HILBERT MATRIX OPERATOR ON SPACES OF ANALYTIC FUNCTIONS

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Abstract. We consider the action of the Hilbert matrix operator, $H$, on the Hardy space $H^1$, weighted Hardy spaces $H^p_\alpha$ ($\alpha \geq 0$), Bergman spaces with logarithmic weights, etc. In particular, we extend Diamantopoulos–Siskakis result by proving that $H$ maps $H^p_\alpha$ into $H^p_\alpha$ if and only if $\alpha + 1/p < 1$. A criterion for $Hf$ to belong to $H^1$ is given provided the coefficients of $f$ are nonnegative. Also, $H$ maps the $A^2$-space with weight $\log(2/(1-|z|^2))$ into the ordinary Bergman space $A^2$ if $\alpha > 3$. Similarly, the Bloch space with logarithmic weight is mapped by $H$ into the ordinary Bloch space.

1. Introduction

The Hilbert matrix is an infinite matrix $H$ whose entries are $a_{n,k} = (n+k+1)^{-1}$. This matrix induces a linear operator on sequences:

$$H : (a_k)_{k \in \mathbb{N}_0} \mapsto \left( \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right)_{n \in \mathbb{N}_0}.$$

The following Hilbert’s inequality implies that this operator is well defined and bounded on the space $l^p$ of all $p$-summable sequences ($p > 1$).

**Theorem 1.1.** (Hilbert’s inequality [5, Chapter IX]) Suppose $1 < p < \infty$. If $(a_k)_{k \in \mathbb{N}_0} \in l^p$, then

$$\left( \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right)^p \right)^{\frac{1}{p}} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{k=0}^{\infty} |a_k|^p \right)^{\frac{1}{p}}.$$  

Moreover, the constant $\frac{\pi}{\sin \frac{\pi}{p}}$ is best possible.

Apart from sequence spaces, the Hilbert matrix can be viewed as an operator on spaces of analytic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$$

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is a holomorphic function in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$, then we define a transformation $H$ by

$$
Hf(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n.
$$

Let $H(D)$ be the algebra of holomorphic functions in $D$. For $0 < p \leq \infty$ Hardy space $H^p$ is the space of all holomorphic functions $f \in H(D)$ for which

$$
\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty,
$$

where

$$
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty;
$$

$$
M_\infty(r, f) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.
$$

It follows from the Hardy’s inequality ([4], p. 48)

$$
\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \leq \pi \|f\|_1
$$

that $H$ is well defined for each $f \in H^p$, $p \geq 1$. It was proved by Diamantopoulos and Siskakis ([1]) that the operator $H$ is bounded on $H^p$, $1 < p < \infty$, and not bounded on $H^1$ and $H^\infty$. In [3] the following formula for $H$ acting on $H^p$, $p \geq 1$, was noticed

$$
Hf = P_+(M_b Cf),
$$

where $Cf(e^{it}) = f(e^{-it})$ is an isometry from $H^p$ into $L^p(T)$, $M_b(u) = bu$, $b(t) = ie^{-it}(\pi - t), 0 \leq t < 2\pi$ and $P_+$ is the Szegö projection given by

$$
P_+u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(t)}{(1 - ze^{-it})} dt, \quad z \in D.
$$

Recall that the space BMOA consists of the functions $f \in H^1$ whose boundary values $f(e^{it})$ are of bounded mean oscillation on $T$, that is

$$
\sup_I \int_I |f(e^{it}) - I(f)| dt < \infty,
$$

where supremum is taken over all intervals $I \subset T$ and

$$
I(f) = \frac{1}{|I|} \int_I f(e^{it}) dt.
$$

If

$$
\lim_{|I| \to 0} \int_I |f(e^{it}) - I(f)| dt = 0,
$$

then we say that $f \in \text{VMOA}$.

Since the space BMOA is the Szegö projection of $L^\infty(T)$, we have also the following

**Theorem 1.2.** The Hilbert matrix operator $H$ acts as a bounded operator from $H^\infty$ into BMOA.

The next theorem describes the polynomials that are mapped by $H$ into VMOA.
Theorem 1.3. Let \( w \) be a polynomial of degree at least 1. Then \( Hw \in VMOA \) if and only if \( w(1) = 0 \).

