OPTIMAL WEAK TYPE ESTIMATES FOR
DYADIC-LIKE MAXIMAL OPERATORS

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Abstract. We provide sharp weak estimates for the distribution function of $M\phi$ when on $\phi$ we impose $L^1$, $L^q$ and $L^p,\infty$ restrictions. Here $M$ is the dyadic maximal operator associated to a tree $T$ on a non-atomic probability measure space. As a consequence we produce that the inequality

$$|||M_T\phi|||_{p,\infty} \leq |||\phi|||_{p,\infty}$$

is sharp allowing every possible value for the $L^1$ and the $L^q$ norm for a fixed $q$ such that $1 < q < p$, where $||| \cdot |||_{p,\infty}$ is the integral norm on and $|| \cdot ||_{p,\infty}$ the usual quasi norm on $L^p,\infty$.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$(1.1) \quad M_d\phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : x \in Q, \, Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}$$

for every $\phi \in L^1_{loc}(\mathbb{R}^n)$ where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 1, 2, \ldots$ and $|A|$ is the Lebesgue measure of any measurable subset $A$ of $\mathbb{R}^n$.

It is easy to prove by using the definition of $M_d$ that it satisfies the following weak type $(1, 1)$ inequality

$$(1.2) \quad \{ \{x \in \mathbb{R}^n : M_d\phi(x) \geq \lambda \} \leq \frac{1}{\lambda} \int_{\{M_d\phi \geq \lambda\}} |\phi(u)| \, du$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$. This inequality is sharp as can be easily seen by considering characteristic functions over dyadic cubes. Using the fact that

$$||M_d\phi||_p^p = \int_0^\infty p\lambda^{p-1}||\{M_d\phi \geq \lambda\}|| \, d\lambda$$

and in the sequel inequality (1.2) along with Fubini’s theorem we easily get the following $L^p$ inequality known as Doob’s inequality

$$(1.3) \quad ||M_d\phi||_p \leq \frac{p}{p-1}||\phi||_p$$

for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$, which is proved to be best possible (see [2, 3] for the general martingales and [10] for the dyadic ones).

A way of studying the dyadic maximal operator is the introduction of the so called Bellman functions (see [8]). Actually, we define for every $p > 1$

$$(1.4) \quad B_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (M_d\phi)^p : \frac{1}{|Q|} \int_Q \phi^p = F, \frac{1}{|Q|} \int_Q \phi = f \right\}$$

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where $Q$ is a fixed dyadic cube, $\phi$ is nonnegative in $L^p(Q)$ and $f, F$ are such that $0 < f^p \leq F$. $B_p(f, F)$ has been computed in [5]. In fact it has been shown that $B_p(f, F) = F\omega_p(f^p/F)^p$ where $\omega_p \colon [0, 1] \to \left[1, \frac{p}{p-1}\right]$ is the inverse function of $H_p(z) = -(p-1)z^p + pz^{p-1}$.

This has been proved in a much more general setting of tree like maximal operators on non-atomic probability spaces. The result turns out to be independent of the choice of the measure space. The study of these operators has been continued in [7] where the Bellman functions of them in the case $p < 1$ have been computed. As in [5] and [7] we will follow the more general approach. So for a tree $T$ on a non atomic probability measure space $(X, \mu)$, we define the associated dyadic maximal operator, namely

$$M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in T \right\}$$

for every $\phi \in L^1(X, \mu)$.

As it can be seen in [9], $M_T : L^{p,\infty} \rightarrow L^{p,\infty}$ is a continuous operator and satisfies the following inequality

$$||M_T \phi||_{p,\infty} \leq ||\phi||_{p,\infty},$$

where $|| \cdot ||_{p,\infty}$ is the usual quasi-norm on $L^{p,\infty}$ defined by

$$||\phi||_{p,\infty} = \sup \left\{ \lambda \mu(\{ \phi \geq \lambda \})^{1/p} : \lambda > 0 \right\},$$

and $|| \cdot ||_{p,\infty}$ is the integral norm on $L^{p,\infty}$ given by

$$||\phi||_{p,\infty} = \sup \left\{ \mu(E)^{-1/\beta} \int_E \phi \, d\mu : E \text{ measurable subset of } X \text{ such that } \mu(E) > 0 \right\}.$$

$|| \cdot ||_{p,\infty}$ and $\| \phi \|_{p,\infty}$ are equivalent because of the following

$$\|\phi\|_{p,\infty} \leq ||\phi||_{p,\infty} \leq \frac{p}{p-1}||\phi||_{p,\infty}, \forall \phi \in L^{p,\infty},$$

which can be seen in [4]. In this paper we prove that inequality (1.5) is sharp and independent of the $L^1$ and $L^q$ norm of $\phi$, for a fixed $q$ such that $1 < q < p$. In fact we prove a stronger result, by evaluating the following function of $\lambda > 0$

$$S(f, A, F, \lambda) = \sup \left\{ \mu(\{ M_T \phi \geq \lambda \}) : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \|\phi\|_{p,\infty} = F \right\},$$

where $(f, A, F)$ is on the domain of the extremal problem. That is we prove the following

**Theorem 1.1.** For $f, A$ such that $f^q < A \leq F f^{p-q/p-1} F^{q-1/p-1}$ and $0 < f \leq F$ the following hold

$$S(f, A, F, \lambda) = \min \left\{ 1, G_{f, A}(\lambda), \frac{F^p}{\lambda^p} \right\},$$

where

$$G_{f, A}(\lambda) = \sup \left\{ \mu(\{ M_T \phi \geq \lambda \}) : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A \right\}.$$
In fact, $G_{f,A}(\lambda)$ has been precisely computed in [6] by using sharp inequalities on a certain class of functions which is enough to describe the related problem. In this paper we avoid the technique used in [6] and refine this result by proving the theorem mentioned using a different approach. As a corollary we obtain the following

**Corollary 1.1.** The following is true

(1.8) $\sup \left\{ ||M_T \phi||_{p,\infty} : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, ||\phi||_{p,\infty} = F \right\} = F,$

that is, (1.5) is sharp allowing every value of the integral and the $L^q$-norm of $\phi$.

