FACTORIZING DERIVATIVES OF FUNCTIONS 
IN THE NEVANLINNA AND SMIRNOV CLASSES

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Abstract. We prove that, given a function \( f \) in the Nevanlinna class \( \mathcal{N} \) and a positive integer \( n \), there exist \( g \in \mathcal{N} \) and \( h \in \text{BMOA} \) such that \( f^{(n)} = gh^{(n)} \). We may choose \( g \) to be zero-free, so it follows that the zero sets for the class \( \mathcal{N}^{(n)} := \{ f^{(n)} : f \in \mathcal{N} \} \) are the same as those for \( \text{BMOA}^{(n)} \).

Furthermore, while the set of all products \( gh^{(n)} \) (with \( g \) and \( h \) as above) is strictly larger than \( \mathcal{N}^{(n)} \), we show that the gap is not too large, at least when \( n = 1 \). Precisely speaking, the class \( \{ gh' : g \in \mathcal{N}, h \in \text{BMOA} \} \) turns out to be the smallest ideal space containing \( \{ f' : f \in \mathcal{N} \} \), where “ideal” means invariant under multiplication by \( \text{H}^\infty \) functions. Similar results are established for the Smirnov class \( \mathcal{N}^+ \).

1. Introduction and results

Let \( \mathcal{H}(\mathbf{D}) \) stand for the set of holomorphic functions on the disk \( \mathbf{D} := \{ z \in \mathbb{C} : |z| < 1 \} \). Given a class \( X \subset \mathcal{H}(\mathbf{D}) \) and an integer \( n \in \mathbb{N} := \{ 1, 2, \ldots \} \), we write

\[
X^{(n)} := \{ f^{(n)} : f \in X \},
\]

where \( f^{(n)} \) is the \( n \)th derivative of \( f \). When \( n = 1 \), we also use the notation \( X' \) instead of \( X^{(1)} \). Further, we denote by \( \mathcal{Z}(X) \) the collection of zero sets for \( X \); a (discrete) subset \( E \) of \( \mathbf{D} \) will thus belong to \( \mathcal{Z}(X) \) if and only if \( E = \{ z \in \mathbf{D} : f(z) = 0 \} \) for some non-null function \( f \in X \). Now, if \( X \) and \( Y \) are subclasses of \( \mathcal{H}(\mathbf{D}) \), we put

\[
X \cdot Y := \{ fg : f \in X, g \in Y \}.
\]

Finally, a vector space \( X \) contained in \( \mathcal{H}(\mathbf{D}) \) is said to be ideal if

\[
\text{H}^\infty \cdot X \subset X,
\]

where, as usual, \( \text{H}^\infty \) is the space of bounded holomorphic functions on \( \mathbf{D} \).

Our starting point is a result of Cohn and Verbitsky [3] which asserts, or rather implies, that

\[
(H^p)^{(n)} = H^p \cdot \text{BMOA}^{(n)}
\]

whenever \( 0 < p < \infty \) and \( n \in \mathbb{N} \). Here, we write \( H^p \) for the classical (holomorphic) Hardy spaces on the disk, and BMOA for the “analytic subspace” of BMO = BMO(\( T \)), the space of functions with bounded mean oscillation on the circle \( T := \partial \mathbf{D} \). Precisely speaking, BMOA can be defined as \( H^1 \cap \text{BMO} \); as to the definitions of (and background information on) \( H^p \) and BMO, the reader will find these standard matters in [5, Chapters II and VI].