Proof. We know that the operator \( Hw = P_+(w(e^{-i\theta})b(\theta)) \), where \( b(\theta) = ie^{-i\theta}(\pi - \theta) \) for \( 0 \leq \theta < 2\pi \). The function \( b \) is continuous on the unit circle \( T \) except for 1. If \( w(1) = 0 \), then the function \( w(e^{-i\theta})b(\theta) \) can be continuously extended on the whole unit circle and \( Hw \) is the Szegö projection of this continuous function which means that \( Hw \in VMOA \). It is also clear that if the function \( w(e^{-i\theta})b(\theta) \) is continuous on \( T \) then \( w(1) = 0 \). \( \square \)

In the next section we show that if \( f \in H^1 \), then \( Hf \) extends to a continuous function on \( \mathbb{D} \setminus \{1\} \) and give a sufficient condition for \( Hf \in H^1 \). In the case of positive Taylor coefficients we obtain a sufficient and necessary condition for \( Hf \in H^1 \). Section 3 is devoted to the weighted Hardy spaces \( H^p_\alpha \), \( 0 < p \leq \infty \), \( \alpha > 0 \), consisting of those \( f \in H(D) \) for which \( M_p(r,f) = O((1-r)^{-\alpha}) \). We prove that the Hilbert matrix operator is bounded on \( H^p_\alpha \) if and only if \( \alpha + 1/p < 1 \). It is known that the operator \( H \) cannot be defined on the Bergman space \( A^2 \) of analytic functions that are square integrable over the unit disk with respect to the Lebesgue area measure. Here we find the subspace of \( A^2 \) which is mapped by \( H \) boundedly into \( A^2 \). Finally, we study the acting of the operator \( H \) on the Bloch space and Besov spaces.

Throughout the paper the notion \( A \asymp B \) means that there exists a positive constant \( C \) such that \( B/C \leq A \leq CB \).

2. Hilbert matrix operator acting on \( H^1 \)

This section contains results on the Hilbert matrix operator acting on \( H^1 \) that are analogous to that obtained for the Libera operator in [12]. The proofs presented here are slightly different from the proofs given in [12].

We start with the following

Lemma 2.1. If \( f \in H^1 \), then \( Hf \) extends to a continuous function on \( \mathbb{D} \setminus \{1\} \).

Proof. By (1.2),

\[
Hf(z) = \frac{1}{1-z} F_f(z),
\]

where

\[
F_f(z) = (1-z) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n.
\]

We will show that the function \( F_f \) can be continuously extended to \( \overline{D} \). For \( z \in D \) we have

\[
F_f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^{n+1}
\]

\[
= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k} z^n
\]

\[
= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} - \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{(n+k)(n+k+1)} z^n.
\]
To see that the last double series converges absolutely and uniformly on $D$ it is enough to note that

$$\sum_{k=0}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{(n+k)(n+k+1)} \right) |\hat{f}(k)| = \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1}. \quad \square$$

Consequently, we also get the following

**Corollary 2.2.** The operator $H$ acts as a bounded operator from $H^1$ into $H^p$, $0 < p < 1$.

**Theorem 2.3.** If $f \in H^1$ is such that

$$\int_{-\pi}^{\pi} |f(e^{it})| \log \frac{\pi}{|t|} dt < \infty,$$

then $Hf \in H^1$.

**Proof.** We first show that if $f$ satisfies the assumptions, then the function $g(z) = f(z) \log \frac{1}{1-z}$ is in $H^1$. To this end, we note that

$$\int_{-\pi}^{\pi} |f(e^{it})| \log \frac{2}{|1-e^{it}|} dt = \int_{-\pi}^{\pi} |f(e^{it})| \log \frac{1}{|\sin \frac{t}{2}|} dt \leq \int_{-\pi}^{\pi} |f(e^{it})| \log \frac{\pi}{|t|} dt,$$

which implies that $g(e^{it})$ is in $L^1(\partial D)$. Since $g \in H^p$, $0 < p < 1$, the Smirnov theorem (see, e.g., [9] p. 74) implies that $g$ is in $H^1$. Now using the formula (see [2]),

$$Hf(z) = \int_{0}^{1} \frac{f(r)}{1-rz} dr, \quad z \in D,$$

and Fubini theorem we get

$$\frac{1}{2\pi} \int_{0}^{2\pi} |Hf(e^{it})| dt \leq \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{1} \frac{|f(r)|}{|1-re^{it}|} dr dt = \int_{0}^{1} \frac{f(r)}{2\pi} \int_{0}^{2\pi} \frac{dt}{|1-re^{it}|} dr \leq C \int_{0}^{1} |f(r)| \log \frac{2}{1-r} dr.$$

Applying the Fejér–Riesz inequality to $g$, we see that $Hf \in L^1(\partial D)$. Since $Hf$ is in $H^p$ for $0 < p < 1$, the Smirnov theorem implies that $Hf$ is in $H^1$. \quad \square

