ON THE LENGTH SPECTRUM METRIC IN INFINITE DIMENSIONAL TEICHMÜLLER SPACES

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Abstract. We consider the length spectrum metric $d_L$ in infinite dimensional Teichmüller space $T(R_0)$. It is known that $d_L$ defines the same topology as that of the Teichmüller metric $d_T$ on $T(R_0)$ if $R_0$ is a topologically finite Riemann surface. In 2003, Shiga proved that $d_L$ and $d_T$ define the same topology on $T(R_0)$ if $R_0$ is a topologically infinite Riemann surface which can be decomposed into pairs of pants such that the lengths of all their boundary components except punctures are uniformly bounded by some positive constants from above and below. In this paper, we extend Shiga’s result to Teichmüller spaces of Riemann surfaces satisfying a certain geometric condition.

1. Introduction

Let $R_0$ be a Riemann surface of infinite topological type. We consider a pair $(R, f)$ of a Riemann surface $R$ and a quasiconformal mapping $f : R_0 \to R$. Two such pairs $(R_1, f_1)$ and $(R_2, f_2)$ are called equivalent if $f_2 \circ f_1^{-1} : R_1 \to R_2$ is homotopic to some conformal mapping, where the homotopy map does not necessarily keep points of ideal boundary $\partial R_0$ fixed. We denote the equivalence class of $(R, f)$ by $[R, f]$. The set of all equivalence classes is called the Teichmüller space of $R_0$; we denote it by $T(R_0)$.

The Teichmüller space $T(R_0)$ has a complete metric $d_T$ called the Teichmüller metric which is defined by

$$d_T([R_1, f_1], [R_2, f_2]) = \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal mappings from $R_1$ to $R_2$ that is homotopic to $f_2 \circ f_1^{-1}$ and $K(f)$ is the maximal dilatation of $f$.

We introduce another metric on $T(R_0)$. Let $C(R_0)$ be the set of non-trivial and non-peripheral closed curves in $R_0$. We define the length spectrum metric $d_L$ by

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in C(R_0)} \left| \log \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))} \right|,$$

where $\ell_{R_i}(f_i(\alpha))$ is the hyperbolic length of the closed geodesic on $R_i$ which is freely homotopic to $f_i(\alpha)$.

Proposition 1.1. [15, Proposition 3.5] Let $S(R_0)$ be the set of simple closed curves in $R_0$. Then

$$d_L([R_1, f_1], [R_2, f_2]) = \sup_{\alpha \in S(R_0)} \left| \log \frac{\ell_{R_1}(f_1(\alpha))}{\ell_{R_2}(f_2(\alpha))} \right|. $$

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In 1972, Sorvali [14] defined $d_L$, and showed the following.

**Lemma 1.2.** [14] For any $[R_1, f_1], [R_2, f_2] \in T(R_0)$,
\[
d_L([R_1, f_1], [R_2, f_2]) \leq d_T([R_1, f_1], [R_2, f_2])
\]
holds.

Sorvali conjectured that $d_L$ defines the same topology as that of $d_T$ on $T(R_0)$ if $R_0$ is a topologically finite Riemann surface. In 1986, Li [9] proved that the statement holds in the case where $R_0$ is a compact Riemann surface with genus $\geq 2$. In 1999, Liu [10] proved that Sorvali’s conjecture is true and he asked whether or not the statement holds for any Riemann surface of infinite type. To this question, Shiga [13] gave a negative answer, that is, he showed that there exists a Riemann surface $R_0$ of infinite type such that $d_L$ and $d_T$ do not define the same topology on $T(R_0)$.

Also, he gave a sufficient condition for these metrics to define the same topology on $T(R_0)$ as follows.

**Theorem 1.3.** [13] Let $R_0$ be a Riemann surface. Assume that there exists a pants decomposition $R_0 = \bigcup_{k=1}^{\infty} P_k$ satisfying the following conditions.

1. Each connected component of $\partial P_k$ ($k = 1, 2, 3 \ldots$) is either a puncture or a simple closed geodesic of $R_0$.
2. There exists a constant $M > 0$ such that if $\alpha$ is a boundary curve of some $P_k$ then
\[
0 < M^{-1} < l_{R_0}(\alpha) < M
\]
holds.

Then $d_L$ defines the same topology as that of $d_T$ on $T(R_0)$.

In our previous paper [8], we showed that the converse of Shiga’s theorem is not true, that is, there exists a Riemann surface $R_0$ such that $R_0$ does not satisfy Shiga’s condition, but the two metrics define the same topology on $T(R_0)$. In this paper, we generalize the example and extend Shiga’s theorem as follows.

