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WEIGHTED ESTIMATES FOR BELTRAMI EQUATIONS

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Abstract. We obtain a priori estimates in $L^p(\omega)$ for the generalized Beltrami equation, provided that the coefficients are compactly supported VMO functions with the expected ellipticity condition, and the weight ω lies in the Muckenhoupt class A_p. As an application, we obtain improved regularity for the jacobian of certain quasiconformal mappings.

1. Introduction

In this paper, we consider the inhomogeneous, Beltrami equation

$$\bar\partial f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = g(z), \quad \text{a.e. } z \in \mathbf{C}, \tag{1}$$

where μ, ν are $L^\infty(\mathbf{C};\mathbf{C})$ functions such that $\||\mu|+|\nu|\|_\infty \le k < 1$, and g is a measurable, $\mathbf{C}$-valued function. The derivatives $\partial f, \overline{\partial} f$ are understood in the distributional sense. In the work [3], the L^p theory of such equation was developed. More precisely, it was shown that if $1+k < p < 1+\frac{1}{k}$ and $g \in L^p(\mathbf{C})$ then (1) has a solution f, unique modulo additive constants, whose differential Df belongs to $L^p(\mathbf{C})$, and furthermore, the estimate

$$\|Df\|_{L^p(\mathbf{C})} \le C\,\|g\|_{L^p(\mathbf{C})} \tag{2}$$

holds for some constant $C = C(k,p) > 0$. For other values of p, (1) the claim may fail in general. However, in the previous work [9], Iwaniec proved that if $\mu \in VMO(\mathbf{C})$, then for any $1 < p < \infty$ and any $g \in L^p(\mathbf{C})$ one can find exactly one solution f to the $\mathbf{C}$-linear equation

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) = g(z)$$

with $Df \in L^p(\mathbf{C})$. In particular, (2) holds whenever $p \in (1,\infty)$. Recently, Koski [11] has extended this result to the generalized equation (1). For results in other spaces of functions, see [5].

In this paper, we deal with weighted spaces, and so we assume $g \in L^p(\omega)$, $1 < p < \infty$. Here ω is a measurable function, and $\omega > 0$ at almost every point. By checking the particular case $\mu = \nu = 0$, one sees that, for a weighted version of the estimate (2) to hold, the Muckenhoupt condition $\omega \in A_p$ is necessary. It turns out that, for compactly supported $\mu \in VMO$, this condition is also sufficient.

Theorem 1. *Let $1 < p < \infty$. Let μ be a compactly supported function in $VMO(\mathbf{C})$, such that $\|\mu\|_\infty < 1$, and let $\omega \in A_p$. Then, the equation*

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) = g(z)$$

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has, for $g \in L^p(\omega)$, a solution f with $Df \in L^p(\omega)$, which is unique up to an additive constant. Moreover, one has

$$\|Df\|_{L^p(\omega)} \leq C\,\|g\|_{L^p(\omega)}$$

for some $C > 0$ depending on μ, p and $[\omega]_{A_p}$.

The proof copies the scheme of [9]. In particular, our main tool is the following compactness Theorem, which extends a classical result of Uchiyama [18] about commutators of Calderón–Zygmund singular integral operators and VMO functions.

Theorem 2. *Let T be a Calderón–Zygmund singular integral operator. Let $\omega \in A_p$ with $1 < p < \infty$, and let $b \in VMO(\mathbf{R}^n)$. The commutator $[b,T]\colon L^p(\omega) \to L^p(\omega)$ is compact.*

Theorem 2 is obtained from a sufficient condition for compactness in $L^p(\omega)$. When $\omega = 1$, this sufficient condition reduces to the classical Frechet–Kolmogorov compactness criterion. Theorem 1 is then obtained from Theorem 2 by letting T be the Beurling–Ahlfors singular integral operator.

A counterpart to Theorem 1 for the *generalized Beltrami equation*,

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = g(z), \tag{3}$$

can also be obtained under the ellipticity condition $\||\mu| + |\nu|\|_\infty \leq k < 1$ and the VMO smoothness of the coefficients (see Theorem 8 below). Theorem 2 is again the main ingredient. However, for (3) the argument in Theorem 1 needs to be modified, because the involved operators are not $\mathbf{C}$-linear, but only $\mathbf{R}$-linear. In other words, $\mathbf{C}$-linearity is not essential. See also [11].

It turns out that any linear, elliptic, divergence type equation can be reduced to equation (3) (see e.g. [2, Theorem 16.1.6]). Therefore the following result is no surprise.

Corollary 3. *Let $K \geq 1$. Let $A\colon \mathbf{R}^2 \to \mathbf{R}^{2\times 2}$ be a matrix-valued function, satisfying the ellipticity condition*

$$\frac{1}{K} \leq v^t\, A(z)\, v \leq K, \quad \text{whenever } v \in \mathbf{R}^2,\ |v| = 1,$$

at almost every point $z \in \mathbf{R}^2$, and such that $A - \mathbf{Id}$ has compactly supported VMO entries. Let $p \in (1,\infty)$ be fixed, and $\omega \in A_p$. For any $g \in L^p(\omega)$, the equation

$$\operatorname{div}(A(z)\,\nabla u) = \operatorname{div}(g)$$

has a solution u with $\nabla u \in L^p(\omega)$, unique up to an additive constant, and such that

$$\|\nabla u\|_{L^p(\omega)} \leq C\,\|g\|_{L^p(\omega)}$$

for some constant $C = C(A,\omega,p)$.

Other applications of Theorem 1 are found in connection to planar K-quasiconformal mappings. Remember that a $W^{1,2}_{\text{loc}}$ homeomorphism $\phi\colon \Omega \to \Omega'$ between domains $\Omega, \Omega' \subset \mathbf{C}$ is called K-quasiconformal if

$$|\overline{\partial}\phi(z)| \leq \frac{K-1}{K+1}\,|\partial\phi(z)| \quad \text{for a.e. } z \in \Omega.$$

In general, jacobians of K-quasiconformal maps are Muckenhoupt weights belonging to the class A_p for any $p > K$ (see [2, Theorem 13.4.2], or also [3]), and this is sharp. As a consequence of Theorem 1, we obtain the following improvement.

Corollary 4. *Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let $\phi\colon \mathbf{C} \to \mathbf{C}$ be a quasiconformal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

Then, for every $1 < p < \infty$ there exists a constant $C = C(p) \geq 1$ such that the estimate

$$\left(\fint_D J(z,\phi)^p\,\mathrm{d}z\right)^{\frac{1}{p}} \leq C_p \fint_D J(z,\phi)\,\mathrm{d}z, \tag{4}$$

holds for every disk $D \subset \mathbf{C}$.

By quasiconformality, the above result is equivalent to say that the inverse mapping ϕ^{-1} has jacobian determinant $J(\cdot,\phi^{-1}) \in A_p$ for every $p > 1$. In turn, Johnson and Neugebauer [10] proved that this is equivalent to the fact that the composition with ϕ^{-1} quantitatively preserves the Muckenhoupt class A_2, and this is what we actually prove. The above Corollary improves the results in [9], which assert that $J(\cdot,\phi) \in L^p_{loc}$ for every finite $p > 1$. Note also that general K-quasiconformal maps need not satisfy the estimate (4) if $p \geq \frac{K}{K-1}$ [3].

The paper is structured as follows. In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 1 and its counterpart for the generalized Beltrami equation. In Section 4 we study some applications. By C we denote a positive constant that may change at each occurrence. $B(x,r)$ denotes the open ball with center x and radius r, and $2B$ means the open ball concentric with B and having double radius.

2. Compactness of commutators

By singular integral operator T, we mean a linear operator on $L^p(\mathbf{R}^n)$ that can be written as

$$Tf(x) = \int_{\mathbf{R}^n} f(y)\,K(x,y)\,\mathrm{d}y.$$

Here $K\colon \mathbf{R}^n \times \mathbf{R}^n \setminus \{x = y\} \to \mathbf{C}$ obeys the bounds

(1) $|K(x,y)| \leq \frac{C_1}{|x-y|^n}$,

(2) $|K(x,y) - K(x,y')| \leq C_2 \frac{|y-y'|}{|x-y|^{n+1}}$ whenever $|x-y| \geq 2|y-y'|$,

(3) $|K(x,y) - K(x',y)| \leq C_3 \frac{|x-x'|}{|x-y|^{n+1}}$ whenever $|x-y| \geq 2|x-x'|$.

One then calls $\|T\|_{CZ} = \max\{C_1, C_2, C_3\}$ the Calderón–Zygmund constant of T. Given a singular integral operator T, we define the *truncated singular integral* as

$$T_\epsilon f(x) = \int_{|x-y|\geq\epsilon} K(x,y) f(y)\mathrm{d}y$$

and the *maximal singular integral* by the relationship

$$T_* f(x) = \sup_{\epsilon>0} |T_\epsilon f(x)|\,.$$

As usually, we denote $\fint_E f(x)\,\mathrm{d}x = \frac{1}{|E|}\int_E f(x)\,\mathrm{d}x$. A weight is a function $\omega \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ such that $\omega(x) > 0$ almost everywhere. A weight ω is said to belong to the Muckenhoupt class A_p, $1 < p < \infty$, if

$$[\omega]_{A_p} := \sup \left(\fint_Q \omega(x)\,\mathrm{d}x\right)\left(\fint_Q \omega(x)^{-\frac{p'}{p}}\,\mathrm{d}x\right)^{\frac{p}{p'}} < \infty, \tag{5}$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$, and where $\frac{1}{p} + \frac{1}{p'} = 1$. One may equivalently consider balls instead of cubes. By $L^p(\omega)$ we denote the set of measurable functions f that satisfy

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbf{R}^n} |f(x)|^p \omega(x)\,\mathrm{d}x \right)^{\frac{1}{p}} < \infty. \tag{6}$$

The quantity $\|f\|_{L^p(\omega)}$ defines a complete norm in $L^p(\omega)$. It is well know that if T is a Calderón–Zygmund operator, then T and also T_* are bounded in $L^p(\omega)$ whenever $\omega \in A_p$ (see for instance [7, Cap. IV, Theorems 3.1 and 3.6]). Also the Hardy–Littlewood maximal operator M is bounded in $L^p(\omega)$. For more about A_p classes and weighted spaces $L^p(\omega)$, we refer the reader to [7].

We first show the following sufficient condition for compactness in $L^p(\omega)$, $\omega \in A_p$. Remember that a metric space X is *totally bounded* if for every $\epsilon > 0$ there exists a finite number of open balls of radius ϵ whose union is the space X. In addition, a metric space is compact if and only if it is complete and totally bounded.