**2.1. The case of positive coefficients.** If $\hat{f}(k) \geq 0$ for all $k$, then $Hf$ is well defined by (1.2) or by (2.3) if and only if

$$\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} < \infty.$$ 

To see the “only if” part it is enough to take $z = 0$. Furthermore, it is shown in [14] that if $\hat{f}(k) \downarrow 0$, then $f$ is in $H^1$ if and only if (2.4) holds. We use this fact to prove:

**Theorem 2.4.** If $\hat{f}(k) \geq 0$, then $Hf \in H^1$ if and only if

$$\sum_{n=0}^{\infty} \frac{\hat{f}(n) \log(n+2)}{n+1} < \infty.$$
Proof. The coefficients of \( h = Hf \) are given by
\[
\hat{h}(n) = \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n + k + 1}
\]
and obviously \( \hat{h}(n) \downarrow 0 \) as \( n \to \infty \). Hence, by what we mentioned above, \( h \in H^1 \) if and only if
\[
\sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{(n + k + 1)} < \infty.
\]
Now note that this double sum is equal to
\[
\sum_{k=0}^{\infty} \hat{f}(k) \sum_{n=0}^{\infty} \frac{1}{(n + 1)(n + k + 1)} = \hat{f}(0) \sum_{n=0}^{\infty} \frac{1}{(n + 1)^2} + \sum_{k=1}^{\infty} \frac{\hat{f}(k)}{k} \sum_{n=0}^{\infty} \left( \frac{1}{n + 1} - \frac{1}{n + k + 1} \right)
\]
which implies the result. \( \square \)

Now let us consider the space \( \mathcal{B}^1 \subsetneq H^1 \) defined by
\[
\mathcal{B}^1 = \left\{ f \in H(D) : \int_D |f'(z)| \, dA(z) < \infty \right\}.
\]
It was also shown in [14] that if \( \hat{f}(k) \downarrow 0 \) then \( f \) belongs to \( \mathcal{B}^1 \) if and only if (2.4) holds. This can be used to strengthen the statement that \( H \) does not map \( H^1 \) into itself. More exactly we have

**Proposition 2.5.** The operator \( H \) does not map \( \mathcal{B}^1 \) into \( H^1 \).

**Proof.** By the above, the function
\[
f(z) = \sum_{n=2}^{\infty} \frac{z^n}{\log^{3/2} n}
\]
belongs to \( \mathcal{B}^1 \), while \( Hf \), by Theorem 2.4, does not belong to \( H^1 \). \( \square \)

### 3. Weighted Hardy spaces

For \( \alpha > 0 \) and \( 0 < p \leq \infty \), we define the weighted Hardy spaces \( H^p_\alpha \) as follows.
\[
H^p_\alpha = \{ f \in H(D) : M_p(r, f) = O(1 - r)^{-\alpha} \},
\]
The norm in these spaces is defined by
\[
\|f\|_{p,\alpha} = \sup_{0<r<1} (1 - r)^\alpha M_p(r, f).
\]
We start with the following

**Theorem 3.1.** If \( \alpha + 1/p < 1 \), then the operator \( H \) maps \( H^p_\alpha \) into \( H^p_\alpha \).
Proof. Let $h = Hf$, $f \in H^p_\alpha$. Then we have

$$h'(z) = \int_0^1 \frac{rf(r) \, dr}{(1 - rz)^2}.$$ 

Using Minkowski’s inequality, the inequality

$$\int_0^{2\pi} |1 - \rho e^{it}|^{-2p} \, dt \asymp (1 - \rho)^{1-2p},$$

and the inequality

$$|f(r)| \leq C(1 - r)^{-\alpha - 1/p} \quad \text{(implied by } f \in H^p_\alpha\text{)}$$

we get

$$M_p(\rho, h') \leq C \int_0^1 |f(r)|(1 - \rho r)^{1/p-2} \, dr \leq C \int_0^1 (1 - r)^{-\alpha - 1/p}(1 - \rho r)^{1/p-2} \, dr$$

$$\leq C(1 - \rho)^{-\alpha - 1/p} \int_0^\rho (1 - r)^{1/p-2} \, dr + C(1 - \rho)^{1/p-2} \int_\rho^1 (1 - r)^{-\alpha - 1/p} \, dr.$$

Now the desired result is obtained by simple computation. It is enough to observe that $1/p - 2 < -1$ and that $-\alpha - 1/p > -1$. \hfill \Box