**Theorem 1.4.** Let $R_0$ be a Riemann surface. Assume that there exists a constant $M > 0$ and a decomposition $R_0 = S \cup (R_0 - S)$ such that

1. $S$ is an open subset of $R_0$ whose relative boundary consists of simple closed geodesics and each connected component of $S$ has a pants decomposition satisfying the same condition as that of Shiga’s theorem for $M$, and
2. $R_0 - S$ is of genus 0 and $d_{R_0}(x, S) < M$ for any $x \in R_0 - S$, where $d_{R_0}(\cdot, \cdot)$ is the hyperbolic distance in $R_0$.

Then $d_L$ defines the same topology as that of $d_T$ on $T(R_0)$.

In Section 2, we show that there exists a Riemann surface such that it satisfies the condition of Theorem 1.4 but it does not satisfy that of Theorem 1.3. In Section 3, we introduce lemmas to prove Theorem 1.4. In Section 4, we prove Theorem 1.4.

In Section 5, we consider Riemann surfaces with bounded geometry. Here we say that a Riemann surface $R_0$ has $(M)$-bounded geometry if it satisfies the following condition: There exists a constant $M > 0$ such that any closed geodesic has the length greater than $1/M$ and for any $x \in R_0$, there exists a closed curve based on $x$ with the length less than $M$.

As a corollary of Theorem 1.4, we obtain the following:
Corollary 1.5. Suppose that $R_0$ is of finite genus and $R_0$ has bounded geometry. Then $d_L$ define the same topology as that of $d_T$ on $T(R_0)$.

Remark 1. We consider surfaces after cutting the flares if such cylindrical ends exist. We assume that the unique complete hyperbolic metric on $R_0 \setminus \partial R_0$ that uniformizes the complex structure on the surface satisfies the following condition: Let $\{P_k\}$ be a pants decomposition of $R_0$. If we replace each boundary component of $P_k$ ($k = 1, 2, \ldots$) with the closed geodesic in its homotopy class, then $P_k$ becomes a sphere with three holes, where a hole is either a boundary component which is a closed geodesic or a cusp.

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2. Examples

First, we give examples of Riemann surfaces satisfying the conditions in Theorem 1.4 and Corollary 1.5.

Example 1. Any Riemann surface satisfying Shiga’s condition satisfies the condition in Theorem 1.4. Hence, in particular, any Riemann surface of finite topological type satisfies it.

Example 2. The Riemann surface $R_0$ constructed in our previous paper [8] satisfies conditions in both Theorem 1.4 and Corollary 1.5. For convenience of the reader, we show the construction.

Let $\Gamma$ be a hyperbolic triangle group of signature $(2,4,8)$ acting on the unit disk $D$ and let $P$ be a fundamental domain for $\Gamma$ with angles $(\pi, \pi/4, \pi/4, \pi/4)$. (See the left in Figure 1.) Let $O, a, b, c$ denote the vertices of $P$, where the angle at $O$ is $\pi$. Now, take a sufficiently small number $\varepsilon > 0$. Let $b'$ the point on the segment $[Ob]$ whose hyperbolic distance from $b$ is $\varepsilon$. Similarly, we take $a'$ and $c'$ in $P$. (See the middle in Figure 1.)

We define a Riemann surface $R_0$ by removing the $\Gamma$-orbits of $a', b', c'$ from the unit disk $D$. (See the right in Figure 1.)

Figure 1. Left: Tessellation by the $(2,4,8)$ group. Middle: Points $a', b', c'$ in $P$. Right: A Riemann surface $R_0 = D - \{\gamma(a'), \gamma(b'), \gamma(c') \mid \gamma \in \Gamma\}$.

It is not hard to see that the surface $R_0$ does not satisfy Shiga’s condition (cf. Section 2 in [8]). However, it satisfies the above conditions. Indeed, we decompose $R_0$ into eight times punctured disks and a multiply-connected domain. (See Figure 2.)
Let $S_i$ be a punctured disk and put $S = \bigcup_{i=1}^{\infty} S_i$. Then we obtain a decomposition in Theorem 1.4: $R_0 = S \cup (R_0 - S)$. On the other hand, $R_0$ satisfies the condition in Corollary 1.5 obviously.

Also, we can construct a Riemann surfaces satisfying Theorem 1.4 and Corollary 1.5 by replacing a hyperbolic triangle group $\Gamma$ with an arbitrary Fuchsian group with a compact fundamental region.

**Example 3.** In Example 2, $R_0$ is a Riemann surface of genus 0 with $\infty$ punctures and 1 flare. By tinkering with $R_0$, we can construct Riemann surfaces of genus $\geq 1$ with two or more flares which satisfies the conditions of Theorem 1.4. For example, in Example 2, we replace a punctured disk $S_i$ with a pair of pants. (See the left in Figure 3.) We regard it as a block and make a copy of it and glue them. (See the right in Figure 3.) Then we obtain a Riemann surface $X_0$ of genus 1 with $\infty$ punctures and two flares. Obviously $X_0$ satisfies Theorem 1.4 and Corollary 1.5. Hence, in the similar way, we can construct Riemann surfaces of genus $\infty$ with $\infty$ flares which satisfies Theorem 1.4.

