ON WEIGHTED POINCARÉ INEQUALITIES

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Abstract. The aim of this note is to show that Poincaré inequalities imply corresponding weighted versions in a quite general setting. Fractional Poincaré inequalities are considered, too. The proof is short and does not involve covering arguments.

1. Introduction

Let $(X, \rho)$ be a metric space with a positive $\sigma$-finite Borel measure $dx$, we will write $|E| = \int_E dx$ for the measure of a Borel set $E \subset X$. We fix some point $x_0 \in X$ and set $B_r = \{x \in X : \rho(x, x_0) < r\}$, $\overline{B}_r = \{x \in X : \rho(x, x_0) \leq r\}$. We call a function $\phi : B_1 \rightarrow [0, \infty)$ a radially decreasing weight, if $\phi$ is a radial function, i.e. $\phi = \Phi(\rho(\cdot, x_0))$ and its profile $\Phi$ is nonincreasing and right-continuous with left-limits.

We assume that $\phi$ is not identically zero on $B_1 \setminus \overline{B}_{1/2}$. For any such weight $\phi$ there exists a positive, non-zero $\sigma$-finite Borel measure $\nu$ on $(\frac{1}{2}, 1]$, such that

\begin{equation}
\phi(x) = \int_{\frac{1}{\rho(x, x_0)}}^{1/2} \nu(dt) = \int_{1/2}^{1} \chi_{B_t}(x) \nu(dt), \quad x \in B_1 \setminus \overline{B}_{1/2}.
\end{equation}

Note that we put $\int_{a}^{b} f(t) \nu(dt) = \int_{(a,b]} f(t) \nu(dt)$. For a function $u$ we denote by

$$u_E = \frac{1}{|E|} \int_E u(x) \, dx$$

the mean of $u$ over the set $E$, and by

$$u^\phi_E = \frac{\int_E u(x) \phi(x) \, dx}{\int_E \phi(x) \, dx}$$

the mean of $u$ over the set $E \subset B_1$ with respect to the weight function $\phi$.

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Our main result is the following:

**Theorem 1.** Let $1 \leq p < \infty$ and let $\phi$ be a radially decreasing weight with $\phi = \Phi(\rho(\cdot, x_0))$. Let $F: L^p(X) \times (\frac{1}{2}, 1] \to [0, \infty]$ be a functional satisfying

\begin{equation}
F(u + a, r) = F(u, r), \quad a \in \mathbb{R},
\end{equation}

\begin{equation}
\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq F(u, r),
\end{equation}

for every $r \in (\frac{1}{2}, 1]$ and every $u \in L^p(X)$. Then for $M = \frac{8p|B_1|}{|B_{1/2}|} \frac{\Phi(0)}{\Phi(1/2)}$,

\begin{equation}
\int_{B_1} |u(x) - u_{B_1}|^p \phi(x) \, dx \leq M \int_{1/2}^1 F(u, t) \nu(dt)
\end{equation}

for every $u \in L^p(B_1)$, where $\nu$ is as in (1).

By choosing the functional $F$ appropriately, (4) becomes a Poincaré inequality with weight $\phi$, see Section 3. Such inequalities have been studied extensively because of their importance for the regularity theory of partial differential equations, see the exposition in [5].

**2. Proof**

**Lemma 2.** Let $\Omega$ be a finite measure space and $p \geq 1$. Assume $f \in L^p(\Omega)$ with $\int_{\Omega} f = 0$. Then

$$
\|f + a\|_{L^p(\Omega)} \geq \frac{1}{2} \|f\|_{L^p(\Omega)}
$$

for every $a \in \mathbb{R}$.

