ON THE MATTILA–SJÖLIN THEOREM FOR DISTANCE SETS

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Abstract. We extend a result, due to Mattila and Sjölin, which says that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d+1}{2}$, then the distance set $\Delta(E) = \{|x-y|: x, y \in E\}$ contains an interval. We prove this result for distance sets $\Delta_B(E) = \{\|x-y\|_B: x, y \in E\}$, where $\| \cdot \|_B$ is the metric induced by the norm defined by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature. We also obtain some detailed estimates pertaining to the Radon–Nikodym derivative of the distance measure.

1. Introduction

The classical Falconer distance conjecture, originated in 1985, [2], says that if the Hausdorff dimension of a compact subset of $\mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the set of distances, $\Delta(E) = \{|x-y|: x, y \in E\}$ is positive. In [2], Falconer proved the first result in this direction by showing that $L^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$. See also [3] and [7] for a thorough description of the problem and related ideas. The best currently known results are due to Wolff in two dimensions, and to Erdogan [1] in dimensions three and greater. They prove that $L^1(\Delta(E)) > 0$ if the Hausdorff dimension of $E$ is greater than $\frac{d}{2} + \frac{1}{3}$.

An important addition to this theory is due to Mattila and Sjölin [8] who proved that if the Hausdorff dimension of $E$ is greater than $\frac{d+1}{2}$, then $\Delta(E)$ not only has positive Lebesgue measure, but also contains an interval. This is accomplished by showing that the natural measure on the distance set has a continuous density. However, this set need not contain an interval with left end-point at the origin as illustrated by an example in ([7], p. 165). It was previously shown by Mattila [6] that if the ambient dimension is two or three, then the density of the distance measure is not in general bounded if the Hausdorff dimension of the underlying set $E$ is smaller than $\frac{d+1}{2}$. In higher dimensions, this question is still open for the Euclidean metric, but has been resolved if the Euclidean metric is replaced by a metric induced by a norm defined by a suitably chosen paraboloid. See [5].

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In this paper we give an alternative proof of the Mattila–Sjölin result and extend it to more general distance sets $\Delta_B(E) = \{\|x - y\|_B : x, y \in E\}$, where $\| \cdot \|_B$ is the norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.

Our main result is the following.

**Theorem 1.1.** Let $E$ be a compact subset of $\mathbb{R}^d$, $d \geq 2$, with Hausdorff dimension, denoted by $s$, greater than $\frac{d+1}{2}$. Let $\mu$ be a Frostman measure on $E$. Let $\sigma$ denote the Lebesgue measure on $\partial B$. Define the distance measure $\nu$ by the relation

$$\hat{h}(t) d\nu(t) = \hat{\|x - y\|_B} d\mu(x) d\mu(y),$$

where $\| \cdot \|_B$ is the norm generated by a symmetric bounded convex body $B$ with a smooth boundary and everywhere non-vanishing Gaussian curvature.

(i) Then the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure.

(ii) We have

$$\nu((t - \epsilon, t + \epsilon)) = M(t) + R^\epsilon(t),$$

where

$$M(t) = \int |\hat{\mu}(\xi)|^2 \hat{\sigma}(t\xi)t^{d-1} d\xi$$

is the density of $\nu$ and

$$\sup_{0 < \epsilon < \epsilon_0} |R^\epsilon(t)| \lesssim \epsilon^s - \frac{d+1}{2}.$$

(iii) Moreover, $M \in C^{[s - \frac{d+1}{2}]}(I)$ for any interval $I$ not containing the origin, where $[u]$ denotes the smallest integer greater than or equal to $u$. In particular, $M$ is continuous away from the origin if $s > \frac{d+1}{2}$ and therefore $\Delta_B(E)$ contains an interval in view of (i).

(iv) Suppose that $s > k + \alpha$, where $k$ is a non-negative integer and $0 < \alpha < 1$. Then the $k$th derivative of the density function of $\nu$ is Hölder continuous of order $\alpha$.