This paper is organized as follows: In Section 2 we provide some lemmas and facts concerning non-atomic probability measure spaces and trees on them. In Section 3 we find the domain of the extremal problem for the case $F = 1$. This is done by finding sharp inequalities relating the $L^1$ and $L^q$ norm of a measurable function $\phi$ under the weak condition $||\phi||_{p,\infty} = 1$. Krein–Milman theorem is a tool for us in order to find these sharp inequalities. At last in section 4 we precisely evaluate $S(f, A, 1, \lambda)$. We need also to mention that all the estimates are independent of the measure space $(X, \mu)$ and the tree $T$.

## 2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability measure space. We state the following lemma which can be found in [1].

**Lemma 2.1.** Let $\phi : (X, \mu) \to R^+$ and $\phi^*$ the decreasing rearrangement of $\phi$, defined on $[0, 1]$. Then

$$\int_0^t \phi^*(u) \, du = \sup \left\{ \int_E \phi \, d\mu : E \text{ measurable subset of } X \text{ with } \mu(E) = t \right\}$$

for every $t \in [0, 1]$, with the supremum attained.

We prove now the following

**Lemma 2.2.** Let $\phi : X \to R^+$ be measurable and $I \subseteq X$ be measurable with $\mu(I) > 0$. Suppose that $\frac{1}{\mu(I)} \int_I \phi \, d\mu = s$. Then for every $t$ such that $0 < t \leq \mu(I)$ there exists a measurable set $E_t \subseteq I$ with $\mu(E_t) = t$ and $\frac{1}{\mu(E_t)} \int_{E_t} \phi \, d\mu = s$.

**Proof.** Consider the measure space $(I, \mu/I)$ and let $\psi : I \to R^+$ be the restriction of $\phi$ on $I$ that is $\psi = \phi/I$. Then, if $\psi^* : [0, \mu(I)] \to R^+$ is the decreasing rearrangement of $\psi$, we have that

$$\frac{1}{t} \int_0^t \psi^*(u) \, du \geq \frac{1}{\mu(I)} \int_0^{\mu(I)} \psi^*(u) \, du = s \geq \frac{1}{t} \int_{\mu(I)-t}^{\mu(I)} \psi^*(u) \, du.$$

Since $\psi^*$ is decreasing, we get the inequalities in (2.1), while the equality is obvious since

$$\int_0^{\mu(I)} \psi^*(u) \, du = \int_I \phi \, d\mu.$$

From (2.1) it is easily seen that there exists $r \geq 0$ such that $t + r \leq \mu(I)$ with

$$\frac{1}{t} \int_r^{t+r} \psi^*(u) \, du = s.$$
It is also easily seen that there exists an \( E_t \) measurable subset of \( I \) such that
\[
\mu(E_t) = t \quad \text{and} \quad \int_{E_t} \phi \, d\mu = \int_r^{t+r} \psi^*(u) \, du,
\]
since \((X, \mu)\) is non-atomic. From (2.2) and (2.3) we get the conclusion of the lemma. \( \square \)

We now call two measurable subsets of \( X \) almost disjoint if \( \mu(A \cap B) = 0 \). We give now the following

**Definition 2.1.** A set \( T \) of measurable subsets of \( X \) will be called a tree if the following conditions are satisfied:

(i) \( X \in T \) and for every \( I \in T \) we have that \( \mu(I) > 0 \).

(ii) For every \( I \in T \) there corresponds a finite or countable subset \( C(I) \subseteq T \) containing at least two elements such that
   (a) the elements of \( C(I) \) are pairwise almost disjoint subsets of \( I \),
   (b) \( I = \bigcup C(I) \).

(iii) \( T = \bigcup_{m \geq 0} T(m) \) where \( T_0 = \{X\} \) and \( T_{(m+1)} = \bigcup_{I \in T_m} C(I) \).

(iv) \( \lim_{m \to +\infty} \mu(I) = 0 \).

From [5] we get the following

**Lemma 2.3.** For every \( I \in T \) and every \( \alpha \) such that \( 0 < \alpha < 1 \) there exists a subfamily \( F(I) \subseteq T \) consisting of pairwise almost disjoint subsets of \( I \) such that
\[
\mu\left( \bigcup_{J \in F(I)} J \right) = \sum_{J \in F(I)} \mu(J) = (1 - \alpha)\mu(I).
\]

Let now \((X, \mu)\) be a non-atomic probability measure space and \( T \) a tree as in Definition 1.1. We define the associated maximal operator to the tree \( T \) as follows: For every \( \phi \in L^1(X, \mu) \) and \( x \in X \), then
\[
M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in T \right\}.
\]

3. The domain of the extremal problem

Our aim is to find the exact allowable values of \((f, A, F)\) for which there exists \( \phi: (X, \mu) \to \mathbb{R}^+ \) measurable such that
\[
(3.1) \quad \int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A \quad \text{and} \quad |||\phi|||_{p, \infty} = F.
\]

We find it in the case where \( F = 1 \). For the beginning assume that \((f, A)\) are such that there exist \( \phi \) as in (3.1). We set \( g = \phi^* : [0,1] \to \mathbb{R}^+ \). Then
\[
\int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad |||g|||_{[0,1], \infty} = 1
\]
where
\[
|||g|||_{[0,1], \infty} = \sup \left\{ |E|^{-1 + \frac{1}{q}} \int_E g : E \subset [0,1] \text{ Lebesgue measurable such that } |E| > 0 \right\}.
\]
This is true because of the definition of the decreasing rearrangement of \( \phi \) and Lemma 2.1. In fact since \( g \) is decreasing \( \| |g||_{p,\infty} \) is equal to
\[
\sup \left\{ t^{-1+\frac{1}{p}} \int_0^t g : 0 < t \leq 1 \right\}.
\]

Of course, we should have that \( 0 < f \leq 1 \) and \( f^q \leq A \). We give now the following

**Definition 3.1.** If \( n \in \mathbb{N} \), and \( h : [0, 1) \to \mathbb{R}^+ \), \( h \) will be called \( \frac{1}{2^n} \)-step if it is constant on each interval
\[
\left[ \frac{i - 1}{2^n}, \frac{i}{2^n} \right), \quad i = 1, 2, \ldots, 2^n.
\]

Now for \( n \in \mathbb{N} \) and \( 0 < f \leq 1 \) fixed, we set
\[
\Delta_n(f) = \left\{ h : [0, 1) \to \mathbb{R}^+ : h \text{ is a } \frac{1}{2^n} \text{-step function, } \int_0^1 h = f, \| |h||_{p,\infty}^{[0,1]} \leq 1 \right\}.
\]
Then
\[
\Delta_n = \Delta_n(f) \subset L^{p,\infty}([0, 1])
\]
where we use the \( \| | \cdot ||_{p,\infty}^{[0,1]} \) norm for functions defined on \([0, 1]\). \( \Delta_n \) is also convex, that is,
\[
h_1, h_2 \in \Delta_n \implies \frac{h_1 + h_2}{2} \in \Delta_n.
\]

Additionally, we have the following

**Lemma 3.1.** \( \Delta_n \) is compact subset of \( L^{p,\infty}([0, 1]) = Y \) where the topology on \( Y \) is that endowed by \( \| | \cdot ||_{p,\infty}^{[0,1]} \).