3.1. The case of monotone coefficients. Now our aim is to prove the following

**Theorem 3.2.** If $\{\hat{f}(k)\}$ is a positive monotone sequence, then $f = \sum_{k=0}^\infty \hat{f}(k) z^k \in H^p_\alpha (1 < p < \infty)$ if and only if

(3.1)

$$\hat{f}(k) \leq C(k + 1)^{\alpha + 1/p - 1}.$$

Let

$$\Delta_n(z) = \sum_{k \in I_n} z^k, \quad n \geq 0,$$

where

$$I_0 = \{0, 1\}, \quad I_n = \{2^n \leq k \leq 2^{n+1} - 1\}, \quad n \geq 1.$$

For $f \in H(D)$, let

$$\Delta_n f(z) = \sum_{k \in I_n} \hat{f}(k) z^k.$$

The following fact was proved in [11].

**Lemma 3.3.** Let $1 < p < \infty$. A function $f \in H(D)$ is in $H^p_\alpha$ if and only if

$$K(f) := \sup_n 2^{-n\alpha} \|\Delta_n f\|_p < \infty,$$

and we have $K(f) \asymp \|f\|_{p, \alpha}$.

**Lemma 3.4.** If $1 < p < \infty$ and $\{\lambda_n\}$ is a positive monotone sequence, then

$$C^{-1}\lambda_{2^n} \|\Delta_n\|_p \leq \left\| \sum_{k \in I_n} \lambda_k z^k \right\|_p \leq C\lambda_{2^{n+1}} \|\Delta_n\|_p \quad \text{if } \{\lambda_n\} \text{ is increasing},$$

$$C^{-1}\lambda_{2^{n+1}} \|\Delta_n\|_p \leq \left\| \sum_{k \in I_n} \lambda_k z^k \right\|_p \leq C\lambda_{2^n} \|\Delta_n\|_p \quad \text{if } \{\lambda_n\} \text{ is decreasing}.$$
Proof. Since \( \{z^n\} \) is a Schauder basis in \( H^p \), \( 1 < p < \infty \), by Proposition 1.a.3 in [10], for any sequence \( \{a_k\} \) and \( 0 \leq m \leq j < n \),
\[
\left\| \sum_{k=m}^j a_k z^k \right\|_p \leq C \left\| \sum_{k=m}^n a_k z^k \right\|_p ,
\]
where the constant \( C \) is independent of \( \{a_k\}, m, n \) and \( j \).

By summation by parts,
\[
\sum_{k=m}^n \lambda_k a_k z^k = \sum_{k=m}^{n-1} (\lambda_k - \lambda_{k+1}) s_k + \lambda_n s_n ,
\]
where
\[
s_k = \sum_{j=m}^k a_j z^j .
\]

Consequently,
\[
\left\| \sum_{k=m}^n \lambda_k a_k z^k \right\|_p \leq C \left( \sum_{k=m}^{n-1} |\lambda_k - \lambda_{k+1}| + \lambda_n \right) \left\| \sum_{k=m}^n a_k z^k \right\|_p .
\]

If \( \{\lambda_k\} \) is monotonically decreasing, then the sum in brackets is \( \lambda_m \), if \( \{\lambda_k\} \) is monotonically increasing, this sum is \( (\lambda_n - \lambda_m) + \lambda_n \leq 2\lambda_n \). This proves the right-hand side inequalities. To prove the left inequalities we observe that if, for example, \( \{\lambda_k\} \) increases, then \( 1/\lambda_k \) decreases and by what we have already proved,
\[
\left\| \sum_{k=m}^n a_k z^k \right\|_p = \left\| \sum_{k=m}^n \frac{1}{\lambda_k} (\lambda_k a_k z^k) \right\|_p \leq C \frac{1}{\lambda_m} \left\| \sum_{k=m}^n \lambda_k a_k z^k \right\|_p . \tag*{□}
\]

Proof of Theorem 3.2. Assume first that \( 1 < p < \infty \). Since
\[
\|\Delta_n\|_p = \|1 + z + \cdots + z^{2^n-1}\|_p \sim 2^{n(1-1/p)} ,
\]
Lemmas 3.3 and 3.4 imply that \( f \in H^p_\alpha \) if and only if
\[
\hat{f}(2^n) \leq C' 2^n(\alpha+1/p-1) ,
\]
and our claim follows from the monotonicity of \( \{\hat{f}(k)\} \). \tag*{□}

