Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 39, 2014, 83–95

ON HOLOMORPHIC SECTIONS OF VEECH HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

Yoshihiko Shinomiya

Tokyo Institute of Technology, Department of Mathematics
2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan; shinomiya.y.aa@m.titech.ac.jp

Abstract. We give upper bounds of the numbers of holomorphic sections of Veech holomorphic families of Riemann surfaces. The numbers depend only on the topological types of the base Riemann surfaces and fibers. We also show a relation between signatures of Veech groups and moduli of cylinder decompositions of flat surfaces.

1. Introduction

Let $M(g,n)$ be the moduli space of Riemann surfaces of type (g,n) with $3g-3+n>0$. A triple (M,π,B) of a two-dimensional complex manifold M, a Riemann surface B, and a holomorphic map $\pi\colon M\to B$ is called a holomorphic family of Riemann surfaces of type (g,n) over B if each fiber $X_t=\pi^{-1}(t)$ is a Riemann surface of type (g,n) and the induced map $B\ni t\mapsto X_t\in M(g,n)$ is holomorphic. The Riemann surface B is called the base space of the holomorphic family (M,π,B) of Riemann surfaces. A holomorphic family of Riemann surfaces is called locally non-trivial if the induced map is non-constant. Manin [Man63], Grauert [Gra65] and Miwa [Miw66] independently proved that every locally non-trivial holomorphic family (M,π,B) of Riemann surfaces of finite type has only finitely many holomorphic sections if the base space B is of finite type. Coleman [Col90] found a gap in Manin's proof and fixed it. Imayoshi and Shiga [IS88] also proved the finiteness by using Tiehcmüller theory. Shiga [Shi97] gave upper bounds of the numbers of holomorphic families of Riemann surfaces of type $(g,n)=(0,n)(n\geq 4)$, $(1,2)$ and $(2,0)$. The upper bounds also give upper bounds of the numbers of holomorphic sections of holomorphic families of Riemann surfaces of such types.

In this paper, we study holomorphic sections of *Veech holomorphic families of Riemann surfaces*. Let X be a Riemann surface of type (g,n) with $3g-3+n>0$. A flat structure u is an Euclidean structure on X with finitely many singularities such that every transition function is of the form $w=\pm z+c$. The pair (X,u) of a Riemann surface X and a flat structure u on X is called a flat surface. All punctures of X and singularities of u are called critical points of the flat surface (X,u). We denote by $C(X,u)$ the set of all critical points of (X,u). We assume that the Euclidean areas of flat surfaces are finite. The affine group $\mathrm{Aff}^+(X,u)$ of a flat surface (X,u) is the group of all quasiconformal self-maps of X which preserve $C(X,u)$ and are affine with respect to the flat structure u. An element h of $\mathrm{Aff}^+(X,u)$ is called an affine map. The derivatives $A\in\mathrm{GL}(2,\mathbf{R})$ of the descriptions

doi:10.5186/aasfm.2014.3906
2010 Mathematics Subject Classification: Primary 32G15; Secondary 32G05, 32G08.
Key words: Holomorphic section, flat surfaces, Veech groups.
This work was supported by JSPS KAKENHI Grant Number 12J05650.

$w = Az + c$ of an affine map h is uniquely determined up to the sign and they are in $\mathrm{SL}(2,\mathbf{R})$. Thus, we have a homomorphism $D\colon \mathrm{Aff}^+(X,u) \to \mathrm{PSL}(2,\mathbf{R})$. We call the homomorphism the derivative map. The Veech group $\Gamma(X,u)$ of (X,u) is the image of the derivative map. Veech [Vee89] proved that $\Gamma(X,u)$ is a Fuchsian group and the mirror image $\overline{B}$ of the orbifold $\mathbf{H}/\Gamma(X,u)$ is holomorphically and locally isometrically embedded into the moduli space $M(g,n)$ equipped with the Teichmüller metric. Let B be a Riemann surface obtained from $\overline{B}$ by removing all cone points. The Veech holomorphic family of Riemann surfaces of type (g,n) over B induced by (X,u) is the holomorphic family of Riemann surfaces corresponding to the holomorphic embedding of B into the moduli space $M(g,n)$. We show in [Shi13] that every holomorphic section of Veech holomorphic families of Riemann surfaces is locally the orbit of a point $a \in (X,u)$ for Teichmüller deformations and the point satisfies $\mathrm{Aff}^+(X,u)\{a\} = \mathrm{Ker}(D)\{a\}$. In [Shi13], we also give upper bounds of the numbers of holomorphic sections of Veech holomorphic families of Riemann surfaces such that the corresponding flat surfaces have simple Jenkins–Strebel directions. The upper bounds depend only on the topological types of fibers and base spaces. However, flat surfaces do not have simple Jenkins–Strebel direction in general. In this paper, we give upper bounds of the numbers of holomorphic sections of all Veech holomorphic families of Riemann surfaces which depend only on the topological types of fibers and base spaces. We also give a relation between signatures of Veech groups and moduli of cylinder decompositions of flat surfaces by Jenkins–Strebel directions. It claims that ratios of moduli of cylinders restrict to the signatures of Veech groups.

2. Preliminaries

In this section, we define flat surfaces, Veech groups, and Veech holomorphic families of Riemann surfaces. We also study their properties and some theorems in [Shi13] which are referred to in this paper.

Let $\overline{X}$ be a compact Riemann surface of genus g and X a Riemann surface of type (g,n) with $3g-3+n>0$ which is $\overline{X}$ with n points removed.

Definition 2.1. (Flat structure and flat surface) A flat structure u on X is an atlas of X with finitely many singular points which satisfies the following conditions:

(1) the local coordinates of u are compatible with the orientation of X,
(2) the transition functions are of the form

$$w = \pm z + c$$

in $z(U \cap V)$ for $(U,z),(V,w) \in u$ with $U \cap V \neq \emptyset$,
(3) the atlas u is maximal with respect to (1) and (2).

A pair (X,u) of a Riemann surface X and a flat structure u on X is called a flat surface. All punctures of X and singular points of u are called critical points of (X,u). The set of all critical points is denoted by $C(X,u)$.