**Proof.** \((Y, \| | \cdot ||_{p,\infty}^{[0,1]})\) is a Banach space. So, especially a metric space. As a consequence we just need to prove that \( \Delta_n \) is sequentially compact. Let now \((h_i)_i \subset \Delta_n \). It is now easy to see by a finite diagonal argument that there exists \((h_{i_j})_j \) subsequence and \( h : [0, 1) \to \mathbb{R}^+ \) such that \( h_{i_j} \to h \) uniformly on \([0, 1]\). Then obviously \( \int_0^1 h = f, \| |h||_{p,\infty}^{[0,1]} \leq 1, \) so \( h \in \Delta_n \). Additionally
\[
\| |h_{i_j} - h||_{p,\infty}^{[0,1]} = \sup \left\{ \frac{|E|^{-1+\frac{1}{p}}}{E} \int_E |h_{i_j} - h| : |E| > 0 \right\}
\]
\[
\leq \sup \left\{ \left( |h_{i_j} - h(t)|, t \in [0, 1] \right) \to 0 \right\}
\]
as \( j \to \infty \). That is \( h_{i_j} \overset{Y}{\to} h \in \Delta_n \). Consequently, \( \Delta_n \) is a compact subset of \( L^{p,\infty}([0, 1]) \). \( \square \)

We give now the following known

**Definition 3.2.** For a closed convex subset \( K \) of a topological vector space \( Y \), and for a \( y \in K \) we say that \( y \) is an extreme point of \( K \), if whenever \( y = \frac{x+z}{2} \), with \( x, z \in K \) it is implied that \( y = x = z \). We write \( y \in \text{ext}(K) \).

**Definition 3.3.** For a subset \( A \) of a topological vector space \( Y \) we set
\[
\text{conv}(A) = \left\{ \sum_{i=1}^n \lambda_i x_i : \lambda_i \geq 0, \ x_i \in A, \ n \in \mathbb{N}^*, \ \sum_{i=1}^n \lambda_i = 1 \right\}.
\]
We call \( \text{conv}(A) \) the convex hull of \( A \).
We state now the following well known

**Theorem 3.1.** (Krein–Milman) Let $K$ be a convex, compact subset of a locally convex topological vector space $Y$. Then $K = \text{conv}(\text{ext}(K))^\circ$, that is, $K$ is the closed convex hull of its extreme points.

According now to Lemma 3.1 we have that

$$\Delta_n = \text{conv}[\text{ext}(\Delta_n)]^{L^p,\infty([0,1])}.$$  

We find now the set $\text{ext}(\Delta_n)$.

**Lemma 3.2.** Let $g \in \text{ext}(\Delta_n)$. Then for every $i \in \{1, 2, \ldots, 2^n\}$ such that

$$\left(\frac{1}{2^n}\right)^{1 - \frac{1}{p}} \leq f,$$  

we have that

$$\sup \left\{ |E|^{-1 + \frac{1}{p}} \int_E g : |E| = \frac{i}{2^n} \right\} = 1.$$  

**Proof.** We prove it first when $i = 1$ and $\left(\frac{1}{2^n}\right)^{1 - \frac{1}{p}} \leq f$. It is now easy to see that $g \in \text{ext}(\Delta_n)$ if and only if $g^* \in \text{ext}(\Delta_n)$. So we just need to prove that

$$\int_0^{1/2^n} g^* = \left(\frac{1}{2^n}\right)^{1 - \frac{1}{p}}.$$  

We write

$$g^* = \sum_{i=1}^{2^n} \alpha_i \xi_{I_i} \quad \text{with} \quad I_i = \left[ \frac{i - 1}{2^n}, \frac{i}{2^n} \right]$$  

and $\alpha_i \geq \alpha_{i+1}$ for every $i \in \{1, 2, \ldots, 2^n - 1\}$. Suppose now that $\alpha_1 < 2^{n/p}$, and that $\alpha_1 > \alpha_2$ (the case $\alpha_1 = \alpha_2$ is handled in an analogous way). For a suitable $\varepsilon > 0$ we set

$$g_1 = \sum_{i=1}^{2^n} \alpha_i^{(1)} \xi_{I_i}, \quad g_2 = \sum_{i=1}^{2^n} \alpha_i^{(2)} \xi_{I_i}, \quad \text{where} \quad \begin{cases} 
\alpha_1^{(1)} = \alpha_1 + \varepsilon, \\
\alpha_1^{(2)} = \alpha_1 - \varepsilon,
\end{cases} \quad \begin{cases} 
\alpha_2^{(1)} = \alpha_2 - \varepsilon, \\
\alpha_2^{(2)} = \alpha_2 + \varepsilon.
\end{cases}$$

and $\alpha_k^{(1)} = \alpha_k^{(2)} = \alpha_k$ for every $k > 2$. Since $\alpha_1 < 2^{n/p}$, we can find small enough $\varepsilon > 0$ such that $g_i$ satisfy $\|g_i\|_{[0,1]} \leq 1$, for $i = 1, 2$. Indeed, for $i = 1$, we need to prove that for small enough $\varepsilon > 0$

$$(3.2) \quad \int_0^t g_1 \leq t^{1 - \frac{1}{p}}$$

for every $t \in [0, 1]$, since $g_1$ is decreasing. $(3.2)$ is now obviously true for $t \geq \frac{2}{2^n}$ since