doi:10.5186/aasfm.2012.3728
2010 Mathematics Subject Classification: Primary 30D50, 30D55.
Key words: Nevanlinna class, Smirnov class, BMOA, derivatives, factorization, zero sets.
Supported in part by grant MTM2011-27932-C02-01 from El Ministerio de Ciencia e Innovación (Spain) and grant 2009-SGR-1303 from AGAUR (Generalitat de Catalunya).
For $n = 1$, identity (1.1) was first obtained in Cohn's earlier paper [2]. It was then extended in [3] to higher order (possibly fractional) derivatives and still further; indeed, more general factorization theorems involving tent spaces—and Triebel spaces—were actually established there. It was also shown in [3] that, when factoring $f^{(n)}$ for $f \in H^p$ in the sense of (1.1), one may choose the $H^p$ factor on the right to be an outer function. As a consequence, one sees that

$$ (1.2) \quad \mathcal{Z}((H^p)^{(n)}) = \mathcal{Z}(\text{BMOA}^{(n)}). $$

In particular, for any fixed $n$, the zero sets for $(H^p)^{(n)}$ are the same for all $p \in (0, \infty)$. This last fact was contrasted in [3] with the Bergman space situation, where different $A^p$ spaces happen to have different zero sets; see [7]. We wish to add, in this connection, that a similar Bergman-type phenomenon (different zero sets for different $p$’s) is also encountered in certain “small” $H^p$-related spaces; namely, it occurs [4] for the star-invariant subspaces $H^p \cap \theta \overline{H}_0^p$ associated with an inner function $\theta$.

Also related to (1.1), in the case $n = 1$, is Aleksandrov and Peller’s work from [1]. There, for a given $f \in H^p$, a weak factorization $f' = \sum_{j=1}^{m} g_j h'_j$ was constructed with suitable $g_j \in H^p$ and $h_j \in H^\infty$. This was done with $m = 2$ for $1 < p < \infty$, with $m = 4$ for $p = 1$, and with a certain larger $m$ for $0 < p < 1$. Yet another weak factorization theorem from [1], which establishes a connection between BMOA’ and $(H^\infty)'$, will be employed in Section 4 below.

The purpose of this paper is to find out whether—and/or to which extent—the (strong) factorization theorem (1.1) carries over to the Nevanlinna class $\mathcal{N}$, or the Smirnov class $\mathcal{N}^+$, in place of $H^p$.

Let us recall that $\mathcal{N}$ is defined as the set of functions $f \in \mathcal{H}(D)$ with

$$ \sup_{0 < r < 1} \int_T \log^+ |f(r \zeta)| |d\zeta| < \infty, $$

while $\mathcal{N}^+$ is formed by those $f \in \mathcal{N}$ which satisfy

$$ \lim_{r \to 1^-} \int_T \log^+ |f(r \zeta)| |d\zeta| = \int_T \log^+ |f(\zeta)| |d\zeta|. $$

Equivalently, the elements of $\mathcal{N}$ (resp., $\mathcal{N}^+$) are precisely the ratios $u/v$, with $u, v \in H^\infty$ and with $v$ nonvanishing (resp., outer) on $D$; for this and other characterizations of the two classes, see [5, Chapter II].

As far as factorization theorems of the form (1.1) are concerned, we can hardly expect the behavior of $\mathcal{N}$ or $\mathcal{N}^+$ to mimic that of $H^p$ too closely. In fact, as we shall soon explain, it is the “easy” part of (1.1), i.e., the inclusion

$$ (1.3) \quad (H^p)^{(n)} \supset H^p \cdot \text{BMOA}^{(n)} $$

that admits no extension to the Nevanlinna or Smirnov setting. Meanwhile, we remark that (1.3) is indeed easy to deduce, at least for $p = 2$, from the (not so easy, but readily available) descriptions of $(H^p)^{(n)}$ and BMOA$^{(n)}$ as the appropriate Triebel spaces; see [11]. One of these tells us that, for $\varphi \in \mathcal{H}(D)$,

$$ \varphi \in (H^p)^{(n)} \iff \int_T \left( \int_0^1 |\varphi(r \zeta)|^2 (1 - r)^{2n-1} dr \right)^{p/2} |d\zeta| < \infty $$
for all \( n \in \mathbb{N} \) and \( 0 < p < \infty \), a fact that has no counterpart for \( \mathcal{N} \) or \( \mathcal{N}^+ \). The other, which involves a Carleson measure characterization of BMOA, will be mentioned in Section 2 below.

