SOBOLEV’S THEOREM AND DUALITY FOR HERZ–MORREY SPACES OF VARIABLE EXPONENT

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Abstract. In this paper, we consider the Herz–Morrey space \( H^{p(\cdot),q,\omega}(G) \) of variable exponent consisting of all measurable functions \( f \) on a bounded open set \( G \subset \mathbb{R}^n \) satisfying
\[
\|f\|_{H^{p(\cdot),q,\omega}(G)} = \left( \int_0^{2^2d_G} (\omega(x_0,r) \|f\|_{L^{p(\cdot)}(B(x_0,r) \setminus B(x_0,r/2))})^q \, dr/r \right)^{1/q} < \infty,
\]
and set \( H^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} H^{p(\cdot),q,\omega}(G) \).

Our first aim in this paper is to give the boundedness of the maximal and Riesz potential operators in \( H^{p(\cdot),q,\omega}(G) \) when \( q = \infty \).

In connection with \( H^{p(\cdot),q,\omega}(G) \) and \( H^{p(\cdot),q,\omega}(G) \), let us consider the families \( H^{p(\cdot),q,\omega}(G) \), \( \tilde{H}^{p(\cdot),q,\omega}(G) \), \( \tilde{H}^{p(\cdot),q,\omega}(G) \) and \( \tilde{H}^{p(\cdot),q,\omega}(G) \). Following Fiorenza–Rakotoson [18], Di Fratta–Fiorenza [17] and Gogatishvili–Mustafayev [19], we next discuss the duality properties among these Herz–Morrey spaces.

1. Introduction

Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space. We denote by \( B(x,r) \) the open ball centered at \( x \) of radius \( r \), and by \( |E| \) the Lebesgue measure of a measurable set \( E \subset \mathbb{R}^n \).

It is well known that the maximal operator is bounded in the Lebesgue space \( L^p(\mathbb{R}^n) \) if \( p > 1 \) (see [34]). In [12], the boundedness of the maximal operator is still valid by replacing the Lebesgue space by several Morrey spaces; the original one was introduced by Morrey [30] to estimate solutions of partial differential equations; for Morrey spaces, we also refer to Peetre [32] and Nakai [31].

One of important applications of the boundedness of the maximal operator is Sobolev’s inequality; in the classical case,
\[
\|I_\alpha * f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}
\]
for \( f \in L^p(\mathbb{R}^n) \), \( 0 < \alpha < n \) and \( 1 < p < n/\alpha \), where \( I_\alpha \) is the Riesz kernel of order \( \alpha \) and \( 1/p^* = 1/p - \alpha/n \) (see, e.g., [2, Theorem 3.1.4]). Sobolev’s inequality for Morrey spaces was given by Adams [1] (also [12]). Further, Sobolev’s inequality was also studied on generalized Morrey spaces (see [31]). This result was extended to local and global Morrey type spaces by Burenkov, Gogatishvili, Guliyev and Mustafayev.

doi:10.5186/aasfm.2014.3913
2010 Mathematics Subject Classification: Primary 31B15, 46E35.
Key words: Herz–Morrey spaces of variable exponent, maximal functions, Riesz potentials, Sobolev’s inequality, Trudinger’s inequality, duality.
Herz–Morrey spaces. Introduced by Herz [23]. In our paper, those Morrey type spaces are referred to as Herz–Morrey spaces.

In [13], Diening showed that the maximal operator is bounded on the variable exponent Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) if the variable exponent \( p(\cdot) \), which is a constant outside a ball, satisfies the locally log-Hölder condition and \( \inf p(x) > 1 \) (see condition (P2) in Section 2). In the mean time, variable exponent Lebesgue spaces were used to discuss nonlinear partial differential equations with non-standard growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity and fluid mechanics; see [16, 33]. On the other hand, variable exponent Morrey or Herz versions were discussed in [4, 5, 24, 26, 29].

Let \( G \) be a bounded open set in \( \mathbb{R}^n \), whose diameter is denoted by \( d_G \). Let \( \omega(\cdot,...): G \times (0, \infty) \to (0, \infty) \) be a uniformly almost monotone function on \( G \times (0, \infty) \) satisfying the uniformly doubling condition. For \( x_0 \in G \), \( 0 < q \leq \infty \) and a variable exponent \( p(\cdot) \), we consider the Herz–Morrey space \( \mathcal{H}^{p(\cdot),q,\omega}(G) \) of variable exponent consisting of all measurable functions \( f \) on \( G \) satisfying

\[
\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} = \left( \int_0^{2d_G} (\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))})^q \frac{dr}{r} \right)^{1/q} < \infty;
\]

when \( q = \infty \),

\[
\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} = \sup_{0<r<d_G} \omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} < \infty.
\]

Set

\[
\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G),
\]

whose norm is defined by

\[
\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.
\]

In connection with \( \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \), let us consider the families \( \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \) and \( \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \) of all functions \( f \) on \( G \) satisfying

\[
\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left( \int_0^{2d_G} (\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0,r))})^q \frac{dr}{r} \right)^{1/q} < \infty
\]

and

\[
\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left( \int_0^{2d_G} (\omega(x_0, r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))})^q \frac{dr}{r} \right)^{1/q} < \infty,
\]

respectively. In the paper by Fiorenza and Rakotoson [18], the Herz–Morrey space \( \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \) is referred to as the generalized Lorentz space denoted by \( GT(p,q,\omega) \).

Note here that

\[
\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \cup \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G) \subset \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G).
\]

Similarly we consider the space

\[
\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G),
\]
whose norm is defined by
\[ \|f\|_{\mathcal{H}^{p(\cdot),\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\mathcal{H}^{p(\cdot),\omega}_{(x_0)}(G)}. \]

Our first aim in this paper is to establish the boundedness of the maximal operator and the Riesz potential operator in \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \); when \( q < \infty \), we refer to [27]. In the borderline case, Trudinger’s exponential integrability is discussed.

Next, following Di Fratta–Fiorenza [17] and Gogatishvili–Mustafayev [19], we study the duality properties among those Herz–Morrey spaces. In particular, we show the associate spaces of \( \mathcal{H}^{p(\cdot),\infty,\omega}_{(x_0)}(G) \) and \( \mathcal{H}^{q(\cdot),\infty,\omega}_{(x_0)}(G) \), which give another characterizations of Morrey spaces by Adams–Xiao [3] (see also [20]).

2. Preliminaries

Throughout this paper, let \( C \) denote various constants independent of the variables in question. The symbol \( g \sim h \) means that \( C^{-1}h \leq g \leq Ch \) for some constant \( C > 1 \). Set \( A(x, r) = B(x, r) \setminus B(x, r/2) \).

Consider a function \( p(\cdot) \) on \( G \) such that

\begin{itemize}
  \item [(P1)] \( 1 < p^- := \inf_{x \in G} p(x) \leq \sup_{x \in G} p(x) =: p^+ < \infty \), and
  \item [(P2)] \( p(\cdot) \) is log-Hölder continuous, namely
    \[ |p(x) - p(y)| \leq \frac{c_p}{\log(2d_G/|x - y|)} \quad \text{for } x, y \in G \]
    with a constant \( c_p \geq 0 \); \( p(\cdot) \) is referred to as a variable exponent.
\end{itemize}

We also consider the family \( \Omega(G) \) of all positive functions \( \omega(\cdot, \cdot) : G \times (0, \infty) \rightarrow (0, \infty) \) satisfying the following conditions:

\begin{itemize}
  \item [(\omega 0)] \( \omega(x, 0) = \lim_{r \to +0} \omega(x, r) = 0 \) for all \( x \in G \) or \( \omega(x, 0) = \infty \) for all \( x \in G \);
  \item [(\omega 1)] \( \omega(x, \cdot) \) is uniformly almost monotone on \( (0, \infty) \), that is, there exists a constant \( Q_1 > 0 \) such that \( \omega(x, \cdot) \) is uniformly almost increasing on \( (0, \infty) \), that is,
    \[ \omega(x, r) \leq Q_1 \omega(x, s) \quad \text{for all } x \in G \text{ and } 0 < r < s \]
    or \( \omega(x, \cdot) \) is uniformly almost decreasing on \( (0, \infty) \), that is,
    \[ \omega(x, s) \leq Q_1 \omega(x, r) \quad \text{for all } x \in G \text{ and } 0 < r < s; \]
  \item [(\omega 2)] \( \omega(x, \cdot) \) is uniformly doubling on \( (0, \infty) \), that is, there exists a constant \( Q_2 > 0 \) such that
    \[ Q_2^{-1} \omega(x, r) \leq \omega(x, 2r) \leq Q_2 \omega(x, r) \quad \text{for all } x \in G \text{ and } r > 0; \]
  \item [(\omega 3)] there exists a constant \( Q_3 > 0 \) such that
    \[ Q_3^{-1} \leq \omega(x, 1) \leq Q_3 \quad \text{for all } x \in G. \]
\end{itemize}

Then one can find constants \( a, b > 0 \) and \( C > 1 \) such that

\begin{equation}
C^{-1}r^a \leq \omega(x, r) \leq Cr^{-b}
\end{equation}

for all \( x \in G \) and \( 0 < r \leq d_G \).

For later use, it is convenient to note the following result, which is proved by (P1), (P2) and (2.1).
Lemma 2.1. There exists a constant $C > 0$ such that
\[ \omega(x, r) p(x) \leq C \omega(x, r) p(y) \]
whenever $|x - y| < r \leq d_G$.

For a locally integrable function $f$ on $G$, set
\[ \|f\|_{L^p(G)} = \inf \left\{ \lambda > 0 : \int_G \left( \frac{|f(y)|}{\lambda} \right)^{p(y)} dy \leq 1 \right\} ; \]
in what follows, set $f = 0$ outside $G$. We denote by $L^p(G)$ the family of locally integrable functions $f$ on $G$ satisfying $\|f\|_{L^p(G)} < \infty$.

Lemma 2.2. Let $0 < q < \infty$. Then
1. \[ \int_0^{2d_G} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \sim \sum_{j=1}^{\infty} (\omega(x, 2^{-j+1}d_G) \|f\|_{L^p(A(x, 2^{-j+1}d_G))})^q ; \]
2. \[ \int_0^{2d_G} (\omega(x, r) \|f\|_{L^p(B(x, r))})^q dr/r \sim \sum_{j=1}^{\infty} (\omega(x, 2^{-j+1}d_G) \|f\|_{L^p(B(x, 2^{-j+1}d_G))})^q ; \]
and
3. \[ \int_0^{2d_G} (\omega(x, r) \|f\|_{L^p(G \setminus B(x, r))})^q dr/r \sim \sum_{j=1}^{\infty} (\omega(x, 2^{-j}d_G) \|f\|_{L^p(G \setminus B(x, 2^{-j}d_G))})^q \]
for all $x \in G$ and measurable functions $f$ on $G$.

