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TOWARDS A REGULARITY THEORY FOR INTEGRAL MENGER CURVATURE

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Abstract. We generalize the notion of integral Menger curvature introduced by Gonzalez and Maddocks [14] by decoupling the powers in the integrand. This leads to a new two-parameter family of knot energies $\operatorname{intM}^{(p,q)}$. We classify finite-energy curves in terms of Sobolev-Slobodeckiĭ spaces. Moreover, restricting to the range of parameters leading to a sub-critical Euler-Lagrange equation, we prove existence of minimizers within any knot class via a uniform bi-Lipschitz bound. Consequemtly, $\operatorname{intM}^{(p,q)}$ is a knot energy in the sense of O'Hara. Restricting to the non-degenerate sub-critical case, a suitable decomposition of the first variation allows to establish a bootstrapping argument that leads to C^∞-smoothness of critical points.

Introduction

Imagine a closed curve in Euclidean space. Each triple of distinct points on the curve uniquely defines its circumcircle that passes through these three points. It degenerates to a line if and only if the points are collinear. The reciprocal of the circumcircle radius can be seen as some kind of approximate curvature. How much information on shape and regularity of the curve can be drawn from the L^p-norm of the latter quantity?

Motivated from applications in microbiology, Gonzalez and Maddocks [14] investigated this question for the case $p = \infty$. They were in search of a notion for the *thickness* of an embedded curve that, in contrast to other approaches, e.g. Litherland et al. [28], does not require initial regularity of the respective curves.

Thickness is influenced by both local and global properties of a curve and is additionally related to the regularity of the curve. In fact, the thickness of an arc-length parametrized curve is finite if and only if it is embedded and has a Lipschitz continuous tangent, i.e., it is $C^{1,1}$, see Gonzalez et al. [15]. Consequently, any curve of finite thickness parametrized by arc-length is bi-Lipschitz continuous with a bi-Lipschitz constant only depending on its thickness.

The latter is particularly interesting in the context of applications. Instead of trying to immediately determine the knot type of a given possibly quite entangled curve, one could first "simplify" it in order to obtain a nicely shaped curve, having large distances between distant strands. Such a deformation process could be defined

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by the gradient flow of a suitable functional which should prevent the curve from leaving the ambient knot class, preserving the bi-Lipschitz property.

This idea was formalized into the concept of *knot energies* by O'Hara [33, Def. 1.1]. A functional on a given space of knots is called a knot energy if it is bounded below and self-repulsive, i.e., it blows up on sequences of embedded curves converging to a non-embedded limit curve.

Among other functionals Gonzalez and Maddocks [14] also proposed to investigate the functional

$$\mathscr{M}_p(\gamma) := \iiint\limits_{(\mathbf{R}/\mathbf{Z})^3} \frac{|\gamma'(u_1)|\,|\gamma'(u_2)|\,|\gamma'(u_3)|}{R(\gamma(u_1),\gamma(u_2),\gamma(u_3))^p}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3, \quad p \in (0,\infty),$$

which is called *integral Menger*[1] *curvature*. Here $\gamma\colon \mathbf{R}/\mathbf{Z} \to \mathbf{R}^n$ is an absolutely continuous curve and $R(x,y,z)$ denotes the circumcircle of the three points $x,y,z \in \mathbf{R}^n$ given by

$$R(x,y,z) := \frac{|y-z|\,|y-x|\,|z-x|}{2\,|(y-x)\wedge(z-x)|} = \frac{|y-z|}{2\sin\sphericalangle\,(y-x,z-x)}. \tag{0.1}$$

The functionals $\mathscr{M}_p$ have been investigated by Strzelecki, Szumańska and von der Mosel in [40] wherein further references can be found. Their results cover the case $p > 3$ where $\mathscr{M}_p$ is known to be a knot energy. Especially they have been able to show that finite energy of an arc-length parametrized curve implies $C^{1,1-3/p}$-regularity and its image is C^1-homeomorphic to the circle. The regularity statement has been sharpened in [3].

The element $\mathscr{M}_2$ is referred to as *total Menger curvature*. Interestingly, it plays an important rôle in complex analysis, more precisely in the proof of *Vitushkin's conjecture*, a partial solution to *Painlevé's problem* which asks to determine *removable* sets. These are compact sets $K \subset \mathbf{C}$ such that for any open $U \subset \mathbf{C}$ containing K and for any bounded analytic function $U \setminus K \to \mathbf{C}$, the latter can be extended to an analytic function on U. Vitushkin conjectured that a compact set K with positive finite one-dimensional Hausdorff measure is removable if and only if it is *purely unrectifiable*, i.e. it intersects every rectifiable curve in a set of measure zero.

A central result in this context is the curvature theorem of David and Léger [26] stating that one-dimensional Borel sets in $\mathbf{C}$ with finite total Menger curvature are 1-rectifiable. Hahlomaa generalized this result to the metric setting [16, 17, 18]. Lin and Mattila [27] investigated Menger curvature for Borel sets of fractional dimension. Mel′nikov and Verdera [29, 30, 46] discovered a connection between L^2-boundedness of the Cauchy integral operator on Lipschitz graphs and the Menger curvature. For further details regarding Vitushkin's conjecture for removable sets we refer to the monographs of Dudziak [12] and Tolsa [45] and references therein.

Menger curvature for higher-dimensional objects has been discussed in [22, 24, 25, 4, 23, 38]. It is also interesting to consider variants derived from $\mathscr{M}_p$ by substituting one or two integrals by suprema as proposed by Gonzalez and Maddocks [14], see

[1] Named after Karl Menger, 1902–1985, US-Austrian mathematician, who used the circumcircle for generalizing geometric concepts to general metric spaces [31]. He worked on many fields including distance geometry, dimension theory, graph theory. Menger was a student of Hahn in Vienna where he received a professorship in 1927. Being member of the Vienna Circle, he was also interested in philosophy and social science. After emigrating to the USA in 1937, he obtained a position at Notre Dame, later at Chicago. See Kass [21] for further reading.

Strzelecki et al. [39, 42] for details. Further information on the context of the integral Menger curvature within the field of geometric knot theory and geometric curvature energies can be found in the recent surveys by Strzelecki and von der Mosel [44, 43].

In this article we make a first step towards the regularity theory of stationary points of integral Menger curvature. Regularity theory for minimizers of certain knot energies has been developed in [32, 13, 19, 36, 35, 5, 6]. A summary is given in [7, 8].

Unfortunately the Euler–Lagrange operator of $\mathscr{M}_p$ is not only non-local but also degenerate. In order to produce non-degenerate energies, we embed this family into the two-parameter family of generalized integral Menger curvature

$$\text{intM}^{(p,q)}(\gamma) := \iiint_{(\mathbf{R}/\mathbf{Z})^3} \frac{|\gamma'(u_1)|\,|\gamma'(u_2)|\,|\gamma'(u_3)|}{R^{(p,q)}(\gamma(u_1),\gamma(u_2),\gamma(u_3))}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3, \quad p,q>0, \tag{0.2}$$

where

$$R^{(p,q)}(x,y,z) := \frac{(|y-z|\,|y-x|\,|z-x|)^p}{|(y-x)\wedge(z-x)|^q} = \frac{|y-z|^p\,|y-x|^{p-q}\,|z-x|^{p-q}}{\sin\sphericalangle\,(y-x,z-x)^q}, \tag{0.3}$$

$x,y,z\in\mathbf{R}^n$. Note that the function $R^{(p,q)}$ is symmetric in all components. Of course, $\mathscr{M}_p = 2^p\text{intM}^{(p,p)}$.

The elements of this family are knot energies under certain conditions only. More precisely, we will see in Remark 1.2 that they penalize self-intersections if and only if

$$p \geq \tfrac{2}{3}q + 1. \tag{0.4}$$

On the other hand, these energies can only be finite on closed curves iff

$$p < q + \tfrac{2}{3}, \tag{0.5}$$

see Remark 1.3.

Due to the non-local structure of the energy functionals (0.2) we arrive at an Euler-Lagrange equation of type $Lu = Ru$ where the left-hand side Lu denotes the Euler-Lagrange operator associated to the fractional q-Laplacian and Ru contains the remaining "lower-order" terms. The properties of this equation highly depend on the respective values (p,q).

Not only for the regularity theory it is of essential importance that curves of finite energy are of class $C^{1,\alpha}$ for an $\alpha>0$. This is not the case for all combinations of indices. We will see that for

$$p \in \left(\tfrac{2}{3}q+1, q+\tfrac{2}{3}\right) \quad (q>1), \tag{0.6}$$

such an embedding exists and we will refer to this range as the *sub-critical range*[2]. It is one of the main results of this article, that within this range the equation $Lu = Ru$ behaves in a *sub-critical* manner in the sense that a combination of potential estimates and Sobolev-embeddings leads to full regularity of solutions to this non-linear equation (cf. Theorem 4).

On the other hand, if $q\neq 2$, we expect the standard q-Laplacian to share effects coming from the degeneracy of the operator (or singularity) with its fractional analogue. Due to this vague heuristic, we call the equation for $q=2$ *non-degenerate* in our terminology.

[2] In contrast, the corresponding range is called *super*-critical by Strzelecki et al. [41] as it lies above the respective critical value for which the energy is scale-invariant.

Restricting to the non-degenerate part of the sub-critical range, we consider

$$p \in \left(\tfrac{7}{3}, \tfrac{8}{3}\right), \quad q = 2. \tag{0.7}$$

The areas mentioned above are visualized in Figure 1.

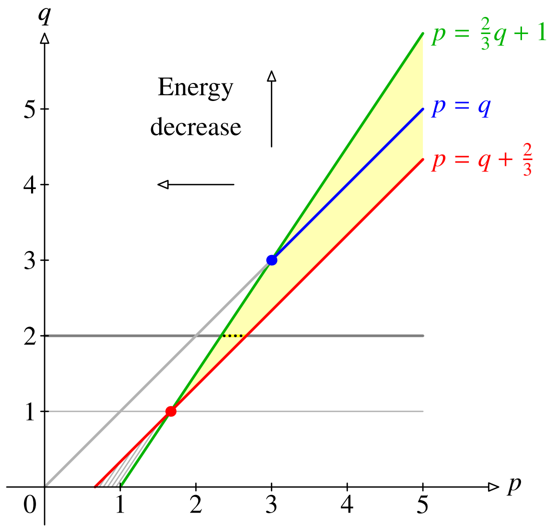

Figure 1. The range of $\mathrm{intM}^{(p,q)}$. Above the line $p = \frac{2}{3}q + 1$ (green), the integrand is not sufficiently singular to penalize self-intersections, thus $\mathrm{intM}^{(p,q)}$ is not a knot energy. On the other hand, below the line $p = q + \frac{2}{3}$ (red), for $q > 1$, the integrand is so singular, that the integral is either equal to zero or infinite, so there are no finite-energy C^1-knots at all. For $q > 1$ these lines bound the sub-critical range (0.6) (yellow). The non-degenerate sub-critical range (0.7) is dotted. The hatched area reveals the strange behavior that there are no finite-energy C^3-knots while it takes finite values on polygons.