**3. Lemmas**

In this section, we present some lemmas to prove Theorem 1.4 and Corollary 1.5.

**Lemma 3.1.** [4, Lemma 3.1] Let $T_1, T_2 \subset \mathbf{D}$ be two hyperbolic triangles with sides $(a_1, b_1, c_1)$ and $(a_2, b_2, c_2)$ respectively. Suppose all their angles are bounded below by $\theta > 0$ and

$$\varepsilon := \max(|\log \frac{a_1}{a_2}|, |\log \frac{b_1}{b_2}|, |\log \frac{c_1}{c_2}|) \leq A.$$ 

Then there is a constant $C = C(\theta, A)$ and a $(1 + C\varepsilon)$-quasiconformal mapping $\varphi: T_1 \to T_2$ such that $\varphi$ maps each vertex to the corresponding vertex and $\varphi$ is affine on the edge of $T_1$. 

\[
\begin{align*}
\text{Figure 2. } R_0 &= \{ \text{punctured disks} \} \cup \{ \text{a multiply-connected domain} \}.
\end{align*}
\]

\[
\begin{align*}
\text{Figure 3. Left: A block obtained by replacing } S_i \text{ in Example 2 with a pair of pants. Right: A Riemann surface } X_0 \text{ of genus 1 with } \infty \text{ punctures and two flares.}
\end{align*}
\]
Lemma 3.2. [4, Corollary 3.3] Let $H, H' \subset D$ be two hyperbolic hexagons with sides $(a_1, \ldots, a_6)$ and $(b_1, \ldots, b_6)$ respectively. Suppose $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$ are $\leq B$ and are comparable with a constant $B$. Also assume that three alternating angles of $H$ and the corresponding angles of $H'$ are $\pi/2$ and the remaining angles are bounded below by $\theta > 0$ and above by $\pi - \theta$. If $\varepsilon = \max_i |\log a_i/b_i| \leq 2$, then there is a constant $C = C(\theta, B)$ and a $(1+C\varepsilon)$-quasiconformal mapping $\varphi: H \rightarrow H'$ such that $\varphi$ maps each vertex to the corresponding vertex and $\varphi$ is affine on the edge of $H$.

Lemma 3.3. [4, Lemma 6.2] Let $P_1$ and $P_2$ be pants with boundary lengths $(a_1, b_1, c_1)$ and $(a_2, b_1, c_1)$ respectively. Suppose $a_1, a_2, b_1, c_1 \leq L$ (punctures count as length zero). Assume that $\varepsilon := |\log a_1/a_2| \leq 2$, where we define $|\log a_1/a_2| = 0$ if $a_1 = a_2 = 0$ and $|\log a_1/a_2| = +\infty$ if one is zero and the other is not. Then there is a constant $C = C(L)$ and a $(1+C\varepsilon)$-quasiconformal mapping $\varphi: P_1 \rightarrow P_2$ such that $\varphi$ is affine on each of the boundary components.

Also we note the following lemma.

Lemma 3.4. Let $R_0$ be a Riemann surface. Suppose $\alpha_1$ and $\alpha_2$ are disjoint simple closed geodesics. Let $\beta_{12}$ be a simple arc connecting $\alpha_1$ and $\alpha_2$. Then there exists a geodesic $\beta^*_{12}$ connecting $\alpha_1$ and $\alpha_2$ such that

1. $\beta_{12}$ and $\beta^*_{12}$ are homotopic, where the homotopy map may not keep end points of $\beta_{12}$ and $\beta^*_{12}$ fixed;
2. $\beta^*_{12}$ is orthogonal to $\alpha_1$ and $\alpha_2$;
3. the length of $\beta^*_{12}$ is determined by lengths of three simple closed geodesics which are homotopic to $\alpha_1$, $\alpha_2$ and $\alpha_{12} := \alpha_1 \cdot \beta_{12} \cdot \alpha_2 \cdot \beta^{-1}_{12}$.

Proof. There exists a closed geodesic in $R_0$ homotopic to $\alpha_{12}$. We denote it by $[\alpha_{12}]$. Consider a pair of pants $P_{12}$ bounded by $\alpha_1$, $\alpha_2$ and $[\alpha_{12}]$. There are three lines which divide $P_{12}$ into two isometric right-hexagons. Let $\beta^*_{12}$ be a line connecting $\alpha_1$ and $\alpha_2$ in those. We denote the length of $\beta^*_{12}$ by $\ell_{R_0}(\beta^*_{12})$. Then, by Theorem 7.19.2 of [3],

$$\cosh \ell_{R_0}(\beta^*_{12}) = \frac{\cosh\left(\frac{1}{2}\ell_{R_0}(\alpha_{12})\right) + \cosh\left(\frac{1}{2}\ell_{R_0}(\alpha_1)\right) \cosh\left(\frac{1}{2}\ell_{R_0}(\alpha_2)\right)}{\sinh\left(\frac{1}{2}\ell_{R_0}(\alpha_1)\right) \sinh\left(\frac{1}{2}\ell_{R_0}(\alpha_2)\right)}$$

holds. \hfill $\square$

In the following lemma, for $R_0 - S$ in Theorem 1.4 we may consider a decomposition by right-hexagons with a bounded condition.