Theorem 5. *Let $p \in (1, \infty)$, $\omega \in A_p$, and let $\mathfrak{F} \subset L^p(\omega)$. Then $\mathfrak{F}$ is totally bounded if it satisfies the next three conditions:*

(1) *$\mathfrak{F}$ is uniformly bounded, i.e.* $\sup_{f\in\mathfrak{F}} \|f\|_{L^p(\omega)} < \infty$.

(2) *$\mathfrak{F}$ is uniformly equicontinuous, i.e.* $\sup_{f\in\mathfrak{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p(\omega)} \xrightarrow{h\to 0} 0$.

(3) *$\mathfrak{F}$ uniformly vanishes at infinity, i.e.* $\sup_{f\in\mathfrak{F}} \|f - \chi_{Q(0,R)} f\|_{L^p(\omega)} \xrightarrow{R\to\infty} 0$, *where $Q(0,R)$ is the cube with center at the origin and sidelength $2R$.*

Let us emphasize that Theorem 5 is a strong sufficient condition for compactness in $L^p(\omega)$, because for a general weight $\omega \in A_p$ the space $L^p(\omega)$ is not invariant under translations. Theorem 5 is proved by adapting the arguments in [8]. In particular, the following result (which can be found in [8, Lemma 1]) is essential.

Lemma 6. *Let X be a metric space. Suppose that for every $\epsilon > 0$ one can find a number $\delta > 0$, a metric space W and an mapping $\Phi\colon X \to W$ such that $\Phi(X)$ is totally bounded, and the implication*

$$d(\Phi(x), \Phi(y)) < \delta \quad \Longrightarrow \quad d(x,y) < \epsilon$$

holds for any $x, y \in X$. Then X is totally bounded.

Proof of Theorem 5. Suppose that the family $\mathfrak{F}$ satisfies the three conditions of Theorem 5, and let $\epsilon > 0$ be fixed. According to the third assumption on $\mathfrak{F}$, we can choose a positive quantity $R > 0$ such that

$$\sup_{f\in\mathfrak{F}} \|f - f\,\chi_{Q(0,R)}\|_{L^p(\omega)} < \frac{\epsilon}{4}. \tag{7}$$

Let us also find $\rho > 0$ small enough so that

$$\sup_{h\in Q(0,2\rho)} \left(\sup_{f\in\mathfrak{F}} \|f(\cdot) - f(\cdot + h)\|_{L^p(\omega)} \right) < \frac{\epsilon}{2^{2+n/p}}. \tag{8}$$

Such a ρ exists due to the equicontinuity assumption on $\mathfrak{F}$. Now, let us choose N cubes $Q_1, \dots, Q_N$ with sidelength 2ρ, having pairwise disjoint interiors, and such that

$$\overline{Q(0,R)} \subset \overline{\bigcup_i Q_i}. \tag{9}$$

Define

$$\Phi f(x) = \begin{cases} \fint_{Q_i} f(z)\,\mathrm{d}z, & x \in Q_i, i = 1, \dots, N, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Since functions in $L^p(\omega)$ are locally integrable, Φf is well defined for any $f \in \mathfrak{F}$. Moreover,

$$\begin{aligned} \int_{\mathbf{R}^n} |\Phi f(x)|^p \omega(x)\,\mathrm{d}x &= \sum_{i=1}^N \left| \fint_{Q_i} f(z)\,\mathrm{d}z \right|^p \int_{Q_i} \omega(x)\,\mathrm{d}(x) \\ &\le \sum_{i=1}^N \left(\fint_{Q_i} |f(z)|^p \omega(z)\,\mathrm{d}z \right) \left(\fint_{Q_i} \omega^{\frac{-p'}{p}}(z)\,\mathrm{d}z \right)^{\frac{p}{p'}} \int_{Q_i} \omega(x)\,\mathrm{d}x \\ &\le [\omega]_{A_p} \|f\|^p_{L^p(\omega)}. \end{aligned}$$

In particular, $\Phi\colon L^p(\omega) \to L^p(\omega)$ is a bounded operator. As $\mathfrak{F}$ is bounded, then $\Phi(\mathfrak{F})$ is a bounded subset of a finite dimensional Banach space, and hence $\Phi(\mathfrak{F})$ is totally bounded.

On the other hand, by (7) and (9) one gets that

$$\|f\,\chi_{\mathbf{R}^n \setminus \cup_i Q_i}\|_{L^p(\omega)} \le \|f\,\chi_{\mathbf{R}^n \setminus Q(0,R)}\|_{L^p(\omega)} < \frac{\epsilon}{4},$$

for any $f \in \mathfrak{F}$. Also, by Jensen's inequality,

$$\begin{aligned} \|f\chi_{\cup_i Q_i} - \Phi f\|^p_{L^p(\omega)} &= \sum_{i=1}^N \int_{Q_i} \left| f(x) - \fint_{Q_i} f(z)\mathrm{d}z \right|^p \omega(x)\,\mathrm{d}x \\ &\le \sum_{i=1}^N \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(z)|^p\,\mathrm{d}z\,\omega(x)\,\mathrm{d}x. \end{aligned}$$

Now, if $x, z \in Q_i$, then $z - x = h \in Q(0, 2\rho)$. Therefore, after a change of coordinates,

$$\begin{aligned} \|f\chi_{\cup_i Q_i} - \Phi f\|^p_{L^p(\omega)} &\le \sum_{i=1}^N \frac{1}{|Q_i|} \int_{Q_i} \int_{Q(0,2\rho)} |f(x) - f(x+h)|^p\,\mathrm{d}h\,\omega(x)\,\mathrm{d}x \\ &= \frac{1}{|Q(0,\rho)|} \int_{Q(0,2\rho)} \sum_{i=1}^N \int_{Q_i} |f(x) - f(x+h)|^p \omega(x)\,\mathrm{d}x\,\mathrm{d}h \\ &\le \frac{1}{|Q(0,\rho)|} \int_{Q(0,2\rho)} \int_{\mathbf{R}^n} |f(x) - f(x+h)|^p \omega(x)\,\mathrm{d}x\,\mathrm{d}h \\ &= 2^n \fint_{Q(0,2\rho)} \|f(\cdot) - f(\cdot + h)\|^p_{L^p(\omega)}\,\mathrm{d}h \\ &\le 2^n \sup_{h \in Q(0,2\rho)} \left(\sup_{f \in \mathfrak{F}} \|f(\cdot) - f(\cdot + h)\|^p_{L^p(\omega)} \right) < \left(\frac{\epsilon}{4} \right)^p. \end{aligned}$$

Summarizing,

$$\|f - \Phi f\|_{L^p(\omega)} \le \|f\,\chi_{\mathbf{R}^n \setminus \cup_i Q_i}\|_{L^p(\omega)} + \|f\,\chi_{\cup_i Q_i} - \Phi f\|_{L^p(\omega)} < \frac{\epsilon}{2},$$

for any $f \in \mathfrak{F}$. Hence

$$\|f\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f\|_{L^p(\omega)}, \quad \text{whenever } f \in \mathfrak{F}. \tag{11}$$

Since Φ is linear, this means that

$$\|f - g\|_{L^p(\omega)} < \frac{\epsilon}{2} + \|\Phi f - \Phi g\|_{L^p(\omega)}, \quad \text{whenever } f, g \in \mathfrak{F}.$$

Set $\delta = \epsilon/2$. The above inequality says that if $f, g \in \mathfrak{F}$ are such that $d(\Phi f, \Phi g) < \delta$, then $d(f, g) < \epsilon$. By the previous Lemma, it follows that $\mathfrak{F}$ is totally bounded. □

In order to prove Theorem 2, we will first reduce ourselves to smooth symbols b. Let us recall that commutators $C_b = [b, T]$ with $b \in BMO(\mathbf{R}^n)$ are continuous in $L^p(\omega)$ [15, Theorem 2.3]. Moreover, in [13, Theorem 1] the following estimate is shown,

$$\|C_b f\|_{L^p(\omega)} \leq C\, \|b\|_*\, \|M^2 f\|_{L^p(\omega)}, \tag{12}$$

where $\|b\|_*$ denotes the BMO norm of b, and the constant C may depend on ω, but not on b. Now, by the boudedness of the Hardy–Littlewood operator M on $L^p(\omega)$, we obtain

$$\|C_b f\|_{L^p(\omega)} \leq C\, \|b\|_*\, \|f\|_{L^p(\omega)}.$$

Since by assumption $b \in VMO(\mathbf{R}^n)$, we can approximate the function b by functions $b_j \in \mathcal{C}^\infty_c(\mathbf{R}^n)$ in the BMO norm, and thus

$$\|C_b f - C_{b_j} f\|_{L^p(\omega)} = \|C_{b-b_j} f\|_{L^p(\omega)} \leq C\, \|b - b_j\|_*\, \|f\|_{L^p(\omega)}.$$

In particular, the commutators with smooth symbol C_{b_j} converge to C_b in the operator norm of $L^p(\omega)$. Therefore it suffices to prove compactness for the commutator with smooth symbol.

Another reduction in the proof of Theorem 2 will be made by slightly modifying the singular integral operator T. This technique comes from Krantz and Li [12]. More precisely, for every $\eta > 0$ small enough, let us take a continuous function K^η defined on $\mathbf{R}^n \times \mathbf{R}^n$, taking values in $\mathbf{R}$ or $\mathbf{C}$, and such that:

(1) $K^\eta(x, y) = K(x, y)$ if $|x - y| \geq \eta$,
(2) $|K^\eta(x, y)| \leq \frac{C_0}{|x-y|^n}$ for $\frac{\eta}{2} < |x - y| < \eta$,
(3) $K^\eta(x, y) = 0$ if $|x - y| \leq \frac{\eta}{2}$,

where C_0 is independent of η. Due to the growth properties of K, it is not restrictive to suppose that condition 2 holds for all $x, y \in \mathbf{R}^n$. Now, let

$$T^\eta f(x) = \int_{\mathbf{R}^n} K^\eta(x, y) f(y)\, \mathrm{d}y,$$

and let us also denote

$$C^\eta_b f(x) = [b, T^\eta] f(x) = \int_{\mathbf{R}^n} (b(x) - b(y)) K^\eta(x, y) f(y)\, \mathrm{d}y.$$

We now prove that the commutators C^η_b approximate C_b in the operator norm.

Lemma 7. *Let $b \in \mathcal{C}^1_c(\mathbf{R}^n)$. There exists a constant $C = C(n, C_0)$ such that*

$$|C_b f(x) - C^\eta_b f(x)| \leq C\, \eta\, \|\nabla b\|_\infty\, M f(x) \quad \textit{almost everywhere},$$

for every $\eta > 0$. As a consequence,

$$\lim_{\eta \to 0} \|C^\eta_b - C_b\|_{L^p(\omega) \to L^p(\omega)} = 0$$

whenever $\omega \in A_p$ *and* $1 < p < \infty$.

