NONLOCALIZATION OF OPERATORS OF SCHRÖDINGER TYPE

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Abstract. Localization properties are studied for operators of Schrödinger type.

1. Introduction

For $f$ belonging to the Schwartz class $\mathcal{S}(\mathbb{R})$ we define the Fourier transform $\hat{f}$ by setting

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) \, dx, \quad \xi \in \mathbb{R}.$$  

For $a > 1$ and $f \in \mathcal{S}(\mathbb{R})$ we also set

$$S_t f(x) = \int_{\mathbb{R}} e^{it\xi} e^{i\xi x} \hat{f}(x) \, dx, \quad x \in \mathbb{R}, \quad t \geq 0.$$  

If we set $u(x, t) = S_t f(x)/2\pi$, then $u(x, 0) = f(x)$ and in the case $a = 2$, $u$ satisfies the Schrödinger equation $i \partial u/\partial t = \partial^2 u/\partial x^2$. We also set

$$m(\xi) = e^{i|\xi|^a}, \quad \xi \in \mathbb{R},$$

and let $K$ denote the Fourier transform of $m$ so that $K \in \mathcal{S}'(\mathbb{R})$. It is known that $K \in C^\infty(\mathbb{R})$ (see Lemma A below) and in the case $t > 0$ it is clear that

$$e^{it|\xi|^a} = m(t^{1/a}\xi)$$

has the Fourier transform

$$K_t(y) = t^{-1/a} K(t^{-1/a}y).$$

One has $S_t f(x) = K_t * f(x)$ for $t > 0$ and $f \in \mathcal{S}(\mathbb{R})$ and we set $S_t f(x) = K_t * f(x)$ for $f \in L^2(\mathbb{R})$ with compact support. We introduce Sobolev spaces $H_s$ by setting

$$H_s = \{ f \in \mathcal{S}'; \| f \|_{H_s} < \infty \}, \quad s \in \mathbb{R},$$

where

$$\| f \|_{H_s} = \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}.$$  

It is well-known (see Sjölin [4] and Vega [5] and in the case $a = 2$ Carleson [1] and Dahlberg and Kenig [2]) that

$$\lim_{t \to 0} \frac{1}{2\pi} S_t f(x) = f(x)$$

almost everywhere if $f \in H_{1/4}$ and $f$ has compact support. Also it is known that $H_{1/4}$ cannot be replaced by $H_s$ if $s < 1/4$. 

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Now assume that \( 0 \leq s < 1/4 \).
Here we shall study the problem if there is localization or localization almost everywhere for the above operators \( S_t \) and functions \( f \in H_s \) with compact support, that is, do we have

\[
\lim_{t \to 0} S_t f(x) = 0
\]
everywhere or almost everywhere in \( \mathbb{R} \setminus (\text{supp } f) \)? We shall prove that there is no localization or localization almost everywhere of this type for \( 0 \leq s < 1/4 \). In fact we shall prove that there exist two disjoint compact intervals \( I \) and \( J \) in \( \mathbb{R} \) and a function \( f \) which belongs to \( H_s \) for all \( s < 1/4 \), with the properties that \( \text{supp } f \subset I \) and for every \( x \in J \) one does not have

\[
\lim_{t \to 0} S_t f(x) = 0.
\]

In the special case \( a = 2 \) this was proved in 2009 by P. Sjölin and F. Soria. The proof for \( a > 1 \) in this paper is a generalization of the proof of Sjölin and Soria for \( a = 2 \). We remark that Sjölin and Soria also obtained the corresponding result for \( a = 2 \) and dimension \( n \geq 2 \).

2. Proofs

We shall use a theorem of Miyachi to obtain some properties of the kernel \( K \) defined in the introduction.