**Remark 1.2.** Metric properties of $\| \cdot \|_B$ are not used in the proof of Theorem 1.1. Let $\Gamma$ be a star shaped body in the sense that for every $\omega \in S^{d-1}$ there exists $1 < r_0(\omega) < 2$ such that $\{r\omega : 0 \leq r \leq r_0(\omega)\} \subset \Gamma$ and $\{r\omega : r > r_0(\omega)\} \cap \Gamma = \emptyset$. Define $\|x\|_{\Gamma} = \inf\{t > 0 : x \in t\Gamma\}$ and let $\Delta_{\Gamma}(E) = \{\|x - y\|_{\Gamma} : x, y \in E\}$. Let $\sigma_{\Gamma}$ denote the Lebesgue measure on the boundary of $\Gamma$. Then if $|\hat{\sigma}_{\Gamma}(\xi)| \lesssim |\xi|^{-\frac{d+1}{2}}$, the conclusion of Theorem 1.1 holds with the same exponents.

**1.1. Sharpness of results.** As we note above, Mattila’s construction [6] shows that if the Hausdorff dimension of $E$ is smaller than $\frac{d+1}{2}$, $d = 2, 3$, then the density of distance measure is not in general bounded in the case of the Euclidean metric. Moreover, Mattila construction can be easily extended to all metrics generated by a bounded convex body $B$ with a smooth boundary and non-vanishing Gaussian curvature.

In dimensions four and higher, all we know at the moment (see the main result in [5]) is that there exists a bounded convex body $B$ with a smooth boundary and non-vanishing curvature, such that the density of the distance measure is not in general
bounded if the Hausdorff dimension of the underlying set \(E\) is less than \(\frac{d+1}{2}\). We do no know what happens when the Hausdorff dimension of \(E\) equals \(\frac{d+1}{2}\) in any dimension and for any smooth metric.

It would be very interesting if any of these results actually depended on the underlying convex body \(B\) in a non-trivial way. This would mean that smoothness and non-vanishing Gaussian curvature of the level set do not tell the whole story. There is some evidence that this may be the case. See, for example, [4], where connections between problems of Falconer type and distribution of lattice points in thin annuli are explored.

If the Hausdorff dimension of \(E\) is less than \(\frac{d}{2}\), then the density of the distance measure, for any metric induced by a bounded convex body \(B\) with a smooth boundary and non-vanishing curvature is not in general bounded by a construction due to Falconer [2].

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2. Proof of Theorem 1.1

2.1. Proof of items (i) and (ii). The proof of item (i) of Theorem 1.1 is due to Falconer [2] and Mattila [6]. This brings us to item (ii). Recall that every compact set \(E\) in \(\mathbb{R}^d\), of Hausdorff dimension \(s > 0\) possesses a Frostman measure (see e.g. [7], p. 112), which is a probability measure \(\mu\) with the property that given any \(\delta > 0\), for every ball of radius \(r^{-1}\), denoted by \(B_{r^{-1}}\), there exists \(C_\delta > 0\) such that

\[
\mu(B_{r^{-1}}) \leq C_\delta r^{-s+\delta}.
\]

Let

\[
\nu^\epsilon(t) = \frac{\nu((t-\epsilon, t+\epsilon))}{2\epsilon} = \frac{1}{2\epsilon} \mu \times \mu \{(x, y): t-\epsilon \leq ||x-y||_B \leq t+\epsilon\}.
\]

We shall prove that \(\lim_{\epsilon \to 0} \nu^\epsilon(t)\) exists and is a \(C^{s-\frac{d+1}{2}}\) function.