On a flat surface (X,u), we may consider Euclidean geometry. For instance, area, segments, lengths or directions of the segments are considered on (X,u). In this paper, we assume that the Euclidean area of (X,u) is finite. A θ-closed geodesic on (X,u) is a closed geodesic with direction $\theta \in [0,\pi)$ which does not contain critical points. A segment connecting critical points with direction θ is called a θ-saddle connection. If $\theta = 0$, we also call a θ-closed geodesic a horizontal closed geodesic, and a θ-saddle connection a horizontal saddle connection.

Definition 2.2. (Jenkins–Strebel direction) A direction $\theta \in [0, \pi)$ is called a Jenkins–Strebel direction if almost all points in (X, u) are contained in θ-closed geodesics.

Let $\theta \in [0, \pi)$ be a Jenkins–Strebel direction. If $z \in (X, u)$ is not contained in θ-closed geodesics, then z is on a θ-saddle connection. Let us remove all θ-saddle connections from X. Then X has finitely many connected components. Every connected component R is a cylinder foliated by θ-closed geodesics and the boundary of R consists of saddle connections. The core curves of the cylinders are not homotopic to each other and not homotopic to a point or a puncture. See [Str84].

Definition 2.3. If a Jenkins–Strebel direction θ decomposes X into m cylinders $R_1, \cdots, R_m$, then the direction θ is called a m-Jenkins–Strebel direction. In particular, if $m = 1$, then we call the direction θ a simple Jenkins–Strebel direction. The decomposition $\{R_i\}$ is called a cylinder decomposition of (X, u) by a Jenkins–Strebel direction θ.

Remark. Since the core curves of R_i's are not homotopic to each other, m is not greater than $3g - 3 + n$.

Let $u = \{(U, z)\}$ be a flat structure on X. For every $A \in \mathrm{SL}(2, \mathbf{R})$, a flat surface $A \cdot (X, u) = (X, u_A)$ is defined by $u_A = \{(U, A \circ z)\}$. The set $C(X, u_A)$ of critical points of (X, u_A) coincides with $C(X, u)$ and u_A gives a new complex structure of X. The $\mathrm{SL}(2, \mathbf{R})$-orbit of the flat surface (X, u) in the Teichmüller space $T(X)$ is defined by $\Delta = \{[A \cdot (X, u), \mathrm{id}] \colon A \in \mathrm{SL}(2, \mathbf{R})\}$. It is easy to show that $[A \cdot (X, u), \mathrm{id}] = [UA \cdot (X, u), \mathrm{id}]$ in $T(X)$ for all $A \in \mathrm{SL}(2, \mathbf{R})$ and $U \in \mathrm{SO}(2)$. Thus, the bijection $\phi \colon \mathrm{SO}(2) \setminus \mathrm{SL}(2, \mathbf{R}) \to \mathbf{H}$ defined by $\phi(\mathrm{SO}(2) \cdot A) = -\overline{A^{-1}(i)}$ induces a map $\iota \colon \mathbf{H} \to T(X)$. Here, $A^{-1}(\cdot)$ acts on $\mathbf{H}$ as a Möbius transformation.

Proposition 2.4. [EG97, HS07] *The map $\iota \colon \mathbf{H} \to T(X)$ is a holomorphic and isometric embedding of the hyperbolic plane $\mathbf{H}$ into the Teichmüller space $T(X)$ equipped with the Teichmüller metric. Every holomorphic and isometric embedding from $\mathbf{H}$ into $T(X)$ is constructed from a flat surface as above.*

Let $\iota \colon \mathbf{H} \to T(X)$ be a holomorphic and isometric embedding constructed from a flat surface (X, u) of type (g, n). The image $\Delta = \iota(\mathbf{H})$ is called a Teichmüller disk. We consider the image of the Teichmüller disk Δ in the moduli space $M(g, n)$. Since $M(g, n) = T(X)/\operatorname{Mod}(X)$, the image of the Teichmüller disk is described as $\Delta/\mathrm{Stab}(\Delta)$. Here, $\operatorname{Mod}(X)$ is the mapping class group of X and $\mathrm{Stab}(\Delta)$ is the subgroup of $\operatorname{Mod}(X)$ consisting of all mapping classes which preserve Δ.

Definition 2.5. (Affine groups) A quasiconformal self-map h of X is called an affine map of (X, u) if h preserves $C(X, u)$ and, for (U, z) and $(V, w) \in u$ with $h(U) \subset V$, the composition $w \circ h \circ z^{-1}$ is of the from $w = Az + c$ for some $A \in \mathrm{GL}(2, \mathbf{R})$ and $c \in \mathbf{C}$. The group of all affine maps of (X, u) is denoted by $\mathrm{Aff}^+(X, u)$ and we call it the affine group of (X, u).

By the definition of flat structures, the derivative A of the affine map $w \circ h \circ z^{-1}$ is uniquely determined up to the sign. Moreover, the assumption that the area of

(X,u) is finite implies that A is in $\mathrm{SL}(2,\mathbf{R})$. Therefore, we obtain a homomorphism $D\colon \mathrm{Aff}^+(X,u) \to \mathrm{PSL}(2,\mathbf{R})$. The homomorphism is called the derivative map.

Definition 2.6. (Veech groups) We call the image $\Gamma(X,u) = D(\mathrm{Aff}^+(X,u))$ of the derivative map D the Veech group of (X,u).

Proposition 2.7. [Vee89] *Affine maps of (X,u) are not homotopic to each other. Hence, the group $\mathrm{Aff}^+(X,u)$ is considered as a subgroup of $\mathrm{Mod}(X)$.*

Veech proved that Veech groups give the images of Teichmüller disks into the moduli spaces as follows.