Proof. Let $f \in L^p(\omega)$. For every $x \in \mathbf{R}^n$ we have

$$\begin{aligned} C_b f(x) - C_b^\eta f(x) &= \int_{|x-y|<\eta} (b(x)-b(y)) K(x,y) f(y)\,\mathrm{d}y \\ &\quad - \int_{\frac{\eta}{2}\le|x-y|<\eta} (b(x)-b(y)) K^\eta(x,y) f(y)\,\mathrm{d}y \\ &= I_1(x) + I_2(x). \end{aligned}$$

Using the smoothness of b and the size estimates of K^η, we have that

$$\begin{aligned} |I_1(x)| &\le \int_{|x-y|<\eta} |b(y)-b(x)||K(x,y)||f(y)|\,\mathrm{d}y \\ &\le C_0\,\|\nabla b\|_\infty \sum_{j=0}^\infty \int_{\frac{\eta}{2^{j+1}}<|x-y|<\frac{\eta}{2^j}} \frac{|f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y \\ &\le 2^n C_0\,\|\nabla b\|_\infty \sum_{j=0}^\infty \frac{\eta\,|B(0,1)|}{2^{j+1}} \fint_{|x-y|<\frac{\eta}{2^j}} |f(y)|\,\mathrm{d}y \\ &\le \eta\, 2^n\, C_0\,\|\nabla b\|_\infty\,|B(0,1)|\,Mf(x) \end{aligned}$$

for almost every x. For the other term, similarly

$$\begin{aligned} |I_2(x)| &\le \eta\,\|\nabla b\|_\infty \int_{\frac{\eta}{2}<|x-y|<\eta} |K^\eta(x,y)|\,|f(y)|\,\mathrm{d}y \\ &\le \eta\, C_0\,\|\nabla b\|_\infty \int_{\frac{\eta}{2}<|x-y|<\eta} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \\ &\le \eta\, 2^n\, C_0\,\|\nabla b\|_\infty\,|B(0,1)| \fint_{|x-y|<\eta} |f(y)|\,\mathrm{d}y \\ &\le \eta\, 2^n\, C_0\,\|\nabla b\|_\infty\,|B(0,1)|\,Mf(x). \end{aligned}$$

Therefore, the pointwise estimate follows. Now, the boundedness of M in $L^p(\omega)$ for any A_p weight ω implies that

$$\|C_b f - C_b^\eta f\|_{L^p(\omega)} \le C\,\eta\,\|\nabla b\|_\infty\,\|Mf\|_{L^p(\omega)} \le C\,\eta\,\|\nabla b\|_\infty\,\|f\|_{L^p(\omega)} \to 0,$$

as $\eta \to 0$. This finishes the proof of Lemma 7. □

We are now ready to conclude the proof of Theorem 2. From now on, $\eta > 0$ and $b \in \mathcal{C}_c^1(\mathbf{R}^n)$ are fixed, and we have to prove that the commutator $C_b^\eta = [b, T^\eta]$ is compact. Thus, the constants that will appear may depend on b and η.

We denote $\mathfrak{F} = \{C_b^\eta f;\ f \in L^p(\omega),\ \|f\|_{L^p(\omega)} \le 1\}$. Then $\mathfrak{F}$ is uniformly bounded, because C_b^η is a bounded operator on $L^p(\omega)$. To prove the uniform equicontinuity of $\mathfrak{F}$, we must see that

$$\lim_{h\to 0} \sup_{f\in\mathfrak{F}} \|C_b^\eta f(\cdot) - C_b^\eta f(\cdot + h)\|_{L^p(\omega)} = 0.$$

To do this, let us write

$$\begin{aligned} C_b^\eta f(x) - C_b^\eta f(x+h) &= (b(x)-b(x+h))\int_{\mathbf{R}^n} K^\eta(x,y)f(y)\,\mathrm{d}y \\ &\quad + \int_{\mathbf{R}^n} (b(x+h)-b(y))(K^\eta(x,y)-K^\eta(x+h,y))f(y)\,\mathrm{d}y \\ &= \int_{\mathbf{R}^n} I_1(x,y,h)\mathrm{d}y + \int_{\mathbf{R}^n} I_2(x,y,h)\,\mathrm{d}y. \end{aligned}$$

For $I_1(x,y,h)$, using the regularity of the function b and the definition of the operator T_*,

$$\begin{aligned} &\left|\int_{\mathbf{R}^n} I_1(x,y,h)\,\mathrm{d}y\right| \\ &\le \|\nabla b\|_\infty |h| \left| \int_{|x-y|>\frac{\eta}{2}} (K^\eta(x,y)-K(x,y))\,f(y)\,\mathrm{d}y + \int_{|x-y|>\frac{\eta}{2}} K(x,y)f(y)\mathrm{d}y \right| \\ &\le \|\nabla b\|_\infty |h| \left(\int_{|x-y|>\frac{\eta}{2}} |K^\eta(x,y)-K(x,y)|\,|f(y)|\,\mathrm{d}y + T_*f(x) \right) \\ &\le \|\nabla b\|_\infty |h| \,(C\,Mf(x) + T_*f(x)) \end{aligned}$$

for some constant $C>0$ that may depend on η, but not on h. Therefore

$$\left(\int \left|\int_{\mathbf{R}^n} I_1(x,y,h)\,\mathrm{d}y\right|^p \omega(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \le C\,|h|\,\|f\|_{L^p(\omega)}, \tag{13}$$

for C independent of f and h. Here we used the boundedness of M and T_* on $L^p(\omega)$ (see [7, Chap. IV, Th. 3.6]). We will divide the integral of $I_2(x,y,h)$ into three regions:

$$\begin{aligned} A &= \left\{ y\in\mathbf{R}^n\colon |x-y|>\frac{\eta}{2},\ |x+h-y|>\frac{\eta}{2}\right\}, \\ B &= \left\{ y\in\mathbf{R}^n\colon |x-y|>\frac{\eta}{2},\ |x+h-y|<\frac{\eta}{2}\right\}, \\ C &= \left\{ y\in\mathbf{R}^n\colon |x-y|<\frac{\eta}{2},\ |x+h-y|>\frac{\eta}{2}\right\}. \end{aligned}$$

Note that $I_2(x,y,h)=0$ for $y\in\mathbf{R}^n\setminus A\cup B\cup C$. Now, for the integral over A, we use the smoothness of b and K^η,

$$\begin{aligned} \left|\int_A I_2(x,y,h)\,\mathrm{d}y\right| &\le C\|\nabla b\|_\infty |h| \int_{|x-y|>\frac{\eta}{4}} \frac{|f(y)|}{|x-y|^{n+1}}\,\mathrm{d}y \\ &\le C\|\nabla b\|_\infty \frac{|h|}{\eta} \sum_{j=0}^{\infty} 2^{-j} \fint_{|x-y|<\frac{2^j\eta}{4}} |f(y)|\,\mathrm{d}y \le C\|\nabla b\|_\infty \frac{|h|}{\eta}\quad Mf(x), \end{aligned}$$

thus

$$\left(\int_{\mathbf{R}^n} \left|\int_A I_2(x,y,h)\,\mathrm{d}y\right|^p \omega(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \le C|h|\,\|f\|_{L^p(\omega)}.$$

for some constant C that may depend on η and b, but not on h. In particular, the term on the right hand side goes to 0 as $|h|\to 0$.

The integrals of $I_2(x,y,h)$ over B and C are symmetric, so we only give the details once. For the integral over the set B, let us assume that $|h|$ is very small. We can first choose $R_0 > \eta/2 + |h|$ such that b vanishes outside the ball $B_0 = B(0,R_0)$. It then follows that $b(\cdot + h)$ has support in $2B_0$. Then, since $B \subset B(x, |h| + \eta/2)$, we have for $|x| < 3R_0$ that $B \subset 4B_0$ and therefore

$$\begin{aligned}\left|\int_B I_2(x,y,h)\,\mathrm{d}y\right| &\le C_0\|\nabla b\|_\infty \int_{B\cap 4B_0} \frac{|x+h-y|\,|f(y)|}{|x-y|^n}\,\mathrm{d}y\\ &\le C_0\|\nabla b\|_\infty \int_{B\cap 4B_0} \frac{|f(y)|}{|x-y|^{n-1}}\,\mathrm{d}y\\ &\le C_0\,\|\nabla b\|_\infty\,(2/\eta)^{n-1}\int_{B\cap 4B_0}|f(y)|\omega(y)^{\frac1p}\omega(y)^{-\frac1p}\,\mathrm{d}y\\ &\le C_0\,\|\nabla b\|_\infty\,(2/\eta)^{n-1}\|f\|_{L^p(\omega)}\left(\int_{B\cap 4B_0}\omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{1}{p'}}\end{aligned}$$

whence

$$\int_{3B_0}\left|\int_B I_2(x,y,h)\,\mathrm{d}y\right|^p\omega(x)\,\mathrm{d}x \le C\,\|f\|^p_{L^p(\omega)}\left(\int_{3B_0}\omega(x)\,\mathrm{d}x\right)\left(\int_{B\cap 4B_0}\omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{p}{p'}}$$

for some constant C that might depend on η, but not on h. If, instead, we have $|x| \ge 3R_0$, then $b(x+h) = 0$ (because $|h| < R_0$ so that $|x+h| > 2R_0$). Note also that for $y \in B$ one has $|x| \le C|x-y|$ where C depends only on η. Therefore

$$\begin{aligned}\left|\int_B I_2(x,y,h)\,\mathrm{d}y\right| &\le C\|b\|_\infty\int_{B\cap 4B_0}\frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \le \frac{C\|b\|_\infty}{|x|^n}\int_{B\cap 4B_0}|f(y)|\,\mathrm{d}y\\ &\le \frac{C\|b\|_\infty}{|x|^n}\,\|f\|_{L^p(\omega)}\left(\int_{B\cap 4B_0}\omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{1}{p'}}.\end{aligned}$$

This implies that

$$\begin{aligned}&\int_{\mathbf{R}^n\setminus 3B_0}\left|\int_B I_2(x,y,h)\mathrm{d}y\right|^p\omega(x)\,\mathrm{d}x\\ &\le C\|b\|^p_\infty\|f\|^p_{L^p(\omega)}\left(\int_{\mathbf{R}^n\setminus 3B_0}\frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x\right)\left(\int_{B\cap 4B_0}\omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{p}{p'}}.\end{aligned}$$