Lemma 3.5. Let $R_0$ be a Riemann surface satisfying the condition of Theorem 1.4. Then $R_0 - S$ can be decomposed hyperbolic right-hexagons $\{H_j\}_{j=1}^{\infty}$ with sides of lengths less than $2M$.

Proof. By assumption, we can decompose $S$ into domains $\{S_i\}_{i=1}^{\infty}$ such that $\partial S_i \cap (R_0 - S)$ $(i = 0, 1, 2, \ldots)$ is a closed geodesic with the length less than $M$. For $S_i$, we consider the following domain:

$$D_i := \{x \in R_0 - S \mid d_{R_0}(x, S_i) \leq d_{R_0}(x, S_j) (\forall j \neq i)\}.$$ 

$D_i$ is contained in $M$-neighborhood of $S_i$, and $D_i \cup S_i$ is convex. The boundary of $D_i$ consists of two kinds of connected components: the boundary of $S_i$ and the boundary
of geodesic polygon with finitely many sides. We denote the polygon with a hole by $W_i$ (i.e. $W_i := \partial D_i$).

$W_i$ is bounded on each side by another $W_j$. We show that two vertices of $W_i$ coincides with those of $W_j$ if $W_i$ is bounded by $W_j$. Assume that there exists a side $w_i$ with vertices $v_i, v'_i$ of $W_i$ such that, for a side $w_j$ with vertices $v_j, v'_j$ of $W_j$, $w_j \subset w_i$ but $0 < d_{R_0}(v_i, v_j) < d_{R_0}(v_i, v'_j)$. (See Figure 5.) Then there exists a polygon with a hole $W_k$ which has a side $w_k \cap w_i - w_j \neq \emptyset$. For a domain $S_k$ in $\{S_i\}_{i=0}^\infty$ with $\partial S_k \cap W_k \neq \emptyset$, we take a line $b_{i,k} := \{x \in R_0 \mid d_{R_0}(x, S_i) = d_{R_0}(x, S_k)\}$. $b_{i,k}$ is a perpendicular bisector of the shortest geodesic segment $[l_{i,k}]$ connecting $\partial S_i$ and $\partial S_k$. (Note that $[l_{i,k}]$ is orthogonal to $\partial S_i$ and $\partial S_k$.) By the definition of domains $\{D_i\}$, $w_k \subset b_{i,k}$. Similarly we take another perpendicular bisector $b_{i,j}$ of the shortest geodesic segment $[l_{i,j}]$ connecting $\partial S_i$ and $\partial S_j$. Then $b_{i,k} = b_{i,j}$ since they are geodesics and $w_k \subset w_i \subset b_{i,j}$. Take four points $p_{i,k} := \partial S_i \cap [l_{i,k}], m_{i,k} := [l_{i,k}] \cap b_{i,k}, p_{i,j} := \partial S_i \cap [l_{i,j}], m_{i,j} := [l_{i,j}] \cap b_{i,j}$ and consider a quadrilateral with vertices $\{p_{i,k}, m_{i,k}, m_{i,j}, p_{i,j}\}$. Then we obtain a right-angled quadrilateral. However there does not exist such a hyperbolic quadrilateral. Hence $d_{R_0}(v_i, v_j) = 0$, i.e. $v_i = v_j$. Similarly $v'_i = v'_j$.

Now, take an arbitrary vertex $v_0$ of an arbitrary polygon with a hole $W_0$ and put $W_j$ $(j = 0, 1, 2, \ldots, n)$; in counterclockwise direction) the polygon with a hole which contains $v_0$. Connect $\partial S_j$ and $\partial S_{j+1}$ by the shortest geodesic segment $[l_{j,j+1}]$ for each $j = 0, 1, 2, \ldots, n$, where $S_j$ is a domain in $\{S_i\}_{i=0}^\infty$ with $\partial S_j \cap W_j \neq \emptyset$ and $S_{n+1} := S_0$. So we obtain a $2n$-sided polygon $P$ which consists of $[l_{j,j+1}]$ and subarcs of $\partial S_j$ $(j = 0, 1, 2, \ldots, n)$. 
Also we join $\partial S_0$ and $\partial S_j$ $(j = 2, 3, \ldots, n - 1)$ by the shortest geodesic in $P$ which is orthogonal to $\partial S_0$ and $\partial S_j$, respectively. (By Lemma 3.4, there exist such geodesics.) Then we obtain $n - 1$ right-hexagons. (See Figure 6.) Each right-hexagon has three alternating sides $[l_{0,j}], [l_{j,j+1}], [l_{j+1,0}]$ with the lengths bounded by $2M$ since $|l_{j,j'}| \leq d_{R_0}(v_0, \partial S_j) + d_{R_0}(v_0, \partial S_{j'}) < M + M = 2M$ for $j, j' = 0, 1, 2, \ldots, n, j \neq j'$. Hence all lengths of sides are bounded by $2M$.