Theorem 2.8. [Vee89, EG97, HS07] *The affine group $\mathrm{Aff}^+(X,u)$ coincides with $\mathrm{Stab}(\Delta)$. For $t \in \mathbf{H}$ and $h \in \mathrm{Aff}^+(X,u)$, we have $h_*(\iota(t)) = \iota(RAR^{-1}(t))$. Here, $A = D(h)$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and RAR^{-1} is a Möbius transformation which acts on $\mathbf{H}$.*

Corollary 2.9. [Vee89, EG97, HS07] *The Veech group $\Gamma(X,u)$ is a Fuchsian group. Let $\overline{\Gamma}(X,u) = R\Gamma(X,u)R^{-1}$. The orbifold $\mathbf{H}/\overline{\Gamma}(X,u)$ is holomorphically and locally isometrically embedded into the moduli space $M(g,n)$. The embedded orbifold is the image of the Teichmüller disk Δ into the moduli space $M(g,n)$.*

Veech holomorphic families of Riemann surfaces are given by such holomorphic and locally isometric embeddings $\Phi\colon \mathbf{H}/\overline{\Gamma}(X,u) \to M(g,n)$. Let $\mathbf{H}^*$ be $\mathbf{H}$ with elliptic fixed points of $\overline{\Gamma}(X,u)$ removed. Since every point $t \in B = \mathbf{H}^*/\overline{\Gamma}(X,u)$ corresponds to a Riemann surface $X_t = \Phi(t)$, we may construct a two-dimensional complex manifold M by

$$M = \{(t,z)\colon t \in B, z \in X_t = \Phi(t)\}.$$

Let $\pi\colon M \to B$ be the projection $\pi(t,z) = t$. Then the triple (M,π,B) is a holomorphic family of Riemann surfaces. See [Shi13], for more details of the construction of the holomorphic families of Riemann surfaces.

Definition 2.10. (Veech holomorphic families of Riemann surfaces) A holomorphic family of Riemann surfaces constructed as above is called a Veech holomorphic family of Riemann surfaces of type (g,n) over B.

Through this paper, we assume that the Veech groups $\Gamma(X,u)$ of flat surfaces (X,u) are co-finite Fuchsian groups. If the Veech group $\Gamma(X,u)$ of a flat surface (X,u) is of signature $(p,k;\nu_1,\cdots,\nu_k)$ ($\nu_i \in \{2,3,\cdots,\infty\}$), the base space B of the corresponding Veech holomorphic family of Riemann surfaces is of type (p,k). In this case, the map $\Phi\colon \mathbf{H}/\overline{\Gamma}(X,u) \to M(g,n)$ induced by the flat surface (X,u) is a Teichmüller curve.

In [Shi13], we characterize holomorphic sections of Veech holomorphic families of Riemann surfaces. Let (X,u) be a flat surface of type (g,n) and (M,π,B) the Veech holomorphic family of Riemann surfaces defined by (X,u). A holomorphic section of (M,π,B) is a holomorphic map $s\colon B \to M$ such that $\pi \circ s = \mathrm{id}_B$. Let $\widetilde{\rho}\colon \mathbf{H} \to \mathbf{H}/\overline{\Gamma}(X,u)$ be the universal covering map. For each $\widetilde{t} \in \mathbf{H}$, let $f_{\widetilde{t}}\colon X \to f_{\widetilde{t}}(X)$ be the Teichmüller map whose Beltrami coefficient $\mu_{\widetilde{t}}$ satisfies $\mu_{\widetilde{t}} = \frac{i-\widetilde{t}}{i+\widetilde{t}}\frac{d\bar{z}}{dz}$ for all $(U,z) \in u$. Then the Riemann surface $f_{\widetilde{t}}(X)$ coincides with $X_{\rho(\widetilde{t})} = \pi^{-1}(\rho\,(\widetilde{t}))$.

Theorem 2.11. [Shi13] *Let $s\colon B \to M$ be a holomorphic section of the Veech holomorphic family (M,π,B) of Riemann surfaces induced by (X,u). There exists*

$a \in (X,u)$ *such that* $s \circ \widetilde{\rho}(\widetilde{t}) = (\widetilde{\rho}(\widetilde{t}), f_{\widetilde{t}}(a))$ *for all* $\widetilde{t} \in \mathbf{H}^*$. *Moreover, the point* $a \in (X,u)$ *satisfies* $\mathrm{Aff}^+(X,u)\{a\} = \mathrm{Ker}(D)\{a\}$.

Let Γ be a finite index subgroup of $\Gamma(X,u)$ and $\overline{\Gamma} = R\Gamma R^{-1}$. Let $\rho\colon \mathbf{H}/\overline{\Gamma} \to \mathbf{H}/\overline{\Gamma}(X,u)$ be the covering map. The holomorphic and locally isometric map $\Phi \circ \rho\colon \mathbf{H}/\overline{\Gamma} \to M(g,n)$ constructs a holomorphic family of Riemann surfaces. Let $\mathbf{H}^*_{\overline{\Gamma}}$ be the upper half-plane $\mathbf{H}$ with elliptic fixed points of $\overline{\Gamma}$ removed. We set $B' = \mathbf{H}^*_{\overline{\Gamma}}/\overline{\Gamma}$, $M' = \{(t,z)\colon t \in B', z \in X_t = \Phi \circ \rho(t)\}$, and $\pi'\colon M' \to B'$ to be a projection $\pi'(t,z) = t$. Then the triple (M', π', B') is a holomorphic family of Riemann surfaces of type (g,n) over B'.

Corollary 2.12. [Shi13] *A holomorphic section* $s'\colon B' \to M'$ *of the holomorphic family* (M', π', B') *of Riemann surfaces as above corresponds to a point* $a \in X$ *satisfying* $D^{-1}(\Gamma)\{a\} = \mathrm{Ker}(D)\{a\}$.

In [Shi13], we estimate the number of points $a \in X$ satisfying $\mathrm{Aff}^+(X,u)\{a\} = \mathrm{Ker}(D)\{a\}$ in case that (X,u) has a simple Jenkins–Strebel direction. The estimation gives upper bounds of the numbers of holomorphic sections of Veech holomorphic families of Riemann surfaces induced by such flat surfaces.

Theorem 2.13. [Shi13] *Let* (X,u) *be a flat surface of type* (g,n) *with a simple Jenkins–Strebel direction. Let* (M,π,B) *be the Veech holomorphic family of Riemann surfaces induced by* (X,u). *Suppose that the base space* B *is of type* (p,k). *Then the number of holomorphic sections of* (M,π,B) *is at most*

$$32\pi(2p-2+k)(3g-3+n)^2(3g-2+n) - 2g + 2.$$

In Section 3, we give upper bounds of the numbers of holomorphic sections of all Veech holomorphic families of Riemann surfaces by extending the proof of Theorem 2.13 in [Shi13]. We also apply the following theorem from [Shi13].