In [3], a characterization of curves with finite $\mathscr{M}_p$ energy was given in terms of function spaces. Using this technique we infer

Theorem 1. (Classification of finite-energy curves) *Consider the sub-critical case* (0.6) *and let* $\gamma \in C^1(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ *be an injective curve parametrized by arc-length. Then* $\mathrm{intM}^{(p,q)}(\gamma) < \infty$ *if and only if* $\gamma \in W^{(3p-2)/q-1,q}$. *Moreover, one then has, for constants* $C, \beta > 0$ *depending on* p, q *only,*

$$[\gamma]_{W^{(3p-2)/q-1,q}} \le C \left(\mathrm{intM}^{(p,q)}(\gamma) + \mathrm{intM}^{(p,q)}(\gamma)^{\beta}\right)^{1/q}. \tag{0.8}$$

We will use the last theorem to show

Theorem 2. (Existence of minimizers within knot classes) *In the sub-critical case* (0.6), *there is a minimizer of* $\mathrm{intM}^{(p,q)}$ *among all injective, regular curves* $\gamma \in C^1(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ *in any knot class.*

To shorten notation we use the abbreviation

$$\triangle_{v,w}\bullet := \bullet(u+v) - \bullet(u+w) \tag{0.9}$$

throughout this paper. Furthermore, we sometimes omit the argument of a function if it is precisely the variable u, i.e. $\gamma = \gamma(u)$ etc.

The first variation of $\mathscr{M}_p$, $p \ge 2$, has been derived by Hermes [20, Thm. 2.33, Rem. 2.35]. Here we use a different approach to prove

Theorem 3. (Differentiability) *In the sub-critical case* (0.6) *the functional* $\operatorname{intM}^{(p,q)}$ *is* C^1 *on the subspace of all regular embedded* $W^{(3p-2)/q-1,q}$*-curves. For any arc-length parameterized embedded* $\gamma \in W^{(3p-2)/q-1,q}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)$ *and* $h \in W^{(3p-2)/q-1,q}$ $(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)$*, the first variation of* $\operatorname{intM}^{(p,q)}$ *at* γ *in direction* h *amounts to*

$$\begin{aligned}
&\delta\operatorname{intM}^{(p,q)}(\gamma,h)\\
&= \iiint_{(\mathbf{R}/\mathbf{Z})^3} \left\{ 2q\frac{|\triangle_{v,0}\gamma \wedge \triangle_{w,0}\gamma|^{q-2}}{(|\triangle_{v,w}\gamma|\,|\triangle_{v,0}\gamma|\,|\triangle_{w,0}\gamma|)^p} \cdot \langle \triangle_{v,0}\gamma \wedge \triangle_{w,0}\gamma, \triangle_{v,0}\gamma \wedge \triangle_{w,0}h\rangle \right.\\
&\qquad - 3p\frac{|\triangle_{v,0}\gamma \wedge \triangle_{w,0}\gamma|^{q}}{(|\triangle_{v,w}\gamma|\,|\triangle_{v,0}\gamma|\,|\triangle_{w,0}\gamma|)^p} \cdot \frac{\langle \triangle_{v,w}\gamma, \triangle_{v,w}h\rangle}{|\triangle_{v,w}\gamma|^2}\\
&\qquad \left. + 3\frac{|\triangle_{v,0}\gamma \wedge \triangle_{w,0}\gamma|^{q}}{(|\triangle_{v,w}\gamma|\,|\triangle_{v,0}\gamma|\,|\triangle_{w,0}\gamma|)^p} \cdot \langle \gamma', h'\rangle \right\} \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned} \tag{0.10}$$

Using this formula, we will see that stationary points of the energies $\operatorname{intM}^{(p,2)}$ restricted to fixed length satisfy a non-local uniformly elliptic pseudo-differential equation. If furthermore $p \in \left(\frac{7}{3},\frac{8}{3}\right)$, the non-linearity turns out to be sub-critical and we can finally use the Euler–Lagrange equation to prove the following main result of this article:

Theorem 4. (Regularity of stationary points) *For* $p \in (\frac{7}{3},\frac{8}{3})$*, let* $\gamma \in W^{3p/2-2,2}$ $(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)$ *be a stationary point of* $\operatorname{intM}^{(p,2)}$ *with respect to fixed length, injective and parametrized by arc-length. Then* $\gamma \in C^\infty$.

In a sense this concludes our study of the non-degenerate, subcritical cases of the most prominent knot energies for curves. Regularity theory for the non-degenerate sub-critical case has already been performed for O'Hara's energies [5] and for the generalized tangent-point energies [6]. The treatment of the critical case however turns out to be far more involved and has yet only be done for O'Hara's knot energies [9].

We briefly introduce *Sobolev–Slobodeckiĭ spaces* in the form we will use them in this text. Let $f \in W^{1,1}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)$. For $s \in (0,1)$ and $\varrho \in [1,\infty)$ we define the seminorm

$$[f]_{W^{1+s,\varrho}} := \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/2}^{1/2} \frac{|f'(u+w)-f'(u)|^\varrho}{|w|^{1+\varrho s}}\,\mathrm{d}w\,\mathrm{d}u \right)^{1/\varrho}. \tag{0.11}$$

On $W^{1,\varrho}$ this seminorm is equivalent to

$$[\![f]\!]_{W^{1+s,\varrho}} := \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w)-2f(u)+f(u-w)|^\varrho}{|w|^{1+\varrho(1+s)}}\,\mathrm{d}w\,\mathrm{d}u \right)^{1/\varrho}, \tag{0.12}$$

see Appendix B.

Now let $W^{k,\varrho}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)$, $k \in \mathbf{N}$, denote the usual Sobolev space (recall $W^{0,\varrho} := L^\varrho$) and

$$W^{k+s,\varrho}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n) := \left\{ f \in W^{k,\varrho}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n) \,\middle|\, \|f\|_{W^{k+s,\varrho}} < \infty \right\}$$

which we equip, depending on the situation, either with the norm $\|f\|_{W^{k,\varrho}} + [f^{(k-1)}]_{W^{1+s,\varrho}}$ or with $\|f\|_{W^{k,\varrho}} + [\![f^{(k-1)}]\!]_{W^{1+s,\varrho}}$, respectively.

Without further notice we will frequently use the embedding

$$W^{k+s,\varrho}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n) \hookrightarrow C^{k,s-1/\varrho}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n), \quad \varrho \in (1,\infty), \quad s \in (\varrho^{-1},1). \tag{0.13}$$

We will denote by C_{ia} resp. W_{ia} injective (embedded) curves parametrized by arc-length. As usual, a curve is said to be *regular* if there is some $c > 0$ such that $|\gamma'| \geq c$ a.e. Constants may change from line to line.

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1. Classification of finite-energy curves

Before we begin the discussion of the first variation, let us rewrite the integral Menger curvature using the symmetry and a suitable covering of the domain of integration $(\mathbf{R}/\mathbf{Z})^3$ by domains, on which it is easier to estimate the terms that will appear. The general idea here is quite similar to [3] and Hermes [20], but we will show that it is actually enough to integrate over a certain subdomain of $(\mathbf{R}/\mathbf{Z})^3$.

To this end we define the range of integration

$$D := \left\{ (v,w) \in (-\tfrac{1}{2},0) \times (0,\tfrac{1}{2}) \,\middle|\, w \leq 1+2v, v \geq -1+2w \right\}, \tag{1.1}$$

which is depicted in Figure 2.

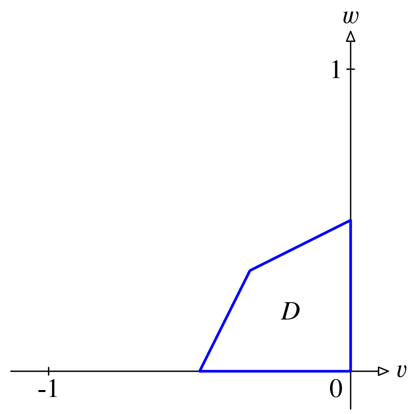

Figure 2. The range of integration D.

Lemma 1.1. (Domain decomposition) *Let $f \in L^1\left((\mathbf{R}/\mathbf{Z})^3\right)$ be symmetric in all components, i.e. $f = f \circ \sigma$ for all permutations $\sigma \in \mathfrak{S}_3$. Then*

$$\iiint_{(\mathbf{R}/\mathbf{Z})^3} f(u_1,u_2,u_3)\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 = 6 \iiint_{\mathbf{R}/\mathbf{Z}\times D} f(u,u+v,u+w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.$$

Proof. Let $P_\sigma \in \mathbf{R}^{3\times 3}$ denote the permutation matrix corresponding to $\sigma \in \mathfrak{S}_3$. We first show that the images $\{ P_\sigma (\mathbf{R}/\mathbf{Z} \times D) \,|\, \sigma \in \mathfrak{S}_3 \}$ cover $(\mathbf{R}/\mathbf{Z})^3$.

Consider $(u_1,u_2,u_3) \in (\mathbf{R}/\mathbf{Z})^3$. Then after a suitable permutation we can assume that

$$d_{\mathbf{R}/\mathbf{Z}}(u_1,u_3) = \max\left(d_{\mathbf{R}/\mathbf{Z}}(u_1,u_2), d_{\mathbf{R}/\mathbf{Z}}(u_2,u_3), d_{\mathbf{R}/\mathbf{Z}}(u_1,u_3)\right),$$

where $d_{\mathbf{R}/\mathbf{Z}}(x,y) = \min\{|x-y-k| \colon k \in \mathbf{Z}\} \in [0,\frac{1}{2}]$ denotes the distance in $\mathbf{R}/\mathbf{Z}$. Hence, interchanging u_1 and u_3 if necessary, there are $(v,w) \in \left[-\frac{1}{2},0\right] \times \left[0,\frac{1}{2}\right]$ with

$$u_1 = u_2 + v, \quad u_3 = u_2 + w,$$

and

$$\begin{aligned}\max(-v,w) &= \max\left(d_{\mathbf{R}/\mathbf{Z}}(u_1,u_2), d_{\mathbf{R}/\mathbf{Z}}(u_2,u_3)\right)\\ &\le d_{\mathbf{R}/\mathbf{Z}}(u_1,u_3) = \min\left(w-v, 1-(w-v)\right) \le 1-w+v,\end{aligned}$$

so $(v,w)\in\overline{D}$. Since furthermore

$$\#\mathfrak{S}_3\cdot|\mathbf{R}/\mathbf{Z}\times D| = 6\cdot\tfrac{1}{6} = \left|(\mathbf{R}/\mathbf{Z})^3\right|$$

the sets $\{P_\sigma(\mathbf{R}/\mathbf{Z}\times D)\,|\,\sigma\in\mathfrak{S}_3\}$ form (up to sets of measure zero) a disjoint partition of $(\mathbf{R}/\mathbf{Z})^3$: If this was not the case, there would exist a set S of positive measure belonging to the image of $P_{\sigma_j}(\mathbf{R}/\mathbf{Z}\times D)$ for two different values $\sigma_1,\sigma_2\in\mathfrak{S}_3$. However, as $(\mathbf{R}/\mathbf{Z})^3\subset\{P_\sigma(\mathbf{R}/\mathbf{Z}\times D)\,|\,\sigma\in\mathfrak{S}_3\}$, this would result in $|\{P_\sigma(\mathbf{R}/\mathbf{Z}\times D)\mid\sigma\in\mathfrak{S}_3\}|\ge 1+|S|$. □

Following Lemma 1.1 we derive using (0.9)

$$\begin{aligned}&\operatorname{intM}^{(p,q)}(\gamma)\\ &= 6\iiint_{\mathbf{R}/\mathbf{Z}\times D}\frac{|\triangle_{v,0}\gamma\wedge\triangle_{0,w}\gamma|^q}{\left(|\triangle_{v,0}\gamma|\,|\triangle_{0,w}\gamma|\,|\triangle_{v,w}\gamma|\right)^p}|\gamma'(u)|\,|\gamma'(u+v)|\,|\gamma'(u+w)|\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.\end{aligned}\tag{1.2}$$

