A GENERAL DIFFERENTIAL INEQUALITY OF THE $k$TH DERIVATIVE THAT LEADS TO NORMALITY

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Abstract. Let $k \geq 0$ be an integer and $\alpha > 1$. Let $F$ be a family of functions meromorphic in a domain $D \subset \mathbb{C}$. If \( \left\{ \frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} : f \in F \right\} \) is locally uniformly bounded away from zero, then $F$ is normal.

1. Introduction

Recently, there has been renewed activity in the study of the connection between differential inequalities and normality. A natural point of departure for this subject is the well-known theorem due to Marty.

Marty’s Theorem. [10, p. 75] A family $F$ of functions meromorphic in a domain $D$ is normal if and only if $\{f^\# : f \in F\}$ is locally uniformly bounded in $D$.

Following Marty’s Theorem, Royden proved the following generalization.

Theorem R. [9] Let $F$ be a family of functions meromorphic in a domain $D$ with the property that for each compact set $K \subset D$, there is a positive increasing function $h_K$ such that $|f'(z)| \leq h_K(|f(z)|)$ for all $f \in F$ and $z \in K$. Then $F$ is normal in $D$.

This result has been significantly extended further in various directions; see [4], [11] and [13]. Li and Xie established a different kind of generalization of Marty’s Theorem, which involves higher derivatives.

Theorem LX. [5] Let $F$ be a family of functions meromorphic in a domain $D$ such that each $f \in F$ has zeros only of multiplicities $\geq k$, $k \in \mathbb{N}$. Then $F$ is normal in $D$ if and only if the family

$$\left\{ \frac{|f^{(k)}(z)|}{1 + |f(z)|^{k+1}} : f \in F \right\}$$

is locally uniformly bounded in $D$.

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In [7], the second and the third authors gave a counterexample to the validity of Theorem LX, without the condition on the multiplicities of zeros for the case $k = 2$.

Concerning differential inequalities with the reversed sign of the inequality, Grahl, and the second author proved the following result, which may be considered a counterpart to Marty’s Theorem.

**Theorem GN.** [2] Let $F$ be a family of functions meromorphic in $D$ and $C > 0$. If $f^{\#}(z) > C$ for every $f \in F$ and $z \in D$, then $F$ is normal in $D$.

Steinmetz [12] gave a shorter proof of Theorem GN, using the Schwarzian derivative and some well-known facts on linear differential equations.

Then in [6], Liu together with the second and third authors generalized Theorem GN and proved the following result.

**Theorem LNP.** Let $1 \leq \alpha < \infty$ and $C > 0$. Let $F$ be the family of all meromorphic functions $f$ in $D$ such that

$$\frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} > C$$

for every $z \in D$.

Then the following hold:

1. If $\alpha > 1$, then $F$ is normal in $D$;
2. If $\alpha = 1$, then $F$ is quasi-normal in $D$ but not necessarily normal.

Observe that (2) of Theorem LNP is a differential inequality that distinguishes between quasi-normality to normality.

In this paper, we continue to study differential inequalities with the reversed sign (“$\geq$”) and prove the following general theorem.

**Theorem 1.** Let $D$ be a domain in $C$. Let $k \geq 0$ be an integer, $C > 0, \alpha > 1$ constants. Then the family $F$ of all functions $f$ meromorphic in $D$ such that

$$(1) \quad \frac{|f^{(k)}(z)|}{1 + |f(z)|^\alpha} > C, \quad z \in D,$$

is normal.

Let us set some notation. For $z_0 \in C$ and $r > 0$ we put $\Delta(z_0, r) = \{ z : |z - z_0| < r \}$ and $\bar{\Delta}(z_0, r) = \{ z : |z - z_0| \leq r \}$. We write $f_n(z) \Rightarrow f(z)$ on $D$ to indicate that the sequence $\{f_n(z)\}$ converges to $f(z)$ in the spherical metric, uniformly on compact subsets of $D$, and $f_n(z) \Rightarrow f(z)$ on $D$ if the convergence is also in the Euclidean metric.

We need two lemmas for the proof.

