V) \to \mathrm{Mod}(0, 4)$ is trivial.

We shall give an answer to the above question if the parameter space V is certain kind of planar domains. Before giving the answer, we state a general theorem on the extendability of holomorphic motions.

Theorem C. Let $\phi\colon R \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion, where R is a hyperbolic Riemann surface (with a basepoint). Suppose the restriction of ϕ to $R \times E'$ extends to a holomorphic motion of $\widehat{\mathbf{C}}$, whenever $\{0, 1, \infty\} \subset E' \subset E$ and E' is finite. Then, ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$.

Remark 3. A weaker version of this result was proved in [12] for holomorphic motions over infinite-dimensional parameter spaces; see Theorem 3 in that paper.

Definition 2.14. Let Δ denote the open unit disk $\{z \in \mathbf{C}\colon |z| < 1\}$. A compact subset K of Δ is called *AB-removable* if every bounded holomorphic function on $\Delta \setminus K$ can be extended to a holomorphic function on Δ.

In [13], the following theorem was proved.

Theorem 2.15. *Let K be a compact subset of Δ. Suppose that K is AB-removable. For a holomorphic motion $\phi\colon (\Delta \setminus K) \times E \to \widehat{\mathbf{C}}$, the following are equivalent:*

(1) *ϕ can be extended to a continuous motion $\widetilde{\phi}\colon (\Delta \setminus K) \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$.*
(2) *ϕ can be extended to a holomorphic motion $\widehat{\phi}\colon (\Delta \setminus K) \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$.*
(3) *ϕ can be extended to a holomorphic motion $\phi_0\colon \Delta \times E \to \widehat{\mathbf{C}}$.*

Statement (3) means that $\phi_0(t, z) = \phi(t, z)$ for all $(t, z) \in (\Delta \setminus K) \times E$.

Let K be any AB-removable subset of Δ. For a holomorphic motion of a finite subset E over $\Delta_K := \Delta \setminus K$, we give an answer to the question for monodromy:

Theorem D. Let $K \subset \Delta$ be an AB-removable compact subset of Δ and $E \supset \{0, 1, \infty\}$ a finite subset of $\widehat{\mathbf{C}}$. Let $\phi : \Delta_K \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion of E over Δ_K. Then ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over Δ_K if and only if the monodromy ρ_ϕ is trivial.

From Theorems C and D, we have the following;

Corollary 4. *Let $K \subset \Delta$ be an AB-removable compact subset of Δ and $E \supset \{0, 1, \infty\}$ a closed subset of $\widehat{\mathbf{C}}$. Let $\phi\colon \Delta_K \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. Suppose that the monodromy of the restriction of ϕ to $\Delta_K \times E'$ is trivial for any finite subset E' of E, containing $0, 1$ and ∞. Then, ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$.*

As for the trace monodromy, we give an answer to the question when V is the complement of an AB-removable compact subset of Δ.

Theorem E. Let E be a closed subset of $\widehat{\mathbf{C}}$; $0, 1, \infty$ belong to E. Suppose that E contains at least five points and K is a nonempty AB-removable compact subset of Δ. We write $\Delta_K := \Delta \setminus K$, and the punctured unit disk $\Delta^* = \{z \in \mathbf{C}\colon 0 < |z| < 1\}$.

(1) If there exists a connected component of $\widehat{\mathbf{C}} \setminus E$ such that it is neither simply connected nor conformally equivalent to Δ^*, then there exists a holomorphic motion ϕ_0 of E over Δ_K such that the trace monodromy $(\phi_0^z)_*$ of ϕ_0 is trivial for every z in E but ϕ_0 has no extension as a holomorphic motion of $\widehat{\mathbf{C}}$ over Δ_K.

(2) If every connected component of $\widehat{\mathbf{C}} \setminus E$ is simply connected or conformally equivalent to Δ^*, then every holomorphic motion of E over Δ_K can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over Δ_K.