Proof of Theorem 1. Recall that any embedded $W^{1+s,\varrho}_{\mathrm{ia}}$-curve, $s>\frac{1}{\varrho}$, is bi-Lipschitz continuous [2, Lemma 2.1], so

$$\operatorname{intM}^{(p,q)}(\gamma)\le C\iiint_{\mathbf{R}/\mathbf{Z}\times D}\frac{\left|\int_0^1\gamma'(u+\theta_1 v)\,\mathrm{d}\theta_1\wedge\int_0^1\gamma'(u+\theta_2 w)\,\mathrm{d}\theta_2\right|^q}{|v|^{p-q}\,|w|^{p-q}\,|v-w|^p}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u,$$

where C depends on p,q and γ. Using $|a\wedge b| = |a\wedge(a\pm b)|\le|a|\,|a\pm b|$ for $a,b\in\mathbf{R}^3$, we obtain

$$\operatorname{intM}^{(p,q)}(\gamma)\le C\iiint_{\mathbf{R}/\mathbf{Z}\times D}\frac{\int_0^1|\gamma'(u+\theta v)-\gamma'(u+\theta w)|^q\,\mathrm{d}\theta}{|v|^{p-q}\,|w|^{p-q}\,|v-w|^p}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.$$

We may substitute $u\mapsto u-\theta w$ due to periodicity and apply Fubini's theorem which gives

$$\operatorname{intM}^{(p,q)}(\gamma)\le C\int_0^1\iiint_{\mathbf{R}/\mathbf{Z}\times D}\frac{|\gamma'(u+\theta(v-w))-\gamma'(u)|^q}{|v|^{p-q}\,|w|^{p-q}\,|v-w|^p}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}\theta.$$

Substituting $\Phi\colon(v,w)\mapsto(t,\tilde{w}) := \left(\frac{v}{v-w},\theta(v-w)\right)$, $|\det D\Phi(v,w)| = \frac{\theta}{|v-w|}$, $\Phi(D)\subset[0,1]\times[-1,0]$, we arrive at

$$\begin{aligned}&\operatorname{intM}^{(p,q)}(\gamma)\\ &\le C\int_0^1\theta^{-1}\int_{\mathbf{R}/\mathbf{Z}}\int_0^1\int_{-1}^0\frac{|\gamma'(u+\tilde{w})-\gamma'(u)|^q}{\left|t\frac{\tilde{w}}{\theta}\right|^{p-q}\left|(t-1)\frac{\tilde{w}}{\theta}\right|^{p-q}\left|\frac{\tilde{w}}{\theta}\right|^{p-1}}\,\mathrm{d}\tilde{w}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}\theta\\ &\le C\int_0^1\theta^{3p-2q-2}\,\mathrm{d}\theta\int_0^1\frac{\mathrm{d}t}{|t(1-t)|^{p-q}}\int_{\mathbf{R}/\mathbf{Z}}\int_{-1}^0\frac{|\gamma'(u+\tilde{w})-\gamma'(u)|^q}{|\tilde{w}|^{3p-2q-1}}\,\mathrm{d}\tilde{w}\,\mathrm{d}u\\ &\le C\left([\gamma]^q_{W^{(3p-2)/q-1,q}}+\|\gamma'\|^q_{L^\infty}\right)\le C\,\|\gamma\|^q_{W^{(3p-2)/q-1,q}}.\end{aligned}\tag{1.3}$$

For the other implication, we first derive for given vectors $a,b\in\mathbf{R}^n$, $|a|=|b|=1$, $\langle a,b\rangle\ge 0$,

$$|a\wedge b|^2 = |a|^2\,|b|^2-\langle a,b\rangle^2\ge 1-\langle a,b\rangle = \tfrac{1}{2}\,|a-b|^2.\tag{1.4}$$

By uniform continuity of γ' we may choose $\delta = \delta(\gamma) \in (0, \frac{1}{2})$ such that

$$|\gamma'(u+v) - \gamma'(u+w)| \le \tfrac{1}{10} \quad \text{for all } u \in \mathbf{R}/\mathbf{Z}, \quad v, w \in [-\delta, \delta]. \tag{1.5}$$

In fact, we may choose δ to be maximal, i.e. we assume that there are $\tilde{u} \in \mathbf{R}/\mathbf{Z}$, $\tilde{v}, \tilde{w} \in [-\delta, \delta]$ with

$$|\gamma'(\tilde{u}+\tilde{v}) - \gamma'(\tilde{u}+\tilde{w})| = \tfrac{1}{10}. \tag{1.6}$$

We fix $u_0 \in \mathbf{R}/\mathbf{Z}$. As $\gamma \in C^1(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ we may apply a suitable translation and rotation of the ambient space $\mathbf{R}^n$ such that $\gamma(u_0) = 0$ and there is a function $f \in C^1(\mathbf{R}, \mathbf{R}^{n-1})$ with $\|f'\|_{L^\infty} \le 1$ and $f(0) = 0$ such that $\tilde{\gamma}(u) := (u, f(u))$ satisfies $\tilde{\gamma}(B_{2\delta}(0)) \subset \gamma(\mathbf{R}/\mathbf{Z})$. Then

$$\tfrac{1}{2} |\triangle_{v,0}\tilde{\gamma}| \le |v| \le |\triangle_{v,0}\tilde{\gamma}| \quad \text{for } v \in [-2\delta, 2\delta]. \tag{1.7}$$

Arc-length parametrization of γ gives

$$\begin{aligned}
\mathrm{intM}^{(p,q)}(\tilde{\gamma}) &\ge c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{\left| \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} \wedge \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|\triangle_{v,0}\tilde{\gamma}|^{p-q} \, |\triangle_{0,w}\tilde{\gamma}|^{p-q} \, |\triangle_{v,w}\tilde{\gamma}|^p} \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u \\
&\overset{(1.4)}{\ge} c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{\left| \operatorname{sign} v \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|\triangle_{v,0}\tilde{\gamma}|^{p-q} \, |\triangle_{0,w}\tilde{\gamma}|^{p-q} \, |\triangle_{v,w}\tilde{\gamma}|^p} \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u \\
&\overset{(1.7)}{\ge} c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \frac{\left| \operatorname{sign} v \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{p-q} \, |w|^{p-q} \, |v-w|^p} \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u.
\end{aligned}$$

Using $\int_{-\delta}^{\delta} \phi(v) \, \mathrm{d}v = \frac{1}{2} \int_{-\delta}^{\delta} \phi(v) \, \mathrm{d}v + \frac{1}{2} \int_{-\delta}^{\delta} \phi(-v) \, \mathrm{d}v$ for any integrable function ϕ we arrive at

$$\begin{aligned}
\mathrm{intM}^{(p,q)}(\tilde{\gamma}) &\ge c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \Bigg(\frac{\left| \operatorname{sign} v \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{p-q} \, |w|^{p-q} \, |v-w|^p} \\
&\qquad + \frac{\left| -\operatorname{sign} v \frac{\triangle_{-v,0}\tilde{\gamma}}{|\triangle_{-v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{p-q} \, |w|^{p-q} \, |v+w|^p} \Bigg) \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u \\
&\ge c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-|v|}^{|v|} \Bigg(\frac{\left| \operatorname{sign} v \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{p-q} \, |w|^{p-q} \, |v+w|^p} \\
&\qquad + \frac{\left| -\operatorname{sign} v \frac{\triangle_{-v,0}\tilde{\gamma}}{|\triangle_{-v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{p-q} \, |w|^{p-q} \, |v-w|^p} \Bigg) \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u \\
&\ge c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{-|v|}^{|v|} \Bigg(\frac{\left| \operatorname{sign} v \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{3p-2q}} \\
&\qquad + \frac{\left| -\operatorname{sign} v \frac{\triangle_{-v,0}\tilde{\gamma}}{|\triangle_{-v,0}\tilde{\gamma}|} + \operatorname{sign} w \frac{\triangle_{0,w}\tilde{\gamma}}{|\triangle_{0,w}\tilde{\gamma}|} \right|^q}{|v|^{3p-2q}} \Bigg) \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u
\end{aligned} \tag{1.8}$$

$$\geq c \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \frac{\left| \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \frac{\triangle_{-v,0}\tilde{\gamma}}{|\triangle_{-v,0}\tilde{\gamma}|} \right|^q}{|v|^{3p-2q-1}} \, dv \, du,$$

where $c > 0$ only depends on p and q. The last line in (1.8) is bounded below by

$$\begin{aligned} c\int_{u_0-\delta}^{u_0+\delta}\int_{-\delta}^{\delta} &\frac{|\triangle_{v,0}\tilde{\gamma} + \triangle_{-v,0}\tilde{\gamma}|^q}{|v|^{3p-q-1}} \, dv \, du - C\int_{u_0-\delta}^{u_0+\delta}\int_{-\delta}^{\delta} \frac{|\triangle_{-v,0}\tilde{\gamma}|^q \left| \frac{1}{|\triangle_{v,0}\tilde{\gamma}|} - \frac{1}{|\triangle_{-v,0}\tilde{\gamma}|} \right|^q}{|v|^{3p-2q-1}} \, dv \, du \\ &\overset{(1.7)}{\geq} c\int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \frac{|\tilde{\gamma}(u+v) - 2\tilde{\gamma}(u) + \tilde{\gamma}(u-v)|^q}{|v|^{3p-q-1}} \, dv \, du \\ &\quad - C\int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \frac{\left| \frac{v}{|\triangle_{v,0}\tilde{\gamma}|} - \frac{v}{|\triangle_{-v,0}\tilde{\gamma}|} \right|^q}{|v|^{3p-2q-1}} \, dv \, du. \end{aligned}$$

By

$$\left| \frac{v}{|\triangle_{v,0}\tilde{\gamma}|} - \frac{v}{|\triangle_{-v,0}\tilde{\gamma}|} \right| \leq \left| \frac{(v, \triangle_{v,0}f)}{|\triangle_{v,0}\tilde{\gamma}|} + \frac{(-v, \triangle_{-v,0}f)}{|\triangle_{-v,0}\tilde{\gamma}|} \right| = \left| \frac{\triangle_{v,0}\tilde{\gamma}}{|\triangle_{v,0}\tilde{\gamma}|} + \frac{\triangle_{-v,0}\tilde{\gamma}}{|\triangle_{-v,0}\tilde{\gamma}|} \right|$$

we may use (1.8) to absorb the last term which finally leads to

$$\mathrm{intM}^{(p,q)}(\gamma) \geq c\int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \frac{|\tilde{\gamma}(u+v) - 2\tilde{\gamma}(u) + \tilde{\gamma}(u-v)|^q}{|v|^{3p-q-1}} \, dv \, du.$$

Since reparametrization to arc-length preserves regularity, we arrive at

$$\mathrm{intM}^{(p,q)}(\gamma) \geq c\int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \frac{|\gamma(u+v) - 2\gamma(u) + \gamma(u-v)|^q}{|v|^{3p-q-1}} \, dv \, du. \tag{1.9}$$

As u_0 was chosen arbitrarily, we obtain

$$[\![\gamma]\!]^q_{W^{(3p-2)/q-1,q}} \leq C \left(\mathrm{intM}^{(p,q)}(\gamma) + \|\gamma'\|^q_{L^\infty} \delta^{-3p+2q+2} \right) \tag{1.10}$$

uniformly on $\mathbf{R}/\mathbf{Z}$. Since the exponent $-3p + 2q + 2$ is negative, we have to show that δ is uniformly bounded away from zero in order to finish the proof. To this end we will establish the Morrey-type estimate

$$\|\gamma'(\cdot + w) - \gamma'(\cdot)\|_{L^\infty} \leq C\mathrm{intM}^{(p,q)}(\gamma)^{1/q}|w|^\alpha \quad \text{for all } w \in [-2\delta, 2\delta] \tag{1.11}$$

where $\alpha = 3(p-1)/q - 2 > 0$. As δ was chosen to be maximal with respect to (1.6), we arrive at

$$\tfrac{1}{10} \leq C\mathrm{intM}^{(p,q)}(\gamma)^{1/q}\delta^\alpha$$

which, applied to (1.10), gives (0.8) with $\beta = 1 + 1/(\alpha q)$.