**Lemma A.** One has \( K \in C^\infty(\mathbb{R}) \) and there exists a number \( \alpha \geq 0 \) such that

\[
|K(x)| \leq C \left(1 + |x|^\alpha\right) \quad \text{for } x \in \mathbb{R}.
\]

**Proof.** Let \( \psi \in C^\infty(\mathbb{R}) \) with

\[
\psi(\xi) = 1, \quad |\xi| \geq 2, \quad \text{and} \quad \psi(\xi) = 0, \quad |\xi| \leq 1.
\]

We have \( m = m_1 + m_2 \), where

\[
m_1(\xi) = (1 - \psi(\xi)) e^{i|\xi|^\alpha} \quad \text{and} \quad m_2(\xi) = \psi(\xi) e^{i|\xi|^\alpha}.
\]

Let \( m_1 \) and \( m_2 \) have Fourier transforms \( K_1 \) and \( K_2 \) respectively. We have

\[
K_1(x) = \int_{|\xi| \leq 2} e^{-ix\xi} (1 - \psi(\xi)) e^{i|\xi|^\alpha} d\xi, \quad x \in \mathbb{R},
\]

and it is easy to see that \( K_1 \) is bounded and belongs to \( C^\infty \).

Also Miyachi [3] has proved that \( K_2 \in C^\infty \) and that

\[
|K_2(x)| \leq C |x|^{(1-a/2)/(a-1)}
\]

for \( |x| \) large. It follows that \( K \in C^\infty \) and that (1) holds with \( \alpha = 0 \) for \( a \geq 2 \) and \( \alpha = (1 - a/2)/(a - 1) \) for \( 1 < a < 2 \). Hence Lemma A is proved. \[\square\]

We shall use the inverse Fourier transform defined by

\[
\hat{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} f(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}).
\]

Now choose \( g \in \mathcal{S}(\mathbb{R}) \) such that \( \text{supp } \hat{g} \subset (-1, 1) \), \( \hat{g}(0) \neq 0 \), and set

\[
f_v(x) = e^{-ix/v^2} \hat{g}(x/v), \quad 0 < v < 1.
\]

It follows that \( \text{supp } f_v \subset (-v, v) \) and \( f_v \in \mathcal{S}(\mathbb{R}) \). We shall use the functions \( f_v \) to construct the counter-example mentioned in the introduction. We remark that similar
functions were used by Dahlberg and Kenig [2]. We need the following lemma, which is essentially contained in [2].

**Lemma 1.** One has $\hat{f}_v(\xi) = v g(v \xi + 1/v)$ for $0 < v < 1$ and $\|f_v\|_{H^s} \leq C v^{1/2 - 2s}$ for $0 < v < 1$ and $0 < s < 1/4$.

In our counter-example we shall use the following estimate.

**Lemma 2.** There exist positive numbers $c_0$, $\delta$ and $v_0$ such that

$$|S_{x/v^{2a-2/a}f_v(x)}| \geq c_0$$

for $0 < v < v_0$ and $0 < x < \delta$.

**Proof.** We have $\int g(\xi) d\xi \neq 0$ and we choose a large number $L$ such that

$$\int_{|\xi| \geq L} |g(\xi)| d\xi \leq \frac{1}{100} \int g(\xi) d\xi.$$ 

Setting $\eta = v \xi + 1/v$ we obtain

$$S_t f_v(x) = \int e^{ix(\xi)} e^{it|\xi|^a} v g(v \xi + 1/v) d\xi$$

$$= \int e^{ix(\eta/v-1/v^2)} e^{it(\eta/v-1/v^2)^a} g(\eta) d\eta = \int e^{iF} g d\xi,$$

where

$$F = F(x, \xi, t, v) = \frac{x}{v}(\xi - \frac{1}{v}) + \frac{t}{v^a} \left( \frac{1}{v} - \eta \right)^a.$$ 

We now take $v_0 = 1/(2L)$ and $v$ such that $0 < v < v_0$. One has

$$S_t f_v(x) = \int_{-L}^{L} e^{iF} g d\xi + \int_{|\xi| \geq L} e^{iF} g d\xi$$

and

$$|S_t f_v(x)| \geq \left| \int_{-L}^{L} e^{iF} g d\xi \right| - \left| \int_{|\xi| \geq L} e^{iF} g d\xi \right| \geq \left| \int_{-L}^{L} e^{iF} g d\xi \right| - \frac{1}{100} \int g d\xi.$$ 