$$(3.3) \quad \int_0^t g_1 = \int_0^t g^* \quad \text{for every such} \ t.$$  

$(3.2)$ is also true for $t = 0, \frac{1}{2^n}$ for a suitable $\varepsilon > 0$. But then it remains true for every $t \in \left(0, \frac{1}{2^n}\right)$ since the function $t \mapsto \int_0^t g_1$ represents a straight line on $[0, \frac{1}{2^n}]$ and $t^{1 - \frac{1}{p}}$ is concave there, analogously for the interval $\left[\frac{1}{2^n}, \frac{2}{2^n}\right]$. That is we proved $\|g_1\|_{[0,1]} \leq 1$. For $i = 2$ we use the same arguments and the hypothesis $\alpha_1 > \alpha_2$ in order to ensure that for small enough $\varepsilon > 0$, $g_2$ is decreasing. Obviously now,
$\int_0^1 g_i = f$, so that $g_i \in \Delta_n$, for $i = 1, 2$. But $g^* = \frac{g_1 + g_2}{2}$, with $g_i \neq g$ and $g_i \in \Delta_n$, $i = 1, 2$, a contradiction since $g^* \in \text{ext}(\Delta_n)$. So,

$$\alpha_1 = 2^{n/p} \quad \text{and} \quad \int_0^{1/2} g^* = \left(\frac{1}{2^n}\right)^{1 - \frac{1}{p}},$$

that is what we wanted to prove. In the same way we prove that for $i \in \{1, 2, \ldots, 2^n - 1\}$ such that 

$$\left(\frac{i + 1}{2^n}\right)^{1 - \frac{1}{p}} \leq f,$$

if $\int_0^{i/2^n} g^* = \left(\frac{i}{2^n}\right)^{1 - \frac{1}{p}}$, then $\int_0^{(i+1)/2^n} g^* = \left(\frac{i + 1}{2^n}\right)^{1 - \frac{1}{p}}$.

The lemma is now proved by induction. \qed

Let now $g \in \text{ext}(\Delta_n)$ and $k = \max\left\{i \leq 2^n : \left(\frac{i}{2^n}\right)^{1 - \frac{1}{p}} \leq f\right\}$, so if we suppose that $f < 1$, we have that 

$$\left(\frac{k}{2^n}\right)^{1 - \frac{1}{p}} \leq f < \left(\frac{k + 1}{2^n}\right)^{1 - \frac{1}{p}}.$$

By Lemma 3.2,

$$\int_0^{k/2^n} g^* = \left(\frac{k}{2^n}\right)^{1 - \frac{1}{p}}.$$

But by using the reasoning of the previous lemma it is easy to see that

$$\int_0^{(k+1)/2^n} g^* = f,$$

which gives

$$\int_{k/2^n}^{k+1/2^n} g^* = f - \left(\frac{k}{2^n}\right)^{1 - \frac{1}{p}} \implies \alpha_{k+1} = 2^n \cdot f - 2^{n/p} \cdot k^{1 - \frac{1}{p}}.$$

Additionally, $\alpha_i = 0$ for $i > k + 1$. From the above we obtain the following

**Corollary 3.1.** Let $g \in \text{ext}(\Delta_n)$. Then $g^* = \sum_{i=1}^{2^n} \alpha_i I_i$, where

$$\alpha_i = 2^{n/p} \left(\frac{1}{p} - (i - 1)^{1 - \frac{1}{p}}\right) \quad \text{for} \quad i = 1, 2, \ldots, k$$

and

$$\alpha_{k+1} = 2^n f - 2^{n/p} \cdot k^{1 - \frac{1}{p}}, \quad \alpha_i = 0, \quad i > k + 1,$$

where

$$k = \max\left\{i \leq 2^n : \left(\frac{i}{2^n}\right)^{1 - \frac{1}{p}} \leq f\right\}.$$

We estimate now the $L^q$-norm of every $g \in \text{ext}(\Delta_n)$. We state it as

**Lemma 3.3.** Let $g \in \text{ext}(\Delta_n)$ and $A = \int_0^1 g^q$. Then $A \leq \Gamma f^{p-q/p-1} + \mathcal{E}_n(f)$, where

$$\Gamma = \left(\frac{p-1}{p}\right)^q \frac{p}{p-q} \quad \text{and} \quad \mathcal{E}_n(f) = \alpha_{k+1}^{q+1} = \frac{(2^n f - 2^{n/p} k^{1 - \frac{1}{p}})^q}{2^n}.$$
Proof. For \( g \) we write \( g^* = \sum_{i=1}^{2^n} \alpha_i \xi_i \), where \( \alpha_i \) are given in Corollary 3.1. Then

\[
A = \int_0^1 (g^*)^q = \left[ \left( \sum_{i=1}^{k} \alpha_i^q \right) + \alpha_{k+1}^q \right] \cdot \frac{1}{2^n}.
\]

Now for \( i \in \{1, 2, \ldots, k\} \)

\[
\alpha_i^q = \left[ 2^{n/p} \left( i^{1/p} - (i - 1)^{1/p} \right) \right]^q = \left\{ 2^n \left[ \left( \frac{i}{2^n} \right)^{1/p} - \left( \frac{i - 1}{2^n} \right)^{1/p} \right] \right\}^q
\]

where \( \psi: (0, 1] \to \mathbb{R}^+ \) is defined by \( \psi(t) = \frac{p-1}{p} t^{1/p} \). By (3.5) and in view of Hölder’s inequality we have that for \( i \in \{1, 2, \ldots, k\} \)

\[
\alpha_i^q \leq 2^n \int_{i-1/2^n}^{i/2^n} \psi^q.
\]

Summing up relations (3.6) we have that

\[
\sum_{i=1}^{k} \alpha_i^q \leq 2^n \int_0^{k/2^n} \psi^q = 2^n \cdot \Gamma \cdot \left( \frac{k}{2^n} \right)^{1-\frac{q}{p}}.
\]

Additionally from the definition of \( k \) we have that

\[
\left( \frac{k}{2^n} \right)^{1-\frac{q}{p}} \leq f \implies k^{1-\frac{q}{p}} \leq (2^n)^{1-\frac{q}{p}} \cdot f^{p-q/p-1}.
\]

From (3.4), (3.7) and (3.8) we obtain

\[
A \leq \left[ 2^n \cdot \Gamma \cdot f^{p-q/p-1} + \alpha_{k+1}^q \right] \cdot \frac{1}{2^n} = \Gamma f^{p-q/p-1} + E_n(f)
\]

and Lemma 3.3 is proved.

\[
\text{Corollary 3.2. For every } g \in \Delta_n,
\]

\[
A \leq \Gamma f^{p-q/p-1} + E_n(f), \text{ where } A = \int_0^1 g^q.
\]

**Proof.** This is true, of course, for \( g \in \text{extr}(\Delta_n) \), and so also for \( g \in \text{conv}(\text{extr}(\Delta_n)) \), since \( t \mapsto t^q \) is convex for \( q > 1 \) on \( \mathbb{R}^+ \). It remains true for \( g \in \text{conv}(\text{extr}(\Delta_n))^{L^p,\infty([0,1])} \) using a simple continuity argument. In fact, we just need the continuity of the identity operator if it is viewed as \( I: L^p,\infty([0,1]) \to L^q([0,1]) \). See [4]. Using now Krein–Milman Theorem the Corollary is proved.