Theorem 2.14. [Shi13] *Let* $\Gamma < \mathrm{PSL}(2,\mathbf{R})$ *be a Fuchsian group of type* $(p,k\colon \nu_1, \cdots, \nu_k)$ $(\nu_i \in \{2,3,\cdots,\infty\})$. *Let* k_0 *be the number of* ν_i*'s which are equal to* ∞. *Assume that* Γ *contains* $\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$ *and it is primitive. Then there exists* $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \Gamma$ *such that*

$$1 \le |c| < \mathrm{Area}(\mathbf{H}/\Gamma) - k_0 + 1.$$

Here, $\mathrm{Area}\,(\mathbf{H}/\Gamma(X,u))$ *is the hyperbolic area of the orbifold* $\mathbf{H}/\Gamma$.

The first inequality is the consequence of the Shimizu lemma (c.f. [IT92]).

Lemma 2.15. (The Shimizu lemma) *Let* $\Gamma < \mathrm{PSL}(2,\mathbf{R})$ *be a Fuchsian group. Assume that* $\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right] \in \Gamma$ *and it is primitive. Then* $c = 0$ *or* $|c| \ge 1$ *for all* $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \Gamma$.

Remark. It is known that

$$\mathrm{Area}\,(\mathbf{H}/\Gamma(X,u)) = 2\pi\left(2p-2+\sum_{i=1}^{k}(1-\frac{1}{\nu_i})\right)$$

for a Fuchsian group Γ of signature $(p,k\colon \nu_1,\cdots,\nu_k)$. See [FK92]. The Shimizu lemma also means that the horodisks centered at punctures of $\mathbf{H}/\Gamma$ whose areas are 1 do not intersect each other.

Proof of Theorem 2.14. Suppose that $|c| \geq \mathrm{Area}(\mathbf{H}/\Gamma) - k_0 + 1$ for all $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \Gamma$ with $c \neq 0$. Let $p_1, \cdots, p_{k_0}$ be punctures of $\mathbf{H}/\Gamma$. We assume that p_1 corresponds to $\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$. Let U_i be the horodisks centered at p_i ($i \in \{2, \cdots, k_0\}$) whose areas are 1. By the Shimizu lemma, $U_i \cap U_j = \emptyset$ ($i \neq j$). Let D be the Ford region of Γ and $\tilde{p}_1 = \infty, \tilde{p}_2, \cdots, \tilde{p}_{k_0}$ the vertices of D corresponding to $p_1, \cdots, p_{k_0}$, respectively. The two edges of D which intersect at $\tilde{p}_i$ are contained in isometric circles, say C_i and C_i', for each $i \in \{2, \cdots, k_0\}$. The radii of C_i and C_i' are equal. Let D_i be the subregion of D which is bounded by C_i, C_i' and the horizontal Euclidean segment connecting the tops of C_i and C_i' for all $i \in \{2, \cdots, k_0\}$. Let $\tilde{U}_i$ be the preimage of U_i in D. Since $\tilde{U}_i$ has area 1 and the region D_i has area $\pi - 2$, the region $\tilde{U}_i$ is contained in D_i for all $i \in \{2, \cdots, k_0\}$. By the assumption, the radii of all isometric circles of Γ are less than or equal to $(\mathrm{Area}(\mathbf{H}/\Gamma) - k_0 + 1)^{-1}$. This implies that p_1 has a horodisk U_1 whose area is at least $\mathrm{Area}(\mathbf{H}/\Gamma) - k_0 + 1$. Moreover, $U_1 \cap U_i = \emptyset$ for all $i \in \{2, \cdots, k_0\}$ and $\mathbf{H}/\Gamma - \bigcup_{i=1}^{k_0} U_i$ has a positive area. Therefore,

$$\mathrm{Area}(\mathbf{H}/\Gamma) > \sum_{i=1}^{k_0} \mathrm{Area}(U_i) \geq \mathrm{Area}(\mathbf{H}/\Gamma).$$

This is a contradiction. $\square$

Finally, we see another property of Veech groups.

Theorem 2.16. (The Veech dichotomy theorem) [Vee89] *Let (X, u) be a flat surface. Suppose that the Veech group $\Gamma(X, u)$ of (X, u) is a co-finite Fuchsian group. Then every direction $\theta \in [0, \pi)$ satisfies one of the following properties:*

- *The direction θ is a Jenkins–Strebel direction. Let $\{R_i\}_{i=1}^m$ be the cylinder decomposition of (X, u) by the direction θ. Then the ratio $\mathrm{mod}(R_i)/\mathrm{mod}(R_j)$ is a rational number for all $i, j \in \{1, \cdots, k\}$.*
- *Every θ-direction geodesic is dense in X and uniquely ergodic. That is, the θ-direction geodesic flow has only one transverse measure μ up to scalar multiples such that the flow is ergodic with respect to μ.*

Here, the modulus $\mathrm{mod}(R_i)$ of the cylinder R_i is the ratio of the circumference to the height.

For details of ergodicity, see [KH95, Nik01].

3. Main theorems

In this section, we prove the following two theorems. The first theorem gives upper bounds of numbers of holomorphic sections of all Veech holomorphic families of Riemann surfaces. The second theorem gives a relation between signatures of Veech groups of flat surfaces and moduli of cylinders of cylinder decompositions of the flat surfaces.

Theorem 3.1. *Let (M, π, B) be a Veech holomorphic family of Riemann surfaces of type (g, n) over B. Suppose that the base space B is a Riemann surface of type (p, k). Then the number of holomorphic sections of (M, π, B) is at most*

$$32\pi(2p - 2 + k) \dim M(g, n)^2 \left\{ 2 \dim M(g, n) + 3 \exp\left(\frac{5}{e} \dim M(g, n)\right) \right\}.$$

Theorem 3.2. *Let (X,u) be a flat surface of type (g,n). Suppose that the Veech group $\Gamma(X,u)$ is of signature $(p,k\colon \nu_1,\cdots,\nu_k)$ ($\nu_i\in\{2,3,\cdots,\infty\}$). Let $\{R_i\}_{i=1}^m$ be the cylinder decomposition of (X,u) by a Jenkins–Strebel direction. Then,*

$$\left(\frac{\mathrm{mod}(R_i)}{\mathrm{mod}(R_j)}\right)^{\frac{1}{2}} < 2\exp\left(\frac{5}{e}\dim M(g,n)\right)\mathrm{Area}(\mathbf{H}/\Gamma(X,u))$$

for all $i,j\in\{1,\cdots,m\}$.