Summarizing,

$$\begin{aligned}&\int_{\mathbf{R}^n}\left|\int_B I_2(x,y,h)\mathrm{d}y\right|^p\omega(x)\,\mathrm{d}x\\ &\le C\,\|f\|^p_{L^p(\omega)}\left(\int_{B\cap 4B_0}\omega(y)^{-\frac{p'}{p}}\,\mathrm{d}y\right)^{\frac{p}{p'}}\left(\int_{3B_0}\omega(x)\,\mathrm{d}x + \int_{\mathbf{R}^n\setminus 3B_0}\frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x\right).\end{aligned}\tag{14}$$

After proving that

$$\int_{|x|>3R_0}\frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x < \infty,$$

the left hand side of (14) will converge to 0 as $|h| \to 0$ since $|B| \to 0$ as $|h| \to 0$. To prove the above claim, let us choose $q < p$ such that $\omega \in A_q$ [7, Theorem 2.6, Ch. IV]. For such q, we have

$$\int_{|x|>R} \frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x = \sum_{j=1}^{\infty} \int_{2^{j-1}<\frac{|x|}{R}<2^j} \frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x \leq \sum_{j=1}^{\infty} (2^{j-1}R)^{-np}\omega(B(0,2^jR)).$$

By [7, Lemma 2.2], we have

$$\int_{|x|>R} \frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x \leq \sum_{j=1}^{\infty} (2^{j-1}R)^{-np}(2^jR)^{nq}\omega(B(0,1)) = \frac{C}{R^{n(p-q)}} < \infty \tag{15}$$

as desired. The equicontinuity of $\mathfrak{F}$ follows.

Finally, we show the decay at infinity of the elements of $\mathfrak{F}$. Let x be such that $|x| > R > R_0$. Then, $x \notin \operatorname{supp} b$, and

$$\begin{aligned} |C_b^\eta f(x)| &= \left| \int_{\mathbf{R}^n} (b(x)-b(y))K^\eta(x,y)f(y)\,\mathrm{d}y \right| \leq C_0\|b\|_\infty \int_{\operatorname{supp} b} \frac{|f(y)|}{|x-y|^n}\,\mathrm{d}y \\ &\leq \frac{C\|b\|_\infty}{|x|^n} \int_{\operatorname{supp} b} |f(y)|\,\mathrm{d}y \leq \frac{C\|b\|_\infty}{|x|^n}\,\|f\|_{L^p(\omega)} \left(\int_{\operatorname{supp} b} \omega(y)^{-\frac{p'}{p}}\,dy \right)^{\frac{1}{p'}} \end{aligned}$$

whence

$$\left(\int_{|x|>R} |C_b^\eta f(x)|^p \omega(x)\,\mathrm{d}x \right)^{\frac{1}{p}} \leq C\|b\|_\infty \|f\|_{L^p(\omega)} \left(\int_{|x|>R} \frac{\omega(x)}{|x|^{np}}\,\mathrm{d}x \right)^{\frac{1}{p}}.$$

The right hand side above converges to 0 as $R \to \infty$, due to (15). By Theorem 5, $\mathfrak{F}$ is totally bounded. Theorem 2 follows.

3. A priori estimates for Beltrami equations

We first prove Theorem 1. To do this, let us remember that the Beurling–Ahlfors singular integral operator is defined by the following principal value

$$\mathcal{B}f(z) = -\frac{1}{\pi}\,P.V. \int \frac{f(w)}{(z-w)^2}\,\mathrm{d}w.$$

This operator can be seen as the formal ∂ derivative of the Cauchy transform,

$$\mathcal{C}f(z) = \frac{1}{\pi} \int \frac{f(w)}{z-w}\,\mathrm{d}w.$$

At the frequency side, $\mathcal{B}$ corresponds to the Fourier multiplier $m(\xi) = \frac{\bar{\xi}}{\xi}$, so that $\mathcal{B}$ is an isometry in $L^2(\mathbf{C})$. Moreover, this Fourier representation also explains the important relation

$$\mathcal{B}(\overline{\partial} f) = \partial f$$

for smooth enough functions f. By $\mathcal{B}^*$ we mean the singular integral operator obtained by simply conjugating the kernel of $\mathcal{B}$, that is,

$$\mathcal{B}^*(f)(z) = -\frac{1}{\pi}\,P.V. \int \frac{f(w)}{(\bar{z}-\bar{w})^2}\,\mathrm{d}w.$$

Note that $\mathcal{B}^*$ has Fourier multiplier $m^*(\xi) = \frac{\xi}{\bar{\xi}}$. Thus,

$$\mathcal{B}\mathcal{B}^* = \mathcal{B}^*\mathcal{B} = \mathbf{Id}.$$

In other words, $\mathcal{B}^*$ is the L^2-inverse of $\mathcal{B}$. It also appears as the $\mathbf{C}$-linear adjoint of $\mathcal{B}$,

$$\int_{\mathbf{C}} \mathcal{B}f(z)\,\overline{g(z)}\,\mathrm{d}z = \int_{\mathbf{C}} f(z)\,\overline{\mathcal{B}^* g(z)}\,\mathrm{d}z.$$

The complex conjugate operator $\overline{\mathcal{B}}$ is the composition of $\mathcal{B}$ with the complex conjugation operator $\mathbf{C}f = \overline{f}$, that is,

$$\overline{\mathcal{B}}(f) = \mathbf{C}\mathcal{B}(f) = \overline{\mathcal{B}(f)}.$$

It then follows that

$$\overline{\mathcal{B}} = \mathbf{C}\mathcal{B} = \mathcal{B}^*\mathbf{C}.$$

Note that $\mathcal{B}$ and $\mathcal{B}^*$ are $\mathbf{C}$-linear operators, while $\overline{\mathcal{B}}$ is only $\mathbf{R}$-linear. See [2, Chapter 4] for more about the Beurling–Ahlfors transform.

Proof of Theorem 1. We follow Iwaniec's idea [9, pp. 42–43]. For every $N = 1, 2, \ldots$, let

$$P_N = \mathbf{Id} + \mu\mathcal{B} + \cdots + (\mu\mathcal{B})^N.$$

Then

$$(\mathbf{Id} - \mu\mathcal{B})P_{N-1} = P_{N-1}(\mathbf{Id} - \mu\mathcal{B}) = \mathbf{Id} - \mu^N\mathcal{B}^N + K_N, \tag{16}$$

where $K_N = \mu^N\mathcal{B}^N - (\mu\mathcal{B})^N$. Each K_N consists of a finite sum of operators that contain the commutator $[\mu, \mathcal{B}]$ as a factor. Thus, by Theorem 2, each K_N is compact in $L^p(\omega)$. On the other hand, the N-th iterate $\mathcal{B}^N$ of the Beurling transform is another convolution-type Calderón–Zygmund operator, whose kernel is

$$b_N(z) = \frac{(-1)^N N}{\pi}\frac{\bar{z}^{N-1}}{z^{N+1}}$$

(see for instance [16, p. 73]). Arguing as in [6, Lemma 7.9 & Theorem 7.11], one sees that the operator norm $\|\mathcal{B}^N\|_{L^p(\omega)}$ depends linearly on both the unweighted norm $\|B^N\|_{L^p(\mathbf{R}^n)}$ and the Calderón–Zygmund constant $\|B^N\|_{CZ}$. Since both quantities are bounded by a constant multiple of N^2, one immediately sees that

$$\|\mathcal{B}^N\|_{L^p(\omega)} \le CN^2, \tag{17}$$

with constant C that depends on $[\omega]_{A_p}$, but not on N. As a consequence,

$$\|\mu^N\mathcal{B}^N f\|_{L^p(\omega)} \le CN^2\|\mu\|_\infty^N\|f\|_{L^p(\omega)},$$

and therefore, for large enough N, the operator $\mathbf{Id} - \mu^N\mathcal{B}^N$ is invertible. This, together with (16), says that $\mathbf{Id} - \mu\mathcal{B}$ is an Fredholm operator. Now apply the index theory to $\mathbf{Id} - \mu\mathcal{B}$. The continuous deformation $\mathbf{Id} - t\mu\mathcal{B}$, $0 \le t \le 1$, is a homotopy from the identity operator to $\mathbf{Id} - \mu\mathcal{B}$. By the homotopical invariance of Index,

$$\mathrm{Index}(\mathbf{Id} - \mu\mathcal{B}) = \mathrm{Index}(\mathbf{Id}) = 0.$$

Since injective operators with 0 index are onto, for the invertibility of $\mathbf{Id} - \mu\mathcal{B}$ it just remains to show that it is injective. So let $f \in L^p(\omega)$ be such that $f = \mu\mathcal{B}f$. Then f has compact support. Now, since belonging to A_p is an open-ended condition (see e.g. [7, Theorem IV.2.6]), there exists $\delta>0$ such that $p-\delta>1$ and $\omega\in A_{p-\delta}$. Then $\omega^{-\frac{1}{p-\delta}}\in L^1_{loc}(\mathbf{C})$. Taking $\epsilon=\frac{\delta}{p-\delta}$, we obtain

$$\begin{aligned}\int_{\mathbf{C}}|f(x)|^{1+\epsilon}\,\mathrm{d}x &\le \left(\int_{\operatorname{supp} f}|f(x)|^p\omega(x)\,\mathrm{d}x\right)^{\frac{1+\epsilon}{p}}\left(\int_{\operatorname{supp} f}\omega(x)^{-\frac{1+\epsilon}{p-(1+\epsilon)}}\,\mathrm{d}x\right)^{\frac{p-(1+\epsilon)}{p}}\\ &\le \|f\|^{1+\epsilon}_{L^p(\omega)}\left(\int_{\operatorname{supp} f}\omega(x)^{-\frac{1+\epsilon}{p-(1+\epsilon)}}\,\mathrm{d}x\right)^{\frac{p-(1+\epsilon)}{p}}<\infty,\end{aligned}\tag{18}$$

therefore $f\in L^{1+\epsilon}(\mathbf{C})$. But $\mathbf{Id}-\mu\mathcal{B}$ is injective on $L^p(\mathbf{C})$, $1<p<\infty$, when $\mu\in VMO(\mathbf{C})$, by Iwaniec's Theorem. Hence, $f\equiv 0$.