We have now the following

**Corollary 3.3.** Let \( \phi: (X, \mu) \to \mathbb{R}^+ \) such that

\[
\int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p,\infty} \leq 1.
\]

Then

\[
f^q \leq A \leq \Gamma f^{p-q/p-1}.
\]
Proof. Let $g = \phi^*: [0, 1] \to \mathbb{R}^+$. There exist a sequence $(g_n)$ of $\frac{1}{2^n}$-simple functions, such that $g_n \leq g_{n+1} \leq g$ and $g_n$ converges almost everywhere to $g$. But then by defining

$$f_n = \int_0^1 g_n, \quad A_n = \int_0^1 g_n^q$$

we have that

$$(3.9) \quad g_n \in \Delta_n(f_n) \text{ so that } A_n \leq \Gamma f_n^{p-q/p-1} + \mathcal{E}_n(f_n).$$

By the monotone convergence theorem $f_n \to f$, $A_n \to A$. Moreover,

$$\mathcal{E}_n(f_n) = \frac{(2^n f_n - k_n^{1-\frac{1}{p}} 2^n)^q}{2^n},$$

where $k_n$ satisfy

$$\left(\frac{k_n}{2^n}\right)^{1-\frac{q}{p}} \leq f_n < \left(\frac{k_n + 1}{2^n}\right)^{1-\frac{1}{p}}.$$

As a consequence

$$\mathcal{E}_n(f_n) = (2^n)^{q-1} \left[ f_n - \left(\frac{k_n}{2^n}\right)^{1-\frac{1}{p}} \right]^q < (2^n)^{q-1} \left[ \left(\frac{k_n + 1}{2^n}\right)^{1-\frac{1}{p}} - \left(\frac{k_n}{2^n}\right)^{1-\frac{1}{p}} \right]^q$$

$$\leq (2^n)^{q-1} \left(\frac{1}{2^{1-\frac{2}{p}}}\right)^q = \left(\frac{1}{2^{1-\frac{2}{p}}}\right)^n \to 0, \text{ as } n \to \infty$$

where in the second inequality we used the known

$$(t + s)^\alpha \leq t^\alpha + s^\alpha \text{ for } t, s \geq 0, 0 < \alpha < 1.$$ 

Now (3.9) gives the corollary. \hfill \Box

In fact the converse of Corollary 3.3 is also true.

**Theorem 3.2.** For $0 < f \leq 1$, $A > 0$ the following are equivalent:

i) $f^q \leq A \leq \Gamma f^{p-q/p-1}$,

ii) $\exists \phi: (X, \mu) \to \mathbb{R}^+$ such that

$$\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A, \quad |||\phi|||_{p, \infty} \leq 1.$$ 

We prove first the following

**Lemma 3.4.** Let $\alpha \in (0, 1)$ and $(f, A)$ such that

$$(3.10) \quad f \leq \alpha^{1-\frac{1}{p}},$$

$$(3.11) \quad f^q \leq \alpha^{q-1} A,$$

$$(3.12) \quad A \leq \Gamma f^{p-q/p-1}.$$ 

Then there exists $g: [0, \alpha] \to \mathbb{R}^+$ such that

$$\int_0^\alpha g = f, \quad \int_0^\alpha g^q = A, \quad |||g|||_{p, \infty} = 1,$$

where

$$|||g|||_{p, \infty} = \sup \left\{ |E|^{-1+\frac{1}{p}} \int_E g: E \text{ measurable subset of } [0, \alpha] \text{ such that } |E| > 0 \right\}.$$
Proof. We search for a \( g \) of the form
\[
g := \begin{cases} 
\frac{p-1}{p} t^{-1/p}, & 0 < t \leq c_1, \\
\mu_2, & c_1 < t \leq \alpha,
\end{cases}
\]
for suitable constant \( c_1, \mu_2 \). We must have that
\[
(3.13) \quad \int_0^\alpha g = f \iff c_1^{1-\frac{1}{p}} + \mu_2 (\alpha - c_1) = f.
\]
Additionally, \( g \) must satisfy
\[
(3.14) \quad \int_0^\alpha g^q = A \iff \Gamma c_1^{1-\frac{q}{p}} + \mu_2^q (\alpha - c_1) = A.
\]
(3.13) gives
\[
(3.15) \quad \mu_2 = \frac{f - c_1^{1-\frac{1}{p}}}{a - c_1},
\]
so (3.14) becomes
\[
(3.16) \quad \Gamma c_1^{1-\frac{q}{p}} + \frac{(f - c_1^{1-\frac{1}{p}})^q}{(a - c_1)^{q-1}} = A.
\]
That is we search for a \( c_1 \in (0, \alpha) \) such that
\[
T(c_1) = A \quad \text{where} \quad T: [0, \alpha) \to \mathbb{R}^+
\]
is defined by
\[
T(t) = \Gamma t^{1-\frac{q}{p}} + \frac{(f - t^{1-\frac{1}{p}})^q}{(a - t)^{q-1}}.
\]
Observe that \( T(0) = \frac{f^q}{a^{q-1}} \leq A \) because of (3.11) and that \( T(f^{p/p-1}) = \Gamma f^{p-q/p-1} \geq A \). Now because of the continuity of \( T \), there exists \( c_1 \in (0, f^{p/p-1}] \) such that \( T(c_1) = A \). Then \( c_1 \in (0, \alpha) \) because of (3.10), and if we define \( \mu_2 \) by (3.15), we guarantee (3.13) and (3.14). We need to prove now that \( |||g|||_{[0,\alpha]}^{0,\alpha} = 1 \). Obviously, because of the form of \( g \), \( |||g|||_{[0,\alpha]}^{0,\alpha} \geq 1 \). So we have to prove that
\[
(3.17) \quad \int_0^t g \leq t^{1-\frac{1}{p}}, \quad \forall t \in (0, \alpha].
\]
This is of course true for \( t \in [0, c_1] \). For \( t \in (c_1, \alpha) \),
\[
\int_0^t g = c_1^{1-\frac{1}{p}} + \mu_2 (t - c_1) =: G(t).
\]
Since \( G(c_1) = c_1^{1-\frac{1}{p}} \), \( G(\alpha) = f < \alpha^{1-\frac{1}{p}} \) and \( t \mapsto t^{1-\frac{1}{p}} \) is concave on \((c_1, \alpha] \), (3.17) is true. Thus Lemma 3.4 is proved. \( \square \)

