COMPLEX RICCATI DIFFERENTIAL EQUATIONS REVISITED

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Abstract. We utilise a new approach via the so-called re-scaling method to derive a thorough theory for polynomial Riccati differential equations in the complex domain.

1. Introduction

The basic features concerning the value distribution of the solutions to Riccati differential equations

\[ w' = a_0(z) + a_1(z)w + a_2(z)w^2 \]

with polynomial coefficients are well understood due to the pioneering work of Wittich (see his book [15], Chapter V, pp. 73–80). The solutions are meromorphic in the complex plane, and every non-rational solution has order of growth

\[ \varrho = \limsup_{r \to \infty} \frac{\log T(r, w)}{\log r} = 1 + n/2 \]

mean type, where the non-negative integer \( n \) depends on the coefficients \( a_\nu \) only. The aim of this paper is to refine the results of Wittich and others (Bank [1], Gundersen [5], Hellerstein and Rossi [7, 8]; see also Laine’s book [9], Chapter 5) on equation (1) and the associated linear differential equation (set \( a_2w = \frac{-u'}{u} \))

\[ u'' - \left( \frac{a_2'(z)}{a_2(z)} + a_1(z) \right) u' + a_0(z)a_2(z)u = 0 \]

by a new approach which has been developed earlier to investigate the solutions of Painlevé differential equations (see [12]). By a simple change of variables (retaining the original notation \( z, w \)) we obtain

\[ w' = a(z) - w^2 \]

with

\[ a(z) = z^n + O(|z|^{n-1}) \quad (z \to \infty). \]

Up to finitely many, all poles are simple with residue 1; \( w \) has counting function

\[ n(r, w) = O(r^\varrho). \]

Our proofs are solely based on the estimate (4), a new existence proof for asymptotic expansions, and the method of re-scaling.

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2. Re-scaling and the distribution of poles

Throughout the whole paper $w$ denotes any non-rational solution to the Riccati equation (R). For $h \neq 0$ we set

$$w_h(z) = h^{-n/2}w(h^2 + h^{-n/2}z),$$

where $h^{-n/2}$ denotes any branch, the same at every occurrence ($h^{-n/2}h^{-n/2} = h^{-n}$).

**Theorem 1.** The re-scaled family $(w_h)_{|h| > 1}$ is normal in the sense of Montel, and every limit function $w = \lim_{h_n \to \infty} w_{h_n}$ satisfies the differential equation

$$w' = 1 - w^2.$$  

We note that the solution $w = \coth \frac{z}{2}$ with pole at the origin has the poles $k\pi i$, $k \in \mathbb{Z}$, and no others. Any sequence $\sigma = (p_k)$ satisfying the approximate recursion

$$p_{k+1} = p_k + \omega p_k^{-n/2} + o(|p_k|^{-n/2})$$

with $\omega = \pm i\pi$ fixed is called a **string**.

**Theorem 2.** Let $w$ be any solution to (R). Then the set of poles on $|z| > r_0$ consists of finitely many strings of poles. Each string $\sigma$ accumulates at some Stokes rays

$$s_\nu: \arg z = \theta_\nu = \frac{(2\nu + 1)\pi}{n + 2}$$

and has counting function

$$n(r, \sigma) = \frac{\rho^\nu}{\pi \rho^2} + o(\rho^\nu).$$

**Remark.** We note that $w$ has Nevanlinna characteristic $T(r, w) = \ell \frac{\rho^\nu}{\pi \rho^2} + o(\rho^\nu)$, where $\ell = \ell(w)$ denotes the number of strings of poles.

3. Stokes sectors and asymptotic expansions

The open sectors

$$S_\nu: \arg z - \frac{2\nu \pi}{n + 2} < -\frac{\pi}{n + 2}$$

are called **Stokes sectors**. They are bounded by the Stokes rays $s_\nu$ and $s_{\nu - 1}$, and will be enumerated as follows:

(a) $0 \leq \nu \leq n + 1$ if $n$ is even, and

(b) $-m - 1 \leq \nu \leq m + 1$ if $n = 2m + 1$ is odd.

In the second case $s_{-m - 2} = s_{m + 1}$ coincides with the negative real axis.