Figure 6. A right-hexagons decomposition around $v_0$.

Continue the above operation, then $R_0 - S$ is divided into right-hexagons with sides of the lengths bounded by $2M$. □

4. Proof of Theorem 1.4

From Lemma 1.2, it is sufficient to show that for any sequence $\{p_n\}_{n=0}^{\infty} \subset T(R_0)$ with $d_L(p_n, p_0) \to 0$ $(n \to \infty)$, $d_T(p_n, p_0)$ converges to 0 as $n \to \infty$. We assume that $p_0 = [R_0, id]$. Put $p_n = [R_n, f_n]$.

In Section 3, we see that $R_0$ is decomposed by pairs of pants and hexagons such that lengths of their boundaries are bounded uniformly; $R_0 = \bigcup_{i=1}^{\infty} S_i \cup \bigcup_{j=1}^{\infty} H_j$. We consider a decomposition of $R_0$ for sufficiently large $n$.

First, for each $j = 1, 2, \ldots$, we replace $f_n(H_j)$ by a right-hexagon in $R_n$ as follows. $H_j \subset R_0$ has edges $a_1, \ldots, a_6$ (in counterclockwise direction). We suppose that $a_1, a_3, a_5$ are subarcs of $\partial S_1, \partial S_2, \partial S_3$ respectively and $a_2$ connects $\partial S_1$ and $\partial S_2$. Put $\alpha_{12} := \partial S_1 \cdot a_2 \cdot \partial S_2 \cdot a_2^{-1} \in C(R_0)$. For a closed curve $f_n(\alpha_{12})$ in $R_n$, we take a closed geodesic $[f_n(\alpha_{12})]$ in $R_n$. If we consider a pair of pants bounded by $[f_n(\partial S_1)]$, $[f_n(\partial S_2)]$ and $[f_n(\alpha_{12})]$, then there exists a geodesic segment connecting $[f_n(\partial S_1)]$ and $[f_n(\partial S_2)]$ as in Lemma 3.4. We denote it by $a_2^n$. The length of $a_2^n$ is determined by the lengths of $[f_n(\partial S_1)]$, $[f_n(\partial S_2)]$ and $[f_n(\alpha_{12})]$. The lengths of them are almost the same as that of preimages of closed geodesics in $R_0$ respectively, so the lengths of $a_2$ and $a_2^n$ are almost the same. Similarly we take geodesic segments $a_6^n$ in $R_n$ for $a_4$ and $a_6$ respectively. Let $H_j^n \subset R_n$ be a right-hexagon bounded by $a_2^n, a_4^n, a_2^n$ and subarcs of $[f_n(\partial S_1)], [f_n(\partial S_2)]$, $[f_n(\partial S_3)]$. (See Figure 7.) Then $H_j^n$ is almost congruous with $H_j$.

Put $R_j^n := \bigcup_{j=1}^{\infty} H_j^n$. By Lemma 3.2, we obtain a quasiconformal mapping $g_n$ from $R_0(= \bigcup_{j=1}^{\infty} H_j)$ to $R_j^n$. We claim that $f_n$ is homotopic to $g_n$ on $R_0$, where the homotopy map does not necessarily keep points of $\partial R_0$ fixed.
Figure 7. Replacement of $f_n(H_j)$ by a right-hexagon $H_j^n$ in $R_n$.

It is enough to see that for an arbitrary simple closed geodesic $\alpha \subset R'_0$, $f_n(\alpha)$ and $g_n(\alpha)$ are homotopic (cf. [5, Lemma 4]). Let $\{H_j^{(k)}\}_{k \in K} \subset R'_0$ be the set of all the right-hexagons such that $H_j^{(k)} \cap \alpha \neq \emptyset$. Since $f_n$ is homeomorphic, $f_n(\alpha) \subset \bigcup_{k \in K} f_n(H_j^{(k)})$. Therefore we see that for each $k \in K$, a curve $g_n(\alpha) \cap H_j^k$ is homotopic to a curve $[f_n(\alpha)] \cap H_j^k$, where the homotopy map does not necessarily fix endpoints. (See Figure 8.) Hence $f_n(\alpha)$ is homotopic to $g_n(\alpha)$, so we verify the claim.

Next, we consider a quasiconformal mapping of $S_i$ for each $i = 1, 2, \ldots$. We decompose $S_i$ into pairs of pants satisfying Shiga’s condition; $S_i = \bigcup_{k=1}^{\infty} P_k$.