3. Proof of Theorem A

Let V be a connected complex Banach manifold with a basepoint x_0. Take any closed curve γ in V. By Theorem 2.9, there exists a quasiconformal motion $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, such that $\widetilde{\phi}$ extends ϕ. By Theorem 2.6, $\widetilde{\phi}_x\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a quasiconformal map, for each x in V. Hence there exists a $\mu(x) \in M(\mathbf{C})$ for each x in V, which is the Beltrami coefficient of the quasiconformal map $\widetilde{\phi}_x$. Furthermore, by Theorem 2.6, the map $x \mapsto \mu(x)$ is continuous on V. Thus, for any z in E, we have $\widetilde{\phi}_x(z) = \phi_x(z) = w^{\mu(x)}(z)$.

For any $z = (z_1, \cdots, z_n) \in E^n \cap Y_n$, define a map $\Gamma\colon I \times I \to X$ as follows: (here, I denotes the unit interval)

$$\Gamma(s,t) := F_n(w^{s\mu(\gamma(t))}(z_1), \cdots, w^{s\mu(\gamma(t))}(z_n)).$$

Clearly, Γ is a continuous map.

Next, note that

$$\Gamma(0,t) = F_n(z_1, \cdots, z_n) = G_n(x_0, z)$$

and

$$\Gamma(1,t) = F_n(w^{\mu(\gamma(t))}(z_1), \cdots, w^{\mu(\gamma(t))}(z_n)) = G_n(\gamma(t), z).$$

It follows that $G_n(\gamma, z)$ is homotopic to $G_n(x_0, z)$. □

Proof of Corollary 1. Here, $H_z(\gamma) = G_4(\gamma, z)$ is homotopic to $G_4(x_0, z) = F_4(\phi_4(x_0, z)) = \phi_{x_0}(z_1, z_2, z_3, z_4)$ (see the notation in Equation (2.2)), which is equal to $F_4(z_1, z_2, z_3, z_4)$. □

Proof of Corollary 2. This follows from Case 1, noting that $F_4(z, 1, 0, \infty) = z$. □

Proof of Corollary 3. Arguing as in the proof of Theorem A, if ϕ can be extended to a continuous motion $\widetilde{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, we have $\widetilde{\phi}_x(z) = \phi_x(z) = w^{\mu(x)}(z)$ for each z in E. For $0 \leq s \leq 1$ and $0 \leq t \leq 1$, define

$$\Gamma(s,t) = F_2\big(\phi_{s\gamma(t)}(z_1), \phi_{s\gamma(t)}(z_2)\big) = F_2\big(w^{s\mu(\gamma(t))}(z_1), w^{s\mu(\gamma(t))}(z_2)\big)$$

which is obviously continuous.

We have

$$\Gamma(0,t) = F_2(z_1, z_2) = F_2(\phi_{x_0}(z_1), \phi_{x_0}(z_2)) = H(x_0)$$

and

$$\Gamma(1,t) = F_2(\phi_{\gamma(t)}(z_1), \phi_{\gamma(t)}(z_2)) = H(\gamma).$$

So, $H(\gamma)$ is homotopic to $H(x_0)$ in X. But $H(x_0) = 0$ and X is discrete. Therefore, $H(\gamma) = 0$. □

4. An application of Theorem A

We first note the following fact that was proved in [13]. A holomorphic motion $\phi\colon V \times E \to \widehat{\mathbf{C}}$ is *trivial* if $\phi(x,z) = z$ for all $(x,z) \in V \times E$.

Theorem 4.1. *Let V be a connected complex Banach manifold with a basepoint. Then there exists a non-trivial holomorphic motion of $\widehat{\mathbf{C}}$ over V if and only if there is a non-constant bounded holomorphic function on V.*

See Theorem 1 in [13].

In this section we discuss an example of Douady, and a theorem of Earle ([5]). A partial generalization of Theorem 1 in [5] is an easy consequence of Corollary 2. In §2 of [5], Earle discusses an example of Douady of a holomorphic motion of a four-point set over $\mathbf{C} \setminus \{0,1\}$, that cannot be extended to a holomorphic motion of $\widehat{\mathbf{C}}$. That fact follows as an easy consequence of our Theorem A. Let $V = \mathbf{C} \setminus \{0,1\}$ and $E = \{0,1,\infty,a\}$ (where $a \neq 0,1,\infty$). Define $\phi\colon V \times E \to \widehat{\mathbf{C}}$ as: $\phi(t,0) = 0, \phi(t,1) = 1, \phi(t,\infty) = \infty$, and $\phi(t,a) = t$ for all t in V. We choose an arbitrary point x_0 in V as the basepoint. Then, it is clear that ϕ is a holomorphic motion of E over V with x_0 as a basepoint. Now, for the point a, we have $\phi^a(t) = \phi(t,a) = t$. Therefore, the induced homomorphism $(\phi^a)_*$ from $\pi_1(V)$ to itself is the identity, which is not trivial. Hence, by Corollary 2, ϕ cannot be extended to a continuous motion of $\widehat{\mathbf{C}}$, and hence cannot be extended to a holomorphic motion of $\widehat{\mathbf{C}}$.