Let \(\rho\) be a smooth cut-off function, identically equal to 1 in the unit ball and vanishing outside the ball of radius 2, with \(\int \rho = 1\). Let \(\rho_\epsilon(x) = \epsilon^{-d} \rho(x/\epsilon)\). Since \(\sigma_\epsilon \ast \rho_\epsilon\) is supported on the annulus of radius \(t\) and width \(\approx \epsilon\), and is \(\approx \epsilon^{-1}\) on this annulus, there is no harm in working with the measure

\[
\int \sigma_\epsilon \ast \rho_\epsilon(x-y) \, d\mu(x) \, d\mu(y),
\]

where \(\sigma_\epsilon\) is the surface measure on the set \(\{x: ||x||_B = t\}\). By abusing notation slightly, we shall refer to this measure as \(\nu^\epsilon(t)\).

By the Fourier inversion formula,

\[
\nu^\epsilon(t) = \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_\epsilon(\xi) \hat{\rho}(\epsilon\xi) \, d\xi
= \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_\epsilon(\xi) \, d\xi - \int |\hat{\mu}(\xi)|^2 \hat{\sigma}_\epsilon(\xi)(1 - \hat{\rho}(\epsilon\xi)) \, d\xi = M(t) + R^\epsilon(t).
\]
We shall prove that $M(t)$ is a $C^{s-d+1/2}$ function and that \( \lim_{\epsilon \to 0} R^c(t) = 0 \). We start with the latter. We shall need the following stationary phase estimate. See, for example, [10], [9] or [11].

**Lemma 2.1.** Let $\sigma$ be the surface measure on a compact piece of a smooth convex surface in $\mathbb{R}^d$, $d \geq 2$, with everywhere non-vanishing Gaussian curvature. Then

\[
\hat{\sigma}(\xi) \lesssim |\xi|^{-d+1/2},
\]

where here, and throughout, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$. Moreover,

\[
|D^\alpha \hat{\sigma}(\xi)| \leq C_{\alpha,d}|\xi|^{-d+1/2},
\]

where $D^\alpha$ is the differential operator with respect to the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$.

We shall also need the following well-known estimate. See, for example, [3] and [7].

**Lemma 2.2.** Let $\mu$ be a Frostman measure on a compact set $E$ of Hausdorff dimension $s > 0$. Then

\[
\int_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 \, d\xi \lesssim 2^{j(d-s)},
\]

and, consequently,

\[
\int |\hat{\mu}(\xi)|^2 |\xi|^{-\gamma} \, d\xi = c \int |x - y|^{-d+\gamma} \, d\mu(x) \, d\mu(y) \lesssim 1
\]

if $\gamma > d - s$. Here, and throughout, $X \lesssim Y$, with the controlling parameter $r$ means that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $X \leq C_\epsilon r^s Y$.

To prove the lemma, observe that

\[
\int_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 \, d\xi \lesssim \int |\hat{\mu}(\xi)|^2 \psi(2^{-j} \xi) \, d\xi,
\]

where $\psi$ is a suitable smooth function supported in $\{x \in \mathbb{R}^d : 1/2 \leq |x| \leq 4\}$ and identically equal to 1 in the unit annulus. By definition of the Fourier transform and the Fourier inversion theorem, this expression is equal to

\[
2^j \int \hat{\psi}(2^j(x - y)) \, d\mu(x) \, d\mu(y) \lesssim 2^{j(d-s)}
\]

since $\hat{\psi}$ decays rapidly at infinity.

By Lemma 2.1 and Lemma 2.2, we have

\[
|R^c(t)| \lesssim \int_{|\xi| > \frac{1}{\epsilon}} |\hat{\mu}(\xi)|^2 |\xi|^{-d+1/2} \, d\xi
\]

\[
\leq \sum_{j > \log_2(1/\epsilon)} \int_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{\mu}(\xi)|^2 |\xi|^{-d+1/2} \, d\xi
\]

\[
\lesssim \sum_{j > \log_2(1/\epsilon)} 2^{j(d-s)} 2^{-d+1/2} \lesssim \epsilon s^{-d+1/2},
\]
and thus

\[ \sup_{0 < t \leq t_0} |R_\epsilon(t)| \leq \epsilon_0^{\frac{d-1}{2}}. \]

In order to handle \(|R'(t)|\) over the integral when \(|\xi| < \frac{1}{\epsilon}\), we notice that \((1 - \hat{\rho}(\epsilon \xi))\) is 0 when \(\xi = (0, \ldots, 0)\) and, by continuity, is small in a neighborhood of the origin.