3.2. Necessity of the condition \( \alpha + 1/p < 1 \). To include in our considerations the cases \( p = 1, \infty \), we use polynomials \( W_n \) (instead of \( \Delta_n \)) constructed in [7] (see also [13]). Let \( \varphi \) be a \( C^\infty \)-function on \( \mathbb{R} \) such that \( \varphi(t) = 1 \) for \( t \leq 1 \), \( \varphi(t) = 0 \) for \( t \geq 2 \), and \( \varphi(t) \) is positive and decreasing on \( (1, 2) \). We set
\[
W_0(z) = 1 + z , \quad W_n(z) = \sum_{k \in J_n} \omega(k/2^{n-1}) z^k , \quad n \geq 1 ,
\]
where
\[
J_n = \{k \in \mathbb{N} : 2^{n-1} \leq k \leq 2^{n+1}\}
\]
and
\[
\omega(t) = \varphi(t/2) - \varphi(t) .
\]
The convolution \( f * g \) of two functions \( f, g \in H(D) \) is defined by

\[
    f * g(z) = \sum_{n=0}^{\infty} \hat{f}(n) \hat{g}(n) z^n,
\]

where \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \) and \( g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \).

The following inequality was proved in [6] and [7].

\[
    (3.2) \quad ||W_n * f||_p \leq C ||f||_p, \quad n = 0, 1, 2, \ldots, \quad 0 < p \leq \infty.
\]

We will also need the following lemmas.

**Lemma 3.5.** [8] Let \( 0 < p \leq \infty \). A function \( f \in H(D) \) is in \( H^p_\alpha \) if and only if \( ||W_n * f||_p = O(2^{n\alpha}) \), and we have

\[
    ||f||_{p,\alpha} \asymp \sup_n 2^{-n\alpha} ||W_n * f||_p.
\]

**Lemma 3.6.** [13, Exercise 7.3.5] Let \( p \in (0, \infty) \), \( P(z) = \sum_{k=m}^{4m} a_k z^k \), \( Q(z) = \sum_{k=m}^{4m} (k + 1)^{\beta} a_k z^k \), where \( m \) is a positive integer and \( \beta \in \mathbb{R} \). Then there is a constant \( C = C(p, \beta) \) such that

\[
    C^{-1} m^{\beta} ||P||_p \leq ||Q||_p \leq C m^{\beta} ||P||_p.
\]

Moreover, for \( |\beta| < \frac{1}{2} \) the constants \( C(p, \beta) \) are uniformly bounded with respect to \( \beta \).

**Proof.** Let \( W_m \) be a trigonometric polynomial such as in Lemma 7.3.2 in [13] with \( \psi(x) = (x + 1/m)^{\beta} \varphi(x) \), where \( \varphi \) is a \( C^\infty \)-function such that \( \text{supp}(\varphi) \subset (\frac{1}{2}, 5) \) and \( \varphi(x) = 1 \) for \( x \in [1, 4] \). Then

\[
    W_m * P(z) = m^{-\beta} Q(z).
\]

Our claim will follow from Theorem 7.3.4 in [13] if we can find the constant \( C_N \) in Lemma 7.3.2 [13] independent of \( \beta \) and \( m \). But, since \( \text{supp}(\varphi) \subset (\frac{1}{2}, 5) \), the Leibniz formula

\[
    \psi^{(N)}(x) = \sum_{j=0}^{N} \binom{N}{j} \beta(\beta - 1) \ldots (\beta - j + 1) \left(x + \frac{1}{m}\right)^{\beta-j} \varphi^{(N-j)}(x)
\]

implies that \( |\psi^{(N)}(x)| \) is bounded uniformly with respect to \( \beta \) and \( m \) and the claim follows. \( \square \)

To prove the last statement it is enough to show that

**Lemma 3.7.** For \( p \in [1, \infty] \) we have

\[
    ||W_n||_p \asymp 2^{n(1 - 1/p)}.
\]

**Proof.** The case \( p = \infty \) is easy. Assume that \( 1 \leq p < \infty \). Since

\[
    M_\infty(r, W_n) \leq C (1 - r)^{-1/p} ||W_n||_p
\]

and \( M_\infty(r, W_n) \geq r^{2n+1} ||W_n||_\infty \), taking \( r = 1 - 2^{-n-1} \), we obtain

\[
    C ||W_n||_p \geq 2^{n(1 - 1/p)}.
\]
To prove the reverse inequality, we take \( f(z) = (1 - z)^{-2} \) and use Lemma 3.1 in [11] and (3.2) to obtain
\[
r^{2n+1} \left\| \sum_{k \in J_n} (k+1)\hat{W}_n(k)z^k \right\|_p \leq M_p(r, W_n * f) = \| W_n * f_r \|_p \leq C(1-r)^{-2+1/p}.
\]
Taking \( r = 1 - 2^{-n-1} \) we get
\[
\left\| \sum_{k \in J_n} (k+1)\hat{W}_n(k)z^k \right\|_p \leq C2^{(2-1/p)n},
\]
and the result follows from Lemma 3.6.\( \square \)