We have now the

Proof of Theorem 3.2. We have to prove the direction i) \( \Rightarrow \) ii). Indeed, if \( f^q \leq A \leq \Gamma f^{p-q/p-1} \) and \( f < 1 \), we apply Lemma 3.4. If \( f^q = A \) with \( 0 < f \leq 1 \), we set \( g \) by \( g(t) = f \), for every \( t \in [0, 1] \), while if \( f = 1 \leq A \leq \Gamma \) a simple modification of Lemma 3.4 gives the result. \( \square \)
We conclude Section 3 with the following theorem which can be proved easily using all the above.

**Theorem 3.3.** For $f, A$ such that $0 < f < 1, A > 0$ the following are equivalent:

i) $f^q \leq A \leq \Gamma f^{p-q/p-1}$,

ii) $\exists \phi: (X, \mu) \to \mathbb{R}^+$ such that $\int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p,\infty} = 1$.

**Remark 3.1.** Theorem 3.3 is completed if we mention that for $f = 1$ the following are equivalent:

i) $f = 1 \leq A \leq \Gamma$,

ii) $\exists \phi: (X, \mu) \to \mathbb{R}^+$ such that $\int_X \phi \, d\mu = 1, \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p,\infty} = 1$.

## 4. The extremal problem

Let $\mathcal{M}_T = \mathcal{M}$ the dyadic maximal operator associated to the tree $T$, on the probability non-atomic measure space $(X, \mu)$. Our aim is to find

$$
T_{f,A,F}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}): \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p,\infty} = F \right\}
$$

for all the allowable values of $f, A, F$. We find it in the case where $F = 1$. We write $T_{f,A}(\lambda)$ for $T_{f,A,1}(\lambda)$. In order to find $T_{f,A}(\lambda)$ we find first the following

$$
T^{(1)}_{f,A}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}): \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p,\infty} \leq 1 \right\}.
$$

The domain of this extremal problem is the following

$$
D = \left\{ (f, A): 0 < f \leq 1, \quad f^q \leq A \leq \Gamma f^{p-q/p-1} \right\}.
$$

Obviously, $T^{(1)}_{f,A}(\lambda) = 1$, for $\lambda \leq f$. Let now $\lambda > f$ and $(f, A) \in D$. Let $\phi$ be as in the definition of $T^{(1)}_{f,A}(\lambda)$ . Consider the decreasing rearrangement of $\phi$, $g = \phi^* : [0, 1] \to \mathbb{R}^+$. Then

$$
\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \|g\|_{p,\infty}^{[0,1]} \leq 1.
$$

Consider also $E = \{M\phi \geq \lambda\} \subseteq X$. Then $E$ is the almost disjoint union of elements of $T$, let $(I_j)_j$. In fact, we just need to consider the elements $I$ of $T$, maximal under the condition

(4.1)

$$
\frac{1}{\mu(I)} \int_I \phi \, d\mu \geq \lambda.
$$

We then have $E = \bigcup_j I_j$ and $\int_E \phi \, d\mu \geq \lambda \mu(E)$ because of (4.1). Then according to Lemma 2.1 we have that $\int_0^\alpha g \geq \alpha \lambda$ where $\alpha = \mu(E)$. That is

(4.2)

$$
T^{(1)}_{f,A}(\lambda) \leq \Delta_{f,A}(\lambda),
$$

where

$$
\Delta_{f,A}(\lambda) = \sup \left\{ \alpha \in (0, 1] : \exists g: [0, 1] \to \mathbb{R}^+: \int_0^1 g = f, \int_0^1 g^q = A, \quad \|g\|_{p,\infty}^{[0,1]} \leq 1, \int_0^\alpha g \geq \alpha \lambda \right\}.
$$
We prove now the converse inequality in (4.2) by proving the following

**Lemma 4.1.** Let \( g \) be as in (4.3) for a fixed \( \alpha \in (0, 1] \). Then there exists \( \phi: (X, \mu) \to \mathbb{R}^+ \) such that

\[
\int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A, \quad \|\phi\|_{p, \infty} \leq 1 \quad \text{and} \quad \mu(\{M\phi \geq \lambda\}) \geq \alpha.
\]

**Proof.** Lemma 2.3 guarantees the existence of a sequence \((I_j)_j\) of pairwise almost disjoint elements of \( T \) such that

\[
\mu \left( \bigcup I_j \right) = \sum \mu(I_j) = \alpha.
\]

Consider now the finite measure space \(([0, \alpha], |\cdot|)\), where \(|\cdot|\) is the Lebesque measure. Then since \( \int_0^\alpha g \geq \alpha \lambda \) and (4.4) holds, applying Lemma 2.2 repeatedly, we obtain the existence of a sequence \((A_j)\) of Lebesque measurable subsets of \([0, \alpha]\) such that the following hold:

\[
(A_j)_j \text{ is a pairwise disjoint family}, \quad \bigcup A_j = [0, \alpha], \quad |A_j| = \mu(I_j), \quad \frac{1}{|A_j|} \int_{A_j} g \geq \lambda.
\]