It remains to prove (1.11) which follows by standard arguments due to Campanato [11]. Let $\gamma'_{B_r(x)}$ denote the integral mean of γ' over $B_r(x)$. We calculate for

$x \in \mathbf{R}/\mathbf{Z}$ and $r \in (0, \delta)$

$$\begin{aligned}\frac{1}{2r}\int_{B_r(x)} |\gamma'(v) - \gamma'_{B_r(x)}| \, \mathrm{d}v &\le \frac{1}{4r^2}\int_{B_r(x)}\int_{B_r(x)} |\gamma'(v) - \gamma'(u)| \, \mathrm{d}u \, \mathrm{d}v \\ &\le \left(\frac{1}{4r^2}\int_{B_r(x)}\int_{B_r(x)} |\gamma'(v) - \gamma'(u)|^q \, \mathrm{d}u \, \mathrm{d}v\right)^{1/q} \\ &\le Cr^\alpha \left(\int_{B_r(x)}\int_{B_r(x)} \frac{|\gamma'(v) - \gamma'(u)|^q}{|u-v|^{3p-2q-1}} \, \mathrm{d}u \, \mathrm{d}v\right)^{1/q} \\ &\overset{(1.9)}{\le} Cr^\alpha \mathrm{intM}^{(p,q)}(\gamma)^{1/q}.\end{aligned}$$

As (1.11) only involves the domain of γ up to a measure zero set, we may restrict to Lebesgue points. We choose two Lebesgue points $u, v \in \mathbf{R}/\mathbf{Z}$ of γ' with $r := |u-v| \in (0, \frac{\delta}{2})$. Then

$$\begin{aligned}&|\gamma'(u) - \gamma'(v)| \\ &\le \sum_{k=0}^{\infty} \left|\gamma'_{B_{2^{1-k}r}(u)} - \gamma'_{B_{2^{-k}r}(u)}\right| + \left|\gamma'_{B_{2r}(u)} - \gamma'_{B_{2r}(v)}\right| + \sum_{k=0}^{\infty} \left|\gamma'_{B_{2^{1-k}r}(v)} - \gamma'_{B_{2^{-k}r}(v)}\right|.\end{aligned}$$

Since

$$\begin{aligned}\left|\gamma'_{B_{2r}(u)} - \gamma'_{B_{2r}(v)}\right| &\le \frac{\int_{B_{2r}(u)} |\gamma'(x) - \gamma'_{B_{2r}(u)}| \, \mathrm{d}x + \int_{B_{2r}(v)} |\gamma'(x) - \gamma'_{B_{2r}(v)}| \, \mathrm{d}x}{|B_{2r}(u) \cap B_{2r}(v)|} \\ &\le C|u-v|^\alpha \mathrm{intM}^{(p,q)}(\gamma)^{1/q}\end{aligned}$$

as $r = |u-v|$ and, for all $y \in \mathbf{R}/\mathbf{Z}$, $R \in (0, \frac{\delta}{2})$,

$$\begin{aligned}\left|\gamma'_{B_{2R}(y)} - \gamma'_{B_R(y)}\right| &\le \frac{\int_{B_R(y)} |\gamma'(x) - \gamma'_{B_{2R}(y)}| \, \mathrm{d}x + \int_{B_R(y)} |\gamma'(x) - \gamma'_{B_R(y)}| \, \mathrm{d}x}{2R} \\ &\le CR^\alpha \mathrm{intM}^{(p,q)}(\gamma)^{1/q},\end{aligned}$$

we deduce $|\gamma'(u) - \gamma'(v)| \le C\left(\sum_{k=0}^{\infty} 2^{-k\alpha} + 1 + \sum_{k=0}^{\infty} 2^{-k\alpha}\right)|u-v|^\alpha \mathrm{intM}^{(p,q)}(\gamma)^{1/q}$. Thus $|\gamma'(u) - \gamma'(v)| \le C|u-v|^\alpha \mathrm{intM}^{(p,q)}(\gamma)^{1/q}$ for all Lebesgue points of γ' with $|u-v| < \frac{\delta}{2}$. The case $\frac{\delta}{2} \le |u-v| \le 2\delta$ follows by the triangle inequality. $\square$

Let us conclude this section by briefly commenting on the other ranges in the (p,q)-domain, see Figure 1.

Remark 1.2. (Non-repulsive energies for $p < \frac{2}{3}q + 1$) A bi-Lipschitz estimate is not guaranteed for injective curves if $p < \frac{2}{3}q + 1$. We briefly give the following example. Consider the curves $u \mapsto (u, 0, 0)$ and $u \mapsto (0, u, \delta)$ for $u \in [-1, 1]$, $\delta \in [0, 1]$. The interaction of these strands leads to the $\mathrm{intM}^{(p,q)}$-value

$$\begin{aligned}&C \iiint_{[-1,1]^3} \frac{(\delta^2 + u^2)^{q/2}}{|v-w|^{p-q}(\delta^2 + u^2 + v^2)^{p/2}(\delta^2 + u^2 + w^2)^{p/2}} \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u \\ &\le C \iiint_{[-1,1]^3} \frac{(\delta^2 + u^2)^{(q-p)/2}}{|v-w|^{p-q}(\delta^2 + u^2 + v^2 + w^2)^{p/2}} \, \mathrm{d}w \, \mathrm{d}v \, \mathrm{d}u.\end{aligned}$$

Introducing polar coordinates $u = r\cos\vartheta,\ v = r\sin\vartheta\cos\varphi,\ w = r\sin\vartheta\sin\varphi$, the former quantity is bounded by

$$C\int_0^{\sqrt{3}}\int_0^{\pi}\frac{(\delta^2+r^2\cos^2\vartheta)^{(q-p)/2}\,r^2\sin\vartheta}{r^{p-q}\sin^{p-q}\vartheta\,(\delta^2+r^2)^{p/2}}\,\mathrm{d}\vartheta\,\mathrm{d}r\underbrace{\int_0^{2\pi}\frac{\mathrm{d}\varphi}{|\cos\varphi-\sin\varphi|^{p-q}}}_{\le C}$$

$$\le C\int_0^{\sqrt{3}}\Bigg(\int_{[0,\frac{\pi}{4}]\cup[\frac{3\pi}{4},\pi]}\frac{(\delta^2+r^2)^{(q-p)/2}\,\mathrm{d}\vartheta}{r^{p-q-2}\underbrace{\sin^{p-q-1}\vartheta}_{\ge 1}(\delta^2+r^2)^{p/2}}$$

$$+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{(\delta^2+r^2\cos^2\vartheta)^{(q-p)/2}\,r\sin\vartheta}{r^{p-q-1}\sin^{p-q}\vartheta\,(\delta^2+r^2)^{p/2}}\,\mathrm{d}\vartheta\Bigg)\,\mathrm{d}r$$

$$\le C\int_0^{\sqrt{3}}\left((\delta+r)^{-3p+2q+2}+r^{-p+q+1}(\delta+r)^{-p}\int_0^r\left(\delta^2+\sigma^2\right)^{(q-p)/2}\mathrm{d}\sigma\right)\mathrm{d}r$$

$$\le C\left(1-\delta^{-3p+2q+3}\right)\le C.$$

Using Theorem 1 and the monotonicity of $\mathrm{intM}^{(\cdot,q)}$ for fixed q, it is easy to produce a family of knots uniformly converging to a non-embedded curve without an energy blow-up as $\delta\searrow 0$, so these energies are not self-repulsive.

Remark 1.3. (Singular energies for $p\ge q+\frac{2}{3}$, $q>1$) For $p\ge q+\frac{2}{3}$, $q>1$, we have $\mathrm{intM}^{(p,q)}(\gamma)\equiv\infty$ for all closed C^1-curves γ. To see this, note that we assumed $p<\frac{2}{3}q+1$ in Theorem 1 mainly because neither (0.11) nor (0.12) is defined for $s\ge 1$. For general $p\ge\frac{2}{3}q+1$ we nevertheless still have

$$\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/2}^{1/2}\frac{|\gamma'(u+w)-\gamma'(u)|^q}{|w|^{3p-2q-1}}\,\mathrm{d}w\,\mathrm{d}u\le C\left(\mathrm{intM}^{(p,q)}(\gamma)+\mathrm{intM}^{(p,q)}(\gamma)^\beta\right).$$

Applying Brezis [10, Prop. 2], the function γ' is constant, hence γ lies on a straight line. Therefore, γ cannot be a closed C^1-curve.

Remark 1.4 (Strange energies for $p\in[q+\frac{2}{3},\frac{2}{3}q+1)$)**.** On $p\in[q+\frac{2}{3},\frac{2}{3}q+1)$, $p,q>0$, see the hatched area in Figure 1, we find the strange behavior that there are no closed finite-energy C^3-curves while self-intersections, and in particular corners, are not penalized. So piecewise linear curves (polygonals) have finite energy.

The latter can be seen by adapting the calculation in Remark 1.2. For the former we recall that a closed arc-length parametrized C^2-curve must have positive curvature $|\gamma''|$ at some point u_0 and by continuity there are $c,\delta>0$ with $|\gamma''|\ge c>0$ on $[u_0-\delta,u_0+\delta]$. As $\gamma''\perp\gamma'$ we obtain $|\gamma''\wedge\gamma'|=|\gamma''|\ge c$. So $\mathrm{intM}^{(p,q)}(\gamma)$ is bounded below by

$$\int_{u_0-\delta}^{u_0+\delta}\int_{-\delta}^{\delta}\int_{\frac{1}{3}|v|}^{\frac{2}{3}|v|}|v|^{-3p+2q}\left|\frac{\triangle_{v,0}\gamma}{v}\wedge\frac{\triangle_{0,w}\gamma}{w}\right|^q\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$

$$
\begin{aligned}
&= \int_{u_0-\delta}^{u_0+\delta} \int_{-\delta}^{\delta} \int_{\frac{1}{3}|v|}^{\frac{2}{3}|v|} |v|^{-3p+2q} \Bigg| \tfrac{v-w}{2}\gamma''(u) \wedge \gamma'(u) \\
&\quad + \tfrac{v^2}{2} \int_0^1 (1-\vartheta_1)^2 \gamma'''(u+\vartheta_1 v)\,\mathrm{d}\vartheta_1 \wedge \big(\gamma'(u) + \tfrac{w}{2}\gamma''(u)\big) \\
&\quad - \tfrac{w^2}{2} \int_0^1 (1-\vartheta_2)^2 \gamma'''(u+\vartheta_2 w)\,\mathrm{d}\vartheta_1 \wedge \big(\gamma'(u) + \tfrac{v}{2}\gamma''(u)\big) \\
&\quad + \tfrac{v^2 w^2}{2} \iint_{[0,1]^2} (1-\vartheta_1)^2(1-\vartheta_2)^2 \gamma'''(u+\vartheta_1 v) \wedge \gamma'''(u+\vartheta_2 w)\,\mathrm{d}\vartheta_1\,\mathrm{d}\vartheta_2 \Bigg|^q \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \\
&\geq \delta \left[\tilde{c} - C\delta \|\gamma'''\|_{L^\infty}^q \left(\|\gamma'''\|_{L^\infty} + \|\gamma''\|_{L^\infty} + 1\right)^q\right] \int_{-\delta}^{\delta} |v|^{-3p+3q+1}\,\mathrm{d}v.
\end{aligned}
$$

Diminishing $\delta > 0$, the square bracket is positive. This gives $\operatorname{intM}^{(p,q)}(\gamma) = \infty$.