This calculation establishes all the claims in part ii) of Theorem 1.1.

We note that the weaker statement showing that \(\lim_{\epsilon \to 0} |R_\epsilon(t)| = 0\) follows in an easier way from the dominated convergence theorem.

2.2. Proof of item (iii). Once again, by Lemma 2.1, we have

\[ |M(t)| \lesssim |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d+1}{2}} d\xi \]

and by the calculation identical to the one in the previous paragraph, we see that this quantity is \(\lesssim 1\) if the Hausdorff dimension of \(E\) is greater than \(\frac{d+1}{2}\). Continuity follows by the Lebesgue dominated convergence theorem. The convergence of the integral allows us to differentiate inside the integral sign. We obtain

\[ M'(t) = \int |\hat{\mu}(\xi)|^2 \frac{d}{dt} \left\{ t^{d-1} \hat{\sigma}(t \xi) \right\} d\xi. \]

We have

\[ \frac{d}{dt} \left\{ t^{d-1} \hat{\sigma}(t \xi) \right\} = (d - 1)t^{d-2} \hat{\sigma}(t \xi) + t^{d-1} \nabla \hat{\sigma}(t \xi) \cdot \xi. \]

Applying (2.2) and (2.3) of Lemma 2.1 once more, it follows that

\[ \left| \frac{d}{dt} \left\{ t^{d-1} \hat{\sigma}(t \xi) \right\} \right| \lesssim |\xi|^{-\frac{d+1}{2}+1}. \]

Repeating the argument in 2.4, we see that \(M'(t)\) exists if the Hausdorff dimension of \(E\) is greater than \(\frac{d+1}{2} + 1\). Proceeding in the same way one establishes that

\[ \frac{d^m}{dt^m} \left\{ t^{d-1} \hat{\sigma}(t \xi) \right\} \lesssim |\xi|^{-\frac{d+1}{2}+m} \]

and the conclusion of Theorem 1.1 follows.

2.3. Proof of item (iv). We shall deal with the case \(k = 0\), as the other cases follow from a similar argument. Let

\[ \lambda(t) = t^{d-1} \hat{\sigma}(t \xi). \]

We must show that

\[ |M(u) - M(v)| \leq C|u - v|^\alpha. \]

We have

\[ M(u) - M(v) = \int |\hat{\mu}(\xi)|^2 (\lambda(u) - \lambda(v)) d\xi = \int |\hat{\mu}(\xi)|^2 (\lambda(u) - \lambda(v))^\alpha (\lambda(u) - \lambda(v))^{1-\alpha} d\xi. \]

Now,

\[ \lambda(u) - \lambda(v) = (u - v)\lambda'(c), \]

where \(c \in (u, v)\), by the mean-value theorem. It follows that

\[ |\lambda(u) - \lambda(v)|^\alpha \leq |u - v|^\alpha |\lambda'(c)|^\alpha. \]
On the other hand,
\[|\lambda(u) - \lambda(v)|^{1-\alpha} \leq |\lambda(u)|^{1-\alpha} + |\lambda(v)|^{1-\alpha}.\]

We have already shown above that
\[|\lambda(u)| \lesssim |\xi|^{-\frac{d-1}{2}} \text{ and } |\lambda'(u)| \lesssim |\xi|^{-\frac{d-1}{2}+1}.
\]

It follows that
\[|M(u)-M(v)| \lesssim |u-v|^\alpha \int |\hat{\mu}(\xi)|^2 |\xi|^{-\frac{d-1}{2}+\alpha} d\xi \lesssim |u-v|^{-\alpha},\]

where the last step follows by Lemma 2.2, and so the item iv) follows.

References


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