Proof. We only prove (1), since the remaining assertions can be proved similarly. Since $A(x, r) \supset B(x, 3t/2) \setminus B(x, t)$ when $3t/2 < r < 2t \leq 2d_G$, we have by $(\omega 1)$ and $(\omega 2)$ that
\[ \int_{3t/2}^{2t} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \geq C (\omega(x, t) \|f\|_{L^p(\partial(B(x, 3t/2) \setminus B(x, t))})^q \]
and similarly, we have
\[ \int_t^{3t/2} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \geq C (\omega(x, t) \|f\|_{L^p(B(x, 3t/2) \setminus B(x, 3t/4))})^q . \]
Thus
\[ \int_t^{2t} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \geq C (\omega(x, t) \|f\|_{L^p(B(x, 3t/2) \setminus B(x, 3t/4))})^q . \]
Therefore, letting $3t/2 = 2^{j+1}d_G$ for a positive integer $j$, we see that
\[ \int_0^{2^{-j+2}d_G} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \geq C (\omega(x, 2^{-j+1}d_G) \|f\|_{L^p(A(x, 2^{-j+1}d_G))})^q , \]
so that
\[ \int_0^{2d_G} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \geq \frac{1}{2} \sum_{j=1}^{\infty} \int_0^{2^{-j+2}d_G} (\omega(x, r) \|f\|_{L^p(A(x, r))})^q dr/r \]
\[ \geq C \sum_{j=1}^{\infty} (\omega(x, 2^{-j+1}d_G) \|f\|_{L^p(A(x, 2^{-j+1}d_G))})^q . \]
The converse inequality is easily obtained. \qed
Further, we obtain the next result.

**Lemma 2.3.** Suppose $0 < q \leq \infty$. If \( \|f\|_{p(\cdot), q; \omega(G)} \leq 1 \), then there exists a constant $C > 0$ such that \( \|f\|_{p(\cdot), \infty; \omega(G)} \leq C \), for $h = \mathcal{H}(x_0), \mathcal{H}(x_0), \mathcal{H}(x_0), \mathcal{H}, \mathcal{H}$.

By Lemma 2.1, we have the following result.

**Lemma 2.4.** There is a constant $C > 0$ such that
\[
\int_{B(x_0, r)} |f(y)|^{p(y)} \, dy \leq C \omega(x_0, r)^{-p(x_0)}
\]
when $x_0 \in G$, $0 < r < d_G$ and $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(B(x_0, r))} \leq 1$.

**Lemma 2.5.** There is a constant $C > 0$ such that
\[
\frac{1}{|A(x_0, r)|} \int_{A(x_0, r)} |f(y)| \, dy \leq C r^{-n/p(x_0)} \omega(x_0, r)^{-1}
\]
when $x_0 \in G$, $0 < r < d_G$ and $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(A(x_0, r))} \leq 1$.

**Proof.** Fix $x_0 \in G$ and $0 < r < d_G$. Let $f$ be a nonnegative measurable function on $G$ satisfying $\omega(x_0, r)\|f\|_{L^{p(\cdot)}(A(x_0, r))} \leq 1$. Then we have by (P2) and Lemmas 2.1 and 2.4,
\[
\frac{1}{|A(x_0, r)|} \int_{A(x_0, r)} f(y) \, dy \\
\leq r^{-n/p(x_0)} \omega(x_0, r)^{-1} + \frac{1}{|A(x_0, r)|} \int_{A(x_0, r)} f(y) \left( \frac{f(y)}{r^{-n/p(x_0)} \omega(x_0, r)^{-1}} \right)^{p(y)-1} \, dy \\
\leq r^{-n/p(x_0)} \omega(x_0, r)^{-1} + C \left( r^{-n/p(x_0)} \omega(x_0, r)^{-1} \right)^{1-p(x_0)} \frac{1}{|A(x_0, r)|} \int_{A(x_0, r)} f(y)^{p(y)} \, dy \\
\leq C r^{-n/p(x_0)} \omega(x_0, r)^{-1},
\]
as required. \( \square \)

3. **Boundedness of the maximal operator for** $q = \infty$

Let us consider the following conditions: let $\eta \in \Omega(G)$ and $x_0 \in G$.

(\(\omega 3.1\)) There exists a constant $Q > 0$ such that
\[
\int_0^r t^{-n/p(x_0)} \omega(x_0, t)^{-1} \frac{dt}{t} \leq Q r^{-n/p(x_0)} \eta(x_0, r)^{-1}
\]
for all $0 < r \leq d_G$; and

(\(\omega 3.2\)) there exists a constant $Q > 0$ such that
\[
\int_r^{2d_G} t^{-n/p(x_0)} \omega(x_0, t)^{-1} \frac{dt}{t} \leq Q r^{-n/p(x_0)} \eta(x_0, r)^{-1}
\]
for all $0 < r \leq d_G$.

By the doubling condition on $\omega$, one notes from (\(\omega 3.1\)) or (\(\omega 3.2\)) that
\[
\omega(x_0, r)^{-1} \leq C \eta(x_0, r)^{-1}.
\]
Lemma 3.1. If \((\omega 3.1)\) and \((\omega 3.2)\) hold for all \(x_0 \in G\) with the same constant \(Q\), then there is a constant \(C > 0\) such that
\[
\int_{B(x,r)} |f(y)| dy \leq C r^{n-n/p(x)} \eta(x, r)^{-1}
\]
and
\[
\int_{G \setminus B(x,r)} |f(y)| |x-y|^{-n} dy \leq C r^{n-n/p(x)} \eta(x, r)^{-1}
\]
for all \(x \in G\), \(0 < r \leq d_G\) and \(f\) with \(\|f\|_{\mathcal{H}^{p(\cdot), \infty, \omega}(G)} \leq 1\).

Proof. Let \(f\) be a nonnegative measurable function on \(G\) satisfying \(\|f\|_{\mathcal{H}^{p(\cdot), \infty, \omega}(G)} \leq 1\). By Lemma 2.5 and \((\omega 3.1)\), we have
\[
\int_{B(x,r)} f(y) dy = \sum_{j=1}^{\infty} \int_{A(x,2^{-j+1}r)} f(y) dy \leq C \sum_{j=1}^{\infty} (2^{-j}r)^{n-n/p(x)} \omega(x, 2^{-j}r)^{-1}
\]
\[
\leq C r^{n-n/p(x)} \eta(x, r)^{-1}.
\]
Similarly, we obtain by use of Lemma 2.5 and \((\omega 3.2)\)
\[
\int_{G \setminus B(x,r)} |f(y)||x-y|^{-n} dy \leq C \sum_{j \geq 1, 2^{-j+1}r \leq d_G} (2^j r)^{-n} \int_{A(x,2^j r)} f(y) dy
\]
\[
\leq C \sum_{j \geq 1, 2^{-j+1}r \leq d_G} (2^j r)^{-n/p(x)} \omega(x, 2^j r)^{-1}
\]
\[
\leq C r^{n-n/p(x)} \eta(x, r)^{-1},
\]
as required. \(\square\)

For a locally integrable function \(f\) on \(G\), the Hardy–Littlewood maximal operator \(\mathcal{M}\) is defined by
\[
\mathcal{M}f(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy;
\]
recall that \(f = 0\) outside \(G\). Now we state the celebrated result by Diening [13].

Lemma 3.2. The maximal operator \(\mathcal{M}\) is bounded in \(L^{p(\cdot)}(G)\), that is, there exists a constant \(C > 0\) such that
\[
\|\mathcal{M}f\|_{L^{p(\cdot)}(G)} \leq C \|f\|_{L^{p(\cdot)}(G)}.
\]

Theorem 3.3. If \((\omega 3.1)\) and \((\omega 3.2)\) hold for all \(x_0 \in G\) with the same constant \(Q\), then the maximal operator \(\mathcal{M}\) is bounded from \(\mathcal{H}^{p(\cdot), \infty, \omega}(G)\) to \(\mathcal{H}^{p(\cdot), \infty, \eta}(G)\).

Guliyev, Hasanov and Samko [21, 22] proved that if \((\omega 3.2)\) holds for all \(x_0 \in G\) with the same constant \(Q\), then the maximal operator \(\mathcal{M}\) is bounded from \(\mathcal{H}^{p(\cdot), \infty, \omega}(G)\) to \(\mathcal{H}^{p(\cdot), \infty, \eta}(G)\) and if \((\omega 3.1)\) holds for \(x_0 \in G\), then the maximal operator \(\mathcal{M}\) is bounded from \(\mathcal{H}^{p(\cdot), \infty, \omega(x_0)}(G)\) to \(\mathcal{H}^{p(\cdot), \infty, \eta(x_0)}(G)\).

Proof of Theorem 3.3. Let \(f\) be a nonnegative measurable function on \(G\) such that \(\|f\|_{\mathcal{H}^{p(\cdot), \infty, \omega}(G)} \leq 1\). For \(x \in G\) and \(0 < r < d_G\), it suffices to show that
\[
\|\mathcal{M}f\|_{L^{p(\cdot)}(A(x, r))} \leq C \eta(x, r)^{-1}.
\]
For this purpose, set
\[
f = f \chi_{G \setminus B(x, 2r)} + f \chi_{B(x, 2r) \setminus B(x, r/4)} + f \chi_{B(x, r/4)} = f_1 + f_2 + f_3,
\]
where \( \chi_E \) denotes the characteristic function of \( E \). We note from Lemma 3.2 that
\[
\| \mathcal{M} f_2 \|_{L^p(A(x,r))} \leq C \| f_2 \|_{L^p(G)} \leq C \| f_2 \|_{L^p(B(x,2r) \setminus B(x,r/4))} \\
\leq C \left\{ \| f_2 \|_{L^p(B(x,2r) \setminus B(x,r))} + \| f_2 \|_{L^p(B(x,r) \setminus B(x,r/2))} \right\} \\
\leq C \omega(x,r)^{-1} \leq C \eta(x,r)^{-1}.
\]

For \( z \in A(x,r) \), Lemma 3.1 gives
\[
\mathcal{M} f_3(z) \leq C r^{-n} \int_{B(x,r/4)} f(y) \, dy \leq C r^{-n/p(x)} \eta(x,r)^{-1},
\]
so that
\[
\| \mathcal{M} f_3 \|_{L^p(A(x,r))} \leq C r^{-n/p(x)} \eta(x,r)^{-1} \left\| 1 \right\|_{L^p(A(x,r))} \leq C \eta(x,r)^{-1}.
\]
Moreover, Lemma 3.1 again gives
\[
\mathcal{M} f_1(z) \leq C \int_{G \setminus B(x,2r)} f(y) \, dy \leq C r^{-n/p(x)} \eta(x,r)^{-1}
\]
and hence
\[
\| \mathcal{M} f_1 \|_{L^p(A(x,r))} \leq C r^{-n/p(x)} \eta(x,r)^{-1} \left\| 1 \right\|_{L^p(A(x,r))} \leq C \eta(x,r)^{-1},
\]
as required. \( \square \)

**Remark 3.4.** If the conditions on \( \omega \) hold at \( x_0 \in G \) only, then one can see that \( \mathcal{M} \) is bounded from \( \mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \) to \( \mathcal{H}^{p(\cdot),\infty,\eta}_{\{x_0\}}(G) \).