We will show that if the condition \( \alpha + 1/p < 1 \) is not satisfied, then the operator \( H \) cannot be extended as a continuous operator even into the space \( H(D) \). More exactly, we have

**Theorem 3.8.** If \( \alpha + 1/p \geq 1, \alpha > 0 \) and \( p \geq 1 \), then the operator \( H \) cannot be extended to a continuous operator from \( H^p_{\alpha} \) into \( H(D) \).

**Proof.** For \( \beta \in (0, \frac{1}{2}) \) set
\[
f_{\beta}(z) = \sum_{k=0}^{\infty} (k+1)^{-\beta}z^k.
\]
Then Lemmas 3.6 and 3.7, and the assumption \( \alpha \geq 1 - 1/p \) imply that
\[
\| W_n * f_{\beta} \|_p \leq C2^{-n\beta}\| W_n \|_p \leq C2^{n(-\beta+1-1/p)} \leq C2^{n(-\beta+\alpha)} \leq C2^{\alpha}.
\]
By Lemma 3.5 this means that
\[
\sup_{0<\beta<\frac{1}{2}} \| f_{\beta} \|_{p,\alpha} < \infty.
\]
If \( H \) could be extended to a bounded operator from \( H^p_{\alpha} \) into \( H(D) \), then we would have \( \sup_{0<\beta<\frac{1}{2}} |Hf_{\beta}(0)| < \infty \), because \( f \mapsto f(0) \) is a continuous linear operator on \( H(D) \). However,
\[
Hf_{\beta}(0) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\beta+1}}
\]
and \( \lim_{\beta \to 0} Hf_{\beta}(0) = \infty \). This contradiction proves the result. \( \square \)

4. Logarithmically weighted Bergman spaces

It is known that the Hilbert matrix does not act on \( A^2 \) (see [3]). For \( \alpha > 0 \) we define the logarithmically weighted Bergman space \( A^2_{\log^\alpha} \subset A^2 \) as follows.
\[
A^2_{\log^\alpha} = \{ f \in H(D) : \| f \|_{\log^\alpha}^2 = \int_D |f(z)|^2 \left( \log \frac{2}{1-|z|^2} \right)^\alpha dA(z) < \infty \},
\]
where \( dA(z) \) is the area measure on \( D \) normalized so that \( \int_D dA(z) = 1 \). The following lemma can be proved in a standard way.
Lemma 4.1. If $f \in A^{2}_{\log^{\alpha}}$, $\alpha > 0$, then there exists a constant $C > 0$ such that

$$|f(z)| \leq \frac{C\|f\|_{\log^{\alpha}}}{(1 - |z|^2) \left(\log\frac{2}{1-|z|^2}\right)^{\frac{\alpha}{2}}}$$

for every $z \in D$.

We claim that $H$ is well defined on $A^{2}_{\log^{\alpha}}$ for $\alpha > 2$. This follows from the following

Lemma 4.2. Let $\alpha > 2$. If $f \in A^{2}_{\log^{\alpha}}$, then

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \leq C\|f\|_{\log^{\alpha}}.$$

Proof. Since the function $s \mapsto M_2(s, f)$ is increasing on $[0, 1)$, the Chebyshev inequality implies

$$\|f\|_{\log^{\alpha}}^2 = \int_0^1 M_2^2(s, f) \left(\log\frac{2}{1-s^2}\right)^{\alpha} s \, ds \geq \frac{1}{2} \int_r^1 M_2^2(s, f) \left(\log\frac{2}{1-s^2}\right)^{\alpha} \, ds \geq \frac{1}{2} (1-r) \left(\log\frac{2}{1-r^2}\right)^{\alpha} M_2^2(r, f).$$

This means that for $r \in [0, 1)$,

$$\sum_{k=0}^{\infty} |\hat{f}(k)|^2 r^{2k} \leq C\|f\|_{\log^{\alpha}}^2 (1-r)^{-1} \left(\log\frac{2}{1-r}\right)^{-\alpha}.$$

Taking $r = 1 - 1/m$, we get

$$\sum_{k=m}^{2m} |\hat{f}(k)|^2 \leq C\|f\|_{\log^{\alpha}}^2 m (\log 2m)^{-\alpha}.$$