### 2. Auxiliary lemmas

The first lemma we need is the lemma of Chen and Gu [1, Thm. 2], see also [8, Lemma 2]. Observe that this is an “if and only if” lemma.

**Lemma 1.** Let $F$ be a family of functions meromorphic in a domain $D \subset C$, all of whose zeros have multiplicity at least $m$, and all of whose poles have multiplicity at least $p$, and let $-p < \alpha < m$. Then $F$ is not normal at some $z_0 \in D$ if and only if there exist sequences $\{f_n\}_{n=1}^\infty \subset F$, $\{z_n\}_{n=1}^\infty \subset D$, $\{\rho_n\}_{n=1}^\infty$ satisfying $z_n \to z_0$,
Lemma 2.

Let \( k \geq 1 \) be an integer. Then the family \( \mathcal{F} \) of all functions meromorphic in a domain \( D \subset \mathbb{C} \) such that \( f(z) \neq 0 \), \( f^{(k)}(z) \neq 1 \) for every \( z \in D \) is normal.

3. Proof of Theorem 1

The case \( k = 0 \) is immediate, so we assume that \( k \geq 1 \). Let \( z_0 \in D \) and let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of functions of \( \mathcal{F} \). We prove that \( \{f_n\}_{n=1}^{\infty} \) is normal at \( z_0 \).

Separate into two cases.

Case (I). There is some \( r > 0 \) and a subsequence of \( \{f_n\}_{n=1}^{\infty} \), all of which are holomorphic in \( \Delta(z_0, r) \).

Without loss of generality, we denote this subsequence also as \( \{f_n\}_{n=1}^{\infty} \). Let us take \( \beta > \frac{k}{\alpha - 1} \).

If \( \{f_n\}_{n=1}^{\infty} \) is not normal at \( z_0 \), then by Lemma 1 there is a subsequence of \( \{f_n\}_{n=1}^{\infty} \) (that will also be denoted by \( \{f_n\}_{n=1}^{\infty} \)), and sequences \( z_n \to z_0, \rho_n \to 0^+ \) such that

\[
\rho_n^\beta f_n(z_n + \rho_n \xi) \to g(\xi) \text{ on } \mathbb{C},
\]

where \( g \) is a nonconstant entire function in \( \mathbb{C} \).

Let \( \xi_0 \in \mathbb{C} \) be such that \( g(\xi_0) \neq 0 \). Differentiating (2) \( k \) times at \( \xi_0 \) gives

\[
g_n(\xi) := \rho_n^\alpha f_n(z_n + \rho_n \xi) \to g^{(k)}(\xi_0) \text{ on } \mathbb{C},
\]

By (2) and the choice of \( \xi_0 \) we have \( f_n(z_n + \rho_n \xi_0) \to \infty \), and thus by (1) we have

\[
|f_n^{(k)}(z_n + \rho_n \xi_0)| > C|f_n(z_n + \rho_n \xi_0)|^\alpha.
\]

Thus \( \rho_n^{\beta + k} |f_n^{(k)}(z_n + \rho_n \xi_0)| \geq C\rho_n^{\beta + k} |f_n(z_n + \rho_n \xi_0)|^\alpha = C(\rho_n^\beta |f_n(z_n + \rho_n \xi_0)|)^\alpha \rho_n^{\beta - \beta\alpha} \).

By the choice of \( \beta \) and \( \xi_0 \) the last expression tends to \( \infty \) as \( n \to \infty \), and this is a contradiction to (3), as \( g^{(k)}(\xi_0) \) is finite.

Case (II). There are \( N \in \mathbb{N} \) and \( \{z_n\}_{n=N}^{\infty} \) such that \( z_n \to z_0 \) and \( f_n(z_n) = \infty \). Without loss of generality \( N = 1 \). Let \( K_n \geq 1 \) denote the multiplicity of the pole \( z_n \) of \( f_n \). We also assume that there is a sequence \( \tilde{z}_n \to z_0 \) as above, there would exist some \( \rho > 0 \) and a subsequence of \( \{f_n\}_{n=1}^{\infty} \) (that we also denote by \( \{f_n\}_{n=1}^{\infty} \)) such that \( f_n \neq 0 \) in \( \Delta(0, \rho) \). Then by Lemma 2, \( \{f_n\}_{n=1}^{\infty} \) would be normal at \( z_0 \), and we are done.