Using Theorem 2.11, a proof of Theorem 3.1 is given by estimating the cardinality of the set

$$S(X,u)=\left\{z\in X\colon\ \mathrm{Aff}^+(X,u)\{z\}=\mathrm{Ker}(D)\{z\}\right\}.$$

Let $\varphi\colon X\to Y=X/\mathrm{Ker}(D)$ be the quotient map. Then Y has a flat structure u' induced by the flat structure u on X. Assume that Y is of type (g',n'). Since $\Gamma(X,u)=\mathrm{Aff}^+(X,u)/\mathrm{Ker}(D)$, we may consider $\Gamma(X,u)$ as a subgroup of the affine group $\mathrm{Aff}^+(Y,u')$. Then, $\varphi(S(X,u))=\{w\in Y\colon \Gamma(X,u)\{w\}=\{w\}\}$. Theorem 2.16 and the assumption that $\Gamma(X,u)$ is a co-finite Fuchsian group imply that the set of all Jenkins–Strebel directions of (X,u) is dense in $[0,\pi)$. We may assume that the direction $\theta=0$ is a m-Jenkins–Strebel direction of (X,u) for some $1\le m\le 3g-3+n$. Then $\theta=0$ is also a m'-Jenkins–Strebel direction of (Y,u') for some $m'\le m$. Let $\{R_i\}_{i=1}^m$ and $\{R'_j\}_{j=1}^{m'}$ be the cylinder decompositions of (X,u) and (Y,u') by the direction $\theta=0$, respectively. We define a map $\sigma\colon\{1,\cdots,m\}\to\{1,\cdots,m'\}$ by $\sigma(i)=j$ if $\varphi(R_i)=R'_j$. By Theorem 2.16, the Veech group $\Gamma(X,u)$ contains elements of the form $\left[\begin{pmatrix}1&b\\0&1\end{pmatrix}\right]$ ($b\ne 0$). As $\Gamma(X,u)$ is a co-finite Fuchsian group, $\Gamma(X,u)$ also contains elements of the form $\left[\begin{pmatrix}*&*\\c&*\end{pmatrix}\right]$ ($c\ne0$). We set

$$b_0=\inf\left\{|b|\colon\ \left[\begin{pmatrix}1&b\\0&1\end{pmatrix}\right]\in\Gamma(X,u),b\ne0\right\}$$

and

$$c_1=\inf\left\{|c|\colon\ \left[\begin{pmatrix}a&b\\c&d\end{pmatrix}\right]\in\Gamma(X,u),c\ne0\right\}.$$

Let $A=\left[\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}\right]$ and $B=\left[\begin{pmatrix}1&b_0\\0&1\end{pmatrix}\right]$ be elements of $\Gamma(X,u)$ which attain the numbers c_1 and b_0, respectively. Denote by h'_A and h'_B the elements of $\mathrm{Aff}^+(Y,u')$ corresponding to A and B, respectively. Let $S_j=\{l_j^1,\cdots,l_j^{n_j}\}$ be the set of all horizontal closed geodesics in R'_j containing a fixed point of h'_B ($j=1,\cdots,m'$). Let L_j^1, L_j^2 be the boundary components of R'_j for each $j\in\{1,\cdots,m'\}$. Then we consider the set

$$\mathrm{Cross}(A)=\left(\bigcup_{j=1}^{m'}\bigcup_{l_j^k\in S_j}\left(l_j^k\cap h'_A\left(l_j^k\right)\right)\right)\bigcup\left(\bigcup_{j=1}^{m'}\bigcup_{r=1,2}\left(L_j^r\cap h'_A(L_j^r)\right)\right).$$

Lemma 3.3. *The set* $\mathrm{Cross}(A)$ *contains* $\varphi(S(X,u))$.

Proof. Since $\varphi\left(S(X,u)\right)$ is the set of all fixed points of $\Gamma(X,u)$ on (Y,u'), the set $\varphi\left(S(X,u)\right)$ is contained in $\mathrm{Cross}(A)$. □

Let H_i,H'_j be heights of the cylinders R_i,R'_j and W_i,W'_j circumferences of the cylinders R_i,R'_j, respectively. Let K_i be the subgroup of $\mathrm{Ker}(D)$ consisting of all maps preserving R_i. Then $R'_{\sigma(i)}$ coincides with the quotient R_i/K_i and the area of

$R'_{\sigma(i)}$ is $H_iW_i/\sharp K_i$. If K_i contains a map permuting two boundary components of R_i, then we have $H'_{\sigma(i)} = H_i/2$ and $W'_{\sigma(i)} = 2W_i/\sharp K_i$. Otherwise, we have $H'_{\sigma(i)} = H_i$ and $W'_{\sigma(i)} = W_i/\sharp K_i$. As a result, we have the following inequalities:

$$W_i/\sharp\,\mathrm{Ker}(D) \le W'_{\sigma(i)} \le W_i,$$

and

$$H_i/2 \le H'_{\sigma(i)} \le H_i.$$

Recall that the moduli of R_i and R'_j are defined by $\mathrm{mod}(R_i) = W_i/H_i$ and $\mathrm{mod}(R'_j) = W'_j/H'_j$. Thus, we have

$$\text{(1)} \qquad \mathrm{mod}(R_i)/\sharp\,\mathrm{Ker}(D) \le \mathrm{mod}(R'_{\sigma(i)}) \le 2\,\mathrm{mod}(R_i).$$

Lemma 3.4. *If $j = \sigma(i)$, then*

$$\sharp S_j \le \left\lceil \frac{b_0 \sharp\,\mathrm{Ker}(D)}{\mathrm{mod}(R_i)} \right\rceil .$$

Here, $\lceil x \rceil$ is the smallest integer which is greater than or equal to x.