Consequently, for $\alpha > 2$, we have

$$\sum_{k=1}^{\infty} \frac{|\hat{f}(k)|}{k+1} = \sum_{k=1}^{\infty} \sum_{j=2k-1}^{2k-1} \frac{|\hat{f}(j)|}{j+1} \leq \sum_{k=1}^{\infty} 2^{1-k} \sum_{j=2k-1}^{2k-1} |\hat{f}(j)| \leq \sum_{k=1}^{\infty} 2^{1-k} \frac{k+1}{k^2} \left(C\|f\|_{\log^{\alpha}}^2 2^{k-1} (\log 2)^{-\alpha}\right)^{\frac{1}{2}} = C\|f\|_{\log^{\alpha}}\sum_{k=1}^{\infty} \frac{1}{k^{\frac{\alpha}{2}}} \leq C\|f\|_{\log^{\alpha}}.$$  \hfill \Box

Moreover, for $\alpha > 2$ the Hilbert matrix operator acting on $A^{2}_{\log^{\alpha}}$ can also be expressed in the integral form (2.3). Furthermore, we have

Theorem 4.3. If $\alpha > 3$, then $H$ acts as a bounded operator from $A^{2}_{\log^{\alpha}}$ to $A^2$. 

Proof. From (2.3) and the integral form of Minkowski’s inequality we obtain
\[
\|Hf\|_{A^2} = \left( \int_D |Hf(z)|^2 \, dA(z) \right)^{\frac{1}{2}} \leq \left( \int_D \left( \int_0^1 \frac{|f(r)|}{|1-rz|} \, dr \right)^2 \, dA(z) \right)^{\frac{1}{2}}
\]
\[
\leq \int_0^1 |f(r)| \left( \int_D \frac{dA(z)}{|1-rz|^2} \right)^{\frac{1}{2}} \, dr \leq C \int_0^1 |f(r)| \left( \log \frac{2}{1-r^2} \right)^{\frac{1}{2}} \, dr.
\]
By Lemma 4.1,
\[
\|Hf\|_{A^2} \leq C \int_0^1 \frac{dr}{(1-r^2) \left( \log \frac{2}{1-r^2} \right)^{\frac{\alpha+1}{2}}} \|f\|_{\log^\alpha},
\]
and the last integral converges for \( \alpha > 3 \).
\[\square\]

5. The Bloch and Besov spaces

For \( 1 < p \leq \infty \), let \( B_p \) denote the analytic Besov space consisting of functions \( f \in H(D) \) for which
\[
\|f\|_{B_p} := |f(0)| + \left( \int_D |f'(z)|^p (1-|z|^2)^{p-2} \, dA(z) \right)^{\frac{1}{p}} < \infty.
\]
In the case \( p = \infty \) this is understood as
\[
\|f\|_{B_\infty} = |f(0)| + \sup_{z \in D} (1-|z|^2) |f'(z)| < \infty,
\]
and hence \( B_\infty = B \) is the Bloch space. The reader is referred to, e.g., [15] for results on these spaces.

It is easy to check that if \( f(z) = \log \frac{1}{1-z} \), then
\[
Hf(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} \right) z^n.
\]
This shows that Bloch space \( B \) is not mapped into itself.

The following lemma describes a space of analytic functions in \( D \) that are mapped by \( H \) into the Bloch space.

**Proposition 5.1.** If \( f \in H(D) \) satisfies the condition
\[
\sup_{z \in D} |f'(z)|(1-|z|) \left( \log \frac{2}{1-|z|} \right)^{1+\varepsilon} < \infty
\]
for an \( \varepsilon > 0 \), then \( Hf \in B \).

**Proof.** Assume that \( f \in H(D) \) satisfies (5.1) and set
\[
F(z) = f(z) - f(0).
\]
It is enough to show that \( HF \in B \). Clearly, \( F \) also satisfies (5.1). Then by Lemma 4.2.8 in [15] we can write
\[
F(z) = \int_D \frac{F'(w)(1-|w|^2)}{\bar{w}(1-\bar{w}z)^2} \, dA(w).
\]
Consequently,

\[ |(HF)'(z)| \leq C \int_0^1 \int_{D} \log^{-1-\varepsilon} \left( \frac{2}{1 - |w|} \right) \frac{1}{|1 - \bar{w}r|^2 |1 - rz|^2} dA(w) dr \]
\[ \leq C \int_0^1 \int_{0}^1 \log^{-1-\varepsilon} \left( \frac{2}{1 - s} \right) \frac{ds dr}{(1 - sr)(1 - r|z|)^2} \]
\[ \leq \frac{C}{1 - |z|} \int_{0}^1 \log^{-1-\varepsilon} \left( \frac{2}{1 - s} \right) \frac{1}{1 - s} ds. \]