2. Existence of minimizers within knot classes

The arguments here are quite similar as for the tangent-point energies [6], however, we provide full proofs for the readers' convenience.

Using Theorem 1 together with the Arzelà–Ascoli theorem, we will see that sets of curves in $C^1_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ with a uniform bound on the energy are sequentially compact in C^1. To this end we need the following result.

Proposition 2.1. (Uniform bi-Lipschitz estimate) *For every $M < \infty$ and* (0.6) *there is a constant $C(M,p,q) > 0$ such that every curve $\gamma \in C^1_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ parametrized by arc-length with*

$$
\operatorname{intM}^{(p,q)}(\gamma) \leq M \tag{2.1}
$$

satisfies the bi-Lipschitz estimate

$$
|u-v| \leq C(M,p,q)\,|\gamma(u)-\gamma(v)| \quad \text{for all } u, v \in \mathbf{R}/\mathbf{Z}. \tag{2.2}
$$

The proof is based on the following lemma. To be able to state it, we set for two arc-length parametrized curves $\gamma_i \colon I_i \to \mathbf{R}$, $i = 1,2$, I_1, I_2 open intervals,

$$
\begin{aligned}
\operatorname{intM}^{(p,q)}(\gamma_1,\gamma_2) := {} & \operatorname{intM}^{(p,q)}(\gamma_1) + \operatorname{intM}^{(p,q)}(\gamma_2) \\
& + \iiint_{I_1^2 \times I_2} \frac{|\gamma_1'(u_1)|\,|\gamma_1'(u_2)|\,|\gamma_2'(u_3)|}{R^{(p,q)}(\gamma_1(u_1), \gamma_1(u_2), \gamma_2(u_3))}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3 \\
& + \iiint_{I_1 \times I_2^2} \frac{|\gamma_1'(u_1)|\,|\gamma_2'(u_2)|\,|\gamma_2'(u_3)|}{R^{(p,q)}(\gamma_1(u_1), \gamma_2(u_2), \gamma_2(u_3))}\,\mathrm{d}u_1\,\mathrm{d}u_2\,\mathrm{d}u_3.
\end{aligned}
$$

Lemma 2.2. *Let $\alpha \in (0,1)$. For $\mu > 0$ we let M_μ denote the set of all pairs (γ_1, γ_2) of curves $\gamma_i \in C^1_{\mathrm{ia}}([-1,1], \mathbf{R}^n)$ satisfying*

(i) $|\gamma_1(0) - \gamma_2(0)| = 1$,
(ii) $\gamma_1'(0) \perp (\gamma_1(0) - \gamma_2(0)) \perp \gamma_2'(0)$,
(iii) $\|\gamma_i'\|_{C^{0,\alpha}} \leq \mu$, $\;i = 1,2$.

Then there is a $c = c(\alpha, \mu) > 0$ *such that*

$$\mathrm{intM}^{(p,q)}(\gamma_1, \gamma_2) \geq c \quad \textit{for all } (\gamma_1, \gamma_2) \in M_\mu.$$

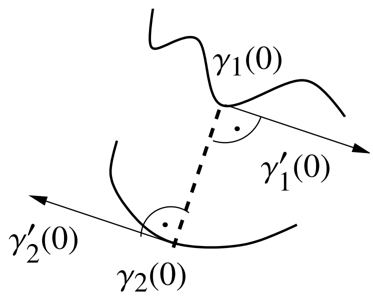

Figure 3. A pair of curves $(\gamma_1, \gamma_2) \in M_\mu$ defined in Lemma 2.2. Note that the arcs γ_1, γ_2 cannot intersect each other.

Proof. It is easy to see that $\mathrm{intM}^{(p,q)}(\gamma_1, \gamma_2)$ is zero if and only if both γ_1 and γ_2 are part of one single straight line. We will show that $\mathrm{intM}^{(p,q)}(\cdot, \cdot)$ attains its minimum on M_μ. As M_μ does not contain straight lines by (i), (ii), this minimum is strictly positive which thus proves the lemma.

Let $(\gamma_1^{(n)}, \gamma_2^{(n)})$ be a minimizing sequence in M_μ, i.e. we have

$$\lim_{n\to\infty} \mathrm{intM}^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \inf_{M_\mu} \mathrm{intM}^{(p,q)}(\cdot, \cdot).$$

Subtracting $\gamma_1(0)$ from *both* curves, i.e. setting

$$\tilde{\gamma}_i^{(n)}(\tau) := \gamma_i^{(n)}(\tau) - \gamma_1(0), \quad i = 1, 2,$$

and using Arzelà–Ascoli we can pass to a subsequence such that

$$\tilde{\gamma}_i^{(n)} \to \tilde{\gamma}_i \quad \text{in } C^1.$$

Furthermore, $(\tilde{\gamma}_1, \tilde{\gamma}_2) \in M_\mu$ since M_μ is closed under convergence in C^1. Since, by Fatou's lemma, the functional $\mathrm{intM}^{(p,q)}$ is lower semi-continuous with respect to C^1 convergence, we obtain

$$\begin{aligned} \mathrm{intM}^{(p,q)}(\tilde{\gamma}_1, \tilde{\gamma}_2) &\leq \lim_{n\to\infty} \mathrm{intM}^{(p,q)}(\tilde{\gamma}_1^{(n)}, \tilde{\gamma}_2^{(n)}) = \lim_{n\to\infty} \mathrm{intM}^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) \\ &= \inf_{M_\mu} \mathrm{intM}^{(p,q)}(\cdot, \cdot). \end{aligned}$$ □

Let us use this lemma to give the

Proof of Proposition 2.1. Applying Theorem 1 to (2.1) we obtain $\|\gamma'\|_{C^{0,\alpha}} \leq C(M)$ for $\alpha = 3\frac{p-1}{q} - 2 \in (0, 1 - \frac{1}{q})$. As an immediate consequence there is a $\delta = \delta(\alpha, M) > 0$ such that

$$|u - v| \leq 2|\gamma(u) - \gamma(v)| \tag{2.3}$$

for all $u, v \in \mathbf{R}/\mathbf{Z}$ with $|u - v| \leq \delta$. Let now

$$S := \inf \left\{ |\gamma(u) - \gamma(v)| \,\middle|\, u, v \in \mathbf{R}/\mathbf{Z}, |u - v| \geq \delta \right\} \leq \tfrac{1}{2}.$$

We will complete the proof by estimating S from below. Using the compactness of $\{u, v \in \mathbf{R}/\mathbf{Z}, |u - v| \geq \delta\}$, there are $s, t \in \mathbf{R}/\mathbf{Z}$ with $|s - t| \geq \delta$ and $|\gamma(s) - \gamma(t)| = S$.

If now $|s-t|=\delta$ we obtain

$$2S = 2|\gamma(s)-\gamma(t)| \overset{(2.3)}{\geq} \delta$$

and hence

$$|u-v| \leq \tfrac{1}{2} \leq \frac{S}{\delta} \leq \frac{|\gamma(u)-\gamma(v)|}{\delta(\alpha, M)}$$

for all $u, v \in \mathbf{R}/\mathbf{Z}$ with $|u-v| \geq \delta$. This proves the proposition in this case. If on the other hand $|s-t| > \delta$ then we infer using the minimality of $|\gamma(s)-\gamma(t)|$

$$\gamma'(s) \perp (\gamma(s)-\gamma(t)) \perp \gamma'(t).$$

We define for $\tau \in [-1,1]$

$$\gamma_1(\tau) := \frac{1}{S}\gamma(s+S\tau) \quad \text{and} \quad \gamma_2(\tau) := \frac{1}{S}\gamma(t+S\tau).$$

Since $\|\gamma_i'\|_{C^{0,\alpha}} \leq \|\gamma'\|_{C^{0,\alpha}} \leq C(M)$ we may apply Lemma 2.2 which yields

$$\mathrm{intM}^{(p,q)}(\gamma_1, \gamma_2) \geq c(\alpha, M) > 0.$$

Together with $\mathrm{intM}^{(p,q)}(\gamma_1,\gamma_2) \leq S^{3p-2q-3}\mathrm{intM}^{(p,q)}(\gamma)$ this leads to

$$S \geq \left(\frac{c(\alpha,M)}{\mathrm{intM}^{(p,q)}(\gamma)}\right)^{\frac{1}{3p-2q-3}} \geq \left(\frac{c(\alpha,M)}{M}\right)^{\frac{1}{3p-2q-3}}.$$

Hence, $|u-v| \leq \frac{1}{2} \leq \frac{|\gamma(u)-\gamma(v)|}{2S} \leq C(M,p,q)\,|\gamma(u)-\gamma(v)|$ for all $u, v \in \mathbf{R}/\mathbf{Z}$ with $|u-v| \geq \delta$. $\square$

We are now in the position to prove the compactness result which is crucial both to the existence of minimizers in any knot class and to the self-avoiding behavior of the energies.

Proposition 2.3. (Sequential compactness) *For each $M < \infty$ the set*

$$A_M := \left\{ \gamma \in C^1_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n) \,\middle|\, \mathrm{intM}^{(p,q)}(\gamma) \leq M \right\}$$

is sequentially compact in C^1 up to translations.

Proof. By Theorem 1 there are $C(M) < \infty$ and $\alpha = \alpha(p,q) > 0$ such that $\|\gamma'\|_{C^\alpha} \leq C(M)$ for all $\gamma \in A_M$ and hence

$$\|\tilde{\gamma}\|_{C^{1,\alpha}} \leq C(M) + 1$$

where $\tilde{\gamma}(u) := \gamma(u) - \gamma(0)$. By Proposition 2.1, the bi-Lipschitz estimate (2.2) holds.