**Proof.** We may assume $a > 0$. Then

$$
\int_{\Omega \cap \{f > 0\}} |f + a|^p \geq \int_{\Omega \cap \{f > 0\}} |f|^p \quad \text{and} \quad \int_{\Omega \cap \{f < -2a\}} |f + a|^p \geq 2^{-p} \int_{\Omega \cap \{f < -2a\}} |f|^p.
$$

Furthermore, since $\int_{\Omega \cap \{f \leq 0\}} |f| = \int_{\Omega \cap \{f > 0\}} |f|$, we obtain

$$
\int_{\Omega \cap \{-2a \leq f \leq 0\}} |f|^p \leq (2a)^{p-1} \int_{\Omega \cap \{-2a \leq f \leq 0\}} |f|
$$

and

$$
\int_{\Omega \cap \{f > 0\}} |f| \leq 2^{p-1} \int_{\Omega \cap \{f > 0\}} |f + a|^p,
$$

where we use $a^{p-1}b \leq (b+a)^{p-1}(b+a)$ for positive $a, b$. Combining these observations we obtain the result. \hfill \Box

**Proof of Theorem 1.** First we observe that it is enough to prove that

\begin{equation}
\int_{B_1} |u(x) - u_{B_1}|^p \tilde{\phi}(x) \, dx \leq \frac{2^{2p}|B_1|}{|B_{1/2}|} \int_{1/2}^1 F(u, t) \nu(dt),
\end{equation}

where $\tilde{\phi}(x) = \phi(x) \wedge \Phi(\frac{1}{2})$. Indeed, we have

$$
\frac{\Phi(\frac{1}{2})}{\Phi(0)} \phi(x) \leq \phi(x) \wedge \Phi(\frac{1}{2}) \leq \phi(x).
$$
Hence if (5) holds, then
\[ \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \]
\[ \geq \frac{\Phi(\frac{1}{2})}{\Phi(0)} 2^{-p} \int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx, \]
where in the last line we have used Lemma 2. Now we prove (5). To simplify the notation, we assume that \( \phi(x) = \Phi(\frac{1}{2}) \) for \( x \in B_{1/2} \), so that \( \tilde{\phi} = \phi \). Because of (2), by subtracting a constant from \( u \), we may and do assume that \( u_{B_1}^\phi = 0 \), which means that
\[ 0 = \int_{B_1} u(x) \phi(x) \, dx = \int_{1/2}^1 \int_{B_1} u(x) \, dx \, \nu(dt) = \int_{1/2}^1 u_{B_1} |B_1| \, \nu(dt). \]
We start from the integral on the right hand side of (4) and use (3)
\[ R := \int_{1/2}^1 F(u, t) \, \nu(dt) \geq \int_{1/2}^1 \int_{B_1} |u(x) - u_{B_1}|^p \, dx \, \nu(dt) \]
\[ = \frac{1}{2} \int_{1/2}^1 \int_{B_1} |u(x) - u_{B_1}|^p \, dx \, \nu(dt) + \frac{1}{2} \int_{1/2}^1 \int_{1/2}^1 |u(x) - u_{B_1}|^p \chi_{B_1}(x) \, \nu(dt) \, dx \]
\[ =: I_1 + I_2 \]
(In fact \( I_1 = I_2 \), but we treat them differently.) We now deal with the inner integral in \( I_2 \). For \( x \in B_{1/2} \) we have
\[ \int_{1/2}^1 |u(x) - u_{B_1}|^p \chi_{B_1}(x) \, \nu(dt) \geq \frac{1}{|B_1|} \int_{1/2}^1 |u(x) - u_{B_1}|^p \, B_1 \, \nu(dt). \]
Since \( \int_{1/2}^1 u_{B_1} |B_1| \, \nu(dt) = 0 \), by Lemma 2 we obtain
\[ \int_{1/2}^1 |u(x) - u_{B_1}|^p \, B_1 \, \nu(dt) \geq 2^{-p} \int_{1/2}^1 u_{B_1} |B_1| \, \nu(dt). \]
Therefore
\[ I_2 \geq \frac{2^{-p}}{2 |B_1|} \int_{1/2}^1 \int_{B_1} |u_{B_1}|^p |B_1| \, \nu(dt) \, dx = \frac{2^{-p} |B_{1/2}|}{2 |B_1|} \int_{1/2}^1 u_{B_1} |B_1| \, \nu(dt). \]
Using the inequality \( |a|^p + |b|^p \geq 2^{1-p} |a + b|^p \) we obtain
\[ I_1 + I_2 \geq \frac{1}{2} \int_{1/2}^1 \int_{B_1} \left( |u(x) - u_{B_1}|^p + \frac{2^{-p} |B_{1/2}|}{|B_1|} u_{B_1} |B_1|^p \right) \, dx \, \nu(dt) \]
\[ \geq \frac{2^{-p} |B_{1/2}|}{2 |B_1|} 2^{1-p} \int_{1/2}^1 \int_{B_1} |u(x)|^p \, dx \, \nu(dt) \]
\[ = |B_{1/2}| \frac{2^{-2p}}{|B_1|} \int_{B_1} |u(x)|^p \phi(x) \, dx \]
and the proof is finished. \( \square \)
3. Applications