Since the last integral is finite, our claim is proved. \[ \square \]

On the other hand, we have

**Proposition 5.2.** If \( f \in B \), then

\[ |(Hf)'(z)| \leq C \frac{1}{1 - |z|} \log \frac{2}{1 - |z|}. \]

**Proof.** For \( f \in B \) set

\[ A_n(f) = \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n + k + 1} = \int_{D} \frac{f(z)}{1 - \bar{z}} |z|^{2n} dA(z). \]

Assuming additionally that \( f(0) = 0 \) and using the Fubini theorem, we obtain

\[ |A_n(f)| \leq \left| \int_{D} \int_{D} \frac{f'(w)(1 - |w|^2)}{\bar{w}(1 - \bar{w}z)^2} dA(w) \frac{|z|^{2n}}{1 - \bar{z}} dA(z) \right| \]
\[ = \left| \int_{D} \frac{f'(w)(1 - |w|^2)}{\bar{w}} \int_{0}^{1} \left( \frac{1}{\pi} \int_{0}^{2\pi} \frac{d\theta}{(1 - \bar{w}r e^{i\theta})^2 (1 - re^{-i\theta})} \right) r^{2n+1} dr dA(w) \right| \]
\[ \leq C \int_{0}^{1} r^{2n+1} \int_{D} \frac{dA(w)}{|1 - r^2 \bar{w}|^2} dr \leq C \int_{0}^{1} \log \frac{2}{(1 - r^2)} r^{2n+1} dr \]
\[ \leq C \int_{0}^{1} r^n \left( \log 2 + \sum_{k=1}^{\infty} \frac{r^k}{k} \right) dr \leq C \frac{1}{n + 1} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n + 1} \right). \]

Hence

\[ |(Hf)'(z)| = \left| \sum_{n=1}^{\infty} n A_n(f) z^{n-1} \right| \leq \sum_{n=1}^{\infty} n|A_n(f)||z|^{n-1} \]
\[ \leq C \sum_{n=0}^{\infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n + 1} \right) |z|^n \leq C \frac{1}{1 - |z|} \log \frac{2}{1 - |z|}. \]

\[ \square \]

The example of \( f(z) = \log \frac{1}{1 - z} \) shows that the inequality in the last lemma cannot be improved.

A little bit more complicated calculations give the following

**Proposition 5.3.** If \( f \in B_p, 1 < p < \infty \), then

\[ |(Hf)'(z)| \leq C \frac{1}{1 - |z|} \left( \log \frac{2}{1 - |z|} \right)^{\frac{1}{p'}}, \]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Proof. Using the notation from the proof of Proposition 5.2 under the assumption that $f(0) = 0$ we get, in much the same way as above,

$$|A_n(f)| \leq C \int_0^1 r^{2n+1} |f'(w)| \frac{(1 - |w|^2)}{|1 - r^2 - w|^2} \, dA(w) \, dr,$$

$$\leq C \|f\|_{B_p} \int_0^1 r^{2n+1} \left( \log \frac{2}{1 - r^2} \right)^{\frac{1}{p'}} \, dr.$$ 

Hence

$$|A_n(f)|^{p'} \leq C \|f\|_{B_p}^{p'} \left( \int_0^1 r^{2n+1} \, dr \right)^{p'-1} \int_0^1 r^{2n+1} \log \frac{2}{1 - r^2} \, dr$$

$$\leq C \|f\|_{B_p}^{p'} \left( \frac{1}{n+1} \right)^{p'} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} \right).$$

Consequently,

$$|(Hf)'(z)|^{p'} \leq \left( \sum_{n=0}^{\infty} (n + 1) |A_{n+1}(f)| |z|^n \right)^{p'}$$

$$\leq \left( \sum_{n=0}^{\infty} (n + 1)^{p'} |A_{n+1}(f)|^{p'} |z|^n \right)^{p'-1} \left( \sum_{n=0}^{\infty} |z|^n \right)^{p'-1}$$

$$\leq C \|f\|_{B_p}^{p'} \left( \frac{1}{1 - |z|} \right)^{p'-1} \left( \sum_{n=0}^{\infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n+1} \right) |z|^n \right)$$

$$\leq C \|f\|_{B_p}^{p'} \left( \frac{1}{1 - |z|} \right)^{p'} \log \frac{2}{1 - |z|},$$

which proves our claim. 

References


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