Finally, since $\mathbf{Id}-\mu\mathcal{B}\colon L^p(\omega)\to L^p(\omega)$ is linear, bounded, and invertible, it then follows that it has a bounded inverse, so the inequality

$$\|g\|_{L^p(\omega)}\le C\,\|(\mathbf{Id}-\mu\mathcal{B})g\|_{L^p(\omega)}$$

holds for every $g\in L^p(\omega)$. Here the constant $C>0$ depends only on the $L^p(\omega)$ norm of $\mathbf{Id}-\mu\mathcal{B}$, and therefore on p,k and $[\omega]_{A_p}$, but not on g. As a consequence, given $g\in L^p(\omega)$, and setting

$$f:=\mathcal{C}(\mathbf{Id}-\mu\mathcal{B})^{-1}g,$$

we immediately see that f satisfies $\overline{\partial}f-\mu\partial f=g$. Moreover, since $\omega\in A_p$,

$$\begin{aligned}\|Df\|_{L^p(\omega)}&\le\|\partial f\|_{L^p(\omega)}+\|\overline{\partial}f\|_{L^p(\omega)}\\ &=\|\mathcal{B}(\mathbf{Id}-\mu\mathcal{B})^{-1}g\|_{L^p(\omega)}+\|(\mathbf{Id}-\mu\mathcal{B})^{-1}g\|_{L^p(\omega)}\le C\|g\|_{L^p(\omega)},\end{aligned}$$

where still C depends only on p,k and $[\omega]_{A_p}$.

For the uniqueness, let us choose two solutions f_1, f_2 to the inhomogeneous equation. The difference $F=f_1-f_2$ defines a solution to the homogeneous equation $\overline{\partial}F-\mu\,\partial F=0$. Moreover, one has that $DF\in L^p(\omega)$ and, arguing as before, one sees that $DF\in L^{1+\epsilon}(\mathbf{C})$. In particular, this says that $(I-\mu\mathcal{B})(\overline{\partial}F)=0$. But for $\mu\in VMO(\mathbf{C})$, it follows from Iwaniec's Theorem that $\mathbf{Id}-\mu\mathcal{B}$ is injective in $L^p(\mathbf{C})$ for any $1<p<\infty$, whence $\overline{\partial}F=0$. Thus $DF=0$ and so F is a constant. □

The $\mathbf{C}$-linear Beltrami equation is a particular case of the following one,

$$\overline{\partial}f(z)-\mu(z)\,\partial f(z)-\nu(z)\,\overline{\partial f(z)}=g(z),$$

which we will refer to as the *generalized Beltrami equation*. It is well known that, in the plane, any linear, elliptic system, with two unknowns and two first-order equations on the derivatives, reduces to the above equation (modulo complex conjugation), whence the interest in understanding it is very big. An especially interesting example is obtained by setting $\mu=0$, when one obtains the so-called *conjugate Beltrami equation*,

$$\overline{\partial}f(z)-\nu(z)\,\overline{\partial f(z)}=g(z).$$

A direct adaptation of the above proof immediately drives the problem towards the commutator $[\nu,\overline{\mathcal{B}}]$. Unfortunately, as an operator from $L^p(\omega)$ onto itself, such commutator is not compact in general, even when $\omega=1$. To show this, let us choose

$$\nu=i\,\nu_0\,\chi_{\mathbf{D}}+\nu_1\chi_{\mathbf{C}\setminus\mathbf{D}},$$

where the constant $\nu_0 \in \mathbf{R}$ and the function ν_1 are chosen so that ν is continuous on $\mathbf{C}$, compactly supported in $2\mathbf{D}$, with $\|\nu\|_\infty < 1$. Let us also consider

$$E = \{f \in L^p;\ \|f\|_{L^p} \le 1,\ \mathrm{supp}(f) \subset \mathbf{D}\},$$

which is a bounded subset of L^p. For every $f \in E$, one has

$$\begin{aligned}\nu\,\overline{\mathcal{B}(f)} - \overline{\mathcal{B}(\nu f)} &= \chi_{\mathbf{D}} i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbf{C}\setminus\mathbf{D}}\,\nu_1\,\overline{\mathcal{B}(f)} - \overline{\mathcal{B}(i\nu_0 f)}\\ &= \chi_{\mathbf{D}} i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbf{C}\setminus\mathbf{D}}\,\nu_1\,\overline{\mathcal{B}(f)} + i\nu_0\overline{\mathcal{B}(f)}\\ &= \chi_{\mathbf{D}} 2i\nu_0 \overline{\mathcal{B}(f)} + \chi_{\mathbf{C}\setminus\mathbf{D}}\,(i\nu_0+\nu_1)\,\overline{\mathcal{B}(f)}.\end{aligned}$$

In view of this relation, and since $\mathcal{B}$ is not compact, we have just cooked a concrete example of function $\nu \in VMO$ for wich the commutator $[\nu, \overline{\mathcal{B}}]$ is not compact. Nevertheless, it turns out that still a priori estimates hold, even for the generalized equation.

Theorem 8. *Let $1 < p < \infty$, $\omega \in A_p$, and let $\mu, \nu \in VMO(\mathbf{C})$ be compactly supported, such that $\||\mu| + |\nu|\|_\infty < 1$. Let $g \in L^p(\omega)$. Then the equation*

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = g(z)$$

has a solution f with $Df \in L^p(\omega)$ and

$$\|Df\|_{L^p(\omega)} \le C\,\|g\|_{L^p(\omega)}.$$

This solution is unique, modulo an additive constant.

A previous proof for the above result has been shown in [11] for the constant weight $\omega = 1$. For the weighted counterpart, the arguments are based on a Neumann series argument similar to that in [11], with some minor modification. We write it here for completeness. The following Lemma will be needed.

Lemma 9. *Let $\mu, \nu \in L^\infty(\mathbf{C})$ be measurable, bounded with compact support, such that $\||\mu| + |\nu|\|_\infty < 1$. If $1 < p < \infty$ and $p' = \frac{p}{p-1}$, then the following statements are equivalent:*

(1) *The operator $\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}}\colon L^p(\mathbf{C}) \to L^p(\mathbf{C})$ is bijective.*
(2) *The operator $\mathbf{Id} - \overline{\mu}\,\mathcal{B}^* - \nu\,\overline{\mathcal{B}^*}\colon L^{p'}(\mathbf{C}) \to L^{p'}(\mathbf{C})$ is bijective.*

Proof. When $\nu = 0$, the above result is well known, and follows as an easy consequence of the fact that, with respect to the dual pairing

$$\langle f, g\rangle = \int_{\mathbf{C}} f(z)\,\overline{g(z)}\,\mathrm{d}z, \tag{19}$$

the operator $\mathbf{Id} - \mu\mathcal{B}\colon L^p(\mathbf{C}) \to L^p(\mathbf{C})$ has precisely $\mathbf{Id} - \mathcal{B}^*\overline{\mu}\colon L^{p'}(\mathbf{C}) \to L^{p'}(\mathbf{C})$ as its $\mathbf{C}$-linear adjoint. Unfortunately, when ν does not identically vanish, $\mathbf{R}$-linear operators do not have an adjoint with respect to this dual pairing. An alternative proof can be found in [11]. We will think the space of $\mathbf{C}$-valued L^p functions $L^p(\mathbf{C})$ as an $\mathbf{R}$-linear space,

$$L^p(\mathbf{C}) = L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C}),$$

by means of the obvious identification $u + iv = (u, v)$. According to this product structure, every bounded $\mathbf{R}$-linear operator $T\colon L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C}) \to L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ has an obvious matrix representation

$$T(u+iv) = T\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix},$$

where every $T_{ij}\colon L^p_{\mathbf{R}}(\mathbf{C}) \to L^p_{\mathbf{R}}(\mathbf{C})$ is bounded. Similarly, bounded linear functionals $U\colon L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C}) \to \mathbf{R}$ are represented by

$$U\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

where every $U_j\colon L^p_{\mathbf{R}}(\mathbf{C}) \to \mathbf{R}$ is bounded. By the Riesz Representation Theorem, we get that $L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ has precisely $L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C})$ as its topological dual space. In fact, we have an $\mathbf{R}$-bilinear dual pairing,

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle = \int u(z)\, u'(z)\, \mathrm{d}z + \int v(z)\, v'(z)\, \mathrm{d}z,$$

whenever $(u,v) \in L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ and $(u',v') \in L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C})$, and which is nothing but the real part of (19). Under this new dual pairing, every $\mathbf{R}$-linear opeartor $T\colon L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C}) \to L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ can be associated another operator

$$T'\colon L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C}) \to L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C}),$$

called the $\mathbf{R}$-adjoint operator of T, defined by the common rule

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, T'\begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle = \left\langle T\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle.$$

If T is a $\mathbf{C}$-linear operator, then T' is the same as the $\mathbf{C}$-adjoint T^* (i.e. the adjoint with respect to (19)) so in particular for the Beurling–Ahlfors transform $\mathcal{B}$ we have an $\mathbf{R}$-adjoint $\mathcal{B}'$, and moreover $\mathcal{B}^* = \mathcal{B}'$. Similarly, the pointwise multiplication by μ and ν are also $\mathbf{C}$-linear operators. Thus their $\mathbf{R}$-adjoints μ', ν' agree with their respectives $\mathbf{C}$-adjoints μ^*, ν^*. But these are precisely the pointwise multiplication with the respective complex conjugates. Symbollically, $\mu' = \overline{\mu}$ and $\nu' = \overline{\nu}$. In contrast, general $\mathbf{R}$-linear operators need not have a $\mathbf{C}$-adjoint. For example, for the complex conjugation,

$$\mathbf{C} = \begin{pmatrix} \mathbf{Id} & 0 \\ 0 & -\mathbf{Id} \end{pmatrix}$$

one simply has $\mathbf{C}' = \mathbf{C}$. Putting all these things together, one easily sees that

$$\begin{aligned}(\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}})' &= (\mathbf{Id} - \mu\mathcal{B} - \nu\mathbf{C}\mathcal{B})' = \mathbf{Id} - (\mu\mathcal{B})' - (\nu\mathbf{C}\mathcal{B})' \\ &= \mathbf{Id} - \mathcal{B}'\mu' - \mathcal{B}'\mathbf{C}'\nu' = \mathbf{Id} - \mathcal{B}^*\overline{\mu} - \mathcal{B}^*\mathbf{C}\overline{\nu} \\ &= \mathcal{B}^*\,(\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \mathbf{C}\overline{\nu}\mathcal{B}^*)\,\mathcal{B} = \mathcal{B}^*\,(\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*)\,\mathcal{B},\end{aligned}$$

where we used the fact that $\mathcal{B}^*\mathcal{B} = \mathcal{B}\mathcal{B}^* = \mathbf{Id}$. As a consequence, and using that both $\mathcal{B}$ and $\mathcal{B}^*$ are bijective in $L^p(\mathbf{C})$, we obtain that the bijectivity of the operator $\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}}$ in $L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ is equivalent to that of $\mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*$ in the dual space $L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C})$. Similarly, one proves that

$$(\mathbf{Id} - \mu\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*)' = \mathcal{B}(\mathbf{Id} - \overline{\mu}\,\mathcal{B} - \nu\,\overline{\mathcal{B}})\mathcal{B}^*.$$