Let us discuss some corollaries. Corollary 3 is well-known [5]. However, our approach allows for very general weights. Proposition 4 allows to deduce a weighted Poincaré inequality for fractional Sobolev norms from an unweighted version. Corollaries 5 and 6 give a more general result for fractional Sobolev norms. The first allows for more general kernels and exponents $p$. Corollary 6 improves [2, Theorem 5.1] because the result is robust for $s \to 1^-$ and allows for general weights and exponents $p$.

Corollary 3. Let $p \geq 1$ and $\phi$ be a radially decreasing weight. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. There exists a positive constant $C$ depending on $p, d$ and $\phi$ such that

\[
\int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \leq C \int_{B_1} |\nabla u(x)|^p \phi(x) \, dx,
\]

for every $u \in W^{1,p}(B_1)$.

Proposition 4. Let $p \geq 1$ and let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$. Assume that for some kernel $k: B_1 \times B_1 \to [0, \infty)$ and some positive constant $C$ the following inequality holds

\[
\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx,
\]

whenever $r \in \left(\frac{1}{2}, 1\right]$ and $u \in L^p(X)$. Then with $M = \frac{8^p |B_1| \Phi(0)}{|B_{1/2}| \Phi(1/2)}$

\[
\int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \leq CM \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x)) \, dy \, dx
\]

for $u \in L^p(X)$.

Corollary 5. Let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(\rho(\cdot, x_0))$ and $p \geq 1$. Let $k: B_1 \times B_1 \to [0, \infty)$ be a kernel satisfying $k \geq c$ for some constant $c > 0$. There is a positive constant $M$ depending on $d, p$ and $\Phi$ such that for $u \in L^p(X)$

\[
\int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx \leq \frac{M}{c} \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x)) \, dy \, dx
\]

for $u \in L^p(X)$.

Corollary 6. Let $p \geq 1$, $R \geq 1$ and $0 < s_0 \leq s < 1$. Consider $X = \mathbb{R}^d$ equipped with the Lebesgue measure and the Euclidean metric. Let $\phi$ be a radially decreasing weight of the form $\phi = \Phi(|\cdot|)$. Then there exists a positive constant $C$ depending on $p, d, s_0$ and $\Phi$ such that

\[
\int_{B_1} |u(x) - u_{B_1}^\phi|^p \phi(x) \, dx
\]

\[
\leq C(1 - s) R^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} \chi_{\{|x-y| \leq \frac{1}{R}\}}(\phi(y) \wedge \phi(x)) \, dy \, dx
\]

for all $u \in L^p(B_1)$. 
Proof of Corollary 3. It is well-known that the following Poincaré inequality holds
\begin{equation}
\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq c \, r^p \int_{B_r} |\nabla u(x)|^p \, dx
\end{equation}
for every \( u \in W^{1,p}(B_r) \) and \( r > 0 \) where \( c > 0 \) depends on \( p \) and \( d \). Set
\[ F(u, r) = c \, r^p \int_{B_r} |\nabla u(x)|^p \, dx, \]
for \( u \in W^{1,p}(B_1) \) and \( F(u, r) = \infty \) otherwise. Then for \( u \in W^{1,p}(B_1) \)
\[ \int_{1/2}^1 F(u, t) \, \nu(dt) = c \int_{1/2}^1 t^p \int_{B_1} |\nabla u(x)|^p \chi_{B_t}(x) \, dx \, \nu(dt) \]
\[ \leq c \int_{B_1} |\nabla u(x)|^p \int_{1/2}^1 \chi_{B_t}(x) \, \nu(dt) \, dx = c \int_{B_1} |\nabla u(x)|^p \phi(x) \, dx. \]
By Theorem 1 the assertion follows with \( C = 2^{3p+d} \frac{\Phi(0)}{\Phi(1/2)^2} c. \)