For $|\xi| \leq L$ we have

$$F = \frac{x}{v} \left( \xi - \frac{1}{v} \right) + \frac{t}{v^a} \left( \frac{1}{v} - \xi \right)^a$$

and using a Taylor expansion one obtains

$$\left( \frac{1}{v} - \xi \right)^a = \frac{1}{v^a}(1 - v \xi)^a = \frac{1}{v^a} \left( 1 - av \xi + \frac{1}{2} a(a-1) v^2 \xi^2 + O(v^3|\xi|^3) \right)$$

$$= \frac{1}{v^a} - a \xi v^{1-a} + \frac{1}{2} a(a-1) v^{a-2} \xi^2 + O(v^{3-a}).$$

Hence

$$F = \frac{x \xi}{v} - \frac{x}{v^2} + \frac{t}{v^{2a}} - a \xi v^{1-2a} + \frac{1}{2} a(a-1) v^{2-2a} \xi^2 + O(v^{3-2a}).$$
Setting $t = x v^{2a-2}/a$ we get

$$F = \frac{x \xi}{v} - \frac{x}{v^2} + \frac{x}{av^2} - \xi x v^{2a-2} v^{1-2a} + \frac{1}{2}(a-1) x v^{2a-2} v^{2-2a} \xi^2 + O(xv)$$

$$= \frac{x \xi}{v} - \frac{x}{v^2} + \frac{x \xi}{v} + \frac{1}{2}(a-1) x \xi^2 + O(xv)$$

for $x > 0$. It follows that

$$F = \frac{x}{av^2} - \frac{x}{v^2} + \frac{1}{2}(a-1) x \xi^2 + O(xv)$$

and hence

$$\left| \int_{-L}^{L} e^{itF} g d\xi \right| = \left| \int_{-L}^{L} e^{i \frac{1}{2}(a-1) x \xi^2} e^{iO(xv)} g(\xi) d\xi \right|$$

$$= \left| \int_{-L}^{L} e^{i \frac{1}{2}(a-1) x \xi^2} g(\xi) d\xi + \int_{-L}^{L} e^{i \frac{1}{2}(a-1) x \xi^2} (e^{iO(xv)} - 1) g(\xi) d\xi \right|$$

$$\geq \left| \int_{-L}^{L} e^{i \frac{1}{2}(a-1) x \xi^2} g(\xi) d\xi \right| - C x \geq \frac{1}{2} \left| \int_{-L}^{L} g(\xi) d\xi \right|$$

for $0 < x < \delta$ if $\delta$ is small.

We conclude that

$$\left| S_{x v^{2a-2}/a} f_v(x) \right| \geq \frac{1}{2} \left| \int_{-L}^{L} g d\xi \right| - \frac{1}{100} \left| \int_{-L}^{L} g d\xi \right|$$

$$\geq \frac{1}{2} \left| \int_{-L}^{L} g d\xi \right| - \frac{1}{100} \left| \int_{-L}^{L} g d\xi \right| - \frac{1}{100} \left| \int_{-L}^{L} g d\xi \right| \geq \frac{1}{4} \left| \int_{-L}^{L} g d\xi \right|$$

for $0 < v < v_0$ and $0 < x < \delta$. Hence Lemma 2 is proved. \hfill \Box

In the remaining part of this paper $\delta$ and $v_0$ are given by Lemma 2 and we may also assume that $\delta < 1$. We need two more lemmas.

**Lemma 3.** For $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, and $\delta/2 < x < \delta$ one has

$$\left| S_t f_v(x) \right| \leq C \frac{v}{t^{7}}$$

where $\gamma = (1 + \alpha)/a > 0$.

**Proof.** Using the estimate in Lemma A we obtain

$$\left| K_t(y) \right| \leq t^{-1/a} C \left( 1 + t^{-1/a} y^a \right) \leq C t^{-1/a} \left( 1 + t^{-1/a} \right) \leq C t^{-(1+\alpha)/a}$$

for $0 < t < 1$ and $|y| \leq 2$.