**Corollary 3.5.** For bounded functions \( \nu(x) : G \to (-\infty, \infty) \) and \( \beta(x) : G \to (-\infty, \infty) \), set \( \omega(x,r) = r^{\nu(x)} (\log(2d_G/r))^{\beta(x)} \). If \( -n/p^+ < \nu^- \leq \nu^+ \leq n (1 - 1/p^-) \), then the maximal operator \( \mathcal{M} \) is bounded in \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

Define
\[
\omega_*(x,r) = \left( \int_0^r \omega(x,t)^{-1} \frac{dt}{t} \right)^{-1}
\]
and
\[
\omega^*(x,r) = \left( \int_r^{2d_G} \omega(x,t)^{-1} \frac{dt}{t} \right)^{-1}
\]
for \( x \in G \) and \( 0 < r \leq d_G \).

**Theorem 3.6.** (1) If \( \omega_*, d_G \) is bounded in \( G \), then \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \subset \mathcal{H}^{p(\cdot),\infty,\omega^*}(G) \).

(2) For each \( x_0 \in G \), \( \mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \subset \mathcal{H}^{p(\cdot),\infty,\omega^*}_{\{x_0\}}(G) \).

**Proof.** Let \( f \) be a measurable function on \( G \) such that \( \| f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \leq 1 \). We show only (1), because (2) can be proved similarly.

For (1), we see that
\[
\| f \|_{L^p(B(x,r))} \leq \sum_{j=1}^{\infty} \| f \|_{L^p(A(x,2^{-j}r))} \leq \sum_{j=1}^{\infty} \omega(x,2^{-j}r)^{-1} \leq C \omega_*(x,r)^{-1}
\]
for all \( x \in G \) and \( 0 < r \leq d_G \), as required. \( \square \)
Remark 3.7. Let \( \omega(x, r) = (\log(2d_G/r))^{\beta(x)+1} \) for a bounded function \( \beta(\cdot): G \to (-\infty, \infty) \).

1. If \( \text{ess inf}_{x \in G} \beta(x) > 0 \), then

\[
\omega_*(x, r) \sim \left( \log \frac{2d_G}{r} \right)^{\beta(x)}
\]

for all \( x \in G \) and \( 0 < r < d_G \); and

2. if \( \beta(x_0) < 0 \) for \( x_0 \in G \), then

\[
\omega^*(x_0, r) \sim \left( \log \frac{2d_G}{r} \right)^{\beta(x_0)}
\]

for all \( 0 < r < d_G \).

Remark 3.8. Let \( \omega(x, r) = r^{\nu(x)} \) for a bounded function \( \nu(\cdot): G \to (-\infty, \infty) \).

1. If \( \text{ess sup}_{x \in G} \nu(x) < 0 \), then

\[
\omega_*(x, r) \sim \omega(x, r)
\]

for all \( x \in G \) and \( 0 < r < d_G \); and

2. if \( \nu(x_0) > 0 \) for \( x_0 \in G \), then

\[
\omega^*(x_0, r) \sim \omega(x_0, r)
\]

for all \( 0 < r < d_G \).

Corollary 3.9. (1) Suppose \( (\omega 3.1) \) and \( (\omega 3.2) \) hold for all \( x_0 \in G \) with the same constant \( Q \). If \( \omega_*(\cdot, d_G) \) is bounded in \( G \), then the maximal operator \( M \) is bounded from \( \mathcal{H}^{p(\cdot), \infty, \omega}(G) \) to \( \mathcal{H}^{p(\cdot), \infty, \omega_*}(G) \).

(2) If \( (\omega 3.1) \) and \( (\omega 3.2) \) hold for \( x_0 \in G \), then the maximal operator \( M \) is bounded from \( \mathcal{H}^{p(\cdot), \infty, \omega}(x_0) \) to \( \mathcal{H}^{p(\cdot), \infty, \omega^*}(x_0) \).

Remark 3.10. Let us consider a singular integral operator \( T \) associated with a standard kernel \( k(x, y) \) in [15, Section 6.3] such that

\[
|k(x, y)| \leq K_1|x - y|^{-n}
\]

for all \( x, y \in \mathbb{R}^n \) and

\[
\|Tf\|_{L^p(\mathbb{R}^n)} \leq K_2\|f\|_{L^p(\mathbb{R}^n)}
\]

for all \( f \in L^p(\mathbb{R}^n) \).

If \( (\omega 3.1) \) and \( (\omega 3.2) \) hold for all \( x_0 \in G \) with the same constant \( Q \), then every singular integral operator \( T \) is bounded from \( \mathcal{H}^{p(\cdot), \infty, \omega}(G) \) to \( \mathcal{H}^{p(\cdot), \infty, \omega^*}(G) \).

4. Sobolev’s inequality for \( q = \infty \)

We consider the following condition: let \( \eta \in \Omega(G) \) and \( x_0 \in G \).

\( (\omega 4.1) \) For \( 0 < \alpha < n \), there exists a constant \( Q > 0 \) such that

\[
\int_r^{2d_G} t^{\alpha - n/p(x)} \omega(x, t)^{-1} \frac{dt}{t} \leq Q t^{\alpha - n/p(x)} \eta(x, r)^{-1}
\]

for all \( 0 < r < d_G \).

As in the proof of Lemma 3.1, we have the following result.
**Lemma 4.1.** If \((\omega 4.1)\) holds for all \(x_0 \in G\) with the same constant \(Q\), then there is a constant \(C > 0\) such that

\[
\int_{G \setminus B(x, r)} |x - y|^{\alpha - n} f(y) \, dy \leq C r^{\alpha - n/p} \eta(x, r)\^{-1}
\]

for all \(x \in G\), \(0 < r < d_G\) and \(f\) with \(\|f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)} \leq 1\).

For \(0 < \alpha < n\), the Riesz potential \(I_\alpha f\) is defined by

\[
I_\alpha f(x) = I_\alpha \ast f(x) = \int_G |x - y|^{\alpha - n} f(y) \, dy
\]

for measurable functions \(f\) on \(G\); and define

\[
\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.
\]

Let us begin with Sobolev's inequality proved by Diening [14, Theorem 5.2]:

**Lemma 4.2.** If \(0 < \alpha < n/p^\sharp\), then there exists a constant \(C > 0\) such that

\[
\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(G)} \leq C \|f\|_{L^{p^\sharp(\cdot)}(G)}
\]

for all \(f \in L^{p^\sharp(\cdot)}(G)\).

Our result is stated in the following:

**Theorem 4.3.** Let \(0 < \alpha < n/p^\sharp\). If \((\omega 3.1)\) and \((\omega 4.1)\) hold for all \(x_0 \in G\) with the same constant \(Q\), then there exists a constant \(C > 0\) such that

\[
\|I_\alpha f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)} \leq C \|f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)}
\]

for all \(f \in \mathcal{H}^{p(\cdot, \infty)}(G)\).

In view of Guliyev, Hasanov and Samko [21, 22], if \((\omega 4.1)\) holds for all \(x_0 \in G\) with the same constant \(Q\), then there exists a constant \(C > 0\) such that

\[
\|I_\alpha f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)} \leq C \|f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)}
\]

for all \(f \in \mathcal{H}^{p(\cdot, \infty)}(G)\) and if \((\omega 3.1)\) holds for \(x_0 \in G\), then there exists a constant \(C > 0\) (which may depend on \(x_0\)) such that

\[
\|I_\alpha f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)} \leq C \|f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)}
\]

for all \(f \in \mathcal{H}^{p(\cdot, \infty)}(G)\).

**Proof of Theorem 4.3.** Let \(f\) be a nonnegative measurable function on \(G\) such that \(\|f\|_{\mathcal{H}^{p(\cdot, \infty)}(G)} \leq 1\). For \(x \in G\) and \(0 < r < d_G\), we have only to show the inequality

\[
\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(A(x, r))} \leq C \eta(x, r)^{-1}.
\]

Set

\[
f = f \chi_{G \setminus B(x, 2r)} + f \chi_{B(x, 2r) \setminus B(x, r/4)} + f \chi_{B(x, r/4)} = f_1 + f_2 + f_3,
\]

as before. We note from Lemma 4.2 that

\[
\|I_\alpha f_2\|_{L^{p^\sharp(\cdot)}(A(x, r))} \leq C \|f_2\|_{L^{p^\sharp(\cdot)}(G)} \leq C \|f_2\|_{L^{p^\sharp(\cdot)}(B(x, 2r) \setminus B(x, r/4))} \leq C \omega(x, r)^{-1} \leq C \eta(x, r)^{-1}.
\]
If \( z \in A(x, r) \), then Lemma 3.1 gives
\[
I_\alpha f_3(z) \leq C r^{\alpha-n} \int_{B(x,r)/4} f(y) \, dy \leq C r^{\alpha-n/p(x)} \eta(x,r)^{-1},
\]
so that
\[
\| I_\alpha f_3 \|_{L^{p(\cdot)}(A(x,r))} \leq C r^{\alpha-n/p(x)} \eta(x,r)^{-1} \| 1 \|_{L^{p(\cdot)}(A(x,r))} \leq C \eta(x,r)^{-1}.
\]
Moreover, Lemma 4.1 gives
\[
I_\alpha f_1(z) \leq \int_{G\setminus B(x,2r)} |x-y|^{-\alpha-n} f(y) \, dy \leq C r^{\alpha-n/p(x)} \eta(x,r)^{-1},
\]
so that
\[
\| I_\alpha f_1 \|_{L^{p(\cdot)}(A(x,r))} \leq C r^{\alpha-n/p(x)} \eta(x,r)^{-1} \| 1 \|_{L^{p(\cdot)}(A(x,r))} \leq C \eta(x,r)^{-1},
\]
as required. \( \square \)

Corollary 4.4. Let \( 0 < \alpha < n/p^+ \) and let \( \nu, \beta \) and \( \omega \) be as in Corollary 3.5. If \( \alpha - n/p^+ < \nu^- \leq \nu^+ < n(1-1/p^-) \), then there exists a constant \( C > 0 \) such that
\[
\| I_\alpha f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}
\]
for all \( f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

Corollary 4.5. Assume that \( 0 < \alpha < n/p^+ \).