Let $f$ be meromorphic on some sector $S: \phi_1 < \arg z < \phi_2$. Then $f$ is said to have the asymptotic expansion $f \sim \sum_{k=0}^\infty c_k z^{-k/q}$ for some $q \in \mathbb{N}$, if for every $\delta > 0$ and every $n \in \mathbb{N}_0$

$$f(z) - \sum_{k=0}^n c_k z^{-k/q} = o(|z|^{-n/q}) \quad (z \to \infty)$$

is valid, uniformly on every sub-sector $S(\delta): \phi_1 + \delta < \arg z < \phi_2 - \delta$. Obviously, the sector $S$ is ‘pole-free’ for $f$ in the following sense: to every $\delta > 0$ there exists $r(\delta) > 0$, such that $f$ has no poles on $S(\delta)$, $|z| > r(\delta)$. It follows from Theorem 2 that the Stokes sectors $S_\nu$ are ‘pole-free’ for every solution to equation (R). By $\sqrt{z}$
we denote the branch of the square root with \( \text{Re} \sqrt{z} > 0 \) on \( |\arg z| < \pi \), and set 
\[ z^{n/2} = (\sqrt{z})^n \] if \( n \) is odd.

**Theorem 3.** The function \( z^{-n/2}w(z) \) has an asymptotic expansion

(a) \( \varepsilon + \sum_{k=1}^{\infty} c_k z^{-k} \) if \( n \) is even, and

(b) \( \varepsilon + \sum_{k=1}^{\infty} c_k z^{-k/2} \) if \( n \) is odd

on every ‘pole-free’ sector \( S \), with \( \varepsilon = \varepsilon(w) \in \{-1, 1\} \) and coefficients \( c_k \) only depending on \( \varepsilon \), but neither on \( w \) nor the sector \( S \). The solution \( w \) is uniquely determined by its asymptotic expansion if \( S \) contains some sub-sector \( S' \) such that

\[ \varepsilon \text{ Re } z^\theta < 0 \quad \text{on } S'. \]

**Remark.** In particular, Theorem 3 holds on Stokes sectors \( S_\nu \) with \( \varepsilon = \varepsilon_\nu = \varepsilon_\nu(w) \). If (8) is valid on \( S_\nu \), then the corresponding solution is uniquely determined and is denoted by \( w_\nu \). With every solution \( w \) we associate its symbol \( \Sigma \).

(a) \( \Sigma = \Sigma(w) = [\varepsilon_0, \ldots, \varepsilon_{n+1}] \) if \( n \) is even, and

(b) \( \Sigma = \Sigma(w) = [\varepsilon_{m-1}, \ldots, \varepsilon_{m+1}] \) if \( n = 2m + 1 \) is odd.

Solutions having the symbol \( \Sigma(w) \) with entries \( \varepsilon_\nu = (-1)^\nu \) are called generic. Noting that \( (-1)^\nu \text{Re } z^\theta > 0 \) holds on \( S_\nu \), we obtain from Theorem 3:

**Theorem 4.** Any generic solution \( w \) has counting function of poles

\[ n(r, w) = \frac{2r^\theta}{\pi} + o(r^\theta). \]

**Theorem 5.** Suppose \( w \) has symbol \( \Sigma \). Then \( w \) has

(a) no string of poles asymptotic to the Stokes ray \( s_\nu \) if \( \varepsilon_\nu = \varepsilon_{\nu+1} \),

(b) exactly one such string if \( (-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = 2 \), while

(c) \( (-1)^\nu(\varepsilon_\nu - \varepsilon_{\nu+1}) = -2 \) is impossible.

If \( n = 2m + 1 \) is odd and \( \nu = m + 1 \), the term \( \varepsilon_{\nu+1} \) has to be replaced by \( -\varepsilon_{m-1} \). In case (a), \( w \) has an asymptotic expansion on \( \theta_{\nu-1} < \arg z < \theta_{\nu+1} \). Generic solutions have exactly one string of poles along every Stokes ray, and in any case we have

\[ n(r, w) = \frac{r^\theta}{\pi \theta} \sum_{\nu} (-1)^\nu \varepsilon_\nu + o(r^\theta). \]

4. Exceptional solutions

The non-generic solutions are called exceptional. Exceptional solutions \( w_\nu \) have the ‘false’ asymptotics

\[ w_\nu \approx (-1)^\nu z^{n/2} \quad \text{on } S_\nu \]

and are uniquely determined by that condition.