One has

$$S_t f_v(x) = \int e^{i |\xi|^a} \hat{f}_v(\xi) e^{i x \xi} d\xi = \int K_t(y) f_v(y + x) dy.$$  

If $\delta/2 < x < \delta$ and $|y| \geq 2$, we obtain $|y + x| \geq |y| - |x| \geq 2 - 1 = 1$ and $f_v(y + x) = 0$ and hence

$$S_t f_v(x) = \int_{|y| \leq 2} K_t(y) f_v(y + x) dy$$
for $\delta/2 < x < \delta$. It follows that
\[
|S_tf_v(x)| \leq \int_{|y| \leq 2} |K_t(y)| |f_v(y + x)| \, dy \leq C t^{-(1+\alpha)/\alpha} \int |f_v(y)| \, dy = C t^{-\gamma/\alpha} \int |\hat{g}(y/v)| \, dy = C \frac{v}{t^\gamma}
\]
where $\gamma = (1 + \alpha)/\alpha$.

\[\square\]

**Lemma 4.** For $0 < v < \min(v_0, \delta/4)$, $0 < t < 1$, and $\delta/2 < x < \delta$ one has
\[
|S_tf_v(x)| \leq C \frac{t}{v^{\beta}}
\]
where $\beta = 2\alpha$.

**Proof.** We have
\[
S_tf_v(x) = \int (e^{it|\xi|^a} - 1) e^{ix\hat{\xi}} \hat{f}_v(\xi) \, d\xi + \int e^{ix\hat{\xi}} \hat{f}_v(\xi) \, d\xi.
\]
The second integral on the above right hand side equals $2\pi f_v(x)$ which vanishes since $x > \delta/2$ and $\text{supp } f_v \subset (-v, v) \subset (-\delta/4, \delta/4)$. Setting $\eta = v\xi$ we obtain
\[
|S_tf_v(x)| \leq \int t|\xi|^a |\hat{f}_v(\xi)| \, d\xi = t \int |\xi|^a g(v\xi + 1/v) \, d\xi
\]
\[
= t \int \frac{\eta}{v} \frac{|\eta|}{v} \left| g(\eta + \frac{1}{v}) \right| \, d\eta = \frac{t}{v^{\alpha}} \int |\xi|^a \left| \xi - \frac{1}{v} \right|^a \, d\xi
\]
\[
\leq \frac{t}{v^{\alpha}} \left( C \int |g(\xi)| \left| \xi \right|^a d\xi + C \int |g(\xi)| \left( \frac{1}{v^{\alpha}} \right) d\xi \right) \leq C \frac{t}{v^{2\alpha}},
\]
and the proof of Lemma 4 is complete. \[\square\]

Now take $v_1$ such that $0 < v_1 < \min(v_0, \delta/4)$ and set $\varepsilon_k = 2^{-k}$ for $k = 1, 2, 3, \ldots$. Also set
\[
v_k = \varepsilon_k v_{k-1}^\mu, \quad k = 2, 3, 4, \ldots,
\]
where
\[
\mu = \max((2a - 2)\gamma, \beta/(2a - 2)).
\]
Since $\beta = 2\alpha$ it is clear that $\mu > 1$. By induction we prove that $v_k < 1$ for $k = 1, 2, 3, \ldots$. It follows that $0 < v_k \leq \varepsilon_k$, $k = 1, 2, 3, \ldots$.

Also we have $v_k \leq \varepsilon_k v_{k-1} \leq \frac{1}{2} v_{k-1}$ for $k = 2, 3, 4, \ldots$. It follows that
\[
\sum_{j=k+1}^{\infty} v_j \leq 2v_{k+1}, \quad k = 1, 2, 3, \ldots,
\]
and
\[
\sum_{j=1}^{k-1} \frac{1}{v_j^\beta} \leq C \frac{1}{v_{k-1}^{\beta}}, \quad k = 2, 3, 4, \ldots.
\]

Now set $f = \sum_{k=1}^{\infty} f_{v_k}$. Then $f \in H_s$ for $s < 1/4$, since
\[
\|f\|_{H_s} \leq \sum_{k=1}^{\infty} \|f_{v_k}\|_{H_s} \leq C \sum_{k=1}^{\infty} v_k^{1/2-2s} \leq C \sum_{k=1}^{\infty} \varepsilon_k^{1/2-2s} < \infty
\]
for $0 < s < 1/4$. It is clear that $\text{supp } f \subset (-\delta/4, \delta/4)$.
We can now formulate our theorem.