1. Suppose (\( \omega3.1 \)) and (\( \omega4.1 \)) hold for all \( x_0 \in G \) with the same constant \( Q \). If \( \omega_*(\cdot,d_G) \) is bounded in \( G \), then the operator \( I_\alpha \) is bounded from \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \) to \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

2. If (\( \omega3.1 \)) and (\( \omega4.1 \)) hold for \( x_0 \in G \), then the operator \( I_\alpha \) is bounded from \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \) to \( \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

5. Exponential integrability for \( q = \infty \)

Set
\[
E_1(x,t) = \exp \left( t^{q(x)} \right) - 1,
\]
where \( 1/p(x) + 1/q(x) = 1 \). For a locally integrable function \( f \) on \( G \), set
\[
\| f \|_{L^{E_1}(G)} = \inf \left\{ \lambda > 0 : \int_G E_1 \left( x, \frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}.
\]
We denote by \( L^{E_1}(G) \) the class of locally integrable functions \( f \) on \( G \) satisfying \( \| f \|_{L^{E_1}(G)} < \infty \).

In connection with \( \mathcal{H}^{p(\cdot),q,\omega}(G) \), let us consider \( \mathcal{H}^{E_1,q,\omega}(G) \) of all functions \( f \) satisfying
\[
\| f \|_{\mathcal{H}^{E_1,q,\omega}(G)} = \sup_{x_0 \in G} \left( \int_0^{2d_G} \left( \omega(x_0,r) \| f \|_{L^{E_1}(A(x_0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty.
\]
Similarly, we define \( \overline{\mathcal{H}}^{E_1,q,\omega}(G) \) and \( \overline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G) \).

Lemma 5.1. \( \| 1 \|_{L^{E_1}(B(x,r))} \sim (\log(1+1/r))^{-1/q(x)} \)
for all \( x \in G \) and \( 0 < r < d_G \).
Lemma 5.2. [28, Theorem 4.1, Corollary 4.2] If \( \alpha \geq n/p^- \), then there exists a constant \( C > 0 \) such that
\[
\| I_\alpha f \|_{L^{p_1}(G)} \leq C \| f \|_{L^{p}(G)}
\]
for all \( f \in L^{p}(G) \).

Our result is stated in the following:

Theorem 5.3. Let \( \alpha \geq n/p^- \).

1. If \((\omega_3.1)\) and \((\omega_4.1)\) hold for all \( x_0 \in G \) with the same constant \( Q \), then there exists a constant \( C > 0 \) such that
\[
\| I_\alpha f \|_{\mathcal{H}^{p_1}\omega(G)} \leq C \| f \|_{\mathcal{H}^{p}\omega(G)}
\]
for all \( f \in \mathcal{H}^{p}\omega(G) \).

2. If \((\omega_4.1)\) holds for all \( x_0 \in G \) with the same constant \( Q \), then there exists a constant \( C > 0 \) such that
\[
\| I_\alpha f \|_{\mathcal{H}^{p_1,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p},\omega(G)}
\]
for all \( f \in \mathcal{H}^{p},\omega(G) \).

3. If \((\omega_3.1)\) holds for \( x_0 \in G \), then there exists a constant \( C > 0 \) (which may depend on \( x_0 \)) such that
\[
\| I_\alpha f \|_{\mathcal{H}^{p_1,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p},\omega(G)}
\]
for all \( f \in \mathcal{H}^{p},\omega(G) \).

Proof. We give only a proof of assertion (1). Let \( f \) be a nonnegative measurable function on \( G \) such that \( \| f \|_{\mathcal{H}^{p},\omega(G)} \leq 1 \). We have only to show the inequality
\[
\| I_\alpha f \|_{L^{p_1}(A(x,r))} \leq C \eta(x,r)^{-1}
\]
for all \( x \in G \) and \( 0 < r < d_G \). Set
\[
f = f \chi_{G \setminus B(x,r/2)} + f \chi_{B(x,2r) \setminus B(x,r/4)} + f \chi_{B(x,r/4)} = f_1 + f_2 + f_3,
\]
as before. We note from Lemma 5.2 that
\[
\| I_\alpha f_2 \|_{L^{p_1}(A(x,r))} \leq C \| f_2 \|_{L^{p}(B(x,2r) \setminus B(x,r/4))} \leq C \eta(x,r)^{-1}.
\]
If \( z \in A(x,r) \), then Lemma 3.1 gives
\[
I_\alpha f_3(z) \leq C r^{\alpha - n} \int_{B(x,r/4)} f(y) \, dy \leq C \eta(x,r)^{-1}
\]
since \( \alpha \geq n/p^- \), so that
\[
\| I_\alpha f_3 \|_{L^{p_1}(A(x,r))} \leq C \eta(x,r)^{-1} \| 1 \|_{L^{p_1}(A(x,r))} \leq C \eta(x,r)^{-1}
\]
by Lemma 5.1. Moreover, Lemma 4.1 gives
\[
I_\alpha f_1(z) \leq C \int_{G \setminus B(x,2r)} |x - y|^{\alpha - n} f(y) \, dy \leq C \eta(x,r)^{-1}
\]
since \( \alpha \geq n/p^- \), so that
\[
\| I_\alpha f_1 \|_{L^{p_1}(A(x,r))} \leq C \eta(x,r)^{-1} \| 1 \|_{L^{p_1}(A(x,r))} \leq C \eta(x,r)^{-1},
\]
as required. \( \square \)
Corollary 5.4. Let \( \alpha \geq n/p^- \) and let \( \nu, \beta \) and \( \omega \) be as in Corollary 3.5.

(1) When \( \alpha - n/p^+ < \nu^- \leq \nu^+ < n(1 - 1/p^-) \), there exists a constant \( C > 0 \) such that

\[
\| I_\alpha f \|_{\mathcal{H}^{p_1,\infty,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}
\]

for all \( f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

(2) When \( \alpha - n/p^+ < \nu^- \), there exists a constant \( C > 0 \) such that

\[
\| I_\alpha f \|_{\mathcal{H}^{p_1,\infty,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}
\]

for all \( f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

(3) When \( \nu(x_0) < n(1 - 1/p(x_0)) \) for \( x_0 \in G \), there exists a constant \( C > 0 \) (which may depend on \( x_0 \)) such that

\[
\| I_\alpha f \|_{\mathcal{H}^{p_1,\infty,\omega}(G)} \leq C \| f \|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}
\]

for all \( f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G) \).

6. Associate spaces of \( \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \)

Recall that for \( x_0 \in G \) and measurable functions \( f \) on \( G \),

\[
\| f \|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<t<d_G} \omega(x_0, t) \| f \|_{L^p(G \cap B(x_0, t))}
\]

and

\[
\| f \|_{\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)} = \int_0^{d_G} \omega(x_0, t) \| f \|_{L^p(B(x_0, t))} \frac{dt}{t}.
\]

Remark 6.1. Let \( x_0 \in G \). Note here that if \( \omega(x_0, 0) = \infty \), then \( \| f \|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty \) if and only if \( f = 0 \) a.e. Hence we may assume that \( \omega(x_0, 0) = 0 \) and then \( \omega(x_0, \cdot) \) is uniformly almost increasing on \( (0, \infty) \) when \( \| f \|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty \).

By the above remark, in this section, suppose

\[
\omega(x, 0) = 0 \quad \text{for all } x \in G.
\]

For \( x \in G \) and \( 0 < t < d_G \), we set

\[
p^+(B(x, t)) = \sup_{y \in B(x, t)} p(y),
\]

as before. We define \( 1/q(x) = 1 - 1/p(x) \).

Following Di Fratta and Fiorenza [17], we have the following Hölder type inequality for log-type weights.

Theorem 6.2. For \( x_0 \in G \), suppose 
\( (\omega 6.1) \) there exist constants \( b, Q > 0 \) such that

\[
\int_0^t \left( \log \frac{2d_G}{r} \right)^{-bp(x_0) - 1} \omega(x_0, r)^{-p^+(B(x_0, t))} \frac{dr}{r} \leq Q \left( \log \frac{2d_G}{t} \right)^{-bp(x_0)} \omega(x_0, t)^{-p(x_0)}
\]

for all \( 0 < t < d_G \).
Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)| \, dx \leq C \|f\|_{L^p(x_0, 1, \eta(G))} \|g\|_{H^p(x_0, \infty, \omega(G))}$$

for all measurable functions $f$ and $g$ on $G$, where

$$\eta(x_0, r) = \left( \log \frac{2d_G}{r} \right)^{-1} \omega(x_0, r)^{-1}.$$

Proof. Let $x_0 \in G$. Let $f$ and $g$ be nonnegative measurable functions on $G$ such that $\|f\|_{L^p(x_0, 1, \eta(G))} \leq 1$ and $\|g\|_{H^p(x_0, \infty, \omega(G))} \leq 1$. We have by Fubini’s theorem and Hölder’s inequality

$$\int_G f(x)g(x) \, dx$$

$$= \int_G f(x)g(x) \left( b \left( \log \frac{2d_G}{|x - x_0|} \right)^{-b} \frac{2d_G}{t} \right) dt dx$$

$$= b \int_0^{2d_G} \left( \int_{B(x_0, t)} f(x)g(x) \left( \log \frac{2d_G}{|x - x_0|} \right)^{-b} dx \right) \left( \log \frac{2d_G}{t} \right)^{-1} dt dx$$

$$\leq C \int_0^{2d_G} \|f\|_{L^p(x_0, t)} \|g\|_{L^p(x_0, t)} \left( \log \frac{2d_G}{t} \right)^{-b} \omega(x_0, t)^{-1} dt dx.$$

Here it suffices to show

$$\left\| g \left( \log \frac{2d_G}{|x - x_0|} \right)^{-b} \right\|_{L^p(B(x_0, t))} \leq C \left( \log \frac{2d_G}{t} \right)^{-b} \omega(x_0, t)^{-1}$$

for $0 < t < d_G$. In fact, we obtain

$$\int_{B(x_0, r)} \frac{g(x)}{\left( \log(2d_G/t) \right)^{-b} \omega(x_0, t)^{-1}} \left( \log \frac{2d_G}{|x - x_0|} \right)^{-bp(x)} \, dx$$

$$\leq C \int_{B(x_0, r)} \frac{g(x)}{\left( \log(2d_G/t) \right)^{-b} \omega(x_0, t)^{-1}} \left( \log \frac{2d_G}{|x - x_0|} \right)^{-bp(x)} \, dx$$

$$\leq C \int_{B(x_0, r)} \frac{g(x)}{\left( \log(2d_G/t) \right)^{-b} \omega(x_0, t)^{-1}} \left( \log \frac{2d_G}{r} \right)^{-bp(x)} dr \, dx$$

$$\leq C \int_0^t \left( \int_{B(x_0, t) \setminus B(x_0, r)} g(x)^{p(x)} \left( \log \frac{2d_G}{t} \right)^{-bp(x)} \omega(x_0, t)^{p(x)} \left( \log \frac{2d_G}{r} \right)^{-bp(x)} dx \right) \frac{dr}{r}.$$
Then there exists a constant $C > 402$.