Since $V = \mathbf{C} \setminus \{0,1\}$ admits no non-constant bounded holomorphic functions, Theorem 4.1 gives an alternative proof for the non-extendability of the above holomorphic motion ϕ.

Here, we show the following generalization; it also gives an example for which Theorem 4.1 cannot apply.

Theorem 4.2. *Let $V = \mathbf{C}\setminus\{0,1\}$ and $E = \{0,1,\infty,a\}$ (where $a \neq 0,1,\infty$). Let $\phi\colon V\times E \to \widehat{\mathbf{C}}$ be the holomorphic motion given by $\phi(t,0) = 0$, $\phi(t,1) = 1$, $\phi(t,\infty) = \infty$, and $\phi(t,a) = t$ for all t in V. (Let an arbitrary point $x_0 \in V$ be a basepoint.) Let W be any connected complex manifold such that there exists a holomorphic map $f\colon W \to V$ satisfying the condition $f_*\colon \pi_1(W) \to \pi_1(V)$ is nontrivial. Then, there exists a holomorphic motion $\psi\colon W \times E \to \widehat{\mathbf{C}}$ which cannot be extended to a continuous motion of $\widehat{\mathbf{C}}$.*

Proof. Without loss of generality, we can assume that $f(W)$ contains x_0. Choose $y_0 \in W$ such that $f(y_0) = x_0$. Define $\psi = f^*(\phi)$. Then, ψ is the required holomorphic motion of E over W with basepoint y_0. We have $\psi^a(s) = f^*(\phi)(s,a) = \phi(f(s),a) = f(s)$. Therefore, $(\psi^a)_* = f_*$, which is nontrivial by hypothesis. Hence, by Corollary 2, we are done. □

An example. Let R be a Belyi surface. That is, the Riemann surface R is defined over $\overline{\mathbf{Q}}$ as an algebraic curve. Then it follows from a theorem of Belyi ([2]) that there exists a holomorphic covering $f\colon R \to \widehat{\mathbf{C}}$ such that it is branched over only the points 0, 1, and ∞. Taking mutually disjoint simply connected neighborhoods U_0, U_1 and U_∞ of $0, 1$ and ∞, respectively, we choose $W = R \setminus f^{-1}(U_0 \cup U_1 \cup U_\infty)$. Then, W and f satisfy the conditions in Theorem 4.2. Obviously, W admits a non-constant bounded holomorphic function. Thus, we cannot apply Theorem 4.1 for this example.

Remark 4.3. In Theorem 1 of [5], Earle discusses three particular examples for W; when W is the punctured disk, an annulus, an once-punctured annulus. For these cases, he shows the non-extendability of the holomorphic motion ψ. Our Theorem 4.2 strengthens that part of Theorem 1 of [5]. For the above three cases for W, Earle actually proves more. He shows that the holomorphic motions are maximal. The same is true for Douady's example. See [5] for the definition of *maximal* holomorphic motions. Theorem 4.2 of our paper shows only the non-extendability of the holomorphic motion ψ to a holomorphic motion of $\widehat{\mathbf{C}}$. In fact, we show that ψ cannot be extended even to a continuous motion of $\widehat{\mathbf{C}}$.

5. Proof of Theorem B

Let $E = \{0, 1, \infty, a\}$ where $a \neq 0, 1, \infty$ and $(\phi^a)_*\colon \pi_1(V) \to \pi_1(\mathbf{C} \setminus \{0, 1\})$ the trace monodromy for a holomorphic motion $\phi\colon V \times E \to \widehat{\mathbf{C}}$.

If ϕ can be extended to a holomorphic motion $\widehat{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, then by Proposition 2.13 and Corollary 2, both ρ_ϕ and $(\phi^a)_*$ are trivial. For the other direction, we follow the argument in the first part of the proof of Theorem 2 of [5].