Now, to see that the \( \mathcal{N} \) and \( \mathcal{N}^+ \) versions of (1.3) actually break down, already for \( n = 1 \), one may recall results of Hayman \[6\] and Yanagihara \[12\] saying that neither \( \mathcal{N} \) nor \( \mathcal{N}^+ \) is invariant with respect to integration. More precisely, Hayman gave an example of a function \( f \in \mathcal{N} \) whose antiderivative \( F(z) := \int_0^z f(\zeta) \, d\zeta \) is not in \( \mathcal{N} \), and Yanagihara refined this by showing that \( F \) need not be in \( \mathcal{N} \) even for \( f \in \mathcal{N}^+ \). Consequently, \( \mathcal{N}^+ \) is not contained in \( \mathcal{N}' \), whence a fortiori
\[
\mathcal{N} \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \not\subset (\mathcal{N}^+)' .
\]

Since \( \mathcal{N} \cdot \text{BMOA}' \) (resp., \( \mathcal{N}^+ \cdot \text{BMOA}' \)) contains \( \mathcal{N} \) (resp., \( \mathcal{N}^+ \)), we readily deduce from (1.4) that
\[
\mathcal{N} \cdot \text{BMOA}' \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \cdot \text{BMOA}' \not\subset (\mathcal{N}^+)' .
\]

A similar conclusion holds for higher order derivatives as well.

We prove, however, that the “difficult” part of (1.1), i.e., the inclusion
\[
(h^p)^{(n)} \subset H^p \cdot \text{BMOA}^{(n)}
\]
does remain valid with either \( \mathcal{N} \) or \( \mathcal{N}^+ \) in place of \( H^p \).

**Theorem 1.1.** For each \( n \in \mathbb{N} \), we have
\[
\mathcal{N}^{(n)} \subset \mathcal{N} \cdot \text{BMOA}^{(n)} \quad \text{and} \quad (\mathcal{N}^+)^{(n)} \subset \mathcal{N}^+ \cdot \text{BMOA}^{(n)} .
\]
Moreover, given \( f \in \mathcal{N} \) (resp., \( f \in \mathcal{N}^+ \)), one can find a zero-free function \( g \in \mathcal{N} \) (resp., an outer function \( g \in \mathcal{N}^+ \)) and an \( h \in \text{BMOA} \) such that \( f^{(n)} = gh^{(n)} \).

It should be mentioned that our method also applies to the meromorphic Nevanlinna class \( \mathcal{N}_{\text{mer}} \), defined as the set of quotients \( u/v \), where \( u, v \in H^\infty \) and \( v \) is merely required to be non-null. In fact, a glance at our proof of Theorem 1.1 will reveal that if the original function \( f \) is of the form \( F/I \), with \( F \in \mathcal{N}^+ \) and \( I \) inner, then we may take \( g = G/I^{n+1} \), with \( G \) outer. And again, just as in the \( H^p \) setting, our factorization theorem yields information on the zero sets.

**Corollary 1.2.** We have
\[
\mathcal{Z} \left( \mathcal{N}^{(n)} \right) = \mathcal{Z} \left( \text{BMOA}^{(n)} \right), \quad n \in \mathbb{N} .
\]

Indeed, Theorem 1.1 shows that every zero set for \( \mathcal{N}^{(n)} \) is a zero set for \( \text{BMOA}^{(n)} \), while the converse is immediate from the fact that \( \text{BMOA} \subset \mathcal{N} \). Furthermore, since \( \mathcal{N}^+ \) lies between \( \text{BMOA} \) and \( \mathcal{N} \), as does every \( H^p \) with \( 0 < p < \infty \), Corollary 1.2 obviously implies the identity
\[
\mathcal{Z} \left( (\mathcal{N}^+)^{(n)} \right) = \mathcal{Z} \left( \text{BMOA}^{(n)} \right)
\]
and also (1.2).