Yoshihiro Mizuta and Takao Ohno

First, we show that

\[ \| f \|_{L^p(B(x_0, r))} \leq C \left( \log \frac{2d_G}{r} \right)^{\frac{bp(x_0)}{b}} \omega(x_0, t)^{p(x_0)} \int_0^t \left( \log \frac{2d_G}{r} \right)^{-bp(x_0)-1} \]

\[ \cdot \left( \int_{B(x_0, t) \setminus B(x_0, r)} \frac{g(x)}{\| \|_{L^p(G \setminus B(x_0, r))}^{p(x)}} \right) \frac{\| g \|_{L^p(G \setminus B(x_0, r))}^{p(x)}}{r} \, dr \]

\[ \leq C \left( \log \frac{2d_G}{t} \right)^{\frac{bp(x_0)}{b}} \omega(x_0, t)^{p(x_0)} \int_0^t \left( \log \frac{2d_G}{r} \right)^{-bp(x_0)-1} \omega(x_0, r)^{-p^+(B(x_0, t))} \frac{dr}{r} \leq C \]

by (P2), Lemma 2.1 and (ω6.1).

Power weights can be treated simpler than Theorem 6.2 in the following manner.

**Theorem 6.3.** For $x_0 \in G$, suppose

(ω6.2) there exist constants $b, Q > 0$ such that

\[ \int_0^t s^b \omega(x_0, r)^{-1} \frac{dr}{r} \leq Qt^b \omega(x_0, t)^{-1} \]

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

\[ \int_G |f(x)g(x)| \, dx \leq C \| f \|_{\mathcal{A}^{(1,1)}(G)} \cdot \| g \|_{\mathcal{P}^{(\infty,\omega)}(G)} \]

for all measurable functions $f$ and $g$ on $G$, where $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

**Proof.** Let $x_0 \in G$. Let $f$ and $g$ be nonnegative measurable functions on $G$ such that $\| f \|_{\mathcal{A}^{(1,1)}(G)} \leq 1$ and $\| g \|_{\mathcal{P}^{(\infty,\omega)}(G)} \leq 1$. For $b > 0$, we have by Fubini’s theorem and Hölder’s inequality

\[ \int_G f(x)g(x) \, dx \leq C \int_0^{2d_G} \left( \int_{B(x_0, t)} f(x)g(x)|x - x_0|^{b} \, dx \right) t^{-b} \frac{dt}{t} \]

\[ \leq C \int_0^{2d_G} \| f \|_{L^p(B(x_0, t))} \| g \|_{L^p(B(x_0, t))} t^{-b} \frac{dt}{t}. \]

First, we show that

\[ \| g \|_{L^p(B(x_0, 2s) \setminus B(x_0, s))} \leq C s^b \omega(x_0, s)^{-1} \leq C s^b \omega(x_0, s)^{-1} \]

for all $0 < s < d_G$. In fact, we obtain

\[ \int_{B(x_0, 2s) \setminus B(x_0, s)} \frac{g(x)}{s^b \omega(x_0, s)^{-1}} \left( \frac{g(x)}{s^b \omega(x_0, s)^{-1}} \right)^{bp(x)} \, dx \]

\[ \leq C \int_{B(x_0, 2s) \setminus B(x_0, s)} \frac{g(x)}{\| g \|_{L^p(B(x_0, 2s) \setminus B(x_0, s))}^{p(x)}} \]

\[ \cdot \left( \omega(x_0, s)^{-p^+(B(x_0, 2s) \setminus B(x_0, s))} \right)^{p(x)} \, dx \leq C. \]
by (P2) and Lemma 2.1, which gives
\[ \| g \cdot x_n \|_{L^p(B(x_0, t))} \leq \sum_{j=1}^\infty \| g \cdot x_n \|_{L^p(B(x_0, 2^{-j+1}t) \setminus B(x_0, 2^{-j}t))} \]
\[ \leq C \int_0^t r^b\omega(x_0, r)^{-1}dr \leq Ct^b\omega(x_0, t)^{-1} \]
by (\omega.6.2). Thus we obtain the required result. \( \square \)

**Theorem 6.4.** Let \( \eta(\cdot, \cdot) \in \Omega(G) \). For \( x_0 \in G \), suppose
(\omega.6.3) there exists a constant \( Q > 0 \) such that
\[ \int_0^{2d_G} \eta(x_0, r) \frac{dr}{r} \leq Q\omega(x_0, t)^{-1} \]
for all \( 0 < t < d_G \).
Then there exists a constant \( C > 0 \) such that
\[ \| f \|_{L^{p(\cdot), 1, \eta}(G)} \leq C \sup_G \int_G |f(x)g(x)| dx \]
for all measurable functions \( f \) on \( G \), where the supremum is taken over all measurable functions \( g \) on \( G \) such that \( \| g \|_X \leq 1 \) with \( X = \overline{\mathcal{H}}^{p(\cdot), \infty, \omega}_{\{x_0\}}(G) \).

**Proof.** Let \( x_0 \in G \). Let \( f \) be a nonnegative measurable function on \( G \). To show the claim, we may assume that
\[ \sup_G \int_G |f(x)g(x)| dx \leq 1, \]
where the supremum is taken over all measurable functions \( g \) on \( G \) such that \( \| g \|_X \leq 1 \). Take a compact set \( K \subset G \setminus \{x_0\} \). Since \( L^{p(\cdot)}(K) = \{g\chi_K: g \in L^{p(\cdot)}(G)\} \subset X \), \( f\chi_K \in L^{q(\cdot)}(G) \), in view of [25] or [16, Theorem 3.2.13]. By (\omega.6.3), we find
\[ \| f\chi_K \|_{L^{q(\cdot), 1, \eta}(G)} < \infty \]
and, moreover, we have by Lemma 2.2
\[ \sum_{j \in N_0} \eta(x_0, 2^{-j+1}d_G)F_j \sim \| f\chi_K \|_{L^{q(\cdot), 1, \eta}(G)} \]
where \( F_j = \|f_j\|_{L^{q(\cdot)}(G)} \), \( f_j = f\chi_{K \cap B(x_0, 2^{-j+1}d_G)} \) and \( N_0 \) is the set of positive integers \( j \) such that \( F_j > 0 \). Set
\[ g(x) = \sum_{j \in N_0} \eta(x_0, 2^{-j+1}d_G)|f_j(x)/F_j|^q(\cdot)-2f_j(x)/F_j. \]
Then we see that
\[ \| g \|_{L^{p(\cdot)}(G \setminus B(x_0, r))} \leq \sum_{j \in N_0, 2^{-j+1}d_G > r} \eta(x_0, 2^{-j+1}d_G)\|f_j/F_j|^q(\cdot)-2f_j/F_j\|_{L^{p(\cdot)}(G)} \]
\[ \leq \sum_{j \geq 1, 2^{-j+1}d_G > r} \eta(x_0, 2^{-j+1}d_G) \leq C\omega(x_0, r)^{-1} \]
for all \( 0 < r < d_G \) by (\omega.6.3) and hence
\[ \| g \|_{\overline{\mathcal{H}}^{p(\cdot), \infty, \omega}_{\{x_0\}}(G)} \leq C. \]
Consequently it follows that
\[
\int_G f(x) g(x) \, dx = \sum_{j \in \mathbb{N}_0} \eta(x_0, 2^{-j+1} d_G) \int_G f(x) |f_j(x)/f_j|^{q(x)} \frac{f(x)}{F_j} \, dx
\]
\[
= \sum_{j \in \mathbb{N}_0} \eta(x_0, 2^{-j+1} d_G) F_j \geq C \|f\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)}^{q(x)}.
\]
Hence, by the monotone convergence theorem, we have
\[
\sup_g \int_G f(x) g(x) \, dx \geq C \|f\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)}^{q(x)},
\]
which gives the required inequality.

Let \( X \) be a family of measurable functions on \( G \) with a norm \( \| \cdot \|_X \). Then the associate space \( X' \) of \( X \) is defined as the family of all measurable functions \( f \) on \( G \) such that
\[
\|f\|_{X'} = \sup_{g \in X : \|g\|_X \leq 1} \int_G |f(x)g(x)| \, dx < \infty.

Theorems 6.2, 6.3 and 6.4 give the following result.

**Corollary 6.5.** For \( x_0 \in G \), suppose (\( \omega 6.1 \)) and (\( \omega 6.3 \)) hold. Then
\[
\left( \mathcal{H}_{\{x_0\}}^{p(\cdot), \infty, \omega}(G) \right)' = \mathcal{H}_{\{x_0\}}^{q(\cdot), 1, \eta}(G),
\]
where \( \eta(x_0, r) = (\log \frac{2d_G}{r})^{-1} \omega(x_0, r)^{-1} \). If (\( \omega 6.2 \)) and (\( \omega 6.3 \)) hold, then the same conclusion is fulfilled with \( \eta(x_0, r) = \omega(x_0, r)^{-1} \).

For \( 0 < q \leq \infty \), set
\[
\mathcal{H}_{\{x_0\}}^{p(\cdot), q, \omega}(G) = \sum_{x_0 \in G} \mathcal{H}_{\{x_0\}}^{p(\cdot), q, \omega}(G),
\]
whose quasi-norm is defined by
\[
\|f\|_{\mathcal{H}_{\{x_0\}}^{p(\cdot), q, \omega}(G)} = \inf_{|f| = \sum_{i} \lambda_i f_i, \{f_i\} \subset G} \sum_i \|f_i\|_{\mathcal{H}_{\{x_0\}}^{p(\cdot), \omega}(G)}.
\]
The Hölder type inequality in Theorem 6.2 or 6.3, under the same assumptions, implies
\[
\int_G |f(x)g(x)| \, dx = \sum_j \int_G |f(x)g_j(x)| \, dx \leq C \|f\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)} \sum_j \|g_j\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)};
\]
so that
\[
\int_G |f(x)g(x)| \, dx \leq C \|f\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)} \|g\| \left\| \frac{\psi_{\omega}}{\psi_{\eta}} \right\|_{L^p(x_0, \omega)}.
\]
Theorem 6.4 gives the converse inequality.

Theorems 6.2, 6.3 and 6.4 give the following result.

**Corollary 6.6.** If (\( \omega 6.1 \)) and (\( \omega 6.3 \)) hold for all \( x_0 \in G \) with the same constant \( Q \), then
\[
\left( \mathcal{H}_{\{x_0\}}^{p(\cdot), \infty, \omega}(G) \right)' = \mathcal{H}_{\{x_0\}}^{q(\cdot), 1, \eta}(G),
\]
where \( \eta(x_0, r) = (\log \frac{2d_G}{r})^{-1} \omega(x_0, r)^{-1} \). If (\( \omega 6.2 \)) and (\( \omega 6.3 \)) hold for all \( x_0 \in G \) with the same constant \( Q \), then the same conclusion is fulfilled with \( \eta(x_0, r) = \omega(x_0, r)^{-1} \).
Remark 6.7. For $0 < q < \infty$, set
\[
\mathcal{H}^{p(\cdot),\eta,\omega}(G) = \bigcap_{x_0 \in G} \mathcal{H}^{p(\cdot),\eta,\omega}_{(x_0)}(G)
\]
and define the norm
\[
\|f\|_{\mathcal{H}^{p(\cdot),\eta,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\mathcal{H}^{p(\cdot),\eta,\omega}_{(x_0)}(G)},
\]
as usual. Then note that
\[
\mathcal{H}^{p(\cdot),\infty,\omega}(G) = \begin{cases} 
L^{p(\cdot)}(G), & \omega(x,0) = 0 \text{ for all } x \in G; \\
\{0\}, & \omega(x,0) = \infty \text{ for all } x \in G.
\end{cases}
\]

For related results, we refer the reader to the paper by Di Fratta and Fiorenza [17] with logarithmic weights, and the paper by Gagatsishvili and Mustafayev [19] with general weights.