Suppose that $(\phi^a)_*$ is trivial. Let $\pi\colon \Delta \to \mathbf{C} \setminus \{0, 1\}$ be a holomorphic universal covering such that $\pi(0) = a$. Since $\phi^a\colon V \to \mathbf{C} \setminus \{0, 1\}$ has the property that $(\phi^a)_*$ is trivial, then $(\phi^a)_*(\pi_1(V)) \subseteq \pi_*(\pi_1(\Delta))$. Therefore, ϕ^a can be lifted to a holomorphic map $\widetilde{\phi^a}\colon V \to \Delta$. Clearly, $\widetilde{\phi^a}$ is basepoint preserving. Let $\psi\colon \mathbf{C} \setminus \{0, 1\} \times E \to \widehat{\mathbf{C}}$ be the holomorphic motion defined as follows: $\psi(t, 0) = 0, \psi(t, 1) = 1, \psi(t, \infty) = \infty$, and $\psi(t, a) = t$ for all t in $\mathbf{C} \setminus \{0, 1\}$. Define $\widehat{\psi} = \pi^*(\psi)$; then, $\widehat{\psi}\colon \Delta \times E \to \widehat{\mathbf{C}}$ is a holomorphic motion. By Slodkowski's theorem (see [15]) there exists a holomorphic motion $\widetilde{\psi}\colon \Delta \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$, such that $\widetilde{\psi}$ extends $\widehat{\psi}$. Finally, define $\widehat{\phi} = (\widetilde{\phi^a})^*(\widetilde{\psi})$. It is easy to see that $\widehat{\phi}\colon V \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ is a holomorphic motion, that extends ϕ.

For the case when the monodromy ρ_ϕ is trivial, the proof is similar. We leave the proof to the reader.

6. Proof of Theorem C

Let R be a hyperbolic Riemann surface with a basepoint p_0, and let $\phi\colon R \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion. Let $\{E_n\}$ be a sequence of finite subsets of E, with the following property:

$$\{0, 1, \infty\} \subset E_1 \subset \cdots \subset E_n \cdots$$

and $\bigcup_{n=1}^\infty E_n$ is dense in E. We are given a holomorphic motion $\phi\colon R \times E \to \widehat{\mathbf{C}}$.

Suppose that the restricted holomorphic motion ϕ_n of ϕ on E_n can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over R. We use the same letter ϕ_n for the extended holomorphic motion. For each $z \in \widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$, we set $F_n^z(\cdot) := \phi_n(\cdot, z)$. Then, $F_n^z\colon R \to \widehat{\mathbf{C}} \setminus \{0, 1, \infty\}$ is a holomorphic function on R. Since F_n^z does not take 0, 1, ∞, $\{F_n^z\}_{n=1}^\infty$ is a normal family.

Now, we consider a countable dense subset $\widehat{E}$ of $\widehat{\mathbf{C}}$ with $E_n \subset \widehat{E}$ $(n = 1, 2, \cdots)$ and put $\widehat{E} = \{z_k\}_{k=1}^{\infty}$. Since $\{F_n^{z_1}\}_{n=1}^{\infty}$ is a normal family, we may find a subsequence $\{n_j^1\}$ of $\mathbf{N}$ such that $\{F_{n_j^1}^{z_1}\}_{j=1}^{\infty}$ converges to a holomorphic function on R. We also find a subsequence $\{n_j^2\}$ of $\{n_j^1\}$ so that $\{F_{n_j^2}^{z_2}\}_{j=1}^{\infty}$ converges to a holomorphic function on R. By using the same method and the diagonal process, we find a subsequence $\{n_p\}$ such that $\{F_{n_p}^{z}\}$ converges to a holomorphic function F^z on R for $z \in \widehat{E}$.

Note that for each $p \in R$ and $z \in E_n$, we have

$$F^z(p) = \lim_{n_p \to \infty} F_{n_p}^z(p) = \lim_{n_p \to \infty} \phi_{n_p}(p, z) = \phi(p, z).$$

On the other hand, $\phi_n(p, \cdot)$ is a normalized K-quasiconformal map for some $K \geq 1$ if p belongs to a compact subset of R. Therefore, $\lim_{n_p \to \infty} \phi_{n_p}(p, \cdot) = w_p(\cdot)$ is also a K-quasiconformal map. Obviously, $w_p(z) = F^z(p)$ for $z \in \widehat{E}$ and $p \in R$.