Then we define \( g_j: [0, |A_j|] \to \mathbb{R}^+ \) by \( g_j = (g/|A_j|)^* \). Define also for every \( j \) a measurable function \( \phi_j: I_j \to \mathbb{R}^+ \) so that \( |\phi_j| = g_j \). The existence of such a function is guaranteed by the fact that \((I_j, \mu/I_j)\) is non-atomic. Since \((I_j)\) is almost pairwise disjoint family we produce a \( \phi^{(1)}: \bigcup I_j \to \mathbb{R}^+ \) measurable such that \( \phi^{(1)}/I_j = \phi_j \). We set now \( Y = X \setminus \bigcup I_j \) and \( h: [0, 1 - \alpha] \to \mathbb{R}^+ \) by \( h = (g/|\alpha, 1|)^* \). Then since \( \mu(Y) = 1 - \alpha \) there exists \( \phi^{(2)}: Y \to \mathbb{R}^+ \) such that \((\phi^{(2)})^* = h\). Set now

\[
\phi = \begin{cases} 
\phi^{(1)}, & \text{on } \bigcup I_j, \\
\phi^{(2)}, & \text{on } Y.
\end{cases}
\]

It is easy to see from the above construction that \( \phi^* = g \) a.e. with respect to Lebesque measure, which gives \( \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A \) and \( \|\phi\|_{p, \infty} \leq 1 \). Additionally,

\[
\frac{1}{\mu(I_j)} \int_{I_j} \phi \, d\mu = \frac{1}{|A_j|} \int_{A_j} g \geq \lambda \quad \text{for every } j,
\]

that is,

\[
\{M\phi \geq \lambda\} \supseteq \bigcup I_j, \quad \text{so} \quad \mu(\{M\phi \geq \lambda\}) \geq \alpha
\]

and the lemma is proved. \( \Box \)

It is now not difficult to see that we can replace the inequality \( \int_0^\alpha g \geq \alpha \lambda \) in the definition of \( \Delta_{f,A}(\lambda) \) by equality, thus defining \( S_{f,A}(\lambda) \), in such a way that

\[
T_{f,A}^{(1)}(\lambda) = \Delta_{f,A}(\lambda) = S_{f,A}(\lambda).
\]

This is true since if \( g \) is as in (4.3) and \( \lambda > f \), there exists \( \beta \geq \alpha \) such that \( \int_0^\beta g = \beta \lambda \). For \((f, A) \in D\) we set

\[
G_{f,A}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}): \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A \right\}.
\]

It is obvious that \( T_{f,A}^{(1)}(\lambda) \leq G_{f,A}(\lambda) \). As a matter of fact \( G_{f,A}(\lambda) \) has been computed in [6] and was found to be
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\[ G_{f,A}(\lambda) = \begin{cases} 
1, & \lambda \leq f, \\
\frac{f}{\lambda}, & f < \lambda < \left(\frac{A}{f}\right)^{1/q-1}, \\
k, & \left(\frac{A}{f}\right)^{1/q-1} \leq \lambda,
\end{cases} \tag{4.6} \]

where \( k \) is the unique root of the equation

\[ \frac{(f - \alpha \lambda)^q}{(1 - \alpha)^q} + \alpha \lambda^q = A \]

on \( \alpha \in \left[0, \frac{f}{\lambda}\right] \), when \( \lambda > \left(\frac{A}{f}\right)^{1/q-1} \).

We have now the following

**Proposition 4.1.** If \((f, A) \in D\), then

\[ T_{f,A}^{(1)}(\lambda) \leq \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}. \]

**Proof.** We just need to see that \( \mu(\{M\phi \geq \lambda\}) \leq \frac{1}{\lambda^p} \) for every \( \phi \) such that \( |||\phi|||_{p,\infty} \leq 1 \). But if \( E = \{M\phi \geq \lambda\} \), we have by the definition of the norm \( |||\cdot|||_{p,\infty} \) that \( \int_E \phi \leq \mu(E)^{1-\frac{1}{p}} \). But by (1.3) \( \int_E \phi \geq \lambda \mu(E) \), so that

\[ \lambda \mu(E) \leq \mu(E)^{1-\frac{1}{p}} \implies \mu(E) \leq \frac{1}{\lambda^p}. \]

So Proposition 4.1 is true. \( \Box \)

We prove now that in Proposition 4.1 we have equality.

**Proposition 4.2.** Let \((f, A) \in D\) and \( \lambda \) such that

\[ \frac{f}{\lambda} = \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}. \tag{4.7} \]

Then \( T_{f,A}^{(1)}(\lambda) = \frac{f}{\lambda} \).

**Proof.** We use Lemma 3.4 and equations (4.5). Because of (4.5) we need to find \( g: [0, 1] \to \mathbb{R}^+ \) such that

\[ \int_0^1 g = f, \quad \int_0^1 g^q = A, \quad |||g|||_{p,\infty} \leq 1 \quad \text{and} \quad \int_0^{f/\lambda} g = \frac{f}{\lambda} \cdot \lambda = f, \]

that is, \( g \) should be defined on \([0, f/\lambda]\). We apply Lemma 3.4, with \( \alpha = \frac{f}{\lambda} \). In fact, since (4.7) is true, we have that \( G_{f,A}(\lambda) = \frac{f}{\lambda} \) so, \( \lambda < \left(\frac{A}{f}\right)^{1/q-1} \) which gives (3.11), while \( \frac{f}{\lambda} \leq \frac{1}{\lambda^p} \) gives (3.10). In fact, Lemma 3.4 works even with equality on (3.10) as it is easily can be seen by continuity reasons. So, in view of (4.5) we have \( T_{f,A}^{(1)}(\lambda) \geq f/\lambda \) and the proposition is proved. \( \Box \)

At the next step we have

**Proposition 4.3.** Let \((f, A) \in D\) and \( \lambda \) such that

\[ k = \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}. \tag{4.8} \]

Then \( T_{f,A}^{(1)}(\lambda) = k \).
Proof. Obviously, (4.8) gives $\lambda \geq \left(\frac{A}{f}\right)^{1/q}$. We prove that there exists $g: [0, 1] \to \mathbb{R}^+$ such that

$$
\int_0^k g = k\lambda, \quad \int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad \||g||_{p,\infty} \leq 1.
$$

For this purpose we define

$$
ge := \begin{cases} 
\lambda, & \text{on } [0, k], \\
\frac{f-k\lambda}{1-k}, & \text{on } (k, 1].
\end{cases}
$$

Then, obviously, the first two conditions in (4.9) are satisfied, while

$$
\int_0^1 g^q = \left(\frac{f-k\lambda}{1-k}\right)^q + k\lambda^q = A,
$$

by the definition of $k$. Moreover, $|||g|||_{p,\infty} \leq 1$. This is true since $k\lambda \leq k^{1-\frac{q}{p}}$, $f \leq 1$ and the fact that $g$ is constant on each of the intervals $[0, k]$ and $(k, 1]$. So the proposition is proved.