Finally, restricting ourselves to the case \( n = 1 \), we wish to take a closer look at the inclusion
\[
\mathcal{N}' \subset \mathcal{N} \cdot \text{BMOA}'
\]
from Theorem 1.1, along with its \( \mathcal{N}^+ \) counterpart. We know from (1.5) that the inclusion is proper, and we now stress an important point of distinction between the two sides. Namely, the right-hand side, \( \mathcal{N} \cdot \text{BMOA}' \), is ideal (i.e., invariant under
multiplication by $H^\infty$ functions), whereas the left-hand side, $N'$, is not. Moreover, the space $N'$ is highly nonideal in the sense that even the identity function $z$ is not a multiplier thereof! (Otherwise, the formula

$$g = (zg)' - zg', \quad g \in N,$$

would imply that $N$ is contained in $N'$, which we know is false.) A similar remark applies to $(N^+)'$.

Our last result states, then, that $N \cdot \text{BMOA}'$ is actually the smallest ideal space containing $N'$, and that the same is true in the $N^+$ setting.

**Theorem 1.3.** (a) The class $N \cdot \text{BMOA}'$ is the ideal hull of $N'$. In other words, $N \cdot \text{BMOA}'$ is an ideal vector space that contains $N'$ and is contained in every ideal space $X$ with $N' \subset X$.

(b) Similarly, $N^+ \cdot \text{BMOA}'$ is the ideal hull of $(N^+)'$.

Now let us turn to the proofs.

2. Preliminaries

A couple of lemmas will be needed.

**Lemma 2.1.** Let $k \geq 0$ and $l \geq 1$ be integers. If $\varphi \in \text{BMOA}(l)$ and $\psi$ is a function in $H^1(D)$ satisfying

$$\psi(z) = O((1 - |z|)^{-k}), \quad z \in D,$$

then $\varphi \psi \in \text{BMOA}(k+l)$.

**Proof.** It is known (see, e.g., [8, 10, 11]) that a function $F \in H^1(D)$ will be in BMOA($n$), with $n \in \mathbb{N}$, if and only if the measure $|F(z)|^2(1 - |z|)^{2n-1} dx \, dy$ (where $z = x + iy$) is a Carleson measure. The required result follows from this immediately, since (2.1) yields

$$|\varphi(z)\psi(z)|^2(1 - |z|)^{2(k+l)-1} \leq \text{const} \cdot |\varphi(z)|^2(1 - |z|)^{2l-1}$$

for all $z \in D$. \hfill \square

When $k = 0$, the above lemma reduces to saying that

$$H^\infty \cdot \text{BMOA}(n) \subset \text{BMOA}(n)$$

for all $n \in \mathbb{N}$; in other words, BMOA($n$) is an ideal space. This in turn leads to the next observation.

**Lemma 2.2.** For each $n \in \mathbb{N}$, the sets $N \cdot \text{BMOA}(n)$ and $N^+ \cdot \text{BMOA}(n)$ are ideal vector spaces.

**Proof.** It is clear that the two sets are invariant under multiplication by $H^\infty$ functions, but maybe not quite obvious that they are vector spaces. It is the linearity property

$$f_1, f_2 \in N \cdot \text{BMOA}(n) \Longrightarrow f_1 + f_2 \in N \cdot \text{BMOA}(n)$$

(and a similar fact with $N^+$ in place of $N$) that should be verified. To this end, we write

$$f_j = \frac{u_j}{v_j} \cdot w_j^{(n)} \quad (j = 1, 2),$$
where \( u_j, v_j \in H^\infty \) and \( w_j \in \text{BMOA} \), and where \( v_j \) is zero-free (resp., outer if the \( f_j \)'s are from \( \mathcal{N}^+ \cdot \text{BMOA}^{(n)} \)). Note that
\[
f_1 + f_2 = \frac{1}{v_1v_2} \left( u_1v_2w_1^{(n)} + u_2v_1w_2^{(n)} \right).
\]
The two terms in brackets, and hence their sum, will be in \( \text{BMOA}^{(n)} \) by virtue of (2.2), while the factor \( 1/(v_1v_2) \) will be in \( \mathcal{N} \) (resp., in \( \mathcal{N}^+ \)). \( \square \)