**Example 1.** The Riccati equation \( w' = z^2 + a_0 - w^2 \) is closely related to the Weber–Hermite equation

\[ y' = y^2 + 2zy - 2 - 2\alpha \quad (w = -y - z, \; a_0 = 1 + 2\alpha). \]

There are four exceptional solutions which may be described by their respective symbols \([-1, -1, 1, -1], [1, 1, 1, -1], [1, -1, -1, -1], \) and \([1, -1, 1, 1]\). The poles are
distributed along two rays: $|\arg z - \pi| = \frac{\pi}{4}$, $|\arg z + \frac{\pi}{2}| = \frac{\pi}{4}$, $|\arg z| = \frac{\pi}{4}$, and $|\arg z - \frac{\pi}{2}| = \frac{\pi}{4}$, respectively.

**Example 2.** The Riccati equation $w' = z + a_0 - w^2$ is closely related to the Airy equation $y'' = 2 + y^2$. It has three exceptional solutions with symbols $[-1, -1, -1]$, $[1, 1, -1]$, and $[-1, 1, 1]$, and strings of poles asymptotic to (actually: on) $\arg z = \pi$, $\arg z = \pi/3$, and $\arg z = -\pi/3$, respectively.

**Theorem 6.** To every Stokes sector $S_\nu$ there exists a unique exceptional solution $w_\nu$. It has the asymptotic expansion (9) also on the Stokes sectors adjacent to $S_\nu$, and no strings of poles along the Stokes rays that form the boundary of $S_\nu$. The number $d_\nu = n - \ell_\nu$, where $\ell_\nu$ denotes the number of strings of poles of $w_\nu$, is even.

**Remark.** The exceptional solutions $w_\nu$ correspond to those solutions to the linear differential equation $y'' = a(z)y$ that are sub-dominant on $S_\nu$; $y_\nu = \exp \int w(z) \, dz$ is called sub-dominant on $S_\nu$, if $y_\nu$ tends to zero exponentially as $z \to \infty$ on $S_\nu$.

**Example 3.** Gundersen and Steinbart [6] considered the linear differential equation $f'' - z^n f = 0$. They proved among others that certain contour integrals

$$f_\nu(z) = \frac{1}{2\pi i} \int_{C_\nu} e^{P(z,w)} \, dw$$

represent solutions having no zeros along given Stokes rays $s_{\nu-1}$ and $s_\nu$. These solutions give rise to exceptional solutions $w_\nu = f'_\nu / f_\nu$ to the special Riccati equation $w' = z^n - w^2$, which is invariant under the transformations $w(z) \mapsto \eta w(\eta z)$, $\eta^{n+2} = 1$. There are exactly two solutions that are invariant under these transformations, namely those which either have a pole or else a zero at the origin. These solutions are generic, hence there are $n + 2$ mutually distinct exceptional solutions. They are obtained from a single one, $w_0$, say, by rotating the plane:

$$w_\nu(z) = e^{\frac{2\nu\pi i}{n+2}} w_0(e^{\frac{2\nu\pi i}{n+2}} z);$$

$w_\nu$ has a single string of poles along every Stokes ray $s_\mu$ except those that bound the Stokes sector $S_\nu$.

In the general case (R) the solutions $w_\nu$ need not be mutually distinct.

**Example 4.** The eigenvalue problem $f'' + (z^4 - \lambda)f = 0$, $f \in L^2(\mathbb{R})$, has infinitely many solutions $(\lambda_k, f_k)$ ($0 < \lambda_k \to \infty$), see Titchmarsh [13]. The eigenfunctions $f_k$ have only finitely many non-real zeros. For every eigenpair $(\lambda, f) = (\lambda_k, f_k)$, $u(z) = f(e^{-i\pi/6} z)$ satisfies $u'' - (z^4 + e^{-i\pi/3} \lambda)u = 0$, and $w = u'/u$ solves

$$w' = z^4 + e^{-i\pi/3} \lambda - w^2.$$  

Up to finitely many the poles of the exceptional solution $w = w_3 = w_5$ belong to the rays $\arg z = \frac{\pi}{6}$ and $\arg z = \frac{\pi}{6} \pi$, hence $w$ has the symbol $[1, -1, -1, -1, 1, 1]$.