Let $G_P$ be the set of closed geodesics which are boundaries of some $P_k$ in $S_i$. For each $\alpha \in G_P$, there exists a closed geodesic $[f_n(\alpha)]$ in $R_n$ homotopic to $f_n(\alpha)$. The set $\{[f_n(\alpha)]\}_{\alpha \in G_P}$ gives a pants decomposition of $f_n(S_i)$. (Indeed, $f_n(S_i)$ is Nielsen-convex; cf. [1, Theorem 4.5].)

Now, we put $\alpha_1, \alpha_2, \alpha_3$ three closed geodesics of $\partial P_1$ and assume that $\alpha_1 \subset R'_0$. By lemmas of Bishop, we obtain a quasiconformal mapping on $\overline{S_1 - P_1}$. However, $g_n$ on $R'_0$ is locally affine on $\alpha_1$, so we construct a quasiconformal mapping on $P_1$.

Let $x_1, \ldots, x_m \in \alpha_1$ be vertices of right-hexagons $\{H_j\}$, and let $y_1, \ldots, y_6 \in \partial P_1$ be the vertices of two symmetric right-hexagons constructing $P_1$ (See Figure 9). Suppose that $y_1$ is on the segment $[x_1 x_2]$, and $y_6$ is on the segment $[x_r x_{r+1}]$ ($1 \leq r \leq m$). Let $d_1$ be the length of $[x_1 x_2]$ and let $d'_1$ be the length of the $[x_1 y_1]$. Then there is a number $t \in [0, 1]$ such that $d_1 = td'_1$. Similarly take $d_r$, $d'_r$ for $[x_r x_{r+1}]$, $[x_r y_6]$, then there is a number $s \in [0, 1]$ such that $d_r = sd'_r$. 
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Figure 9. Points on $\partial P_1$.

On the other hand, let $x_1^n, \ldots, x_m^n \in [f_n(\alpha_1)]$ in $R_n$ be vertices of right-hexagons $H^n_j$. We take the points $g_n(y_1), \ldots, g_n(y_6)$ on $\partial P_1^n$, where $P_1^n$ is a pair of pants corresponding to $f_n(P_1)$.

We consider a hyperbolic hexagon with vertices $g_n(y_1), \ldots, g_n(y_6)$. We claim that the angle formed by $[g_n(y_2)g_n(y_3)]$ and $[g_n(y_3)g_n(y_4)]$ is almost $\pi/2$. Indeed, for $S_i \subset R_0$, let $\hat{S}_i$ be the Nielsen extension of $S_i$. We consider the Fenchel–Nielsen coordinates of the Teichmüller space $T(\hat{S}_i)$. Then the twist parameter along $[f_n(\alpha_2)]$ is almost the same as that along $\alpha_2$ (cf. [13, Lemma 4.1]). Hence we verify the claim.

The remaining angles are almost $\pi/2$, similarly.

Let $d^n_i$ be the hyperbolic length of the segment $[x^n_{i}x^n_{i+1}]$ ($1 \leq i \leq m$), and let $d'^n_{r}$ be the hyperbolic length of the segment $[x^n_{r}g_n(*)]$ ($i = 1, r, * = y_1, y_6$). Then, for $t \in [0, 1]$ and $s \in [0, 1]$ we took above, $d'^n_{r} = td^n_{i}$ and $d'^n_{r} = sd^n_{r}$ hold, because $g_n$ of $R'_0$ is locally affine on $\alpha_1$. Moreover, since the quasiconformal mapping $g_n$ of $S_i - P_1$ is affine on $\alpha_2$ and $\alpha_3$, the lengths of sides $[y_1y_2], \ldots, [y_6y_1]$ and the lengths of sides $[g_n(y_1)g_n(y_2)], \ldots, [g_n(y_6)g_n(y_1)]$ are almost the same respectively. Hence the right-hexagon with vertices $(y_1, \ldots, y_6)$ and the hexagon with vertices $(g_n(y_1), \ldots, g_n(y_6))$ are almost congruous.

Figure 10. Triangulation.

From the First Cosine Rule for hyperbolic geometry (cf. [3]), the length of the new sides are determined by the sides and angles of the hexagon. Quasiconformal mappings of the triangles with vertices $(y_2, y_3, y_4), (y_4, y_5, y_6)$ and $(y_2, y_4, y_6)$ are obtained from Lemma 3.1.

We consider a quasiconformal mapping of the triangle $T$ with vertices $(y_1, y_2, y_6)$. In $T$, connect the points $x_2, \ldots, x_r$ by geodesics segments to $y_2$. Similarly, in the triangle $T_n$ with vertices $(g_n(y_1), g_n(y_2), g_n(y_6))$, connect the points $x^n_2, \ldots, x^n_r$ by
geodesics segments to $g_n(y_2)$. Then we obtain a quasiconformal mapping of the
triangle $T$ from Lemma 3.1.