Hence, the bijectivity of $\mathbf{Id} - \mu\mathcal{B}^* - \nu\mathbf{C}\mathcal{B}^*$ in $L^p_{\mathbf{R}}(\mathbf{C}) \oplus L^p_{\mathbf{R}}(\mathbf{C})$ is equivalent to the bijectivity of $\mathbf{Id} - \overline{\mu}\,\mathcal{B} - \nu\,\overline{\mathcal{B}}$ in $L^{p'}_{\mathbf{R}}(\mathbf{C}) \oplus L^{p'}_{\mathbf{R}}(\mathbf{C})$. □

Lemma 10. *If $1 < p < \infty$, $\omega \in A_p$, $\mu, \nu \in VMO$ have compact support, and $\||\mu| + |\nu|\|_\infty \le k < 1$, then the operators*

$$\mathbf{Id} - \mu\mathcal{B} - \nu\overline{\mathcal{B}} \quad \text{and} \quad \mathbf{Id} - \mu\mathcal{B}^* - \nu\overline{\mathcal{B}^*}$$

are Fredholm operators in $L^p(\omega)$*.*

Proof. We will show the claim for the operator $\mathbf{Id}-\mu\mathcal{B}-\nu\overline{\mathcal{B}}$. For $\mathbf{Id}-\mu\mathcal{B}^*-\nu\overline{\mathcal{B}^*}$ the proof follows similarly. It will be more convenient for us to write $\overline{\mathcal{B}}=\mathbf{C}\mathcal{B}$. As in the proof of Theorem 1, we set

$$P_N=\sum_{j=0}^{N}(\mu\mathcal{B}+\nu\mathbf{C}\mathcal{B})^j.$$

Then

$$\begin{aligned}(\mathbf{Id}-\mu\mathcal{B}-\nu\mathbf{C}\mathcal{B})\circ P_{N-1}&=\mathbf{Id}-(\mu\mathcal{B}+\nu\mathbf{C}\mathcal{B})^N,\\ P_{N-1}\circ(\mathbf{Id}-\mu\mathcal{B}+\nu\mathbf{C}\mathcal{B})&=\mathbf{Id}-(\mu\mathcal{B}+\nu\mathbf{C}\mathcal{B})^N.\end{aligned}$$

We will show that

$$(\mu\mathcal{B}+\nu\mathbf{C}\mathcal{B})^N=R_N+K_N \tag{20}$$

where K_N is a compact operator, and R_N is a bounded, linear operator such that

$$\|R_Nf\|_{L^p(\omega)}\le C\,k^N\,N^3\,\|f\|_{L^p(\omega)}.$$

Then, the Fredholm property follows immediately. To prove (20), let us write, for any two operators T_1, T_2,

$$(T_1+T_2)^N=\sum_{\sigma\in\{1,2\}^N}T_\sigma,$$

where $\sigma\in\{1,2\}^N$ means that $\sigma=(\sigma(1),\dots,\sigma(N))$ and $\sigma(j)\in\{1,2\}$ for all $j=1,\dots,N$, and

$$T_\sigma=T_{\sigma(1)}T_{\sigma(2)}\dots T_{\sigma(N)}.$$

By choosing $T_1=\mu\mathcal{B}$ and $T_2=\nu\mathbf{C}\mathcal{B}$, one sees that every $T_{\sigma(j)}$ can be written as

$$T_{\sigma(j)}=M_{\sigma(j)}C_{\sigma(j)}\mathcal{B}$$

being $M_1=\mu$, $M_2=\nu$, $C_1=\mathbf{Id}$ and $C_2=\mathbf{C}$. Thus

$$T_\sigma=M_{\sigma(1)}C_{\sigma(1)}\mathcal{B}\,M_{\sigma(2)}C_{\sigma(2)}\mathcal{B}\,\dots M_{\sigma(N)}C_{\sigma(N)}\mathcal{B}.$$

Our main task consists of rewriting T_σ as

$$T_\sigma=M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}\dots M_{\sigma(N)}C_{\sigma(N)}\,B_\sigma+K_\sigma. \tag{21}$$

for some compact operator K_σ and some bounded operator $B_\sigma\in\{\mathcal{B},\mathcal{B}^*\}^N$. If this is possible, then one gets that

$$\begin{aligned}(T_1+T_2)^N&=\sum_{\sigma\in\{1,2\}^N}M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}\dots M_{\sigma(N)}C_{\sigma(N)}\,B_\sigma+\sum_{\sigma\in\{1,2\}^N}K_\sigma\\&=R_N+K_N.\end{aligned}$$

It is clear that K_N is compact (it is a finite sum of compact operators). Moreover, from $B_\sigma\in\{\mathcal{B},\mathcal{B}^*\}^N$, one has

$$|B_\sigma f(z)|\le\sum_{j=1}^{N}|\mathcal{B}^nf(z)|+\sum_{j=1}^{N}|(\mathcal{B}^*)^nf(z)|.$$

Thus

$$
\begin{aligned}
|R_N f(z)| &\leq \sum_{\sigma\in\{1,2\}^N} |M_{\sigma(1)}C_{\sigma(1)}\dots M_{\sigma(N)}C_{\sigma(N)}\,B_\sigma f(z)|\\
&\leq \sum_{\sigma\in\{1,2\}^N} |M_{\sigma(1)}(z)|\dots|M_{\sigma(N)}(z)|\left(\sum_{n=1}^N|\mathcal{B}^n f(z)|+\sum_{j=1}^N|(\mathcal{B}^*)^n f(z)|\right)\\
&= \left(\sum_{n=1}^N|\mathcal{B}^n f(z)|+\sum_{j=1}^N|(\mathcal{B}^*)^n f(z)|\right).\,(|M_1(z)|+|M_2(z)|)^N
\end{aligned}
$$

Now, since $\|\mathcal{B}^j f\|_{L^p(\omega)}\leq C_\omega\, j^2\,\|f\|_{L^p(\omega)}$ (and similarly for $(\mathcal{B}^*)^n$, see (17)), one gets that

$$
\|R_N f\|_{L^p(\omega)}\leq \||M_1|+|M_2|\|_\infty^N\, C_\omega\left(\sum_{j=1}^N j^2\right)\|f\|_{L^p(\omega)}=C\,k^N\,N^3\,\|f\|_{L^p(\omega)}
$$

and so (20) follows from the representation (21). To prove that representation (21) can be found, we need the help of Theorem 2, according to which the differences $K_j=\mathcal{B}M_{\sigma(j)}-M_{\sigma(j)}\mathcal{B}$ are compact. Thus,

$$
\begin{aligned}
T_\sigma &= M_{\sigma(1)}C_{\sigma(1)}\mathcal{B}\,M_{\sigma(2)}C_{\sigma(2)}\mathcal{B}\,\dots M_{\sigma(N)}C_{\sigma(N)}\mathcal{B}\\
&= M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}\mathcal{B}\,C_{\sigma(2)}M_{\sigma(3)}\,\dots\mathcal{B}C_{\sigma(N)}\mathcal{B}+K_\sigma
\end{aligned}
$$

where all the factors containning K_j are includded in K_σ. In particular, K_σ is compact. Now, by reminding that

$$
\mathbf{C}\,\mathcal{B}=\mathcal{B}^*\,\mathbf{C},
$$

we have that $\mathcal{B}C_{\sigma(j+1)}=C_{\sigma(j+1)}B_j$ for some $B_j\in\{\mathcal{B},\mathcal{B}^*\}$. Thus

$$
T_\sigma = M_{\sigma(1)}C_{\sigma(1)}M_{\sigma(2)}C_{\sigma(2)}B_1\,M_{\sigma(3)}\,\dots C_{\sigma(N)}B_{N-1}\mathcal{B}+K_\sigma
$$

Now, one can start again. On one hand, the differences $B_j\,M_{\sigma(j+2)}-M_{\sigma(j+2)}B_j$ are again compact, because $B_j\in\{\mathcal{B},\mathcal{B}^*\}$ and $M_{\sigma(j+2)}\in VMO$. Moreover, the composition $B_jC_{\sigma(j+2)}$ can be writen as $C_{\sigma(j+2)}\tilde{B}_j$, where $\tilde{B}_j$ need not be the same as B_j but still $\tilde{B}_j\in\{\mathcal{B},\mathcal{B}^*\}$. So, with a little abbuse of notation, and after repeating this algorythm a total of $N-1$ times, one obtains (21). The claim follows. □

Proof of Theorem 8. The equation we want to solve can be rewritten, at least formally, in the following terms

$$
(\mathbf{Id}-\mu\mathcal{B}-\nu\overline{\mathcal{B}})(\overline{\partial}f)=g,
$$

so that we need to understand the $\mathbf{R}$-linear operator $T=\mathbf{Id}-\mu\mathcal{B}-\nu\overline{\mathcal{B}}$. By Lemma 10, we know that T is a Fredholm operator in $L^p(\omega)$, $1<p<\infty$. Now, we prove that it is also injective. Indeed, if

$$
T(h)=0
$$

for some $h\in L^p(\omega)$ and $\omega\in A_p$, it then follows that

$$
h=\mu\mathcal{B}(h)+\nu\overline{\mathcal{B}}(h)
$$

so that h has compact support, and thus $h\in L^{1+\epsilon}(\mathbf{C})$ for some $\epsilon>0$ (arguing as in (18)). We are then reduced to show that

$$
T\colon L^{1+\epsilon}(\mathbf{C})\to L^{1+\epsilon}(\mathbf{C})\quad\text{is injective.}
$$

Let us first see how the proof finishes. Injectivity of T in $L^{1+\epsilon}(\mathbf{C})$ gives us that $h = 0$. Therefore, T is injective also in $L^p(\omega)$. Being as well Fredholm, it is also surjective, so by the open map Theorem it has a bounded inverse $T^{-1}\colon L^p(\omega) \to L^p(\omega)$. As a consequence, given any $g \in L^p(\omega)$, the function

$$f = \mathcal{C}T^{-1}(g)$$

is well defined, and has derivatives in $L^p(\omega)$ satisfying the estimate

$$\begin{aligned}\|Df\|_{L^p(\omega)} &\le \|\partial f\|_{L^p(\omega)} + \|\overline{\partial} f\|_{L^p(\omega)} = \|\mathcal{B}T^{-1}(g)\|_{L^p(\omega)} + \|T^{-1}(g)\|_{L^p(\omega)} \\ &\le (C+1)\,\|T^{-1}(g)\|_{L^p(\omega)} \le C\,\|g\|_{L^p(\omega)},\end{aligned}$$

because $\omega \in A_p$. Moreover, we see that f solves the inhomogeneous equation

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = g(z).$$

Finally, if there were two such solutions f_1, f_2, then their difference $F = f_1 - f_2$ solves the homogeneous equation, and also $DF \in L^p(\omega)$. Thus

$$T(\overline{\partial} F) = 0.$$

By the injectivity of T we get that $\overline{\partial} F = 0$, and from $DF \in L^p(\omega)$ we get that $\partial F = 0$, whence F must be a constant.