**Example 5.** Eremenko and Gabrielov [2] considered the linear equation

$$y'' - (z^3 - az + \lambda)y = 0.$$  

For certain real parameters $a$ and $\lambda$ it has solutions with infinitely many zeros, only finitely many of them are non-real or real and positive. Thus $w' = z^3 - az + \lambda - w^2$ has a solution $w$ with symbol $[1, 1, 1, 1, 1]$, hence $w = w_1 = w_{-1}$, and mutually distinct solutions $w_0$, $w_{-2}$, and $w_2$ with symbols $[1, -1, -1, -1, 1]$, $[-1, -1, 1, -1, 1]$, and $[1, -1, 1, -1, -1]$, respectively, each having three strings of poles.
5. Poles close to a single line

Several papers (Eremenko and Merenkov [3], Eremenko and Gabrielov [2], Gundersen [4, 5], Shin [11]) are devoted to the question whether or not the linear differential equation
\[
(10) \quad y'' - P(z)y = 0 \quad (P(z) = a_n z^n + \cdots \text{ a polynomial of degree } n, \quad |a_n| = 1)
\]
has solutions with all but finitely many zeros on the real axis. From Theorem 5 we obtain (see also [3, 4]):

**Theorem 7.** Suppose that equation (10) has a solution whose zeros are asymptotic to the real axis. Then the following is true:
If \( n \) is even, then either
- \( y \) has only finitely many zeros, or else
- \( n \equiv 0 \mod 4, a_n = -1, y \) has exactly one string of zeros asymptotic to the negative and positive real axis, and \( y'/y \approx \mp iz^{n/2} \) holds on the upper and lower half-plane, respectively.
If \( n = 2m + 1 \) is odd, then either
- \( a_n = 1, y \) has exactly one string of poles asymptotic to the negative real axis with asymptotics \( y'/y \approx (-1)^{m+1}z^{n/2} \) on \( |\arg z| < \pi \), or else
- \( a_n = -1, y \) has exactly one string of poles asymptotic to the positive real axis with asymptotics \( y'/y \approx (-1)^{m+1}(-z)^{n/2} \) on \( |\arg(-z)| < \pi \).

If \( P \) is real, then in each case all but finitely many zeros are real and \( y \) is a (multiple of a) real entire function.

6. The Schwarzian derivative

In [10] Nevanlinna considered the locally univalent meromorphic functions \( f \) of finite order. They are characterised by the fact that their Schwarzian derivative
\[
S_f = \left( f'' / f' \right)' - \frac{1}{2} \left( f'' / f' \right)^2 \text{ is a polynomial } 2P, \text{ say. Moreover, } f \text{ is the quotient } y(z; 0) / y(z; \infty) \text{ of two linearly independent solutions to the linear differential equation}
\]
\[
y'' + P(z)y = 0,
\]
which is equivalent to the Riccati equation \( w' = -P(z) - w^2 \) via \( w = y'/y \). The generic solutions have counting function of poles and Nevanlinna characteristic \( T(r, w) \sim Cr^\varrho \) with \( \varrho = 1 + \frac{1}{2} \deg P; \quad C > 0 \) is some known constant. Every exceptional solution \( w_{\nu} \), however, has counting function and Nevanlinna characteristic \( T(r, w_{\nu}) \sim C \frac{n+2-2d_{\nu}}{n+2} r^\varrho \), where \( d_{\nu} \) is some positive integer such that \( \sum_{\nu} d_{\nu} = n + 2 \). Since the zeros of \( f - a \) are the same as the zeros of \( y(z; a) = y(z; 0) - ay(z; \infty) \), hence coincide with the poles of \( w(z; a) = y'(z; a)/y(z; a), \) it follows that \( f \) has Nevanlinna deficiencies \( \delta(a_{\nu}) = 2d_{\nu}/(n+2) (w_{\nu}(z) = w(z; a_{\nu})) \) with \( \sum_{\nu} \delta(a_{\nu}) = 2 \).