Proof of Proposition 4. Let
\[ F(u, r) = C \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx. \]
Then
\[ \int_{1/2}^1 F(u, t) \, \nu(dt) = C \int_{1/2}^1 \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \chi_B(y) \chi_{B_t}(x) \, dy \, dx \, \nu(dt) \]
\[ = C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y) \int_{1/2}^1 \chi_B(y) \chi_{B_t}(x) \, \nu(dt) \, dy \, dx \]
\[ = C \int_{B_1} \int_{B_1} |u(x) - u(y)|^p k(x, y)(\phi(y) \wedge \phi(x)) \, dy \, dx. \]
The assertion now follows from Theorem 1.

Proof of Corollary 5. First we use a well-known argument to obtain a non-weighted Poincaré inequality. By calculus and convexity of the function \( x \mapsto |x|^p \) we conclude that \( |a + b|^p \geq |a|^p + |a|^p \text{sgn}(a) \). Thus
\[ \int_{B_r} \int_{B_r} |u(x) - u(y)|^p k(x, y) \, dy \, dx \geq c \int_{B_r} \int_{B_r} |(u(x) - u_{B_r}) + (u_{B_r} - u(y))|^p \, dy \, dx \]
\[ \geq c |B_r| \int_{B_r} |u(x) - u_{B_r}|^p \, dx \]
\[ \geq c |B_1/2| \int_{B_r} |u(x) - u_{B_r}|^p \, dx, \]
whenever \( u \in L^p(B_r) \) and \( \frac{1}{2} < r \leq 1 \). The assertion follows now from Proposition 4.

In the proof of Corollary 6 we use the following auxiliary result.

Lemma 7. Let \( R \geq 1, p \geq 1 \) and \( 0 < s < 1 \). Then
\begin{equation}
\int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d + ps}} \, dy \, dx \leq (3R)^{p(1-s)} \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x - y|^{d + ps}} \chi_{|x-y| \leq \frac{R}{3}} \, dy \, dx
\end{equation}
for all $u \in L^p(B_1)$.

Proof. Let $n$ be a natural number such that $n \geq 2R > n - 1$. We introduce

$$A_k = A_k(x, y) = \frac{k}{n}y + \frac{n-k}{n}x, \quad k = 0, 1, \ldots n.$$

Then

$$I = \int_{B_1} \int_{B_1} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \, dy \, dx = \int_{B_1} \int_{B_1} \frac{\sum_{k=1}^{n} (u(A_{k-1}) - u(A_k))^p}{|x-y|^{d+ps}} \, dy \, dx \leq n^{p-1} \sum_{k=1}^{n} \int_{B_1} \int_{B_1} \frac{|u(A_{k-1}) - u(A_k)|^p}{|x-y|^{d+ps}} \, dy \, dx.$$

Note that $|A_{k-1} - A_k| = \frac{1}{n}|x - y|$. If we substitute $\tilde{x} = A_{k-1}$, $\tilde{y} = A_k$, then $d\tilde{y} \, d\tilde{x} = n^{-d} \, dy \, dx$ (which follows by an elementary calculation, see also [3, p. 570]). Moreover, $\tilde{x}, \tilde{y} \in B_1$ with $|\tilde{x} - \tilde{y}| \leq \frac{2}{n} \leq \frac{1}{R}$. Hence

$$I \leq n^{p-1} \int_{B_1} \int_{B_1} \frac{|u(\tilde{x}) - u(\tilde{y})|^p}{|\tilde{x} - \tilde{y}|^{d+ps}} \chi_{\{|\tilde{x} - \tilde{y}| \leq \frac{1}{4}\}} \, d\tilde{y} \, d\tilde{x}.$$

Since $n < 2R + 1 \leq 3R$, the assertion follows. □

Proof of Corollary 6. From [4] and [1, p. 80] we know that there exists a constant $C = C(p, d, s_0)$, such that for $s_0 \leq s < 1$

$$\int_{B_r} |u(x) - u_{B_r}|^p \, dx \leq C(1-s)^{r^{ps}} \int_{B_r} \int_{B_r} \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} \, dy \, dx,$$

for all $u \in L^p(B_1)$. The assertion now follows from (14), Proposition 4 and Lemma 7. □

References


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