Remark 6.8. If $\omega(t) = (\log(2d_G/t))^{-a}$ with $a > 0$, then (\omega 6.1) and (\omega 6.3) hold for $\eta(t) = (\log(2d_G/t))^{a-1}$; and if $\omega(t) = t^a$ with $a > 0$, then (\omega 6.2) and (\omega 6.3) hold for $\eta(t) = t^{-a}$.

7. Associate spaces of $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$

Recall that for $x_0 \in G$ and measurable functions $f$ on $G$,
\[
\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{(x_0)}(G)} = \sup_{0 < t < d_G} \omega(x_0, t) \|f\|_{L^{p(\cdot)}(B(x_0,t))}
\]
and
\[
\|f\|_{\mathcal{H}^{p(\cdot),1,\omega}_{(x_0)}(G)} = \int_0^{2d_G} \omega(x_0, t) \|f\|_{L^{p(\cdot)}(G \setminus B(x_0,t))} \frac{dt}{t}.
\]

We have the Hölder type inequality for log type weights $\omega$.

Theorem 7.1. For $x_0 \in G$, suppose
(\omega 7.1) there exist constants $r_0, b, Q > 0$ such that
\[
\int_t^{2r_0} \left( \left( \frac{2d_G}{r} \right)^b \omega(x_0, t)^{-1} \right)^{\frac{cp}{\log(2d_G/r)}} \left( \left( \frac{2d_G}{r} \right)^b \omega(x_0, r)^{-1} \right)^{\frac{p(x_0)}{2d_G}} \left( \frac{2d_G}{r} \right)^{-1} \frac{dr}{r} \leq Q \left( \left( \frac{2d_G}{r} \right)^b \omega(x_0, t)^{-1} \right)^{\frac{p(x_0)}{2d_G}}
\]
for all $0 < t < r_0$.

Then there exists a constant $C > 0$ such that
\[
\int_G |f(x)g(x)| \, dx \leq C \|f\|_{\mathcal{H}^{p(\cdot),\eta,\omega}_{(x_0)}(G)} \|g\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{(x_0)}(G)}
\]
for all measurable functions $f$, $g$ on $G$, where $\eta(x_0, r) = (\log(2d_G/r))^{-1} \omega(x_0, r)^{-1}$.

Proof. Let $x_0 \in G$. Let $f$ and $g$ be nonnegative measurable functions on $G$ such that $\|f\|_{\mathcal{H}^{p(\cdot),\eta,\omega}_{(x_0)}(G)} \leq 1$ and $\|g\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{(x_0)}(G)} \leq 1$. For $b > 0$ we have by Fubini’s theorem
and Hölder’s inequality
\[
\int_G f(x)g(x)\,dx 
\leq C \int_0^{d_G} \|f\|_{L^p(G \setminus B(x_0,t))} \left\| g \left( \log \frac{2d_G}{|x-x_0|} \right)^b \right\|_{L^p(G \setminus B(x_0,t))} \left( \log \frac{2d_G}{t} \right)^{-b-1} \frac{dt}{t},
\]
as in the proof of Theorem 6.2. It suffices to show
\[
\left\| g \left( \log \frac{2d_G}{|x-x_0|} \right)^b \right\|_{L^p(G \setminus B(x_0,t))} \leq C \left( \log \frac{2d_G}{t} \right)^b \omega(x_0,t)^{-1} = C \left( \log \frac{2d_G}{t} \right)^{b+1} \eta(x_0,t)
\]
for all \(0 < t < d_G\). In fact, we obtain for \(0 < r_0 < d_G\)
\[
\int_{B(x_0,r_0) \setminus B(x_0,t)} \left( \frac{g(x)}{(\log(2d_G/t))^b \omega(x_0,t)^{-1}} \right)^{p(x)} \left( \log \frac{2d_G}{|x-x_0|} \right)^{bp(x)} \,dx 
\leq C \int_{B(x_0,r_0) \setminus B(x_0,t)} \left( \frac{g(x)}{(\log(2d_G/t))^b \omega(x_0,t)^{-1}} \right)^{p(x)} \left( \log \frac{2d_G}{|x-x_0|} \right)^{bp(x)} \,dx 
\leq C \int_{B(x_0,r_0) \setminus B(x_0,t)} \left( \frac{g(x)}{(\log(2d_G/t))^b \omega(x_0,t)^{-1}} \right)^{p(x)} \left( \int_{|x-x_0|}^{2r_0} \left( \log \frac{2d_G}{r} \right)^{bp(x_0)-1} \frac{dr}{r} \right) \,dx 
\leq C \int_0^{2r_0} \left( \int_{B(x_0,r) \setminus B(x_0,t)} g(x)^{p(x)} \left( \log \frac{2d_G}{r} \right)^{-b} \omega(x_0,t)^{p(x)} \right) \left( \log \frac{2d_G}{r} \right)^{bp(x_0)-1} \frac{dr}{r} 
\leq C \left( \log \frac{2d_G}{t} \right)^{-bp(x_0)} \omega(x_0,t)^{p(x)} \int_t^{2r_0} \left( \log \frac{2d_G}{t} \right)^b \omega(x_0,t)^{-1} \cdot \left( \log \frac{2d_G}{r} \right)^{bp(x_0)-1} \left( \int_{B(x_0,r)} g(x)^{p(x)} \,dx \right) \frac{dr}{r} 
\leq C \left( \log \frac{2d_G}{t} \right)^{-bp(x_0)} \omega(x_0,t)^{p(x)} \int_t^{2r_0} \left( \log \frac{2d_G}{t} \right)^b \omega(x_0,t)^{-1} \cdot \left( \log \frac{2d_G}{r} \right)^{bp(x_0)} \omega(x_0,r)^{-1} \left( \log \frac{2d_G}{r} \right)^{-1} \frac{dr}{r} 
\leq C \left( \log \frac{2d_G}{t} \right)^{-bp(x_0)} \omega(x_0,t)^{p(x)} \left( \log \frac{2d_G}{t} \right)^b \omega(x_0,t)^{-1} \leq C
\]
by (P2), condition \((\omega 7.1)\) and Lemmas 2.1 and 2.4, which gives
\[
\left\| g \left( \log \frac{2d_G}{r} \right)^b \right\|_{L_p(B(x_0, r_0))} \leq C \left( \log \frac{2d_G}{t} \right)^b \omega(x_0, t)^{-1}
\]
for all \(0 < t < r_0\). Moreover,
\[
\left\| g \left( \log \frac{2d_G}{r} \right)^b \right\|_{L_p(G \setminus B(x_0, r_0))} \leq C \left\| g \right\|_{L_p(G \setminus B(x_0, r_0))} \leq C,
\]
which completes the proof.

\[\square\]

**Remark 7.2.** We show that \(\omega(t) = (\log(2d_G/t))^a\) with \(a > 0\) satisfies \((\omega 7.1)\).
To show this, for \(b, c > 0\) one can find constants \(r_0, Q > 0\) such that
\[
\int_t^{2r_0} \left( \log \frac{2d_G}{r} \right)^{c/\log(2d_G/r)} \left( \log \frac{2d_G}{r} \right)^{-b-1} \frac{dr}{r} \leq Q \left( \log \frac{2d_G}{t} \right)^b
\]
for all \(0 < t < r_0\) and \(x_0 \in G\). In fact, first find \(0 < r_0 < d_G/e\) such that \(\varepsilon = 1/\log(d_G/r_0) < b/2c\), and note for \(t = 2d_Ge^{-1/(\log(2d_G/t))^{1/2}}\) that
\[
\int_t^{2r_0} \left( \log \frac{2d_G}{r} \right)^{c/\log(2d_G/r)} \left( \log \frac{2d_G}{r} \right)^{-b-1} \frac{dr}{r} \leq C \int_t^{2r_0} \left( \log \frac{2d_G}{r} \right)^{-b} \frac{dr}{r} \leq Q \left( \log \frac{2d_G}{t} \right)^b
\]
since \((\log(2d_G/t))^{c/\log(2d_G/r)} \leq C\) for all \(t < r < \hat{t}\) and
\[
\int_t^{2r_0} \left( \log \frac{2d_G}{t} \right)^{c/\log(2d_G/r)} \left( \log \frac{2d_G}{r} \right)^{-b} \frac{dr}{r} \leq C \left( \log \frac{2d_G}{t} \right)^c \int_t^{2r_0} \left( \log \frac{2d_G}{r} \right)^{-b} \frac{dr}{r} \leq Q \left( \log \frac{2d_G}{t} \right)^{cx+b/2} \leq Q \left( \log \frac{2d_G}{t} \right)^b,
\]
as required.

For power weights \(\omega\), we obtain the following result.

**Theorem 7.3.** For \(x_0 \in G\), suppose \((\omega 7.2)\) there exist constants \(b, Q > 0\) such that
\[
\int_t^{2d_G} r^{-b} \omega(x_0, r)^{-1} \frac{dr}{r} \leq Qr^{-b} \omega(x_0, t)^{-1}
\]
for all \(0 < t < d_G\).

Then there exists a constant \(C > 0\) such that
\[
\int_G |f(x)g(x)| \, dx \leq C \left\| f \right\|_{L_p(G \setminus B(x_0, r_0))} \left\| g \right\|_{L_p(G \setminus B(x_0, r_0))}
\]
for all measurable functions \(f, g\) on \(G\), where \(\eta(x_0, r) = \omega(x_0, r)^{-1}\).

As in the proof of Theorem 6.4, we have the following result.

**Theorem 7.4.** Let \(\eta(\cdot, \cdot) \in \Omega(G)\). For \(x_0 \in G\), suppose
(ω7.3) there exists a constant $Q > 0$ such that
\[
\int_0^t \eta(x_0, r) \frac{dr}{r} \leq Q \omega(x_0, t)^{-1}
\]
for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that
\[
\|f\|_{\mathcal{H}^{p, \infty}_{\{x_0\}}(G)} \leq C \sup_G \int_G |f(x)g(x)| \, dx
\]
for all measurable functions $f$ on $G$, where the supremum is taken over all measurable functions $g$ on $G$ such that $\|g\|_X \leq 1$ with $X = \mathcal{H}^{p, \infty}_{\{x_0\}}(G)$.