7. Proof of Theorem 1 and Theorem 2

**Proof of Theorem 1.** From
\[
w_h'(3) = h^{-n}a(h + h^{-n/2}3) + w_h(3)^2
\]
and \( z^{-n}a(z) \to 1 \) as \( z \to \infty \) it follows that
\[
|w_h'(3)| \leq 2 + |w_h(3)|^2
\]
holds on $|z| < R$, $|h| > \eta R$. Thus the family $(w^\nu_h)|h|\geq 1$ of spherical derivatives
\[ w^\nu_h = \frac{|w'_h|}{1 + |w_h|^2} \]
is bounded on $|z| < R$ by $M(R) = \sup\{|w^\nu_h(z)| : |z| < R, \ 1 < |h| < \eta R\} + 2$, say. The limit function $w = \lim_{h_k \to \infty} w_{h_k} \equiv \infty$ does not occur since otherwise $u_{h_k} = 1/w_{h_k}$ would tend to zero, this contradicting $u_{h_k}' = 1 - h_k^{-n}a(h_k + h_k^{-n/3})u_{h_k}^2 \to 1$. Thus every limit function $w$ satisfies (5) outside the set $\mathcal{P}$ of poles of $w$. □

**Proof of Theorem 2.** From Theorem 1 and Hurwitz’ Theorem it follows that given $\epsilon > 0$ and $R > 0$ there exists some $r_0 > 0$, such that the disc
\[ \triangle_R(p) = \{z : |z - p| < R|p|^{-n/2}\} \]
about any pole $p$ with $|p| > r_0$ contains the poles $\tilde{p}_k$ with
\[ |\tilde{p}_k - (\frac{p}{k} + k\pi e^{-n/2})| < \epsilon |p|^{-n/2} \quad (-k_1(p) \leq k \leq k_2(p)), \]
and no others; the numbers $k_1$ and $k_2$ are bounded by a number only depending on $R$ (for example, $k_1 = k_2 = 318$ if $R = 1000$ and $r_0$ is sufficiently large). Thus up to finitely many every pole is contained in a unique string of poles $(p_k)$ satisfying (6). Then $z_k = p_k^\nu (\nu = n/2 + 1)$ satisfies
\[ z_{k+1} = z_k + \omega \delta + o(1) \]
with $\omega = \pm \pi i$ fixed, hence $z_k = \omega \delta k + o(k)$, $p_k = (\omega \delta k)^{1/\nu}(1 + o(1))$, and
\[ \frac{n + 2}{2} \arg p_k = \arg \omega + o(1) = \pm \frac{\pi}{2} + o(1) \quad \mod 2\pi, \]
that is, $\arg p_k = \theta_\nu + o(1) = \frac{2\nu + 1}{n + 2} + o(1)$ holds for some $\nu$. The counting function of $\sigma$ equals $n(r, \sigma) = \frac{\nu}{2\pi} + o(r^0)$, and from $n(r, w) = O(r^0)$ it follows that there are only finitely many strings of poles. □

8. **Proof of Theorem 3**

Let $w$ be any solution to (R) and $S$: $|\arg z - \phi_0| < \eta$ any sector that is ‘pole-free’ for $w$. From Theorem 1 then it follows that $w(z)z^{-n/2}$ tends to either $+1$ or else $-1$ as $z \to \infty$; the convergence to $+1$, say, is uniform on each closed sub-sector $S(\delta)$: $|\arg z - \phi_0| \leq \eta - \delta$ (take any sequence $h_k \to \infty$ in $S(\delta)$ such that $\lim_{h_k \to \infty} |w(h_k)h_k^{-n/2} - 1| = \limsup_{z \to \infty} |w(z)z^{-n/2} - 1|$ on $S(\delta)$). If $n = 2m$ is even we set $v(z) = z^{-m}w(z)$ to obtain
\[ z^{-m}v' + mv^{-m-1}v = a(z)z^{-2m} - v^2. \]
If, however, $n = 2m + 1$ is odd we set $v(z) = z^{-n}w(z^2)$ to obtain
\[ z^{-n-1}v' + nz^{-n-2}v = 2a(z^2)z^{-2n} - 2v^2. \]
From (11) resp. (12) and the fact that $v(z) \to \pm 1$ on some sector $S$ we have to conclude $v \sim \pm 1 + \sum_{k=1}^\infty c_k z^{-k}$ on $S$. For definiteness we will consider equation (11) with $v(z) \to 1$ on $S$. If we assume that
\[ v(z) = 1 + \sum_{k=1}^n c_k z^{-k} + o(|z|^{-n}) = \psi_n(z) + o(|z|^{-n}) \]
has already been proved (this is true for \( n = 0 \)) we obtain from

\[
v'(z) = \psi'_n(z) + o(|z|^{-n-1})
\]

and (11)

\[
a(z)z^{-2m} - v^2 = z^{-m}\psi'_n(z) + mz^{-m-1}\psi_n(z) + o(|z|^{-n-m-1}).
\]