Consider now the sequence \( \{\frac{f_n^{(k)}}{f_n}\}_{n=1}^{\infty} \). If \( |f_n(z)| \leq 1 \), then \( |\frac{f_n^{(k)}}{f_n}(z)| \geq |f_n^{(k)}(z)| \geq \frac{\frac{f_n^{(k)}(z)}{1 + |f_n(z)|^\alpha}}{1 + |f_n(z)|^\alpha} \geq C \). If \( |f_n(z)| > 1 \), then \( |\frac{f_n^{(k)}}{f_n}(z)| \geq \frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^\alpha} \geq C \). Hence \( \{\frac{f_n^{(k)}}{f_n}\}_{n=1}^{\infty} \) is normal in \( D \) and so is \( \{\frac{f_n}{f_n}\}_{n=1}^{\infty} \). Thus we can assume, after moving to a subsequence...
(that will also be denoted by \(\{\frac{f_n}{f_n^{(k)}}\}_{n=1}^{\infty}\)) that \(\frac{f_n}{f_n^{(k)}} \xrightarrow{n \to \infty} H\) in \(D\). Since for each \(n\), \(\frac{f_n}{f_n^{(k)}}\) is holomorphic in \(D\), and since \(\frac{f_n}{f_n^{(k)}}(z_n) = \frac{f_n}{f_n^{(k)}}(\tilde{z}_n) = 0\), \(H\) is analytic in \(D\). The point \(\tilde{z}_n\) is a zero of \(\frac{f_n}{f_n^{(k)}}\) of multiplicity at least 1. The point \(z_n\) is a zero of \(\frac{f_n}{f_n^{(k)}}\) of multiplicity exactly \(k\). Thus, if \(H \neq 0\), then by Rouche’s Theorem \(z_0\) is a zero of \(H\) of multiplicity at least \(k + 1\). Thus, in both cases \(H \neq 0\) or \(H \equiv 0\), we have

\[
(4) \quad \left(\frac{f_n}{f_n^{(k)}}(z_n)\right) \xrightarrow{n \to \infty} 0.
\]

In some small neighborhood of \(z_n\) (that depends on \(n\)), we have

\[
(5) \quad f_n(z) = \frac{A_n}{(z - z_n)^{K_n}}(1 + h_n(z))
\]

where \(A_n \neq 0\) is a constant and \(h_n\) is analytic, \(h_n(z_n) = 0\).

Differentiating (5) \(k\) times gives

\[
(6) \quad f_n^{(k)}(z) = \frac{(-1)^k K_n(K_n + 1) \cdots (K_n + k - 1)A_n}{(z - z_n)^{K_n+k}}(1 + h_n^*(z)),
\]

where \(h_n^*\) has the same properties of \(h_n\). Dividing (5) in (6) and differentiating \(k\) times at \(z_n\) gives

\[
(7) \quad \left(\frac{f_n}{f_n^{(k)}}(z_n)\right) = \frac{(-1)^k k!}{K_n(K_n + 1) \cdots (K_n + k - 1)}.
\]

Now, if \(\{K_n\}_{n=1}^{\infty}\) is bounded, then the right hand side of (7) does not tend to 0 as \(n \to \infty\), contradicting (4).

Otherwise, we can choose \(n\) such that \(K_n > \frac{k}{\alpha - 1}\). We then have that both the nominator and the denominator of (1) are infinite at \(z_n\) and by (6) we have

\[
\frac{|f_n^{(k)}(z_n)|}{1 + |f_n(z_n)|^\alpha} = \lim_{z \to z_n} \frac{K_n(K_n+1) \cdots (K_n+k-1)|A_n|}{|z - z_n|^{K_n+k-\alpha} |A_n|^{\alpha}} \cdot \frac{|K_n^{(\alpha - 1)-k}}{|z - z_n|^{K_n^{(\alpha - 1)-k}}}
\]

By the choice of \(K_n\) this limit is 0. This is a contradiction to (1) and the proof of Theorem 1 is completed.

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References


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