Theorems 7.1, 7.3 and 7.4 give the following result.

**Corollary 7.5.** If (ω7.1) and (ω7.3) hold for $x_0 \in G$, then
\[
\left(\mathcal{H}^{p, \infty}_{\{x_0\}}(G)\right)' = \mathcal{H}^{q, 1, \eta}_{\{x_0\}}(G),
\]
where $\eta(x_0, r) = (\log \frac{2d_G}{r})^{-1} \omega(x_0, r)^{-1}$. If (ω7.2) and (ω7.3) hold for $x_0 \in G$, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

**Remark 7.6.** If $\omega(t) = (\log(2d_G/t))^a$ with $a > 0$, then (ω7.1) and (ω7.3) hold for $\eta(t) = (\log(2d_G/t))^{-a-1}$; and if $\omega(t) = t^{-a}$ with $a > 0$, then (ω7.2) and (ω7.3) hold for $\eta(t) = t^a$.

For $0 < q \leq \infty$, we may consider
\[
\mathcal{H}^{p, q, \omega}_{\sim}(G) = \sum_{x_0 \in G} \mathcal{H}^{p, q, \omega}_{\{x_0\}}(G),
\]
whose quasi-norm is defined by
\[
\|f\|_{\mathcal{H}^{p, q, \omega}_{\sim}(G)} = \inf_{|f| = \sum_j |f_j|, \{x_j\} \subset G} \sum_j \|f_j\|_{\mathcal{H}^{p, q, \omega}_{\{x_j\}}(G)}.
\]

One can show that
\[
\mathcal{H}^{p, q, \omega}_{\sim}(G) = L^p_{\sim}(G).
\]

For this, we only show the inclusion $L^p_{\sim}(G) \subset \mathcal{H}^{p, q, \omega}_{\sim}(G)$. Take $f \in L^p_{\sim}(G)$ and $x_1, x_2 \in G$ ($x_1 \neq x_2$). Write
\[
f = f \chi_{B(x_2, |x_1-x_2|/2)} + f \chi_{G \setminus B(x_2, |x_1-x_2|/2)} = f_1 + f_2.
\]

Then
\[
\|f_1\|_{\mathcal{H}^{p, q, \omega}_{\{x_1\}}(G)} \leq \left( \int_{|x_1-x_2|/2}^{2d_G} \omega(x_1, r) \|f_1\|_{L^p(B(x_1, r))}^q \frac{dr}{r} \right)^{1/q} \\
\leq \|f_1\|_{L^p(G)} \left( \int_{|x_1-x_2|/2}^{2d_G} \omega(x_1, r)^q \frac{dr}{r} \right)^{1/q} = A \|f_1\|_{L^p(G)}
\]
and
\[ \|f_2\|_{L^{p(x),q(x)}(G)} \leq \left( \int_{|x_1-x_2|^2/2}^{2d_G} \omega(x_2,r) \|f_2\|_{L^{p(x)}(B(x_2,r))}^q \frac{dr}{r} \right)^{1/q} \]
\[ \leq \|f_2\|_{L^{p(x)}(G)} \left( \int_{|x_1-x_2|^2/2}^{2d_G} \omega(x_2,r)^q \frac{dr}{r} \right)^{1/q} = B\|f_2\|_{L^{p(x)}(G)}. \]
Hence
\[ \|f\|_{L^{p(x),q(x)}(G)} \leq \|f_1\|_{L^{p(x),q(x)}(G)} + \|f_2\|_{L^{p(x),q(x)}(G)} \leq A\|f_1\|_{L^{p(x)}(G)} + B\|f_2\|_{L^{p(x)}(G)} \]
\[ \leq (A + B)\|f\|_{L^{p(x)}(G)} < \infty, \]
as required.

8. Associate spaces of \( \mathcal{H}^{p(x),1,\omega}(G) \)

**Theorem 8.1.** Let \( \eta(\cdot, \cdot) \in \Omega(G) \), \( x_0 \in G \) and \( X = \mathcal{H}^{p(x),1,\omega}(G) \). Suppose \((\omega 8.1)\) there exists a constant \( Q > 0 \) such that
\[ \int_{t}^{2d_G} \omega(x_0,r) \frac{dr}{r} \leq Q\eta(x_0,t)^{-1} \]
for all \( 0 < t < d_G \).
Then there exists a constant \( C > 0 \) such that
\[ \|f\|_{\mathcal{H}^{p(x),\infty,\eta}(G)} \leq C\|f\|_{X'}, \]
for all measurable functions \( f \) on \( G \).

**Proof.** Let \( x_0 \in G \). First we show
\[ (8.1) \quad \int_{G \setminus B(x_0,R)} f(x)g(x) \, dx \leq C\eta(x_0,R)^{-1}\|g\|_{L^{p(x)}(G \setminus B(x_0,R))}\|f\|_{X'}, \]
for \( 0 < R < d_G \) and nonnegative measurable functions \( f, g \) on \( G \). To show this, we consider
\[ h = \eta(x_0,R)g \chi_{G \setminus B(x_0,R)}/\|g\|_{L^{p(x)}(G \setminus B(x_0,R))} \]
when \( 0 < \|g\|_{L^{p(x)}(G \setminus B(x_0,R))} < \infty \). Then we have by \((\omega 8.1)\)
\[ \int_{0}^{2d_G} \omega(x_0,t)\|h\|_{L^{p(x)}(B(x_0,t))} \frac{dt}{t} \leq \eta(x_0,R) \int_{R}^{2d_G} \omega(x_0,t) \frac{dt}{t} \leq C, \]
and hence
\[ \int_{G \setminus B(x_0,R)} f(x)h(x) \, dx \leq C\|f\|_{X'}. \]
Now we obtain
\[ \int_{G \setminus B(x_0,R)} f(x)g(x) \, dx \leq C\eta(x_0,R)^{-1}\|g\|_{L^{p(x)}(G \setminus B(x_0,R))}\|f\|_{X'}. \]
If we take \( g(x) = \|f\|_{L^{p(x)}(G \setminus B(x_0,R))}^{q(x)-1} \chi_{G \setminus B(x_0,R)} \) when \( 0 < \|f\|_{L^{p(x)}(G \setminus B(x_0,R))} < \infty \), then we have by \((8.1)\) that
1 = \int_{G \setminus B(x_0, R)} \{ f(x)/\| f \|_{L^{q}((G \setminus B(x_0, R)))} \}^{q(x)} dx
\leq C \eta(x_0, R)^{-1} \{ \| f \|_{L^{r}((G \setminus B(x_0, R)))} \}^{q^{-1} - 1} \| f \|_{L^{r}((G \setminus B(x_0, R)))} \| x' \|
\leq C \eta(x_0, R)^{-1} \{ \| f \|_{L^{r}((G \setminus B(x_0, R)))} \}^{-1} \| f \|_{x'},
which shows
\eta(x_0, R) \| f \|_{L^{r}((G \setminus B(x_0, R)))} \leq C \| f \|_{x'}.

Thus it follows that
\| f \|_{H^{p(\cdot), \infty, \eta}_{\{x_0\}}(G)} \leq C \| f \|_{x'},
as required.

**Corollary 8.2.** If (ω8.1) holds for \( x_0 \in G \) and (ω6.1) holds for \( x_0 \in G \), \( \eta \) and \( q(\cdot) \), then
\( \left( H^{p(\cdot), \omega}_{\{x_0\}}(G) \right)' = \mathcal{H}^{p(\cdot), \infty, \eta}_{\{x_0\}}(G), \)
where \( \eta(x_0, r) = (\log \frac{2dG}{r})^{-1} \omega(x_0, r)^{-1} \). If (ω8.1) holds for \( x_0 \in G \) and (ω6.2) holds for \( x_0 \in G \), \( \eta \) and \( q(\cdot) \), then the same conclusion is fulfilled with \( \eta(x_0, r) = \omega(x_0, r)^{-1} \).

As in Fiorenza–Rakotoson [18, Corollary 1], we see that the associate and dual spaces of \( \mathcal{H}^{p(\cdot), \omega}_{\{x_0\}}(G) \) coincides with each other.

**Remark 8.3.** If \( \omega(t) = (\log(2dG/t))^{-1/a} \) with \( a > 1 \), then (ω8.1) holds for \( \eta(t) = (\log(2dG/t))^{-1/a'} \); and if \( \omega(t) = t^{-a} \) with \( a > 0 \), then (ω8.1) holds for \( \eta(t) = t^a \).

9. Associate space of \( \mathcal{H}^{p(\cdot), \omega}_{\{x_0\}}(G) \)

As in the proof of Theorem 8.1, we have the following result.

**Theorem 9.1.** Let \( \eta(\cdot, \cdot) \in \Omega(G) \), \( x_0 \in G \) and \( X = \mathcal{H}^{p(\cdot), \omega}_{\{x_0\}}(G) \). Suppose (ω9.1) there exists a constant \( Q > 0 \) such that
\[ \int_{0}^{t} \omega(x_0, r) \frac{dr}{r} \leq Q \eta(x_0, t)^{-1} \]
for all \( 0 < t < dG \).

Then there exists a constant \( C > 0 \) such that
\[ \| f \|_{\mathcal{H}^{p(\cdot), \infty, \eta}_{\{x_0\}}(G)} \leq C \| f \|_{x'} \]
for all measurable functions \( f \) on \( G \).

**Corollary 9.2.** If (ω9.1) holds for \( x_0 \in G \) and (ω7.1) holds for \( x_0 \in G \), \( \eta \) and \( q(\cdot) \), then
\( \left( \mathcal{H}^{p(\cdot), \omega}_{\{x_0\}}(G) \right)' = \mathcal{H}^{p(\cdot), \infty, \eta}_{\{x_0\}}(G), \)
where \( \eta(x_0, r) = (\log \frac{2dG}{r})^{-1} \omega(x_0, r)^{-1} \). If (ω9.1) holds for \( x_0 \in G \) and (ω7.2) holds for \( x_0 \in G \), \( \eta \) and \( q(\cdot) \), then the same conclusion is fulfilled with \( \eta(x_0, r) = \omega(x_0, r)^{-1} \).
Corollary 9.3. If \((ω9.1)\) holds for all \(x_0 \in G\) with the same constant \(Q\) and \((ω7.1)\) holds for \(η, q(\cdot)\) and all \(x_0 \in G\) with the same constant \(Q\), then

\[
\left(\mathcal{H}^{p(\cdot),1,ω(\cdot)}(G)\right)' = \mathcal{H}^{q(\cdot),∞,η}(G),
\]

where \(η(x_0, r) = (\log \frac{2d}{r})^{-1} \omega(x_0, r)^{-1}\). If \((ω9.1)\) holds for all \(x_0 \in G\) with the same constant \(Q\) and \((ω7.2)\) holds for \(η, q(\cdot)\) and all \(x_0 \in G\) with the same constant \(Q\), then the same conclusion is fulfilled with \(η(x_0, r) = \omega(x_0, r)^{-1}\).