Let now $\gamma_n \in A_M$. Then

$$\|\tilde{\gamma}_n\|_{C^{1,\alpha}} \leq C(M) + 1$$

and hence after passing to suitable subsequence we have $\tilde{\gamma}_n \to \gamma_0$ in C^1. Since γ_n was parametrized by arc-length, γ_0 is still parametrized by arc-length and the bi-Lipschitz estimate carries over to γ_0. So, especially, $\gamma_0 \in C^1_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$. From lower semi-continuity with respect to C^1 convergence we infer

$$\mathrm{intM}^{(p,q)}(\gamma_0) \leq \liminf_{n\to\infty} \mathrm{intM}^{(p,q)}(\gamma_n) \leq M.$$

So $\gamma_0 \in A_M$. $\square$

We may now pass to the

Proof of Theorem 2. Let $(\gamma_k)_{k\in\mathbf{N}} \in C^1_{\mathrm{ia}}$ be a minimal sequence for $\mathrm{intM}^{(p,q)}$ in a given knot class K, i.e. let

$$\lim_{k\to\infty} \mathrm{intM}^{(p,q)}(\gamma_k) = \inf_{C^1_{\mathrm{ia}}\cap K} \mathrm{intM}^{(p,q)}.$$

After passing to a subsequence and suitable translations, we hence get by Proposition 2.3 a $\gamma_0 \in C^1_{\mathrm{ia}}$ with $\gamma_k \to \gamma_0$ in C^1. As the intersection of every knot class with C^1 is an open set in C^1 [1, Cor. 1.5] (see [34] for an explicit construction), the curve γ_0 belongs to the same knot class as the elements of the minimal sequence $(\gamma_k)_{k\in\mathbf{N}}$. The lower semi-continuity of $\mathrm{intM}^{(p,q)}$ furthermore implies

$$\inf_{C^1_{\mathrm{ia}}\cap K} \mathrm{intM}^{(p,q)} \le \mathrm{intM}^{(p,q)}(\gamma_0) \le \lim_{n\to\infty} \mathrm{intM}^{(p,q)}(\gamma_n) = \inf_{C^1_{\mathrm{ia}}\cap K} \mathrm{intM}^{(p,q)}.$$

Hence, γ_0 is the minimizer we have been searching for. □

By the same reasoning one derives the existence of a global minimizer of $\mathrm{intM}^{(p,q)}$.

Let us conclude this section by deriving that the generalized integral Menger curvature are in fact knot energies (in the sub-critical range).

Proposition 2.4. ($\mathrm{intM}^{(p,q)}$ is a strong knot energy [41, Cor. 2.3]) *Let* (0.6) *hold.*

(1) *If $(\gamma_k)_{k\in\mathbf{N}}$ is a sequence of embedded $W^{(3p-2)/q-1,q}$-curves uniformly converging to a non-injective curve $\gamma_\infty \in C^{0,1}$ parametrized by arc-length then* $\mathrm{intM}^{(p,q)}(\gamma_k) \to \infty$.

(2) *For given $E, L > 0$ there are only finitely many knot types having a representative with* $\mathrm{intM}^{(p,q)} < E$ *and length* $= L$.

Proof. The first statement immediately follows from the bi-Lipschitz estimate in Proposition 2.1, as a sequence with bounded energy would be sequentially compact in C^1_{ia} and thus cannot uniformly converge to a non-injective curve.

To show the second statement, let us assume that it was wrong, i.e., that there are curves $(\gamma_k)_{k\in\mathbf{N}}$ of length L, all belonging to different knot classes, with energy less than E. Of course we can assume that $L = 1$. After applying suitable transformations and passing to a subsequence, Proposition 2.3 guarantees the existence of $\gamma_0 \in A_M$ with $\gamma_k \to \gamma_0$ in C^1. Again by [1, 34] this implies that almost all γ_k belong to the same knot class as γ_0, which is a contradiction. □

3. Differentiability

Recall that we have for $v, w \in D$

$$|v|, |w| \le |v - w| \le \tfrac{2}{3}.$$

Hence, we obtain for each curve $\gamma \in C^{0,1}_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z})$ with $\mathrm{intM}^{(p,q)}(\gamma) < \infty$ due to the bi-Lipschitz estimate

$$|\gamma(u+v) - \gamma(u)| \simeq |v|, \quad |\gamma(u+w) - \gamma(u)| \simeq |w|, \quad |\gamma(u+v) - \gamma(u+w)| \simeq |v - w|$$

for all $(u, v, w) \in \mathbf{R}/\mathbf{Z} \times D$. (Here $a \simeq b$ is an abbreviation for the existence of uniform constants $0 < c \le C < \infty$ with $cb \le a \le Cb$.) The same estimates hold (with different constants) if γ is merely an injective regular curve which we will assume throughout this section.

We derive the following form for the first variation which is at first site much more complicated than the formula derived by Hermes [20], but due to the special structure of D it is easier to do estimates using this formula. We abbreviate

$$R^{p,q}(u_1, u_2, u_3) := R^{(p,q)}(\gamma(u_1), \gamma(u_2), \gamma(u_3))$$

and we still use

$$\triangle_{v,w}\bullet := \bullet(u+v) - \bullet(u+w). \tag{0.9}$$

In contrast to O'Hara's knot energies, we can use a rather direct argument to deduce that the integral Menger curvature is Gâteaux differentiable by investigating the integrand, i.e. by looking at the Lagrangian

$$L(\gamma)(u,v,w) := \frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^p |\triangle_{v,0}\gamma|^p |\triangle_{v,w}\gamma|^p} |\gamma'(u)||\gamma'(u+v)||\gamma'(u+w)|.$$

For $\gamma, h \in W^{(3p-2)/q-1,q}$, and $\gamma_\tau := \gamma + \tau h$ one calculates

$$\begin{aligned}
&\delta L(\gamma;h)(u,v,w) := \frac{\partial}{\partial\tau}\left(L(\gamma_\tau)(u,v,w)\right)\Big|_{\tau=0} \\
&= \Bigg\{ q\,|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^{q-2}\, \frac{\langle \triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma, \triangle_{w,0}h \wedge \triangle_{v,0}\gamma + \triangle_{w,0}\gamma \wedge \triangle_{v,0}h\rangle}{|\triangle_{w,0}\gamma|^p |\triangle_{v,0}\gamma|^p |\triangle_{v,w}\gamma|^p} \\
&\quad - p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p+2} |\triangle_{v,0}\gamma|^p |\triangle_{v,w}\gamma|^p} \cdot \langle \triangle_{w,0}\gamma, \triangle_{w,0}h\rangle \\
&\quad - p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p} |\triangle_{v,0}\gamma|^{p+2} |\triangle_{v,w}\gamma|^p} \cdot \langle \triangle_{v,0}\gamma, \triangle_{v,0}h\rangle \\
&\quad - p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p} |\triangle_{v,0}\gamma|^p |\triangle_{v,w}\gamma|^{p+2}} \cdot \langle \triangle_{v,w}\gamma, \triangle_{v,w}h\rangle \\
&\quad + R^{p,q}(u, u+w, u+v) \left\langle \frac{\gamma'(u)}{|\gamma'(u)|}, \frac{h'(u)}{|\gamma'(u)|} \right\rangle \\
&\quad + R^{p,q}(u, u+w, u+v) \left\langle \frac{\gamma'(u+v)}{|\gamma'(u+v)|}, \frac{h'(u+v)}{|\gamma'(u+v)|} \right\rangle \\
&\quad + R^{p,q}(u, u+w, u+v) \left\langle \frac{\gamma'(u+w)}{|\gamma'(u+w)|}, \frac{h'(u+w)}{|\gamma'(u+w)|} \right\rangle \Bigg\} |\gamma'(u+w)||\gamma'(u+v)||\gamma'(u)|.
\end{aligned}$$

For future reference, we denote the seven terms one obtains from this formula (after multiplying each one by $|\gamma'(u+w)||\gamma'(u+v)||\gamma'(u)|$) by $\delta L_1, \ldots, \delta L_7$.

Lemma 3.1. *Let* (0.6) *hold and* $\gamma \in W^{(3p-2)/q-1,q}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ *be an injective regular curve. Then* $\mathrm{intM}^{(p,q)}$ *is Gâteaux differentiable in* γ *and the first variation in direction* $h \in W^{(3p-2)/q-1,q}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ *is given by*

$$\delta \mathrm{intM}^{(p,q)}(\gamma; h) = 6 \iiint_{\mathbf{R}/\mathbf{Z}\times D} \delta L(\gamma;h)(u,v,w)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \tag{3.1}$$

and (0.10) *holds.*

Proof. Let U be a neighborhood of γ in $W^{(3p-2)/q-1,q} \subset C^{(3p-3)/q-1} \subset C^1$ consisting only of regular curves with

$$\inf_{\tilde\gamma\in U, u\in\mathbf{R}/\mathbf{Z}} |\tilde\gamma'(u)| =: M_1 > 0$$

and

$$\sup_{\tilde\gamma\in U, u\neq v\in\mathbf{R}/\mathbf{Z}} \frac{|\tilde\gamma(u)-\tilde\gamma(v)|}{|u-v|} =: M_2 < \infty.$$

Using

$$\begin{aligned}
&\left\langle \frac{\triangle_{w,0}\gamma}{w}\wedge\frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0}h}{w}\wedge\frac{\triangle_{v,0}\gamma}{v}+\frac{\triangle_{w,0}\gamma}{w}\wedge\frac{\triangle_{v,0}h}{v}\right\rangle \\
&= \left\langle \left(\frac{\triangle_{w,0}\gamma}{w}-\frac{\triangle_{v,0}\gamma}{v}\right)\wedge\frac{\triangle_{v,0}\gamma}{v}, \left(\frac{\triangle_{w,0}h}{w}-\frac{\triangle_{v,0}h}{v}\right)\wedge\frac{\triangle_{v,0}\gamma}{v}\right. \\
&\quad \left. +\left(\frac{\triangle_{w,0}\gamma}{w}-\frac{\triangle_{v,0}\gamma}{v}\right)\wedge\frac{\triangle_{v,0}h}{v}\right\rangle, \\
&\frac{\triangle_{0,w}\gamma}{w}\wedge\frac{\triangle_{0,v}\gamma}{v} = \left(\frac{\triangle_{0,w}\gamma}{w}-\frac{\triangle_{0,v}\gamma}{v}\right)\wedge\frac{\triangle_{0,v}\gamma}{v},
\end{aligned}$$

and

$$R^{p,q}(u,u+w,u+v) = \frac{\left|\left(\frac{\triangle_{0,w}\gamma}{w}-\frac{\triangle_{0,v}\gamma}{v}\right)\wedge\triangle_{0,v}\gamma\right|^q}{|w|^{-q}\,|\triangle_{0,w}\gamma|^p\,|\triangle_{v,0}\gamma|^p\,|\triangle_{v,w}\gamma|^p}$$

together with the bi-Lipschitz estimate we infer for $\tilde\gamma\in U$

$$\begin{aligned}
&|\delta L(\tilde\gamma;h)(u,v,w)| \\
&\le C\frac{|\frac{\triangle_{w,0}\tilde\gamma}{w}-\frac{\triangle_{v,0}\tilde\gamma}{v}|^q\|h'\|_{L^\infty}+|\frac{\triangle_{w,0}\tilde\gamma}{w}-\frac{\triangle_{v,0}\tilde\gamma}{v}|^{q-1}|\frac{\triangle_{w,0}h}{w}-\frac{\triangle_{v,0}h}{v}|}{|w|^{p-q}|v|^{p-q}|v-w|^p}.
\end{aligned} \tag{3.2}$$

So for $0<|\tau|\le 1$ so small that $\gamma_\tau\in U$ we may let $\tilde\gamma=\gamma_\tau$ which leads to

$$\begin{aligned}
&\left|\frac{\partial}{\partial\tilde\tau}L(\gamma_{\tilde\tau})(u,v,w)\Big|_{\tilde\tau=\tau}\right| \\
&\le C\frac{|\frac{\triangle_{w,0}\gamma}{w}-\frac{\triangle_{v,0}\gamma}{v}|^q\|h'\|_{L^\infty}+|\frac{\triangle_{w,0}h}{w}-\frac{\triangle_{v,0}h}{v}|^q\|h'\|_{L^\infty}}{|w|^{p-q}|v|^{p-q}|v-w|^p} \\
&\quad +\frac{|\frac{\triangle_{w,0}\gamma}{w}-\frac{\triangle_{v,0}\gamma}{v}|^{q-1}|\frac{\triangle_{w,0}h}{w}-\frac{\triangle_{v,0}h}{v}|+|\frac{\triangle_{w,0}h}{w}-\frac{\triangle_{v,0}h}{v}|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} \\
&=: g(u,v,w)
\end{aligned} \tag{3.3}$$

where g does not depend on τ and C depends on M_1, M_2, p, q only.