At last we prove

**Proposition 4.4.** Let $(f, A) \in D$ and $\lambda$ such that

$$
\frac{1}{\lambda^p} = \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}.
$$

Then $T_{f,A}^{(1)}(\lambda) = \frac{1}{\lambda^p}$.

Proof. As before we search for a function $g$ such that

$$
\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \||g||||_{p,\infty} \leq 1 \quad \text{and} \quad \int_0^{1/\lambda^p} g = \frac{1}{\lambda^p} \cdot \lambda = \frac{1}{\lambda^{p-1}}.
$$

We define

$$
\vartheta_{\lambda} = \frac{\Gamma}{\lambda^{p-q}} + \left(\frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}}\right)^q,
$$

and we consider two cases:

i) $\vartheta_{\lambda} > A$. We search for a function of the form

$$
g := \begin{cases} 
\left(1 - \frac{1}{p}\right) t^{-1/p}, & 0 < t \leq c_1, \\
\mu_2, & c_1 < t \leq \frac{1}{\lambda^p}, \\
\mu_3, & \frac{1}{\lambda^p} < t < 1,
\end{cases}
$$

for suitable constants $c_1 \leq \frac{1}{\lambda^p}$, $\mu_2$, $\mu_3$. Then in view of (4.11) the following must hold:

$$
c_1^{1-\frac{1}{p}} + \mu_2 \left(\frac{1}{\lambda^p} - c_1\right) = \frac{1}{\lambda^{p-1}},
$$

$$
c_1^{1-\frac{1}{p}} + \mu_2 \left(\frac{1}{\lambda^p} - c_1\right) + \mu_3 \left(1 - \frac{1}{\lambda^p}\right) = f,
$$

$$
\Gamma c_1^{1-\frac{2}{p}} + \mu_2^2 \left(\frac{1}{\lambda^p} - c_1\right) + \mu_3^2 \left(1 - \frac{1}{\lambda^p}\right) = A.
$$
Notice that the condition $|||g|||_{p,\infty} \leq 1$ is automatically satisfied because of the form of \( g \) and the previous stated relations. Now (4.13) and (4.14) give
\[
\mu_3 = \frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}}
\]
and
\[
\mu_2 = \frac{1}{\lambda^{p-1}} - c_1^{1 - \frac{1}{p}} \lambda^p - c_1
\]
while (4.15) gives \( T(c_1) = A \) where \( T \) is defined on \( [0, \frac{1}{\lambda^p}] \) by
\[
T(c) = \Gamma c^{1 - q} + \left( \frac{1}{\lambda^{p-1}} - c^{1 - \frac{1}{p}} \right)^q + \left( \frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}} \right)^q.
\]
Then
\[
T(0) = \frac{1}{\lambda^{p-q}} + \left( \frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}} \right)^q.
\]
It is now easy to see that \( T(0) \leq A \) by using that \( F: [0, f/\lambda] \to \mathbb{R}^+ \) defined by
\[
F(t) = \frac{(f - t\lambda)^q}{(1 - t)^{q-1}} + t\lambda^q
\]
is increasing, and the definition of \( G_{f, A}(\lambda) \). Moreover \( \lim_{c \to \frac{1}{\lambda^p}} T(c) = \vartheta_\lambda > A \), so by continuity of the function \( t \), we end case i). Now for ii) \( \vartheta_\lambda \leq A \). We search for a function of the form
\[
g := \begin{cases} \left( 1 - \frac{1}{\lambda^p} \right)^{(1-1/p)} & 0 < t \leq c_1, \\ \mu_2, & c_1 < t \leq 1, \end{cases}
\]
where \( \frac{1}{\lambda^p} < c_1 \). Similar arguments as in case i) give the result. \( \square \)

From Propositions 4.1–4.4 we have now

**Theorem 4.1.** For \( (f, A) \in D \),
\[
T_{f, A}^{(1)}(\lambda) = \min \left\{ 1, G_{f, A}(\lambda), \frac{1}{\lambda^p} \right\}.
\]

**Remark 4.1.** Notice that \( T_{f, A}(\lambda) = T_{f, A}^{(1)}(\lambda) \) for every \( f, A \) such that \( f^q < A \leq \Gamma f^{p-1}q/p-1 \) and \( 0 < f \leq 1 \). Indeed, suppose that \( \alpha = T_{f, A}^{(1)}(\lambda) \). Then there exists \( g: [0, 1] \to \mathbb{R}^+ \) such that
\[
(4.18) \quad \int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \int_0^\alpha g = \alpha \lambda \quad \text{and} \quad |||g|||_{p,\infty} \leq 1.
\]
It is easy to see that for every \( \varepsilon > 0 \), small enough we can produce from \( g \) a function \( g_\varepsilon \) satisfying
\[
\int_0^{\alpha - \varepsilon} g_\varepsilon \geq (\alpha - \varepsilon) \lambda, \quad \int_0^1 g_\varepsilon = f, \quad \int_0^1 g_\varepsilon = A + \delta \varepsilon \quad \text{and} \quad |||g_\varepsilon|||_{p,\infty} = 1,
\]
where \( \lim_{\varepsilon \to 0^+} \delta \varepsilon = 0 \). This and continuity reasons shows \( T_{f,A}(\lambda) = \alpha \).

iii) The case \( A = f^q \) can be worked out separately because there is essentially unique function \( g \) satisfying \( \int_0^1 g = f, \int_0^1 g^q = f^q \), namely the constant function with value \( f \).

Scaling all the above we have that

**Theorem 4.2.** For \( f, A \) such that \( f^q < A \leq \Gamma f^{p-q/p-1} F^{p(q-1)/(p-1)} \) and \( 0 < f \leq F \) the following hold

\[
\sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \|\|\phi\||_{p,\infty} = F \right\}
\]

\[
(4.19)
\]

\[
= \min \left\{ 1, G_{f,A}(\lambda), \frac{F^p}{\lambda^p} \right\}
\]

and

\[
\sup \left\{ \|M\phi\|_{p,\infty} : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^q \, d\mu = A, \|\|\phi\||_{p,\infty} = F \right\} = F.
\]

**References**


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