This corollary gives a characterization of Morrey spaces of variable exponents; see also the paper by Gogatishvili and Mustafayev [19] for constant exponents.

Remark 9.4. If \(ω(t) = (\log(2dG/t))^{-a-1}\) with \(a > 0\), then \((ω9.1)\) holds for \(η(t) = (\log(2dG/t))^{a}\); and if \(ω(t) = t^a\) with \(a > 0\), then \((ω9.1)\) holds for \(η(t) = t^{-a}\).

10. Grand and small Lebesgue spaces

Following Capone–Fiorenza [11], for \(0 < θ < 1\) and measurable functions \(f\) on the unit ball \(B = B(0, 1)\), we define the norm

\[
\left\|f\right\|_{\mathcal{P}_θ(\cdot),∞,θ}(B) = \sup_{0 < t < 1} \left(\log \frac{2}{t}\right)^{-θ/p(0)} \left\|f\right\|_{L^p(\cdot)(B \setminus B(0, t))}
\]

and

\[
\left\|f\right\|_{L^p(\cdot)-θ,θ}(B) = \sup_{0 < ε < p^{-1} - 1} \varepsilon^θ/p(0) \left\|f\right\|_{L^p(\cdot)-θ}(B).
\]

Theorem 10.1. There exists a constant \(C > 0\) such that

\[
\left\|f\right\|_{L^p(\cdot)-θ,θ}(B) \leq C \left\|f\right\|_{\mathcal{P}_θ(\cdot),∞,θ}(B)
\]

for all measurable functions \(f\) on \(B\).

Proof. Let \(f\) be a nonnegative measurable function on \(B\) such that \(\left\|f\right\|_{\mathcal{P}_θ(\cdot),∞,θ}(B) \leq 1\) or

\[
\int_{B \setminus B(0, t)} \left(\log \frac{2}{t}\right)^{-θ/p(0)} f(x)^{p(x)} \, dx \leq 1
\]

for all \(0 < t < 1\). For \(0 < ε < p^{-1} - 1\), we take \(0 < s < 1\) such that \(ε = (p^{-1} - 1)\log(2)/\log(2/s)\). We have

\[
\int_{B \setminus B(0, s)} \varepsilon^{θ/p(0)} f(x)^{p(x)-ε} \, dx \leq \int_{B \setminus B(0, s)} 1 \, dx + \int_{B \setminus B(0, s)} \varepsilon^{θ/p(0)} f(x)^{p(x)} \, dx \leq C.
\]

By multiplying (10.1) by \((\log(2/t))^{-b-1}\) for (large) \(b > 1\), integration gives

\[
\int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \frac{dt}{t} \geq \int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \left(\int_{B(0,r) \setminus B(0,t)} \left(\log \frac{2}{t}\right)^{-θ/p(0)} f(x)^{p(x)} \, dx \right) \frac{dt}{t}
\]
\[
\begin{aligned}
&\geq \int_0^r \left( \log \frac{2}{t} \right)^{-b-1} \left( \int_{B(0,r) \setminus B(0,t)} \left( \log \frac{2}{t} \right)^{-\theta - c_p / \log(2/|x|)} f(x)^{p(x)} \, dx \right) \frac{dt}{t} \\
&= \int_{B(0,r)} f(x)^{p(x)} \left( \int_0^{|x|} \left( \log \frac{2}{t} \right)^{-b-1-\theta - c_p / \log(2/|x|)} \frac{dt}{t} \right) \, dx \\
&\geq C \int_{B(0,r)} f(x)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-b-\theta} \, dx,
\end{aligned}
\]

or
\[
\int_{B(0,r)} f(x)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-b-\theta} \, dx \leq C \left( \frac{2}{r} \right)^{-b}
\]

for \(0 < r < 1\).

First consider the case when
\[
A = \int_{B(0,s)} \left( \log \frac{2}{|x|} \right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} \, dx \geq 1.
\]

For \(k > 1\), we obtain
\[
\begin{aligned}
&\int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)-\varepsilon} \, dx \\
&\leq \int_{B(0,s)} \left( \varepsilon^k A^{-1/p(0)} \left( \log \frac{2}{|x|} \right)^{(\theta+b)/\varepsilon} \right)^{p(x)-\varepsilon} \, dx \\
&+ \int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)-\varepsilon} \left( \varepsilon^k A^{-1/p(0)} \left( \log(2/|x|) \right)^{(\theta+b)/\varepsilon} \right)^{\varepsilon} \, dx \\
&\leq C \left\{ \varepsilon^{kp(0)} \int_{B(0,s)} A^{-(p(x)-\varepsilon)/p(0)} \left( \log \frac{2}{|x|} \right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} \, dx \\
&+ \varepsilon^\theta A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-(\theta+b)} \, dx \right\}
\end{aligned}
\]

since \(\varepsilon^{p(x)-\varepsilon} \leq C \varepsilon^{p(0)}\) by (P2) for all \(x \in B(0,s)\). Since \(\log(2/t) \leq (2^a/a)t^{-a}\) for \(0 < t < 1\) and \(a = \varepsilon / \{2(p(0) - \varepsilon)(\theta + b)\}\), we find
\[
A \leq \int_{B(0,s)} \left( \frac{2^a}{a} |x|^{-a} \right)^{1/(2a)} \, dx \leq \left( \frac{2^a}{a} \right)^{1/(2a)} \int_{B} |x|^{-1/2} \, dx \leq Ca^{-1/(2a)}
\]

so that we have by (P2)
\[
A^{-p(x)/p(0)} \leq C A^{-1+c\varepsilon/p(0)} \quad \text{for } x \in B(0,s) \quad \text{and some constant } c > 0
\]

and
\[
A^{\varepsilon/p(0)} \leq C \varepsilon^{-(b+\theta)}.
\]
Hence we have
\[
\int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)-\varepsilon} \, dx \\
\leq C \left\{ \varepsilon^{\theta - (p(0)-(1+c)/p(0))} \int_{B(0,s)} \left( \log \frac{2}{|x|} \right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} \, dx \\
+ \varepsilon^{\theta} A^\varepsilon/p(0) \int_{B(0,s)} f(x)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-(\theta+b)} \, dx \right\} \\
\leq C \left\{ \varepsilon^{\theta - (p(0)-(1+c)/p(0))} + \varepsilon^{\theta} A^\varepsilon/p(0) \right\} \leq C \left\{ \varepsilon^{\theta(p(0)-(b+\theta)(1+c)+1)} \right\}.
\]

If we take \( b \) and \( k \) such that \( kp(0) - (b+\theta)(1+c) \geq 0 \), then the present case is obtained.

If \( A \leq 1 \), then we obtain by (P2)
\[
\int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)-\varepsilon} \, dx \\
\leq \int_{B(0,s)} \left( \log \frac{2}{|x|} \right)^{-(\theta+b)(p(x)-\varepsilon)/\varepsilon} \, dx + \int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)-\varepsilon} \left( \frac{\varepsilon^{\theta/p(0)} f(x)}{(\log(2/|x|))^{(\theta+b)/\varepsilon}} \right)^{\varepsilon} \, dx \\
\leq C + \int_{B(0,s)} \left( \varepsilon^{\theta/p(0)} f(x) \right)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-(\theta+b)} \, dx \\
\leq C \left\{ 1 + \varepsilon^{\theta} \int_{B(0,s)} f(x)^{p(x)} \left( \log \frac{2}{|x|} \right)^{-(\theta+b)} \, dx \right\} \leq C \left\{ 1 + \varepsilon^{\theta} \left( \log \frac{2}{s} \right)^{\varepsilon \theta} \right\} \leq C,
\]
which completes the proof.

Given \( f \) on \( \mathbb{R}^n \), recall the definition of the symmetric decreasing rearrangement of \( f \) by
\[
f^*(x) = \int_0^\infty \chi_{E_f(t)}(x) \, dt,
\]
where \( E_f = \{ x : |B(0,|x|)| < |E| \} \) and \( E_f(t) = \{ y : |f(y)| > t \} \); see Burchard [6].

**Theorem 10.2.** There exists a constant \( C > 0 \) such that
\[
\|f^*\|_{\mathcal{L}^{(1),\infty,\varepsilon}_f(B)} \leq C \|f^*\|_{L^{p(0)-\theta}(B)}
\]
for all measurable functions \( f \) on \( B \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( B \) such that \( \|f^*\|_{L^{p(0)-\theta}(B)} \leq 1 \). Note that
\[
\int_{B \setminus B(0,1/2)} \left( \varepsilon^{\theta/p(0)} f^*(x) \right)^{p(x)-\varepsilon} \, dx \leq 1
\]
for all $0 < t < 1$ and $\varepsilon = (p^- - 1)(\log 2)/\log(2/t)$. We have
\[
\int_{B \setminus B(0,t)} \left( \frac{\varepsilon^{\theta/p(0)} f^*(x)}{x} \right)^{p(x)} \, dx 
\]
\[
\leq C \left( \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x) \, dx \right) ^\varepsilon \int_{B \setminus B(0,t)} \varepsilon^{\theta p(x)/p(0)} f^*(x)^{p(x)-\varepsilon} \, dx
\]
since $f^*$ is radially decreasing. Set
\[
I = \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x) \, dx
\]
and
\[
J = \left( \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x)^{p(x)-\varepsilon} \, dx \right) ^{1/(p(0)-\varepsilon)}.
\]
If $J \geq 1$, then we have by (10.2)
\[
I \leq J + C \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x) \left( \frac{f^*(x)}{J} \right)^{p(x)-\varepsilon-1} \, dx
\]
\[
\leq J + C \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x)^{p(x)-\varepsilon} \, dx \leq CJ
\]
by (P2) since $J \leq C t^{-n/p(0)} (\log(2/t))^{\theta/p(0)}$ for all $0 < t < 1$ and if $J \leq 1$, then
\[
I \leq 1 + \frac{1}{|B(0,t) \setminus B(0,t/2)|} \int_{B(0,t) \setminus B(0,t/2)} f^*(x)^{p(x)-\varepsilon} \, dx \leq C.
\]
Hence
\[
I^\varepsilon \leq C \left( t^{-n \varepsilon/p(0)} (\log(2/t))^{\theta \varepsilon/p(0)} + 1 \right) \leq C,
\]
so that
\[
\int_{B \setminus B(0,t)} \left( \frac{\varepsilon^{\theta/p(0)} f^*(x)}{x} \right)^{p(x)} \, dx \leq C \int_{B \setminus B(0,t/2)} \left( \frac{\varepsilon^{\theta/p(0)} f^*(x)}{x} \right)^{p(x)-\varepsilon} \, dx \leq C,
\]
which completes the proof. \qed

References

Sobolev’s theorem and duality for Herz–Morrey spaces of variable exponent


Received 25 March 2013 • Revised received 29 July 2013 • Accepted 26 August 2013