For $f \in W^{(3p-2)/q-1,q}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$, we have

$$\begin{aligned}
&\iiint_{\mathbf{R}/\mathbf{Z}\times D} \frac{\left|\frac{\triangle_{w,0}f}{w} - \frac{\triangle_{v,0}f}{v}\right|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}u \\
&= \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{\left|\int_0^1 \left(f'(u+\theta w) - f'(u+\theta v)\right)\,\mathrm{d}\theta\right|^q}{|w|^{p-q}|v|^{p-q}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \\
&\le C\int_0^1 \iiint_{\mathbf{R}/\mathbf{Z}\times D} \frac{|f'(u+\theta(w-v)) - f'(u)|^q}{|w|^{p-q}|v|^{p-q}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}\theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} |g(u,v,w)| \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \\
&\le C\,[\gamma]^q_{W^{(3p-q-2)/q,q}} \|h'\|_{L^\infty} + [h]^q_{W^{(3p-q-2)/q,q}} (1 + \|h'\|_{L^\infty}) \\
&\quad + C\left(\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{\left|\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}\right|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u\right)^{1-\frac{1}{q}} \\
&\quad\cdot \left(\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{\left|\frac{\triangle_{w,0}h}{w} - \frac{\triangle_{v,0}h}{v}\right|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u\right)^{\frac{1}{q}} \\
&\le C\left([\gamma]^q_{W^{(3p-2)/q-1,q}} + [h]^q_{W^{(3p-2)/q-1,q}}\right)(1 + \|h'\|_{L^\infty}) \\
&\quad + C\,[\gamma]^{q-1}_{W^{(3p-q-2)/q,q}}\,[h]^q_{W^{(3p-q-2)/q,q}}\,.
\end{aligned} \tag{3.4}$$

So $\delta L(\gamma_\tau; h)$ has a uniform L^1-majorant for τ sufficiently small. Therefore, by Lebegue's theorem of dominanted convergence, we finally can use the fundamental theorem of calculus to write for τ small

$$\begin{aligned}
\frac{\mathrm{intM}^{(p,q)}(\gamma+\tau h) - \mathrm{intM}^{(p,q)}(\gamma)}{\tau} &= 6\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \int_0^1 \delta L(\gamma_{s\tau}; h)(u,v,w)\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \\
&\xrightarrow{\tau\to 0} 6\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \delta L(\gamma; h)(u,v,w)\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.
\end{aligned}$$

Consequently, the first variation exists and has the form (3.1).

Using once more the symmetry of the integrand, we can bring this into the form (0.10) as follows. Due to the symmetry of the integrand we have $L \circ P_\sigma = L$ for any permutation matrix $P_\sigma, \in \mathfrak{S}_3$. So we obtain

$$6\iiint_{\mathbf{R}/Z\times D} \delta L(\gamma; h) = \sum_{\sigma\in\mathfrak{S}_3} \iiint_{P_\sigma(\mathbf{R}/Z\times D)} \delta L(\gamma; h) = \iiint_{(\mathbf{R}/\mathbf{Z})^3} \delta L(\gamma; h).$$

The symmetry of $R^{(p,q)}$ now leads to the desired.

Furthermore, by (3.2), the first variation defines a bounded linear operator on $W^{(3p-2)/q-1,q}$. Hence $\mathrm{intM}^{(p,q)}$ is Gâteaux differentiable. $\square$

In fact, we can even show that $\mathrm{intM}^{(p,q)}$ is C^1, though we will not use this fact in the rest of this article.

Lemma 3.2. *The functional* $\mathrm{intM}^{(p,q)}$ *is* C^1 *on the subspace of all regular embedded* $W^{(3p-2)/q-1,q}$*-curves.*

Proof. We will prove by contradiction that $\delta\mathrm{intM}^{(p,q)}$ is a continuous map from embedded regular $W^{(3p-2)/q-1,q}$-curves to $\left(W^{(3p-2)/q-1,q}\right)^*$. So let us assume that $\delta\mathrm{intM}^{(p,q)}$ was not continuous in γ_0. Consequently, there are some $\varepsilon_0 > 0$ and sequences $(\gamma_k)_{k\in\mathbf{N}}, (h_k)_{k\in\mathbf{N}} \subset W^{(3p-2)/q-1,q}$, $\gamma_k \to \gamma_0$ in $W^{(3p-2)/q-1,q}$, $\|h_k\|_{W^{(3p-2)/q-1,q}} \leq 1$, with

$$|\delta\mathrm{intM}^{(p,q)}(\gamma_k; h_k) - \mathrm{intM}^{(p,q)}(\gamma; h_k)| \geq \varepsilon_0. \tag{3.5}$$

As in the proof of Lemma 3.1, we can exploit the embedding $W^{(3p-2)/q-1,q} \hookrightarrow C^1$ and the openness of the set of regular embedded curves in C^1, to find an open neighborhood U of γ_0 consisting only of embedded curves, such that

$$\inf_{\tilde\gamma\in U, u\in\mathbf{R}/\mathbf{Z}} |\tilde\gamma'(u)| =: M_1 > 0$$

and

$$\sup_{\tilde\gamma\in U} \|\tilde\gamma\|_{W^{(3p-2)/q-1,q}} + \sup_{\tilde\gamma\in U, u\neq v\in\mathbf{R}/\mathbf{Z}} \frac{|\tilde\gamma(u) - \tilde\gamma(v)|}{|u-v|} =: M_2 < \infty.$$

After passing to a subsequence we may assume $(\gamma_k)_{k\in\mathbf{N}} \subset U$ and $h_k \to h_0 \in W^{(3p-2)/q-1,q}$ in C^1 due to the compactness of the embedding $W^{(3p-2)/q-1,q} \hookrightarrow C^1$ which then also gives

$$\delta L(\gamma_k; h_k) - \delta L(\gamma_0; h_0) \to 0$$

pointwise almost everywhere on $\mathbf{R}/\mathbf{Z} \times D$ and hence in measure, i.e., for all $\varepsilon > 0$ we have

$$\lim_{k\to\infty} \mathcal{L}^3(A_{\varepsilon,k}) = 0, \tag{3.6}$$

where $\mathcal{L}^3$ denotes the Lebesgue measure and

$$A_{\varepsilon,k} := \left\{ (u,v,w) \in \mathbf{R}/\mathbf{Z} \times D \,\middle|\, \left|\delta L(\gamma_k; h_k)(u,v,w) - \delta L(\gamma_0; h_0)(u,v,w)\right| \geq \varepsilon \right\}.$$

For all $\varepsilon > 0$ we can deduce from (3.2) using Young's inequality that there is a $C_\varepsilon > 0$ with

$$\begin{aligned} |\delta L(\gamma_k; h_k)(u,v,w)| &\leq C_\varepsilon \frac{\left|\frac{\triangle_{w,0}\gamma_k}{w} - \frac{\triangle_{v,0}\gamma_k}{v}\right|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} + \varepsilon \frac{\left|\frac{\triangle_{w,0}h_k}{w} - \frac{\triangle_{v,0}h_k}{v}\right|^q}{|w|^{p-q}|v|^{p-q}|v-w|^p} \\ &=: C_\varepsilon g_k^{(1)}(u,v,w) + \varepsilon g_k^{(2)}(u,v,w) \end{aligned} \tag{3.7}$$

for all $k \in \mathbf{N} \cup \{0\}$. Since the first summand converges in L^1 as $k \to \infty$, it is uniformly integrable, so there is a $\delta_\varepsilon > 0$ such that $\mathcal{L}^3(E) \leq \delta_\varepsilon$ for any measurable subset $E \subset \mathbf{R}/\mathbf{Z} \times D$ implies

$$\iiint_E g_k^{(1)} \leq \frac{\varepsilon}{C_\varepsilon} \quad \text{for all } k \in \mathbf{N} \cup \{0\}. \tag{3.8}$$

Furthermore we infer from (1.3)

$$\iiint_{\mathbf{R}/\mathbf{Z}\times D} g_k^{(2)} \,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w \leq C \|h_k\|_{W^{(3p-2)/q-1,q}} \leq C.$$

By (3.6) there is some $k_0 = k_0(\varepsilon) \in \mathbf{N}$ with

$$\mathcal{L}^3\left(A_{\varepsilon,k}\right) \le \delta_\varepsilon \quad \text{for all } k \ge k_0$$

which yields for $k \ge k_0$ and $B_{\varepsilon,k} := \mathbf{R}/\mathbf{Z} \times D \setminus A_{\varepsilon,k}$

$$\begin{aligned}
&\tfrac{1}{6}|\delta \mathrm{intM}^{(p,q)}(\gamma_k;h_k) - \delta\mathrm{intM}^{(p,q)}(\gamma_0;h_0)| \\
&\le \iiint_{A_{\varepsilon,k}} |\delta L(\gamma_k;h_k) - \delta L(\gamma_0;h_0)| + \iiint_{B_{\varepsilon,k}} |\delta L(\gamma_k;h_k) - \delta L(\gamma_0;h_0)| \\
&\le \iiint_{A_{\varepsilon,k}} \left(C_\varepsilon g_k^{(1)} + \varepsilon g_k^{(2)}\right) + \iiint_{A_{\varepsilon,k}} \left(C_\varepsilon g_0^{(1)} + \varepsilon g_0^{(2)}\right) + \mathcal{L}^3(B_{\varepsilon,k})\varepsilon \le C\varepsilon.
\end{aligned}$$

Hence,

$$\begin{aligned}
\varepsilon_0 &\overset{(3.5)}{\le} \left|\delta\mathrm{intM}^{(p,q)}(\gamma_k;h_k) - \mathrm{intM}^{(p,q)}(\gamma;h_k)\right| \\
&\le \left|\delta\mathrm{intM}^{(p,q)}(\gamma_k;h_k) - \mathrm{intM}^{(p,q)}(\gamma;h)\right| + \left|\delta\mathrm{intM}^{(p,q)}(\gamma;h_k) - \mathrm{intM}^{(p,q)}(\gamma;h)\right| \\
&\le C\varepsilon + C\left\|h_k - h\right\|_{W^{(3p-2)/q-1,q}}
\end{aligned}$$

for all $\varepsilon > 0$ and $k \ge k_0(\varepsilon)$ which leads to a contradiction. □