We now prove the injectivity of T in $L^p(\mathbf{C})$, $1 < p < \infty$. First, if $p \ge 2$ and $h \in L^p(\mathbf{C})$ is such that $T(h) = 0$, then h has compact support, whence $h \in L^2(\mathbf{C})$. But $\mathcal{B}, \overline{\mathcal{B}}$ are isometries in $L^2(\mathbf{C})$, whence

$$\|h\|_2 \le k\,\|\mathcal{B}h\|_2 = k\|f\|_2$$

and thus $h = 0$, as desired. For $p < 2$, we recall from Lemma 9 that the bijectivity of T in $L^p(\mathbf{C})$ is equivalent to that of $T' = \mathbf{Id} - \overline{\mu}\mathcal{B}^* - \nu\overline{\mathcal{B}^*}$ in the dual space $L^p(\mathbf{C})$. For this, note that the injectivity of T' in $L^{p'}(\mathbf{C})$ follows as above (since $p' \ge 2$). Note also that, by Lemma 10 we know that T' is a Fredholm operator in $L^{p'}(\mathbf{C})$, since $\overline{\mu}$ and ν are compactly supported VMO functions. The claim follows. □

4. Applications

We start this section by recalling that if $\mu, \nu \in L^\infty(\mathbf{C})$ are compactly supported with $\||\mu| + |\nu|\|_\infty \le k < 1$ then the equation

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) - \nu(z)\,\overline{\partial\phi(z)} = 0$$

admits a unique homeomorphic $W^{1,2}_{\mathrm{loc}}(\mathbf{C})$ solution $\phi\colon \mathbf{C} \to \mathbf{C}$ such that $|\phi(z) - z| \to 0$ as $|z| \to \infty$. We call it the *principal* solution, and it defines a global K-quasiconformal map, $K = \frac{1+k}{1-k}$. See the monograph [1].

Applications of Theorem 1 are based in the following change of variables lemma, which is already proved in [3, Lemma 14]. We rewrite it here for completeness.

Lemma 11. *Given a compactly supported function $\mu \in L^\infty(\mathbf{C})$ such that $\|\mu\|_\infty \le k < 1$, let ϕ denote the principal solution to the equation*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

For a fixed weight ω, let us define

$$\eta(\zeta) = \omega(\phi^{-1}(\zeta))\,J(\zeta, \phi^{-1})^{1-\frac{p}{2}}.$$

The following statements are equivalent:

(a) *For every $h \in L^p(\omega)$, the inhomogeneous equation*

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) = h(z) \tag{22}$$

has a solution f with $Df \in L^p(\omega)$ and

$$\|Df\|_{L^p(\omega)} \le C_1\, \|h\|_{L^p(\omega)}. \tag{23}$$

(b) *For every $\tilde h \in L^p(\eta)$, the equation*

$$\overline{\partial} g(\zeta) = \tilde h(\zeta) \tag{24}$$

has a solution g with $Dg \in L^p(\eta)$ and

$$\|Dg\|_{L^p(\eta)} \le C_2\, \|\tilde h\|_{L^p(\eta)}. \tag{25}$$

Proof. Let us first assume that (b) holds. To get (a), we have to find a solution f of (22) such that $Df \in L^p(\omega)$ with the estimate (23). To this end, we make in (22) the change of coordinates $g = f \circ \phi^{-1}$. We obtain for g the following equation

$$\overline{\partial} g(\zeta) = \tilde h(\zeta), \tag{26}$$

where $\zeta = \phi(z)$ and

$$\tilde h(\zeta) = h(z)\, \frac{\partial \phi(z)}{J(z,\phi)}.$$

In order to apply the assumption (b), we must check that $\tilde h \in L^p(\eta)$. However,

$$\begin{aligned}\|\tilde h\|^p_{L^p(\eta)} &= \int |\tilde h(\zeta)|^p\, \eta(\zeta)\, \mathrm{d}\zeta = \int |\tilde h(\phi(z))|^p\, \omega(z)\, J(z,\phi)^{\frac{p}{2}}\, \mathrm{d}z \\ &= \int |h(z)\,|^p\, \frac{\omega(z)}{(1-|\mu(z)|^2)^{\frac{p}{2}}}\, \mathrm{d}z \le \frac{1}{(1-k^2)^{\frac{p}{2}}}\, \|h\|^p_{L^p(\omega)}.\end{aligned}$$

Since $\tilde h \in L^p(\eta)$, (b) applies, and a solution g to (26) can be found with the estimate

$$\|Dg\|_{L^p(\eta)} \le C_2\, \|\tilde h\|_{L^p(\eta)} \le \frac{C_2}{(1-k^2)^{\frac{1}{2}}}\, \|h\|_{L^p(\omega)}.$$

With such a g, the function $f = g \circ \phi$ is well defined, and

$$\begin{aligned}\int |Df(z)|^p\, \omega(z)\, \mathrm{d}z &= \int |Dg(\phi(z))\, D\phi(z)|^p\, \omega(z)\, \mathrm{d}z \\ &= \int |Dg(\zeta)\, D\phi(\phi^{-1}(\zeta))|^p\, \omega(\phi^{-1}(\zeta))\, J(\zeta,\phi^{-1})\mathrm{d}\zeta \\ &\le \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p\, J(\phi^{-1}(\zeta),\phi)^{\frac{p}{2}}\, \omega(\phi^{-1}(\zeta))\, J(\zeta,\phi^{-1})\mathrm{d}\zeta \\ &= \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Dg(\zeta)|^p\, \eta(\zeta)\, \mathrm{d}\zeta.\end{aligned}$$

due to the $\frac{1+k}{1-k}$-quasiconformality of ϕ. Moreover, f satisfies the desired equation, and so (a) follows, with constant $C_1 = \frac{C_2}{1-k}$.

To show that (a) implies (b), for a given $\tilde h \in L^p(\eta)$ we have to find a solution of (24) satisfying the estimate (25). Since this is a $\overline{\partial}$-equation, this could be done by simply convolving $\tilde h$ with the Cauchy kernel $\frac{1}{\pi z}$, but then the desired estimate for the solution g cannot be obtained in this way, because at this point the weight η is not known to belong to A_p. So we will proceed in a different maner. Namely, let $\tilde{h} \in L^p(\eta)$ be fixed, and set $h(z) = \tilde{h}(\phi(z))\, \overline{\partial\phi(z)}\, (1 - |\mu(z)|^2)$. Then

$$\int |h(z)|^p\, \omega(z)\, \mathrm{d}z = \int |\tilde{h}(\zeta)|^p\, (1 - |\mu(\phi^{-1}(\zeta))|^2)^{p/2}\, \eta(\zeta)\, \mathrm{d}\zeta \leq \int |\tilde{h}(\zeta)|^p\, \eta(\zeta)\, \mathrm{d}\zeta,$$

and so $h \in L^p(\omega)$. By (a), the equation

$$\overline{\partial} f(z) - \mu(z)\, \partial f(z) = h(z)$$

has a unique solution f with $Df \in L^p(\omega)$, and furthermore $\|Df\|_{L^p(\omega)} \leq C_1\, \|\tilde{h}\|_{L^p(\eta)}$. Now we simply set $g = f \circ \phi^{-1}$. By the chain rule, one gets that $\overline{\partial} g = \tilde{h}$, and

$$\begin{aligned}
\int |Dg(\zeta)|^p \eta(\zeta)\, \mathrm{d}\zeta &= \int |Dg(\phi^{-1}(z))|^p J(z, \phi^{-1}) \eta(\phi^{-1}(z))\, \mathrm{d}z \\
&= \int |D(g \circ \phi^{-1})(z)\, (D\phi^{-1}(z))^{-1}|^p J(z, \phi^{-1}) \eta(\phi^{-1}(z))\, \mathrm{d}z \\
&\leq \int |Df(z)|^p\, |D\phi(\phi^{-1}(z))|^p J(z, \phi^{-1}) \eta(\phi^{-1}(z))\, \mathrm{d}z \\
&\leq \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Df(z))|^p\, J(\phi^{-1}(z), \phi)^{\frac{p}{2}} J(z, \phi^{-1}) \eta(\phi^{-1}(z))\, \mathrm{d}z \\
&= \left(\frac{1+k}{1-k}\right)^{\frac{p}{2}} \int |Df(z))|^p\, \omega(z)\, \mathrm{d}z.
\end{aligned}$$

Thus, $\|Dg\|_{L^p(\eta)} \leq C_2\, \|\tilde{h}\|_{L^p(\eta)}$ with $C_2 = \left(\frac{1+k}{1-k}\right)^{\frac{1}{2}} C_1$, and (b) follows. □

According to the previous Lemma, a priori estimates for $\overline{\partial} - \mu\, \partial$ in $L^p(\omega)$ are equivalent to a priori estimates for $\overline{\partial}$ in $L^p(\eta)$. However, by Theorem 1, if ω is taken in A_p, the first statement holds, at least, when μ is compactly supported and belongs to VMO. We then obtain the following consequence.

Corollary 12. *Let $\mu \in VMO$ be compactly supported, such that $\|\mu\|_\infty < 1$, and let ϕ be the principal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\, \partial\phi(z) = 0.$$

If $1 < p < \infty$ and $\omega \in A_p$, then the weight

$$\eta(z) = \omega(\phi^{-1}(z))\, J(z, \phi^{-1})^{1-p/2}$$

belongs to A_p. Moreover, its A_p constant $[\eta]_{A_p}$ can be bounded in terms of μ, p and $[\omega]_{A_p}$.