Proof. Suppose that $j = \sigma(i)$. If h'_B does not fix R'_j, then $\sharp S_j = 0$. If h'_B fixes R'_j and permutes two boundary components of R'_j, then $\sharp S_j = 1$. Otherwise, by identifying R'_j with $[0, W'_j) \times (0, H'_j)$, the affine map h'_B can be represented as

$$h'_B \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} W'_j\xi \\ 0 \end{pmatrix}$$

for some $0 \le \xi < 1$. Thus, $(u, v) \in [0, W'_j) \times (0, H'_j)$ corresponds to a fixed point of h'_B if and only if $v = (k - \xi)W'_j/b_0$ for some $k \in \mathbf{Z}$. Since $v \in (0, H'_j)$, such integers k satisfy $\xi < k < \frac{b_0}{\mathrm{mod}(R'_j)} + \xi$. Hence, by the inequality (1), we have $\sharp S_j \le \left\lceil \frac{b_0}{\mathrm{mod}(R'_j)} + \xi \right\rceil - 1 \le \left\lceil \frac{b_0 \sharp\,\mathrm{Ker}(D)}{\mathrm{mod}(R_i)} \right\rceil$. □

Lemma 3.5. *For every $j \in \{1, \cdots, m'\}$ and a horizontal closed geodesic $l_j^k \in S_j$, the inequality*

$$\sharp \left(l_j^k \cap h'_A(l_j^k) \right) \le 2\,\mathrm{mod}(R_i)c_1$$

holds if $j = \sigma(i)$.

Proof. Assume that $j = \sigma(i)$ and $l_j^k \in S_j$. Let us identify the closed geodesic l_j^k with the vector $\begin{pmatrix} W'_j \\ 0 \end{pmatrix}$. Then the closed geodesic $h'_A(l_j^k)$ is identified with the vector $A\begin{pmatrix} W'_j \\ 0 \end{pmatrix} = \begin{pmatrix} W'_j a_1 \\ W'_j c_1 \end{pmatrix}$. Since the closed geodesic $h'_A(l_j^k)$ passes through the cylinder R'_j exactly $\sharp(l_j^k \cap h'_A(l_j^k))$ times, we have $\sharp(l_j^k \cap h'_A(l_j^k))H'_j \le W'_j c_1$. Thus, by the inequality (1), we have

$$\sharp \left(l_j^k \cap h'_A(l_j^k) \right) \le W'_j c_1/H'_j \le 2\,\mathrm{mod}(R_i)c_1. \qquad \square$$

Lemma 3.6. *We have*

$$\sharp \bigcup_{j=1}^{m'} \bigcup_{r=1,2} \left(L_j^r \cap h'_A(L_j^r) \right) \le \sum_{i=1}^{m} 4\,\mathrm{mod}(R_i)c_1.$$

Proof. Assume that $\sigma(i) = j$. By the same argument as in the proof of Lemma 3.5, we have $\sharp\left(L_j^r \cap h'_A(L_j^r)\right) \leq 2\operatorname{mod}(R_i)c_1$. Therefore, the inequality

$$\sharp \bigcup_{j=1}^{m'} \bigcup_{r=1,2} \left(L_j^r \cap h'_A(L_j^r)\right) \leq \sum_{j=1}^{m'} \sum_{r=1,2} \sharp\left(L_j^r \cap h'_A(L_j^r)\right) \leq \sum_{i=1}^{m} 4\operatorname{mod}(R_i)c_1$$

holds. □

Proposition 3.7. *We have*

$$\sharp S(X,u) \leq 2b_0c_1 \sharp \operatorname{Ker}(D) \sum_{i=1}^{m} \left(\sharp \operatorname{Ker}(D) + \frac{3\operatorname{mod}(R_i)}{b_0}\right).$$

Proof. By Lemma 3.4, Lemma 3.5 and Lemma 3.6, we have

$$\begin{aligned}
\sharp S(X,u) &\leq \sharp\varphi^{-1}(\operatorname{Cross}(A)) \leq \sharp \operatorname{Ker}(D) \cdot \sharp \operatorname{Cross}(A) \\
&\leq \sharp \operatorname{Ker}(D) \sum_{i=1}^{m} 2\operatorname{mod}(R_i)c_1 \left(\left\lceil \frac{b_0 \sharp \operatorname{Ker}(D)}{\operatorname{mod}(R_i)} \right\rceil + 2\right) \\
&\leq \sharp \operatorname{Ker}(D) \sum_{i=1}^{m} 2\operatorname{mod}(R_i)c_1 \left(\frac{b_0 \sharp \operatorname{Ker}(D)}{\operatorname{mod}(R_i)} + 3\right) \\
&= 2b_0c_1 \sharp \operatorname{Ker}(D) \sum_{i=1}^{m} \left(\sharp \operatorname{Ker}(D) + \frac{3\operatorname{mod}(R_i)}{b_0}\right).
\end{aligned}$$ □

To prove Theorem 3.1 and 3.2, we estimate b_0c_1, $\sharp \operatorname{Ker}(D)$, $\frac{\operatorname{mod}(R_i)}{b_0}$, and $\frac{b_0}{\operatorname{mod}(R_i)}$.

Lemma 3.8. *We have*

$$b_0c_1 \leq \operatorname{Area}\left(\mathbf{H}/\Gamma(X,u)\right).$$

Proof. We put $P = \left[\begin{pmatrix} 1/\sqrt{b_0} & 0 \\ 0 & \sqrt{b_0} \end{pmatrix}\right]$. Then $PAP^{-1} = \left[\begin{pmatrix} a_1 & b_1/b_0 \\ b_0c_1 & d_1 \end{pmatrix}\right]$ and $PBP^{-1} = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]$. By the definition of the numbers c_1 and b_0, PBP^{-1} is primitive in $P\Gamma(X,u)P^{-1}$ and

$$b_0c_1 = \inf\left\{|c| \colon \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in P\Gamma(X,u)P^{-1}, c \neq 0\right\}.$$

Therefore, applying Theorem 2.14 to the Fuchsian group $P\Gamma(X,u)P^{-1}$, we have

$$b_0c_1 \leq \operatorname{Area}\left(\mathbf{H}/\Gamma(X,u)\right).$$ □

For all $i \in \{1, \cdots, m\}$, let s_i^1, s_i^2 be the numbers of horizontal saddle connections on two boundary components of R_i.