Hence we obtain a quasiconformal mapping $g_n$ of the whole of $R_0$ such that $g_n$ is
homotopic to $f_n$ and $K(g_n) \to 1 \; (n \to \infty)$. Thus $d_T(p_n, p_0) \to 0 \; (n \to \infty)$.

In the case where $p_0 \neq [R_0, id]$, we can show that $d_T(p_n, p_0) \to 0 \; (n \to \infty)$
similarly. Indeed, any Riemann surface which is quasiconformally equivalent to $R_0$
satisfies the condition of Theorem 1.4 for some constant. \hfill $\square$

5. Corollary of Theorem 1.4

In this section, we consider Riemann surfaces with bounded geometry.

**Proposition 5.1.** Let $R_0$ be a Riemann surface of finite genus with $M$-bounded
geometry. Then $R_0$ satisfies the assumption of Theorem 1.4.

**Proof.** On $R_0$, we construct $S$ in the condition of Theorem 1.4 as a union of pairs
of pants. Note that we may construct a pair of pants from two disjoint simple closed
geodesics and an simple arc connecting the two geodesics.

At first, we take a constant $d = d(M) > 0$ as the following:

For any $x \in R_0$, there exists a closed curve $c_x$ passing through $x$ with $1/M <$ the
length of $c_x < M$. We take a geodesic $\alpha_x$ in the homotopy class of $c_x$. We put

$$d_x := \max_{\alpha_x} \{M - \ell_{R_0}(\alpha_x)\} > 0,$$

where $\alpha_x \in \{\alpha_x : \text{a geodesic in the homotopy class of } \alpha_x, \text{there exists a closed curve } c_x \text{ passing through } x \text{ with } 1/M < \text{ the length of } c_x < M.\}$. Moreover we put

$$d := \sup_{x \in R_0} d_x.$$

Then $d$ satisfies the following property: For any $x \in R_0$ and any closed curve $c$
passing through $x$ with $1/M <$ the length of $c < M$, the geodesic $\alpha$ in the homotopy
class of $c$ satisfies $d_{R_0}(x, \alpha) < d$.

Indeed, if $d_{R_0}(x, \alpha) \geq d$ holds, then $d_{R_0}(x, \alpha) \geq d_x \geq M - \ell_{R_0}(\alpha)$ i.e. $d_{R_0}(x, \alpha) + \ell_{R_0}(\alpha) \geq M$. Hence the length of $c(\geq d_{R_0}(x, \alpha) + \ell_{R_0}(\alpha))$ is larger than $M$. This
contradicts. Therefore $d_{R_0}(x, \alpha) < d$.

(Note that, in other words, $d$ is a constant such that if $x$ is a point in $R_0$ and $\alpha$ is
a simple closed geodesic with $1/M < \ell_{R_0}(\alpha) < M$ and $d_{R_0}(x, \alpha) \geq d$, then the length
of any closed curve $c$ passing through $x$ which is homotopic to $\alpha$ is larger than $M$.)

![Figure 11. A closed curve $c$ such that the length $= d + \ell_{R_0}(\alpha)$.](image)

Now let us start to construct pairs of pants. Let $x_0$ be an arbitrary point in $R_0$.
Then there exists a closed curve $c_0$ passing through $x_0$ with the length less than $M$.\hfill $\square$
We take a geodesic $\alpha_0$ in the homotopy class of $c_0$. Then $0 < 1/M < \ell(\alpha_0) < M$ and $d_{R_0}(x_0, \alpha_0) < d$.

Put $D := \max\{d, M\}$. Next, take $y_0 \in R_0$ with $d_{R_0}(y_0, \alpha_0) = D + 1$. Also, take a geodesic $\beta_0$ for $y_0$ in the same way. Then $\beta_0 \neq \alpha_0$ and $d_{R_0}(y_0, \beta_0) < d$. Hence $d_{R_0}(\alpha_0, \beta_0) < D + 1 + d < 2D + 1$. Thus there exists a simple arc $\hat{\gamma}_0$ connecting $\alpha_0$ and $\beta_0$ such that the length of $\gamma_0 \leq 2D + 1$.

If we construct a pair of pants $P_0$ by $\alpha_0, \beta_0$, and $\hat{\gamma}_0$, then the length of each boundary component is bounded by some constant $L = L(M)$ from above and below.