We now put, for $(p, z) \in R \times \widehat{E}$,

$$\Phi(p, z) := F^z(p).$$

Note that:

(1) For the basepoint $p_0 \in R$, $\phi_n(p_0, z) = z$ for any $z \in \widehat{\mathbf{C}}$. Thus, $\Phi(p_0, z) = z$ for any $z \in \widehat{E}$.
(2) $\Phi(\cdot, z)$ is holomorphic on R.
(3) $\Phi(p, z) \neq \Phi(p, z')$ if $z \neq z'$ because $\Phi(p, \cdot) = w_p(\cdot)$ is a K-quasiconformal map.

It follows that $\Phi\colon R \times \widehat{E} \to \widehat{\mathbf{C}}$ is a holomorphic motion of $\widehat{E}$. Finally, since $\widehat{E}$ is dense in $\widehat{\mathbf{C}}$, Φ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ by the λ-lemma (see, for example, Lemma 14.1 in [11].) □

7. Proof of Theorem D

Let $E = \{z_1, \ldots, z_n\}$ and let $\phi\colon \Delta_K \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion.

Suppose that the holomorphic motion $\phi\colon \Delta_K \times E \to \widehat{\mathbf{C}}$ can be extended to a holomorphic motion $\widehat{\phi}$ of $\widehat{\mathbf{C}}$ over Δ_K. Then, by Proposition 2.13, the monodromy is trivial.

Conversely, suppose that the monodromy $\rho_\phi\colon \pi_1(\Delta_K) \to \mathrm{Mod}(0, n)$ is trivial. Let Γ_K be a Fuchsian group acting on Δ which represents Δ_K and $\pi\colon \Delta \to \Delta_K$ a holomorphic universal covering, with $\pi(0) = x_0$. We define a holomorphic motion $\Phi\colon \Delta \times E \to \widehat{\mathbf{C}}$ by $\Phi = \pi^*(\phi)$. So, we have

$$\Phi(x, z) = \phi(\pi(x), z) \quad \text{for all } (x, z) \in \Delta \times E.$$

By Slodkowski's theorem ([15]), there exists a holomorphic motion $\widetilde{\Phi}\colon \Delta \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ such that $\widetilde{\Phi}$ extends Φ. Therefore, by Theorem 2.3, there exists a basepoint preserving holomorphic map $f\colon \Delta \to M(\mathbf{C})$ such that

$$\widetilde{\Phi}(x, z) = \Phi(x, z) = w^{f(x)}(z) \quad \text{for } (x, z) \in \Delta \times E.$$

The correspondence $\Delta \ni x \mapsto [w^{f(x)}]$ defines a holomorphic map $\widetilde{F}$ from Δ to the Teichmüller space $T(0, n)$ (the Teichmüller space of $\widehat{\mathbf{C}}$ with n punctures). Since the monodromy ρ_ϕ is trivial, the holomorphic map $\widetilde{F}$ satisfies

$$\widetilde{F} \circ g = \widetilde{F}$$

for any $g \in \Gamma_K$. Hence, it can be projected to a holomorphic map F from $\Delta_K = \Delta/\Gamma_K$ to $T(0,n)$. Then, the holomorphic map F can be extended a holomorphic map of Δ. Indeed, for any point $\xi \in K$ there exists a sequence $\{x_k\}_k^\infty \subset \Delta_K$ such that $\lim_{k\to\infty} x_k = \xi$ (since K is compact AB- removable, it has no interior points). Since K is AB-removable, the Carathéodory distance c_{Δ_K} on Δ_K which is defined by the space of bounded holomorphic functions, is equal to $c_\Delta|_{\Delta_K}$, which is the restriction of the Carathéodory distance on Δ to Δ_K. Therefore, the sequence $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence with respect to c_{Δ_K}. By the distance decreasing property, we have

$$c_{T(0,n)}(F(x_k), F(x_\ell)) \leq c_{\Delta_K}(x_k, x_\ell) \quad (k, \ell \in \mathbf{N}),$$

where $c_{T(0,n)}$ is the Carathéodory distance on $T(0,n)$. Thus, $\{F(x_k)\}_{k=1}^\infty$ is also a Cauchy sequence in $T(0,n)$ and it converges to a point in $T(0,n)$ because of the completeness of the Carathéodory distance in the Teichmüller space ([4], [14]). Therefore, F can be extended to a holomorphic map of Δ. Since the extended holomorphic map gives a holomorphic motion of E over Δ which extends ϕ, it follows from Theorem 2.15 that ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over Δ_K. □

8. Proof of Theorem E

Proof of (1). We prove the statement by constructing a concrete example.