Lemma 3.9. *We have*

$$\sum_{i=1}^{m} \left(s_i^1 + s_i^2\right) \leq 4(3g - 3 + n).$$

Proof. For each cylinder R_i, we take a segment in R_i connecting critical points in two boundary components of R_i. Then these m segments and horizontal saddle connections decompose X into m polygons. The set of vertices of the polygons coincides with $C(X,u)$ and the number of horizontal saddle connections is $\sum_{i=1}^{m}\left(s_i^1+s_i^2\right)/2$. Hence, by the Euler characteristic, we have the equation

$$2-2g=\sharp C(X,u)-\left(m+\frac{1}{2}\sum_{i=1}^{m}\left(s_i^1+s_i^2\right)\right)+m=\sharp C(X,u)-\frac{1}{2}\sum_{i=1}^{m}\left(s_i^1+s_i^2\right).$$

Therefore, the equation

$$\sum_{i=1}^{m}\left(s_i^1+s_i^2\right)=2(2g-2+\sharp C(X,u))$$

holds. Let q be the holomorphic quadratic differential on (X,u) such that $q=dz^2$ for $(U,z)\in u$. All points of $X\setminus C(X,u)$ are not zeros of q. Every point of $C(X,u)\cap X$ is a zero of q and the orders of the punctures of X with respect to q are greater than or equal to -1. By the Riemann–Roch theorem, the sum of orders of all zeros of q is $4g-4$. Hence, $\sharp C(X,u)\leq 4g-4+2n$ and we obtain the claim. □

Lemma 3.10. *We have*

$$\sharp\operatorname{Ker}(D)\leq 4(3g-3+n).$$

Proof. Each element of $\operatorname{Ker}(D)$ is uniquely determined by the image of a saddle connection and its orientation if it exists. Therefore, we obtain $\sharp\operatorname{Ker}(D)\leq\sum_{i=1}^{m}\left(s_i^1+s_i^2\right)\leq 4(3g-3+n)$. □

Lemma 3.11. *For all $i\in\{1,\cdots,m\}$,*

$$\frac{1}{2}\exp\left(-\frac{5}{e}(3g-3+n)\right)<\frac{b_0}{\operatorname{mod}(R_i)}<2\exp\left(\frac{5}{e}(3g-3+n)\right)\operatorname{Area}\left(\mathbf{H}/\Gamma(X,u)\right)^2$$

holds.

Remark. Theorem 3.2 is immediately proved from Lemma 3.11.

To prove Lemma 3.11, we consider the Landau function $G(n)$. The Landau function $G(n)$ is the greatest order of an element of the symmetric group S_n of degree n.

Remark. Landau [Lan03] showed that

$$\lim_{n\to\infty}\frac{\log(G(n))}{\sqrt{n\log(n)}}=1$$

and Massias [Mas84] showed that

$$\log(G(n))\leq 1.05313\ldots\sqrt{n\log(n)}$$

with equality at $n=1319766$. In this paper, we apply the following inequality which is easily proved

$$G(n)\leq\exp\left(\frac{n}{e}\right). \tag{2}$$

This inequality is better than Massias's if $n\leq 27$.

Proof of Lemma 3.11. Take $h_A, h_B \in \mathrm{Aff}^+(X,u)$ with $D(h_A) = A, D(h_B) = B$. There exists $m_0 \leq G(m)$ such that $h_B^{m_0}(R_i) = R_i$ for all $i \in \{1, \cdots, m\}$. Then $h_B^{2m_0}$ preserves each boundary component of R_i. For each i, we choose $\alpha_i^r \leq s_i^r$ $(r = 1, 2)$ such that $h_B^{2m_0\alpha_i^1\alpha_i^2}$ fixes each boundary component of R_i pointwise. Set

$$\alpha = 2m_0 \prod_{i=1}^{m} \alpha_i^1 \alpha_i^2.$$

By Lemma 3.9 and the inequality (2), we have

$$\alpha \leq 2G(m) \prod_{i=1}^{m} s_i^1 s_i^2 \leq 2\exp\left(\frac{m}{e}\right)\left\{\sum_{i=1}^{m}\left(s_i^1 + s_i^2\right)/2m\right\}^{2m} \tag{3}$$
$$\leq 2\exp\left(\frac{1}{e}(3g-3+n)\right)\left\{\frac{2(3g-3+n)}{m}\right\}^{2m} < 2\exp\left(\frac{5}{e}(3g-3+n)\right).$$

Let C_i be a horizontal closed geodesic in the cylinder R_i for each $i \in \{1, \cdots, m\}$. Then h_B^α is a composition of right hand Dehn twists along C_i $(i = 1, \cdots, m)$. Hence, for every $i \in \{1, \cdots, m\}$, there exists $n_i \geq 1$ such that $\alpha b_0 = n_i \operatorname{mod}(R_i)$. Thus

$$\frac{\operatorname{mod}(R_i)}{b_0} = \frac{\alpha}{n_i} < 2\exp\left(\frac{5}{e}(3g-3+n)\right).$$

Next, let us consider the affine map $h = h_A^{-1} \circ h_B^\alpha \circ h_A$. The derivative of h is

$$D(h) = A^{-1}B^\alpha A = \left[\begin{pmatrix} 1+\alpha b_0 c_1 d_1 & \alpha b_0 d_1^2 \\ -\alpha b_0 c_1^2 & 1-\alpha b_0 c_1 d_1 \end{pmatrix}\right].$$

Identifying the closed geodesic C_i with the vector $\begin{pmatrix} W_i \\ 0 \end{pmatrix}$, the closed geodesic $h(C_i)$ is identified with the vector $D(h)\begin{pmatrix} W_i \\ 0 \end{pmatrix} = \begin{pmatrix} W_i(1+\alpha b_0 c_1 d_1) \\ -W_i \alpha b_0 c_1^2 \end{pmatrix}$. Since the closed geodesic $h(C_i)$ intersects each cylinder R_j exactly $\sharp\left(h(C_i) \cap C_j\right)$ times, we have

$$W_i \alpha b_0 c_1^2 = \sum_{j=1}^{m} \sharp\left(h(C_i) \cap C_j\right) H_j$$

for all $i \in \{1, \cdots, m\}$. Then we have

$$\operatorname{mod}(R_i)\alpha b_0 c_1^2 = \frac{1}{H_i}\sum_{j=1}^{m} \sharp\left(h(C_i) \cap C_j\right) H_j \geq \sharp\left(h(C_i) \cap C_i\right)$$
$$= \sharp\Big(h_B^\alpha\big(h_A(C_i)\big) \cap h_A(C_i)\Big) \geq 1$$

since $h_A(C_i)$ intersects at least one of the C_j's and h_B^α fixes all boundary components of R_j's. Therefore, by Lemma 3.8 and the inequality(3), we have

$$\frac{b_0}{\operatorname{mod}(R_i)} = \frac{(b_0 c_1)^2}{\operatorname{mod}(R_i) b_0 c_1^2} < 2\exp\left(\frac{5}{e}(3g-3+n)\right)\operatorname{Area}\left(\mathbf{H}/\Gamma(X,u)\right)^2. \qquad \square$$