3. Proof of Theorem 1.1

We treat the case of \( \mathcal{N} \) first. Take \( f \in \mathcal{N} \) and write \( f = u/v \), where \( u, v \in H^\infty \) and \( v \) has no zeros in \( D \). We have then
\[
f^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} (1/v)^{(k)}.
\]
For each \( k \in \{0, \ldots, n\} \), Faà di Bruno’s formula (see [9, Chapter 3]) yields
\[
\left( \frac{1}{v} \right)^{(k)} = \sum C(m_1, \ldots, m_k) v^{-m_1-\cdots-m_k-1} \prod_{j=1}^{k} (v^{(j)})^{m_j},
\]
where the sum is over the \( k \)-tuples \((m_1, \ldots, m_k)\) of nonnegative integers satisfying
\[
m_1 + 2m_2 + \cdots + km_k = k
\]
and where
\[
C(m_1, \ldots, m_k) = (-1)^{m_1+\cdots+m_k} \frac{(m_1+\cdots+m_k)!}{m_1! \cdots m_k!} \frac{k!}{1!m_1 \cdots k!m_k}.
\]
For any fixed multiindex \((m_1, \ldots, m_k)\) as above, we clearly have
\[
v^{-m_1-\cdots-m_k-1} = v^{-n-1}, v^{n-m_1-\cdots-m_k},
\]
the last factor on the right being bounded. Indeed,
\[
v^{n-m_1-\cdots-m_k} \in H^\infty,
\]
since it follows from (3.3) that \( n - m_1 - \cdots - m_k \geq 0 \). We further observe that, for \( j \in \mathbb{N} \),
\[
v^{(j)}(z) = O((1-|z|)^{-j}), \quad z \in D
\]
(because \( v \in H^\infty \)), and this implies together with (3.3) that
\[
\prod_{j=1}^{k} [v^{(j)}(z)]^{m_j} = O((1-|z|)^{-k}), \quad z \in D.
\]
Combining (3.2) and (3.4), we see that the \( k \)th summand in (3.1) takes the form \( v^{-n-1}w_k \), where
\[
w_k := \binom{n}{k} \sum C(m_1, \ldots, m_k) u^{(n-k)} v^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} (v^{(j)})^{m_j};
\]
the sum is understood as in (3.2). We want to show that \( w_k \in \text{BMOA}^{(n)} \), and our plan is to check the corresponding inclusion for each individual term in (3.8). Thus, we claim that the function

\[
\Phi_{m_1, \ldots, m_k} := u^{(n-k)}v^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} (v^{(j)})^{m_j}
\]

satisfies

\[
(3.9) \quad \Phi_{m_1, \ldots, m_k} \in \text{BMOA}^{(n)}
\]

whenever \( 0 \leq k \leq n \) and the \( m_j \)’s are related by (3.3).

First let us verify (3.9) in the case \( k = n - 1 \). To this end, we notice that

\[
u^{(n-k)} \in (H^\infty)^{(n-k)} \subset \text{BMOA}^{(n-k)},
\]

where \( n - k \geq 1 \), while

\[
[v(z)]^{n-m_1-\cdots-m_k} \prod_{j=1}^{k} [v^{(j)}(z)]^{m_j} = O((1 - |z|)^{-k}), \quad z \in D,
\]

by virtue of (3.5) and (3.7). The validity of (3.9) is then guaranteed by Lemma 2.1.