The algebraic equation

\[
a(z)z^{-2m} - y^2 = z^{-m}\psi'_n(z) + mz^{-m-1}\psi_n(z)
\]

has a unique solution \( y = 1 + \sum_{k=1}^{\infty} c_k z^{-k} \) about \( z = \infty \), and from \( v + y = 2 + o(1) \) and \((v - y)(v + y) = v^2 - y^2 = o(|z|^{-n-m-1})\) it follows that

\[
v = y + o(|z|^{-n-m-1}) = 1 + \sum_{k=1}^{n+1} c_k z^{-k} + o(|z|^{-n-1}) = \psi_{n+1}(z) + o(|z|^{-n-1}).
\]

It is obvious that \( c_k = c'_k \) holds for \( 0 \leq k \leq n \), and this proves the existence part. The proof is the same in all other cases.

To prove the uniqueness part of Theorem 3 we assume that \( w_1 \) and \( w_2 \) have the same asymptotic expansion on the sector \( S' \). Then \( u = w_1 - w_2 \) solves

\[
u' = -(w_1(z) + w_2(z))u = -2\varepsilon z^{n/2}(1 + O(|z|^{1/2}))u,
\]

hence \( u = C \exp(-2\varepsilon z^\theta + O(|z|^\theta)) \) holds. Our hypothesis \( \varepsilon \Re z^\theta < 0 \) and \( u \to 0 \) on \( S' \subset S \) then gives \( u = C = 0 \), and this proves Theorem 3 completely.

\[\square\]

9. Proof of Theorem 5

Since all but finitely many poles of \( w \) are simple with residue 1, the Residue Theorem gives

\[
n(r, w) = \frac{1}{2\pi i} \int_{\Gamma_r} w(z) \, dz + O(1),
\]

where the simple closed curve \( \Gamma_r \) is obtained from the circle \( C_r : |z| = r \) by replacing the intersection of \( C_r \) with any disc \( \triangle_r(p) = \{ z : |z - p| < \epsilon |p|^{-n/2} \} \) \( (\epsilon > 0 \) sufficiently small, \( p \) any pole of \( w \) by an appropriate sub-arc of \( \partial \triangle_r(p) \). From \( w = O(|z|^{n/2}) = O(|z|^{\theta - \delta}) \) on \( \Gamma_r \) (this following from the normality of the family \( w_h(z) = h^{-n/2}w(h + h^{-n/2}z) \)) and the fact that \( \Gamma_r \cap \{ z : |z - \theta_v| < \delta \} \) has length at most \( 2\pi \delta r \) as \( \delta \to 0 \), it follows that the contribution of the Stokes sector \( S_v \) to the counting function of poles equals

\[
(-1)^\nu \varepsilon_\nu \frac{r^\theta}{\pi \theta} + o(r^\theta) \quad (\theta = n/2 + 1).
\]

In particular, \( w \) has \( \sum_\nu (-1)^\nu \varepsilon_\nu \) strings of poles. Integrating \( w \) along the line segment \( \sigma \) from \( r_0 e^{i(\theta_v - \delta)} \) \((\delta > 0 \small, r_0 > 0 \large)\) to \( r_0 e^{i(\theta_v + \delta)} \) gives

\[
\frac{1}{2\pi i} \int_{\sigma} w(z) \, dz = \frac{\varepsilon_\nu}{2\pi i \theta} e^{i\theta \varepsilon} + o(r^\theta) = (-1)^\nu \frac{\varepsilon_\nu}{2\pi i \theta} e^{-i\theta \varepsilon} r^\theta + o(r^\theta).
\]

Thus, if \( \gamma_v^\theta \) denotes the simple closed curve which consists of the line segment \( \sigma \), the part of \( \Gamma_r \) from \( r_0 e^{i(\theta_v - \delta)} \) to \( r_0 e^{i(\theta_v + \delta)} \), the line segment from \( r_0 e^{i(\theta_v + \delta)} \) to \( r_0 e^{i(\theta_v - \delta)} \), and the circular arc on \( |z| = r_0 \) from \( r_0 e^{i(\theta_v + \delta)} \) to \( r_0 e^{i(\theta_v - \delta)} \) we obtain

\[
\frac{1}{2\pi i} \int_{\gamma_v^\theta} w(z) \, dz = (-1)^\nu \frac{r^\theta}{2\pi i \theta} [\varepsilon_\nu - \varepsilon_{\nu+1}] + O(\delta r^\theta) + o(r^\theta).
\]
(r → ∞, δ → 0). Now the integral on the left hand side equals the number of poles inside γ′, while (−1)νqεν+1 coincides with the number of strings of poles along the Stokes ray sν: arg z = θν. From this the assertions (a), (b), and (c) in Theorem 5 immediately follow.