Proof of Theorem 3.1. Since m is the number of cylinders of a cylinder decomposition of (X,u), we have $m \leq 3g-3+n$. By Proposition 3.7, Lemma 3.8, Lemma 3.10, and Lemma 3.11, we have

$$\begin{aligned}\sharp S(X,u) &\le 2b_0c_1\sharp\operatorname{Ker}(D)\sum_{i=1}^{m}\left(\sharp\operatorname{Ker}(D)+\frac{3\operatorname{mod}(R_i)}{b_0}\right)\\ &< 32\pi(2p-2+k)(3g-3+n)^2\left\{2(3g-3+n)+3\exp\left(\frac{5}{e}(3g-3+n)\right)\right\}. \quad \square\end{aligned}$$

Let Γ be a finite index subgroup of $\Gamma(X,u)$ which is of signature $(p',k':\nu'_1,\cdots,\nu'_{k'})$ $(\nu'_i\in\{2,3,\cdots,\infty\})$. Let (M',π',B') be a holomorphic family of Riemann surfaces corresponding to Γ as in Section 2. By the same argument as above, we obtain the following corollary.

Corollary 3.12. *The number of holomorphic sections of the holomorphic family (M',π',B') of Riemann surfaces of type (g,n) over B' is at most*

$$32\pi(2p'-2+k')\dim M(g,n)^2\left\{2\dim M(g,n)+3\exp\left(\frac{5}{e}\dim M(g,n)\right)\right\}.$$

Acknowledgements. The author would like to thank Hiroshige Shiga for introducing him to this research area. The author is also deeply grateful to the referees for their valuable suggestions and comments.

References

[Col90] Coleman, R. F.: Manin's proof of the Mordell conjecture over function fields. - Enseign. Math. (2) 36:3-4, 1990, 393–427.

[EG97] Earle, C. J., and F. P. Gardiner: Teichmüller disks and Veech's $\mathcal{F}$-structures. - In: Extremal Riemann surfaces (San Francisco, CA, 1995), Contemp. Math. 201, Amer. Math. Soc., Providence, RI, 1997, 165–189.

[FK92] Farkas, H. M., and I. Kra: Riemann surfaces. - Grad. Texts in Math. 71, Springer-Verlag, New York, second edition, 1992.

[Gra65] Grauert, H.: Mordells Vermutung über rationale Punkte auf algebraischen Kurven und Funktionenkörper. - Inst. Hautes Études Sci. Publ. Math. 25, 1965, 131–149.

[GHS03] Gutkin, E., P. Hubert, and T. A. Schmidt: Affine diffeomorphisms of translation surfaces: periodic points, Fuchsian groups, and arithmeticity. - Ann. Sci. Éc. Norm. Supér. (4) 36:6, 2003, 847–866.

[HS07] Herrlich, F., and G. Schmithüsen: On the boundary of Teichmüller disks in Teichmüller and in Schottky space. - In: Handbook of Teichmüller theory. Vol. I, IRMA Lect. Math. Theor. Phys. 11, Eur. Math. Soc., Zürich, 2007, 293–349.

[Ima09] Imayoshi, Y.: A construction of holomorphic families of Riemann surfaces over the punctured disk with given monodromy. - In: Handbook of Teichmüller theory. Vol. II, IRMA Lect. Math. Theor. Phys. 13, Eur. Math. Soc., Zürich, 2009, 93–130.

[IS88] Imayoshi, Y., and H. Shiga: A finiteness theorem for holomorphic families of Riemann surfaces. - In: Holomorphic functions and moduli, Vol. II (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ. 11, Springer, New York, 1988, 207–219.

[IT92] Imayoshi, Y., and M. Taniguchi: An introduction to Teichmüller spaces. - Springer-Verlag, Tokyo, 1992.

[KH95] Katok, A., and B. Hasselblatt: Introduction to the modern theory of dynamical systems. - Encyclopedia Math. Appl. 54, Cambridge Univ. Press, Cambridge, 1995.

[Lan03] Landau, E.: Über die Maximalordnung der Permutationen gegebenen Grades. - Arch. Math. Phys. 5, 1903, 92–103.

[Man63] Manin, Ju. I.: Rational points on algebraic curves over function fields. - Izv. Akad. Nauk SSSR Ser. Mat. 27, 1963, 1395–1440.

[Mas84] Massias, J.-P.: Majoration explicite de l'ordre maximum d'un élément du groupe symétrique. - Ann. Fac. Sci. Toulouse Math. (5) 6:3-4, 1984, 269–281.

[Miw66] Miwa, M.: On Mordell's conjecture for algebraic curves over function fields. - J. Math. Soc. Japan 18, 1966, 182–188.

[Möl06] Möller, M.: Periodic points on Veech surfaces and the Mordell-Weil group over a Teichmüller curve. - Invent. Math. 165:3, 2006, 633–649.

[Nik01] Nikolaev, I.: Foliations on surfaces. - Ergeb. Math. Grenzgeb. (3) 41, Springer-Verlag, Berlin, 2001.

[Shi97] Shiga, H.: On monodromies of holomorphic families of Riemann surfaces and modular transformations. - Math. Proc. Cambridge Philos. Soc. 122:3, 1997, :541–549.

[Shi13] Shinomiya, Y.: Veech holomorphic families of Riemann surfaces, holomorphic sections, and Diophantine problems. - Trans. Amer. Math. Soc. (to appear).

[Str84] Strebel, K.: Quadratic differentials. - Ergeb. Math. Grenzgeb. (3) 35, Springer-Verlag, Berlin, 1984.

[Vee89] Veech, W. A.: Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. - Invent. Math. 97:3, 1989, 553–583.

[Vee91] Veech, W. A.: Erratum: "Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards". - Invent. Math. 103:2, 1991, 447.

Received 16 November 2012 • Accepted 14 June 2013