**Theorem 1.** Let $f$ be the function we have just constructed. With $t_k = t_k(x) = x v_k^{2a-2}/a$ one has

$$|S_{t_k(x)}f(x)| \geq c_0/2$$

for $\delta/2 < x < \delta$ and $k \geq k_0$. Here $c_0$ denotes a positive constant. Hence we do not have $\lim_{t \to 0} S_t f(x) = 0$ in the interval $(\delta/2, \delta)$. Thus we do not have localization or localization almost everywhere for all functions in $H_s$ if $s < 1/4$.

**Proof.** We have

$$S_{t_k(x)}f(x) = \sum_{j=1}^{\infty} S_{t_k(x)}f_{v_j}(x)$$

and

$$|S_{t_k(x)}f(x)| \geq |S_{t_k(x)}f_{v_k}(x)| - \sum_{j \neq k} |S_{t_k(x)}f_{v_j}(x)|$$

and using Lemma 2 we obtain

$$|S_{t_k(x)}f(x)| \geq c_0 - \sum_{j=1}^{k-1} |S_{t_k(x)}f_{v_j}(x)| - \sum_{j=k+1}^{\infty} |S_{t_k(x)}f_{v_j}(x)|.$$ 

We shall estimate the two sums on the right hand side for $\delta/2 < x < \delta$. For $j \geq k+1$ we have

$$|S_{t_k(x)}f_{v_j}(x)| \leq C \frac{v_j}{(t_k(x))^{\gamma}}$$

according to Lemma 3. Hence

$$|S_{t_k(x)}f_{v_j}(x)| \leq C \frac{v_j}{(x v_k^{2a-2})^{\gamma}} \leq C \frac{v_j}{v_k^{(2a-2)\gamma}}$$

and

$$\sum_{j=k+1}^{\infty} |S_{t_k(x)}f_{v_j}(x)| \leq C \frac{1}{v_k^{(2a-2)\gamma}} \sum_{j=k+1}^{\infty} v_j \leq C \frac{v_{k+1}}{v_k^{(2a-2)\gamma}}.$$ 

Since $\mu \geq (2a-2)\gamma$ we have $v_{k+1} \leq \varepsilon_{k+1} v_k^{(2a-2)\gamma}$ and hence

$$\sum_{j=k+1}^{\infty} |S_{t_k(x)}f_{v_j}(x)| \leq C \varepsilon_{k+1}.$$ 

For $1 \leq j \leq k-1$ we have

$$|S_{t_k(x)}f_{v_j}(x)| \leq C \frac{t_k(x)}{v_j^{\beta}} \leq C \frac{v_k^{2a-2}}{v_j^{\beta}}$$

according to Lemma 4. It follows that

$$\sum_{j=1}^{k-1} |S_{t_k(x)}f_{v_j}(x)| \leq C v_k^{2a-2} \frac{1}{v_j^{\beta}} \leq C v_k^{2a-2} \frac{1}{v_{k-1}^{\beta}}.$$ 

Since $\mu \geq \beta/(2a-2)$ we obtain

$$v_k \leq \varepsilon_k v_{k-1}^{\beta/(2a-2)}$$

and

$$v_k^{2a-2} \leq \varepsilon_k^{2a-2} v_{k-1}^{\beta}.$$
We conclude that
\[ \sum_{j=1}^{k-1} |S_{I_k(x)} f_{I_j}(x)| \leq C \varepsilon_k^{2n-2}. \]

Thus for \( k \geq k_0 \) one obtains
\[ |S_{I_k(x)} f(x)| \geq c_0/2 \]
for \( \delta/2 < x < \delta \) and the proof of the theorem is complete. \( \square \)

References


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