Let K be an AB-removable compact subset of Δ. We may assume that K contains the origin. Let Ω be a connected component of $\widehat{\mathbf{C}} \setminus E$ which is neither simply connected nor conformally equivalent to the punctured disk. Since E contains at least five points, there exists a simply connected domain D such that $\partial D \subset \Omega$, D contains at least two points of E, say z_1, z_2, and $D^c \cap E$ contains at least three points. We may assume that $0, 1$ and ∞ are not in D. We take z_0 in $D \setminus E$.

Let $h\colon \Delta \to D$ be a Riemann map with $h(0) = z_0$. Then, there exists a positive number $r < 1$ such that $h(\{r < |x| < 1\}) \cap E = \emptyset$. We construct a holomorphic motion ϕ_0 of E over Δ^* as follows; for $x \in \Delta^*$,

$$\phi_0(x,z) = \begin{cases} z, & z \in E \setminus D, \\ h(xh^{-1}(z)/r), & z \in E \cap D. \end{cases} \tag{8.1}$$

It is easy to see that ϕ_0 is a holomorphic motion of E over Δ^* with basepoint $r \in \Delta^*$ and the trace monodromy $(\phi_0^z)_*\colon \pi_1(\Delta^*) \to \pi_1(\widehat{\mathbf{C}} \setminus \{0,1,\infty\})$ is trivial for any $z \in E$. On the other hand, the holomorphic motion does not satisfy the winding number condition in Corollary 3.

Indeed, let γ be a simple closed curve in Δ^* such that the winding number of γ about the origin is one and $h(\gamma) \ni z_1, z_2$. Then, for a holomorphic function $f(x) := h(xh^{-1}(z_1)/r) - h(xh^{-1}(z_2)/r)$, $H(\gamma)$ defined by (2.5) is the winding number of $f(\gamma)$ about the origin. Hence, by the argument principle we see that it is the number of zeros of f inside γ and it is one since h is a conformal map on Δ.

If we restrict the holomorphic motion ϕ_0 over $\Delta_K \subset \Delta^*$, we obtain the desired holomorphic motion. □

Proof of (2). Let E be a closed subset of $\widehat{\mathbf{C}}$ such that every connected component of $\widehat{\mathbf{C}} \setminus E$ is either simply connected or conformally equivalent to the punctured disk.

Let $\phi\colon \Delta_K \times E \to \widehat{\mathbf{C}}$ be a holomorphic motion of E over Δ_K with a basepoint x_0. By Lemma 2.12, we have a quasiconformal mapping $w_c\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ for each $c \in \pi_1(\Delta_K, x_0)$ which represents $\rho_\phi(c)$ as in Equation (2.4). Since w_c keeps every point of E fixed, it also fixes every component of $\widehat{\mathbf{C}} \setminus E$ (cf. [7] Lemma 2.5). By using the hypothesis that every component of the complement of E is simply connected or conformally equivalent to the punctured disk, we will see from the following Lemma 8.1, that w_c is isotopic to the identity relative to E. Therefore, if we restrict the holomorphic motion of any finite subset E' of E, the monodromy of the restricted holomorphic motion must be trivial. Thus, from Corollary 4, we conclude that ϕ can be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over Δ_K. □

Lemma 8.1. *Let w be a quasiconformal self map of $\widehat{\mathbf{C}}$ and D a subdomain of $\widehat{\mathbf{C}}$ which is simply connected or conformally equivalent to the punctured disk. Suppose that w keeps every point of ∂D fixed. Then, $w|_D$ is isotopic to the identity relative to ∂D.*