Now if \( k = n \), then the multiindices involved are of the form \((m_1, \ldots, m_n)\) with \( \sum_{j=1}^{n} jm_j = n \). For any such multiindex, at least one of the \( m_j \)’s (say, \( m_l \)) must be nonzero, so that \( m_l \geq 1 \) and

\[
l(m_l - 1) + \sum_{1 \leq j \leq n, j \neq l} jm_j = n - l.
\]

Consider the factorization

\[
\Phi_{m_1, \ldots, m_n} = v^{(l)} \left\{ uv^{n-m_1-\cdots-m_n} \left( v^{(l)} \right)^{m_l-1} \prod_{1 \leq j \leq n, j \neq l} (v^{(j)})^{m_j} \right\}.
\]

The first factor, \( v^{(l)} \), is then in \((H^\infty)^{(l)}\) and hence in \text{BMOA}^{(l)}\), while the second factor (the one in curly brackets) is \( O((1 - |z|)^{-n+l}) \). The latter estimate is due to (3.6) and (3.10), coupled with the fact that \( u \) and \( v \) are in \( H^\infty \). Applying Lemma 2.1 to the current factorization, we arrive at (3.9), this time with \( k = n \).

Now that (3.9) is known to be true, we infer that the functions \( w_k \) from (3.8) are all in \text{BMOA}^{(n)}, whence obviously \( \sum_{k=0}^{n} w_k \in \text{BMOA}^{(n)} \). Recalling that

\[
f^{(n)} = v^{-n-1} \sum_{k=0}^{n} w_k,
\]

we finally conclude that \( f^{(n)} \) can be written as \( gh^{(n)} \), where \( g := v^{-n-1} \in \mathcal{N} \) and \( h \) is a \text{BMOA} function satisfying \( h^{(n)} = \sum_{k=0}^{n} w_k \).

The case of \( \mathcal{N}^{+} \) is similar. This time, \( v \) is taken to be an outer function in \( H^\infty \), so \( g = v^{-n-1} \) will be an outer function in \( \mathcal{N}^{+} \).

\[\square\]

4. Proof of Theorem 1.3

We shall only prove (a), the proof of (b) being similar. We know from Lemma 2.2 that \( \mathcal{N} \cdot \text{BMOA} \) is an ideal space. Furthermore, Theorem 1.1 tells us that \( \mathcal{N} \cdot \text{BMOA} \) contains \( \mathcal{N}^{0} \). It remains to verify that, whenever \( X \) is an ideal space with \( \mathcal{N}^{0} \subset X \),
we necessarily have
\begin{equation}
\mathcal{N} \cdot \text{BMOA}^\prime \subset X.
\end{equation}

Take any \( g \in \mathcal{N} \) and \( h \in H^\infty \). Note that
\begin{equation}
gh' = (gh)' - g'h,
\end{equation}
where both terms on the right are in \( X \). Indeed, \((gh)'\) is obviously in \( \mathcal{N}' \) and hence in \( X \), while the inclusion \( g'h \in X \) is due to the facts that \( g' \in \mathcal{N} \subset X \) and \( hX \subset X \) (recall that \( X \) is ideal). It now follows from (4.2) that \( gh' \in X \), and we have thereby checked that
\begin{equation}
\mathcal{N} \cdot \left( H^\infty \right)' \subset X.
\end{equation}

Finally, given \( \eta \in \text{BMOA} \), we invoke a result of Aleksandrov and Peller [1, Theorem 3.4] to find functions \( \varphi_j, \psi_j \in H^\infty \) \((j = 1, 2)\) such that \( \eta' = \varphi_1 \psi_1' + \varphi_2 \psi_2' \). Letting \( g \in \mathcal{N} \) as before, we get
\begin{equation}
g\eta' = g\varphi_1 \psi_1' + g\varphi_2 \psi_2'.
\end{equation}
Here, the two terms of the form \( g\varphi_j \psi_j' \) are in \( \mathcal{N} \cdot \left( H^\infty \right)' \), so we infer from (4.3) that they are also in \( X \). The right-hand side of (4.4) is therefore in \( X \), and so is the left-hand side, \( g\eta' \). Thus we conclude that \( g\eta' \in X \) for all \( g \in \mathcal{N} \) and \( \eta \in \text{BMOA} \). This establishes (4.1) and completes the proof. \(\square\)

References


Received 9 August 2011