Proof. Under the above assumptions, by Theorem 1, we know that if $h \in L^p(\omega)$ then the equation $\overline{\partial} f - \mu\, \partial f = h$ can be found a solution f with $Df \in L^p(\omega)$ and such that $\|Df\|_{L^p(\omega)} \leq C_0\, \|h\|_{L^p(\omega)}$, for some constant $C_0 > 0$ depending on k, p and $[\omega]_{A_p}$. Equivalently, by Lemma 11, for every $\tilde{h} \in L^p(\eta)$ we can find a solution g of the inhomogeneous Cauchy–Riemann equation

$$\overline{\partial} g = \tilde{h},$$

with $Dg \in L^p(\eta)$ and in such a way that the estimate

$$\|Dg\|_{L^p(\eta)} \leq C\, \|\tilde{h}\|_{L^p(\eta)}$$

holds for some constant C depending on C_0, k and p. Now, let us choose $\varphi \in \mathcal{C}_0^\infty(\mathbf{C})$ and set $\tilde{h} = \overline{\partial}\varphi$. Then of course $g = \varphi$ and $\partial\varphi = \mathcal{B}(\overline{\partial}\varphi)$, and the above inequality says that

$$\||\partial\varphi| + |\overline{\partial}\varphi|\|_{L^p(\eta)} \le C\,\|\overline{\partial}\varphi\|_{L^p(\eta)},$$

whence the estimate

$$\|\mathcal{B}(\psi)\|_{L^p(\eta)} \le (C^p - 1)^{\frac{1}{p}}\,\|\psi\|_{L^p(\eta)} \tag{27}$$

holds for any $\psi \in \mathcal{D}^* = \{\psi \in \mathcal{C}_c^\infty(\mathbf{C})\colon \int \psi = 0\}$. It turns out that $\mathcal{D}^*$ is a dense subclass of $L^p(\eta)$, provided that $\eta \in L^1_{\mathrm{loc}}$ is a positive function with infinite mass. But this is actually the case. Indeed, one has

$$\int_{D(0,R)} \eta(\zeta)\,\mathrm{d}\zeta = \int_{\phi^{-1}(D(0,R))} \omega(z)\,J(z,\phi)^{\frac{p}{2}}\,\mathrm{d}z.$$

Above, the integral on the right hand side certainly grows to infinite as $R \to \infty$. Otherwise, one would have that $J(\cdot,\phi)^{\frac{1}{2}} \in L^p(\omega)$. But ϕ is a principal quasiconformal map, hence $J(z,\phi) = 1 + O(1/|z|^2)$ as $|z| \to \infty$. Thus for large enough $N > M > 0$,

$$\int_{M<|z|<N} J(z,\phi)^{\frac{p}{2}}\,\omega(z)\,\mathrm{d}z \ge C \int_{M<|z|<N} \omega(z)\,\mathrm{d}z$$

and the last integral above blows up as $N \to \infty$, because ω is an A_p weight.

Therefore, the estimate (27) holds for all ψ in $L^p(\eta)$. By [17, Ch. V, Proposition 7], this implies that $\eta \in A_p$, and moreover, $[\eta]_{A_p}$ depends only on the constant $(C^p-1)^{\frac{1}{p}}$, that is, on k, p and $[\omega]_{A_p}$. □

The above Corollary is especially interesting in two particular cases. First, for the constant weight $\omega = 1$ the above result says that

$$J(\cdot,\phi^{-1})^{1-p/2} \in A_p, \quad 1 < p < \infty.$$

Without the VMO assumption, this is only true for the smaller range $1 + \|\mu\|_\infty < p < 1 + \frac{1}{\|\mu\|_\infty}$ (see e.g. [2, Theorem 13.4.2]). Secondly, by setting $p = 2$ in Corollary 12 we get the following.

Corollary 13. *Let $\mu \in VMO$ be compactly supported, and assume that $\|\mu\|_\infty < 1$. Let ϕ be the principal solution of*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) = 0.$$

Then, for every $\omega \in A_2$ one has $\omega \circ \phi^{-1} \in A_2$.

The above result drives us to the problem of finding what homeomorphisms ϕ preserve the A_p classes under composition with ϕ^{-1}. Note that preserving A_p forces also the preservation of the space BMO of functions with bounded mean oscillation, and thus such homeomorphisms ϕ must be quasiconformal [14]. However, at level of Muckenhoupt weights, the question is deeper. As an example, simply consider the weight

$$\omega(z) = \frac{1}{|z|^\alpha},$$

and its composition with the inverse of a radial stretching $\phi(z) = z|z|^{K-1}$. It is clear that the values of p for which A_p contains ω and $\omega \circ \phi^{-1}$ are *not* the same, whence preservation of A_p requires something else. This question was solved by Johnson and Neugebauer [10] as follows.

Theorem 14. *Let $\phi\colon \mathbf{C} \to \mathbf{C}$ be K-quasiconformal. Then, the following statements are equivalent:*

(1) *If $\omega \in A_2$ then $\omega \circ \phi^{-1} \in A_2$ quantitatively.*
(2) *For a fixed $p \in (1,\infty)$, if $\omega \in A_p$ then $\omega \circ \phi^{-1} \in A_p$ quantitatively.*
(3) *$J(\cdot,\phi^{-1}) \in A_p$ for every $p \in (1,\infty)$.*

It follows from Corollary 13 and Theorem 14 that, if $\mu \in VMO$ is compactly supported, $\|\mu\|_\infty \le k < 1$ and ϕ is the principal solution to the $\mathbf{C}$-linear equation $\overline{\partial}\phi = \mu\,\partial\phi$, then

$$J(\cdot,\phi^{-1}) \in A_p, \quad \text{for every } p > 1.$$

By quasisymmetry, the A_p condition (5) for $J(\cdot,\phi^{-1})$ also holds if D is quasidisk. But then, after a change of coordinates, one gets for any disk D' and $D = \phi(D')$ that

$$\left(\fint_D J(\cdot,\phi^{-1})\right)\left(\fint_D J(\cdot,\phi^{-1})^{1-p'}\right)^{p-1} = \left(\left(\fint_{D'} J(\cdot,\phi)\right)^{-1}\left(\fint_{D'} J(\cdot,\phi)^{p'}\right)^{\frac{1}{p'}}\right)^{p},$$

where $p' = \frac{p}{p-1}$. As a consequence, we get that $J(\cdot,\phi)$ satisfies the reverse Hölder estimate (4) for any $1 < p' < \infty$. This shows Corollary 4.

It is not clear to the authors what is the role of $\mathbf{C}$-linearity in the above results. In other words, there seems to be no reason for Theorem 13 to fail if one replaces the $\mathbf{C}$-linear equation by the generalized one, while mantainning the ellipticity, compact support and smoothness on the coefficients. Thus one may ask what is the class of weights $\omega > 0$ for which the estimate

$$\|Df\|_{L^2(\omega)} \le C\,\|\overline{\partial} f - \mu\,\partial f - \nu\,\overline{\partial f}\|_{L^2(\omega)}$$

holds for any $f \in \mathcal{C}_0^\infty(\mathbf{C})$. The following result, which is a counterpart of Lemma 11, explains this class contains A_p.

Lemma 15. *To each pair $\mu,\nu \in L^\infty(\mathbf{C})$ of compactly supported functions with $\||\mu| + |\nu|\|_\infty \le k < 1$, let us associate, on one hand, the principal solution ϕ to the equation*

$$\overline{\partial}\phi(z) - \mu(z)\,\partial\phi(z) - \nu(z)\,\overline{\partial\phi(z)} = 0,$$

and on the other, the function λ defined by $\lambda \circ \phi = \frac{-2i\nu}{1-|\mu|^2+|\nu|^2}$. For a fixed weight ω, let us define

$$\eta(\zeta) = \omega(\phi^{-1}(\zeta))\, J(\zeta,\phi^{-1})^{1-\frac{p}{2}}.$$

The following statements are equivalent:

(a) *For every $h \in L^p(\omega)$, the equation*

$$\overline{\partial} f(z) - \mu(z)\,\partial f(z) - \nu(z)\,\overline{\partial f(z)} = h(z)$$

has a solution f with $Df \in L^p(\omega)$ and $\|Df\|_{L^p(\omega)} \le C\,\|h\|_{L^p(\omega)}$.

(b) *For every $\tilde{h} \in L^p(\eta)$, the equation*

$$\overline{\partial} g(\zeta) - \lambda(\zeta)\,\mathrm{Im}(\partial g(\zeta)) = \tilde{h}(\zeta)$$

has a solution g with $Dg \in L^p(\eta)$ and $\|Dg\|_{L^p(\eta)} \le C\,\|\tilde{h}\|_{L^p(\eta)}$.

Although the proof requires quite tedious calculations, it follows the scheme of Lemma 11, and thus we omit it. From this Lemma, the following one is a natural question to ask.

Question 16. Let $\omega \in L^1_{\text{loc}}(\mathbf{C})$ be such that $\omega(z) > 0$ almost everywhere, and let $\lambda \in L^\infty(\mathbf{C})$ be a compactly supported VMO function, such that $\|\lambda\|_\infty < 1$. If the estimate

$$\|Df\|_{L^p(\omega)} \leq C\,\|\overline{\partial} f - \lambda\,\operatorname{Im}(\partial f)\|_{L^p(\omega)}$$

holds for every $f \in \mathcal{C}^\infty_0$, is it true that $\omega \in A_2$?

What we actually want is to find planar, elliptic, first order differential operators, different from the $\overline{\partial}$, that can be used to characterize the Muckenhoupt classes A_p. In this direction, an affirmative answer tho Question 16 would allow us to characterize A_2 weights as follows: given $\mu, \nu \in VMO$ uniformly elliptic and compactly supported, a positive a.e. function $\omega \in L^1_{\text{loc}}$ is an A_2 weight if and only if there is a constant $C \geq 1$ such that

$$\|Df\|_{L^2(\omega)} \leq C\,\|\overline{\partial} f - \mu\,\partial f - \nu\,\overline{\partial f}\|_{L^2(\omega)}, \quad \text{for every } f \in \mathcal{C}^\infty_0(\mathbf{C}). \tag{28}$$

Note that if one assumes $\||\mu| + |\nu|\|_\infty < \epsilon$ for some $\epsilon > 0$, then (28) implies that

$$\|\partial f\|^2_{L^2(\omega)} + \|\overline{\partial} f\|_{L^2(\omega)} \leq C\,\|\overline{\partial} f\|_{L^2(\omega)} + C\,\epsilon\,\|\partial f\|_{L^2(\omega)}.$$

In particular, if for some reason $\epsilon < \frac{1}{C}$ then one gets

$$\|\partial f\|_{L^2(\omega)} \leq \frac{C-1}{1-C\epsilon}\|\overline{\partial} f\|_{L^2(\omega)}.$$

From the above estimate, weighted bounds for $\mathcal{B}$ easily follow, and so in this case such a characterization holds.

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