Proof. First, we consider the case where D is simply connected. We may assume that $D \neq \mathbf{C}, \widehat{\mathbf{C}}$ (otherwise, the statement of the lemma is trivial). Take a point $z_0 \in D$ as a basepoint. If $w(z_0) \neq z_0$, then we take a simply connected subregion U of D bounded by a smooth Jordan curve in D. It is not hard to see there exists a quasiconformal selfmap w_0 of $\widehat{\mathbf{C}}$ such that w_0 is the identity in $\widehat{\mathbf{C}} \setminus U$, $w_0(w(z_0)) = z_0$ and $w_0|D$ is isotopic to the identity relative to ∂D in D. By considering $w_0 \circ w$ instead of w, we may assume that $w(z_0) = z_0$.

Let $f\colon \Delta \to D$ be a Riemann map with $f(0) = z_0$ and put $\omega := f^{-1} \circ w|D \circ f\colon \Delta \to \Delta$. To show the statement we use a result by Earle–McMullen.

Proposition 8.2. (cf. [6], Corollary 2.4) *Let X be a hyperbolic planar region and $w\colon X \to X$ a quasiconformal map. We consider the unit disk as the universal covering of X. Let $\omega\colon \Delta \to \Delta$ denote a lift of w to the universal covering Δ. Then the following are equivalent.*

(1) *ω is the identity on $\partial\Delta$.*
(2) *w is isotopic to the identity relative to ∂X.*

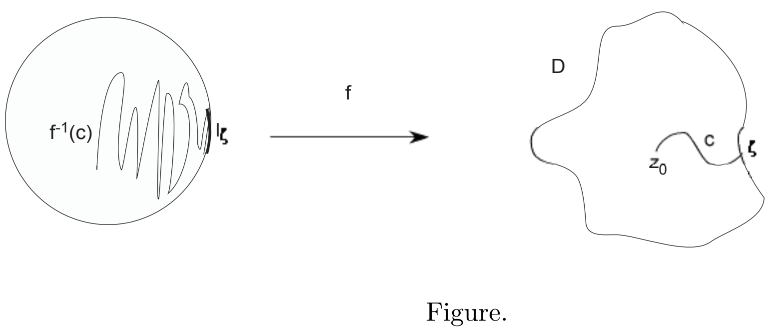

Figure.

Now, we take a point $\zeta \in \partial D$ which is accessible from D. Then, there exists a Jordan curve c in D which connects z_0 and ζ. We may assume that ∂D is a bounded subset of $\mathbf{C}$. Then, f is bounded near $\partial\Delta$. Classical results of bounded holomorphic functions guarantee that $f^{-1}(c)$ lands at a single point on $\partial\Delta$.

Indeed, if the set of the accumulation points of $f^{-1}(c)$ on $\partial\Delta$ consists of more than one points, it contains an open interval I_ζ on $\partial\Delta$ with positive length. Then, for almost all points $x \in I_\zeta$ we may find a sequence $\{p_n\}_{n=1}^{\infty}$ on $f^{-1}(c)$ such that it converges to x non-tangentially (See Figure).

Since $p_n \in f^{-1}(c)$, we have $\lim_{n\to\infty} f(p_n) = \zeta$. On the other hand, it is well known that f has non-tangential limits almost everywhere on $\partial\Delta$ (cf. [8]; Chapter 1, Theorem 5.3). Thus, we conclude that the non-tangential limit of f at $x \in I_\zeta$ is ζ, if the limit exists. Since I_ζ has positive length, it follows from F. and M. Riesz Theorem (cf. [8]; Chapter 2, Corollary 4.2) that f must be a constant and that is a contradiction. Therefore, $f^{-1}(c)$ lands at a single point on $\partial\Delta$.

We claim that the two Jordan curves $f^{-1}(c)$ and $\omega(f^{-1}(c))(= f^{-1}(w(c)))$ terminate at the same point on $\partial\Delta$.

Suppose that $f^{-1}(c)$ and $\omega(f^{-1}(c))$ terminate at distinct points x_1 and x_2 on $\partial\Delta$, respectively. Then, there exists a Jordan domain Ω_0 in Δ bounded by subarcs of $f^{-1}(c)$, $\omega(f^{-1}(c))$ and a subarc I_1 of $\partial\Delta$ between x_1 and x_2. Write $L := f(\partial\Omega_0 \cap \Delta)$. Then, $\widehat{L} := L \cup \{\zeta\}$ is a simple closed curve in $\widehat{\mathbf{C}}$. Let D_0 be a simply connected domain bounded by $\widehat{L}$ with $D_0 \supset f(\Omega_0)$.