10. Proof of Theorem 6

It is easily seen that equation (11) resp. (12), written as

\[ z^{-q}v' = f(z, v) \quad (q = m \text{ resp. } q = n + 1) \]

has a formal solution \( \varepsilon_\nu + \sum_{\nu=1}^\infty c_\nu z^{-\nu} \) with \( \varepsilon_\nu = -(1)^\nu \). Since \( \lim_{\nu \to \infty} f_\nu(z, \varepsilon_\nu) = -2\varepsilon_\nu \neq 0 \), Theorem 12.1 in Wasow’s monograph [14] applies to the corresponding equation for \( v - \varepsilon_\nu \). Hence to every sector \( |\arg z - \theta_0| < \frac{\pi}{2q+2} \) there exists a solution to equation (14) with asymptotic expansion \( v \sim \varepsilon_\nu + \sum_{\nu=1}^\infty c_\nu z^{-\nu} \). In particular, for every \( \nu \) we obtain a (unique) solution \( w = w_\nu \) to (R) with the desired asymptotic expansion (9) on the Stokes sector \( S_\nu \).

11. Proof of Theorem 7

If \( y(z) = P_1(z)e^{P_2(z)} \) has only finitely many zeros, then \( n = 2 \deg P_2 - 2 \) is even, and not much more can be said (of course, \( P \) can be computed explicitly from \( P_1 \) and \( P_2 \)). From now on we assume that \( y \) has infinitely many zeros. The change of variables \( w(z) = \eta y'(\eta z)/y(\eta z) \) with \( \eta^{n+2}a_n = 1 \) transforms equation (10) into equation (R) with \( a(z) = \eta^2P(\eta z) = z^n + \cdots \), hence the question whether or not there are solutions \( y \) to (10) having infinitely many zeros, ‘most’ of them close to the real axis is transformed into the question for solutions \( w \) to (R) having just one string of poles asymptotic to some Stokes ray \( s_\nu \): \( \arg z = \theta_\nu = (2\nu+1)\pi/n+2 \) if \( n \) is odd, and asymptotic to the Stokes rays \( s_\nu \) and \( s_{\nu+m} \) if \( n = 2m \) is even, respectively. This yields \( \eta = \pm e^{i\nu} \) up to an arbitrary root of unity of order \( n + 2 \), and we are free to choose \( \eta = e^{-\pi i/2} \) and \( \nu = 0 \) if \( n \) is even, and \( \eta = \pm 1 \) and \( \nu = m + 1 \) if \( n = 2m + 1 \) is odd. In the first case we obtain \( a_n = -1 \), and from Theorem 5 it follows that \( \epsilon_0 - \epsilon_1 = 2 \) and \( (−1)^{m+1}(\epsilon_{m+1} - \epsilon_{m+2}) = 2 \), hence \( \epsilon_0 = 1 \) and \( \epsilon_1 = -1 \), this implying \( \epsilon_2 = \cdots = \epsilon_{m+1} = \epsilon_1 = -1, \epsilon_{m+2} = \cdots = \epsilon_{2m+1} = \epsilon_0 = 1, m = 2k \) and \( n = 4k \). This proves the first part of Theorem 6.

In the second case we have \( a_n = +1 \) and \( a_n = -1 \) with zeros asymptotic to the negative and positive real axis, respectively, and asymptotic expansions \( y'/y \approx (−1)^{m+1}z^{n/2} \) on \( |\arg z| < \pi \) resp. \( y'/y \approx (−1)^{m+1}(-z)^{n/2} \) on \( |\arg(-z)| < \pi \) (note that \( z^{n/2} \) means \( (\sqrt{z})^n \)).

Now \( y \) is uniquely determined up to a constant factor. Thus if \( P \) is a real polynomial, then the zeros of \( y^*(z) = y(\bar{z}) \) are also asymptotic to the real axis, hence \( y \) and \( y^* \) are linearly dependent, and \( y \) is a multiple of a real function with all but finitely many zeros real.

\[ \Box \]

References


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