Next, we take a point $x_1 \in R_0$ such that $d_{R_0}(P_0, x_1) = 3D + 2$. Also, take a geodesic $\alpha_1$ for $x_1$ in the above way. (Note that $d_{R_0}(\alpha_1, x_1) < D$.) Then we can take a point $y_1$ in $R_0 - P_0$ such that $d_{R_0}(y_1, \alpha_1) = D + 1$. Indeed, since

$$d_{R_0}(\alpha_1, P_0) \geq d_{R_0}(x_1, P_0) - d_{R_0}(\alpha_1, x_1) \geq 2D + 2,$$

$$\{y \in R_0 \mid d_{R_0}(y, \alpha_1) = D + 1\} \cap P_0 = \emptyset.$$

Now, we take a geodesic $\beta_1$ for $y_1$ similarly. $\beta_1 \neq \alpha_1$ and $d_{R_0}(y_1, \beta_1) < d < D$ hold. Hence $d_{R_0}(\alpha_1, \beta_1) < 2D + 1$. Thus $\beta_1 \cap P_0 = \emptyset$ since

$$d_{R_0}(\beta_1, P_0) \geq d_{R_0}(P_0, \alpha_1) - d_{R_0}(\alpha_1, \beta_1) \geq 2D + 2 - (2D + 1) = 1.$$

Also, there exists a simple arc $\hat{\gamma}_1$ connecting $\alpha_1$ and $\beta_1$ with the length of $\gamma_1 < 2D + 1$. Then we see that $\hat{\gamma}_1 \cap P_0 = \emptyset$ since $d_{R_0}(\alpha_1, P_0) \geq 2D + 2$.

We construct a pair of pants $P_1$ by $\alpha_1, \beta_1$ and $\hat{\gamma}_1$. Then $P_0 \cap P_1 = \emptyset$. Indeed, if we take a geodesic $\gamma_1$ which is homotopic to a closed curve $\alpha_1 \cdot \hat{\gamma}_1 \cdot \beta_1 = \gamma_1^{-1}$ then $\gamma_1 \cap P_0 = \emptyset$ by property of geodesics.

Similarly, we take a point $x_2 \in R_0$ such that $d_{R_0}(x_2, P_0 \cup P_1) = 3D + 2$. Let $\alpha_2$ be the geodesic in homotopy class of a simple closed curve $c_2$ passing through $x_2$ with $M^{-1} < \gamma$ less than the length of $c_2 < M$. $d_{R_0}(\alpha_2, x_2) < d < D$. We can take a point $y_2 \in R_0 - (P_0 \cup P_1)$ such that $d_{R_0}(\alpha_2, y_2) = D + 1$ since $d_{R_0}(\alpha_2, P_0 \cup P_1) \geq 2D + 2$. Let $\beta_2$ be the geodesic in homotopy class of a simple closed curve $c_2'$ passing through $y_2$ with $M^{-1} < \gamma$ less than the length of $c_2' < M$. $d_{R_0}(\beta_2, y_2) < d < D$. Hence $d_{R_0}(\alpha_2, \beta_2) < 2D + 1$. Thus $\beta_2 \cap (P_0 \cup P_1) = \emptyset$. Also, there exists a simple arc $\hat{\gamma}_2$ connecting $\alpha_2$ and $\beta_2$ with $\ell(\hat{\gamma}_2) < 2D + 1$. $\hat{\gamma}_2 \cap (P_0 \cup P_1) = \emptyset$ since $d_{R_0}(\alpha_2, P_0 \cup P_1) \geq 2D + 2$. If we construct a pair of pants $P_2$ by $\alpha_2, \beta_2$ and $\hat{\gamma}_2$ then $P_2 \cap (P_0 \cup P_1) = \emptyset$.

Inductively, we construct a sequence of pairs of pants $\{P_i\}$. Then it satisfies the following.

(i) $P_i \cap P_j = \emptyset$ if $i \neq j$.

(ii) For any $i$, the length of each connected component of $\partial P_i$ is bounded by $L$ from above and below.

Finally we consider the set

$$X_n := \{x \in R_0 \mid d_{R_0}(x_0, x) < n\}$$

for a sufficiently large number $n > 0$. Since the closure $\overline{X_n}$ of $X_n$ is relatively compact in $R_0$, there exists a finite sequence of pairs of pants $\{P_i\}_i^{k=0}$ such that $X_n \cap P_i \neq \emptyset$ (for $n = 0, 1, \ldots, k$) and it satisfies (i) and (ii); moreover

$$d_{R_0}(x, \bigcup_{i=0}^{k} P_i) < 3D + 2$$

for any $x \in X_n - \bigcup_{i=0}^{k} P_i$. 

Construct \( \{P_i\}_{i=0}^\infty \) as \( n \to \infty \). Since \( \bigcup_{n=1}^\infty X_n = R_0 \), we obtain a subset \( S = \bigcup_{i=0}^\infty P_i \subset R_0 \) in the condition of Theorem 1.4.

(We note that \( R_0 - S \) is of genus 0. Indeed, the lengths of geodesics cutting genus of \( R_0 \) are bounded by \( M \) since \( R_0 \) is of finite genus. Hence we can choose them as the curves of \( \partial P_i \).)

Hence we obtain Corollary 1.5.

References


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