Since ω is a quasiconformal selfmap of Δ, the map can be homeomorphically extended to $\overline{\Delta}$ and it is orientation preserving. Hence, there exists a small arc $\delta_1 \subset I_1$ near x_1 such that $\omega(\delta_1) \cap I_1 = \emptyset$. Noting that that f has non-tangential limits almost everywhere on $\partial\Delta$, we may find a point $x_0 \in \delta_1$ so that f has a limit along the radius ℓ_0 from the origin to x_0. In other words, $f(\ell_0)$ terminates at a point $\zeta_0 \in \partial D$. Since $f(\omega(\ell_0)) = w(f(\ell_0))$ and w keeps every point of ∂D fixed, $f(\omega(\ell_0))$ also terminates at ζ_0. This implies that $f(\omega(\ell_0))$ eventually belongs to $f(\Omega_0) \subset D_0$. However, it is absurd because $\omega(\ell_0)$ terminates at $\omega(\zeta_0) \in \omega(\delta_1)$. Thus, we verify that $f^{-1}(c)$ and $\omega(f^{-1}(c))$ terminate at the same point on $\partial\Delta$.

The above argument shows that $\omega(x) = x$ if for the radius ℓ_x from the origin to $x \in \partial\Delta$, $f(\ell_x)$ ends at a single point of ∂D. Since $f(\ell_x)$ ends at a single point for almost every $x \in \partial\Delta$ and ω is a homeomorphism on $\Delta \cup \partial\Delta$, ω fixes every point on $\partial\Delta$. Hence, it follows from Proposition 8.2 that w is isotopic to the identity relative to ∂D.

Next, we assume that D is conformally equivalent to the punctured disk Δ^*. Let a be the puncture of D. Then, D is represented by $\mathbf{H}/ < g >$, where $\mathbf{H}$ is the upper half plane and $g(z) = z+1$. Let $\pi\colon \mathbf{H} \to D$ be a canonical projection and $\omega\colon \mathbf{H} \to \mathbf{H}$ a lift of w.

Let $\zeta \in \partial D\setminus\{a\}$ be an accessible point. We take a Jordan arc c in D connecting a and ζ. By using the same argument as above, we see that any connected component $\tilde{c}$ of $\pi^{-1}(c)$ is a Jordan arc in $\mathbf{H}$ connecting ∞ and some point on $\mathbf{R}$. Furthermore, we also see that $\tilde{c}$ and $\omega(\tilde{c})$ end the same points and we conclude that ω is the identity on $\mathbf{R} \cup \{\infty\}$. Hence, it follows from Proposition 8.2 that w is isotopic to the identity relative to ∂D. □

Remark 4. In this lemma, it is crucial that the map w is quasiconformal on $\widehat{\mathbf{C}}$. In fact, there exists a simply connected domain D and an orientation preserving homeomorphism $w\colon D \to D$ such that $\omega\colon \Delta \to \Delta$ given in the above proof is homotopic to the identity relative to $\partial\Delta$ while w does not have a continuous extension to $\overline{D}$ (cf. [6]).

Example 8.3. Let R be a Riemann surface. Suppose that there exists a holomorphic map $f\colon R \to \Delta^*$ such that $f_*(\pi_1(R))$ is non-trivial. Then, for any closed set E satisfying the condition of Theorem E(1) there exists a holomorphic motion ϕ of E over R such that the trace monodromy $(\phi^z)^*$ is trivial for any $z \in E$ but ϕ cannot be extended to a holomorphic motion of $\widehat{\mathbf{C}}$ over R. Indeed, we define a holomorphic motion ϕ_0 of E over Δ^* by (8.1) and $\phi = f^*(\phi_0)$. Since $f_*(\pi_1(R))$ is not trivial, there exists a closed curve c in R such that $f(c)$ is not trivial in Δ^*. Therefore, by the same argument as in the proof of Theorem E(1), we can verify that ϕ cannot be extended to a holomorphic motion of $\widehat{\mathbf{C}}$.

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