4. Regularity of stationary points

For the rest of this section, let us restrict to the case that γ is parametrized by arc-length. Then we get using Lemma 3.1

$$\delta\mathrm{intM}^{(p,q)}(\gamma;h) := 6q\tilde{Q}^{p,q}(\gamma,h) + 6R_1^{p,q}(\gamma h)$$

where

$$\begin{aligned}
\tilde{Q}^{p,q}(\gamma,h) := \iiint_{\mathbf{R}/\mathbf{Z}\times D} &|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^{q-2} \\
&\cdot \frac{\langle \triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma, \triangle_{w,0}h \wedge \triangle_{v,0}\gamma + \triangle_{w,0}\gamma \wedge \triangle_{v,0}h\rangle}{|\triangle_{w,0}\gamma|^p|\triangle_{v,0}\gamma|^p|\triangle_{v,w}\gamma|^p}\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u
\end{aligned}$$

and

$$\begin{aligned}
R_1(\gamma,h) := \iiint_{\mathbf{R}/\mathbf{Z}\times D} \Bigg(&- p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p+2}|\triangle_{v,0}\gamma|^p|\triangle_{v,w}\gamma|^p} \cdot \langle \triangle_{w,0}\gamma, \triangle_{w,0}h\rangle \\
&- p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p}|\triangle_{v,0}\gamma|^{p+2}|\triangle_{v,w}\gamma|^p} \cdot \langle \triangle_{v,0}\gamma, \triangle_{v,0}h\rangle \\
&- p\frac{|\triangle_{w,0}\gamma \wedge \triangle_{v,0}\gamma|^q}{|\triangle_{w,0}\gamma|^{p}|\triangle_{v,0}\gamma|^p|\triangle_{v,w}\gamma|^{p+2}} \cdot \langle \triangle_{v,w}\gamma, \triangle_{v,w}h\rangle \\
&+ R^{p,q}(u,u+w,u+v)\,\langle\gamma'(u), h'(u)\rangle \\
&+ R^{p,q}(u,u+w,u+v)\,\langle\gamma'(u+v), h'(u+v)\rangle \\
&+ R^{p,q}(u,u+w,u+v)\,\langle\gamma'(u+w), h'(u+w)\rangle \Bigg)\,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned}$$

For $q = 2$ we will see that $\tilde{Q}^p := \tilde{Q}^{p,2}$ contains the highest order term of the Euler–Lagrange operator. To see this we use

$$\langle a \wedge b, c \wedge d \rangle = \det \begin{pmatrix} \langle a, c\rangle & \langle a, d\rangle \\ \langle b, c\rangle & \langle b, d\rangle \end{pmatrix}$$

to get

$$\langle a \wedge b, a \wedge c \rangle = \langle a, a\rangle \langle c, b\rangle - \langle a, c\rangle \langle a, b\rangle = |a|^2 \left\langle P_a^\perp b, c \right\rangle, \tag{4.1}$$

where $P_a^\perp b = b - \langle a, b\rangle \frac{a}{|a|^2}$. Hence,

$$\begin{aligned}
\tilde{Q}^p(\gamma; h) &= \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \Bigg(\frac{\left\langle P^\perp_{\triangle_{v,0}\gamma} \triangle_{w,0}\gamma, \triangle_{w,0} h \right\rangle}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |\triangle_{w,0}\gamma|^p} \\
&\qquad + \frac{\left\langle P^\perp_{\triangle_{w,0}\gamma} \triangle_{v,0}\gamma, \triangle_{v,0} h \right\rangle}{|\triangle_{w,v}\gamma|^p \, |\triangle_{v,0}\gamma|^{p} \, |\triangle_{w,0}\gamma|^{p-2}} \Bigg) \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \\
&= \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \left(\frac{\left\langle P^\perp_{\triangle_{v,0}\gamma} \triangle_{w,0}\gamma, \triangle_{w,0} h \right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^p} + \frac{\left\langle P^\perp_{\triangle_{w,0}\gamma} \triangle_{v,0}\gamma, \triangle_{v,0} h \right\rangle}{|v-w|^p \, |v|^{p} \, |w|^{p-2}} \right) \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u \\
&\quad + R_2(\gamma; h) \\
&= \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \Bigg(\frac{\left\langle P^\perp_{\triangle_{v,0}\gamma} \left(\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v} \right), \frac{\triangle_{w,0} h}{w} \right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \\
&\qquad - \frac{\left\langle P^\perp_{\triangle_{w,0}\gamma} \left(\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v} \right), \frac{\triangle_{v,0} h}{v} \right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \Bigg) \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u + R_2(\gamma; h) \\
&= \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \left(\frac{\left\langle \frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0} h}{w} - \frac{\triangle_{v,0} h}{v} \right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \right) \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u + R_2(\gamma; h) - R_3(\gamma; h)
\end{aligned}$$

where

$$\begin{aligned}
R_2(\gamma; h) := \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \Bigg(&\left\langle P^\perp_{\triangle_{v,0}\gamma} \triangle_{w,0}\gamma, \triangle_{w,0} h \right\rangle \\
&\left(\frac{1}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |\triangle_{w,0}\gamma|^p} - \frac{1}{|v-w|^p \, |v|^{p-2} \, |w|^p} \right) \\
&+ \left\langle P^\perp_{\triangle_{w,0}\gamma} \triangle_{v,0}\gamma, \triangle_{v,0} h \right\rangle \\
&\left(\frac{1}{|\triangle_{w,v}\gamma|^p \, |\triangle_{v,0}\gamma|^{p} \, |\triangle_{w,0}\gamma|^{p-2}} - \frac{1}{|v-w|^p \, |v|^{p} \, |w|^{p-2}} \right) \Bigg) \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u
\end{aligned}$$

and

$$R_3(\gamma;h) := \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \left(\frac{\left\langle P^{\top}_{\triangle_{v,0}\gamma}\left(\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}\right), \frac{\triangle_{w,0}h}{w}\right\rangle}{|v-w|^p\,|v|^{p-2}\,|w|^{p-2}} + \frac{\left\langle P^{\top}_{\triangle_{w,0}\gamma}\left(\frac{\triangle_{v,0}\gamma}{v} - \frac{\triangle_{w,0}\gamma}{w}\right), \frac{\triangle_{v,0}h}{v}\right\rangle}{|v-w|^p\,|v|^{p-2}\,|w|^{p-2}} \right) \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u.$$

Using

$$Q^{(p)}(\gamma;h) := \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \left(\frac{\left\langle \frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0}h}{w} - \frac{\triangle_{v,0}h}{v}\right\rangle}{|v-w|^p\,|v|^{p-2}\,|w|^{p-2}} \right) \mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$

we hence get

$$\delta M^{p,2} = 12\left(Q^{(p)} + \tfrac{1}{2}R_1 + R_2 + R_3\right). \tag{4.2}$$

Proposition 4.1. *The functional* $Q^{(p)}$ *is bilinear on* $\left(W^{3p/2-2,2}\right)^2$, *more precisely*

$$Q^{(p)}(f,g) = \sum_{k\in\mathbf{Z}} \varrho_k \left\langle \hat{f}_k, \hat{g}_k\right\rangle_{\mathbf{C}^n} \quad \text{where } \varrho_k = c\,|k|^{3p-4} + o\left(|k|^{3p-4}\right) \text{ as } |k| \nearrow \infty$$

and $c>0$. *Here* $\hat{\cdot}_k$ *denotes the* k*-th Fourier coefficient*

$$\hat{f}_k := \int_0^1 f(x)e^{-2\pi ikx}\,\mathrm{d}x.$$

Proof. Testing with the basis $\mathrm{e}_l\cdot e^{2\pi ikx}$ of L^2, $l=1,\dots,n$, $k\in\mathbf{Z}$, where $\mathrm{e}_1,\dots,\mathrm{e}_n$ is the standard basis of $\mathbf{R}^n$, we get

$$Q^{(p)}(f,g) = \sum_{k\in\mathbf{Z}} \left\langle \hat{f}_k, \hat{g}_k\right\rangle_{\mathbf{C}^d} \varrho_k$$

where

$$\varrho_k := \iint\limits_D \frac{\left|\frac{e^{2\pi ikw}-1}{w} - \frac{e^{2\pi ikv}-1}{v}\right|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p}\,\mathrm{d}w\,\mathrm{d}v.$$

A simple substitution leads to

$$\varrho_k = |k|^{3p-4} \iint\limits_{D_k} \frac{\left|\frac{e^{2\pi iw}-1}{w} - \frac{e^{2\pi iv}-1}{v}\right|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p}\,\mathrm{d}w\,\mathrm{d}v$$

where $D_k := k\cdot D$. We use the fundamental theorem of calculus and Jensen's inequality to get

$$\begin{aligned}\frac{\left|\frac{e^{2\pi iw}-1}{w} - \frac{e^{2\pi iv}-1}{v}\right|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p} &\le 4\pi^2\int_0^1 \frac{|e^{2\pi i\theta w} - e^{2\pi i\theta v}|^2}{|w|^{p-2}|v|^{p-2}|v-w|^p}\,\mathrm{d}\theta \\ &= 4\pi^2\int_0^1 \frac{|e^{2\pi i\theta(v-w)} - 1|^2}{|w|^{p-2}|v|^{p-2}|v-w|^p}\,\mathrm{d}\theta.\end{aligned}$$

Hence, substituting $(u,v) \mapsto (u/\theta, v/\theta)$,

$$\begin{aligned}
\iint\limits_{v<0,w>0} \frac{\left|\frac{e^{2\pi ikw}-1}{w} - \frac{e^{2\pi ikv}-1}{v}\right|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p} &\le C \iint\limits_{v<0,w>0} \int_0^1 \frac{|e^{2\pi i\theta(v-w)}-1|^2}{|w|^{p-2}|v|^{p-2}|v-w|^p} \,\mathrm{d}\theta \,\mathrm{d}w \,\mathrm{d}v \\
&= C\left(\int_0^1 \theta^{3p-6} \,\mathrm{d}\theta\right) \iint\limits_{v<0,w>0} \frac{|e^{2\pi i(v-w)}-1|^2}{|w|^{p-2}|v|^{p-2}|v-w|^p} \,\mathrm{d}w \,\mathrm{d}v \\
&\le C \iint\limits_{v<0,w>0} \frac{|e^{2\pi i(v-w)}-1|^2}{|w|^{p-2}|v|^{p-2}|v-w|^p} \,\mathrm{d}w \,\mathrm{d}v \\
&= C \int_0^\infty \int_0^{\tilde{w}} \frac{|e^{-2\pi i\tilde{w}}-1|^2}{|\tilde{w}-v|^{p-2}|v|^{p-2}|\tilde{w}|^p} \,\mathrm{d}v \,\mathrm{d}\tilde{w} \\
&= C \int_0^\infty \int_0^1 \frac{|e^{-2\pi i\tilde{w}}-1|^2}{|1-t|^{p-2}|t|^{p-2}|\tilde{w}|^{3p-5}} \,\mathrm{d}t \,\mathrm{d}\tilde{w} \\
&\le C \int_0^\infty \frac{1-\cos 2\pi\tilde{w}}{|\tilde{w}|^{3p-5}} \,\mathrm{d}\tilde{w} < \infty.
\end{aligned}$$

Thus, we have shown that

$$\frac{\varrho_k}{|k|^{3k-4}} \xrightarrow{|k|\to\infty} \iint_{v<0,w>0} \frac{\left|\frac{e^{2\pi ikw}-1}{w} - \frac{e^{2\pi ikv}-1}{v}\right|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p} \,\mathrm{d}w \,\mathrm{d}v \in (0,\infty). \qquad \square$$

In the following statement, we use the symbol $\otimes$ to denote any kind of product structure, such as cross product, dot product, scalar or matrix multiplication.

Lemma 4.2. *The term* $R^{(p)} := \frac{1}{2}R_1 + R_2 + R_3$ *is a finite sum of terms of the form*

$$\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \underset{[0,1]^K}{\int\cdots\int} g^p(u,v,w;s_1,\ldots,s_{K-2}) \otimes h'(u+s_{K-1}v+s_Kw) \,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_K \,\mathrm{d}v \,\mathrm{d}w \,\mathrm{d}u$$

where $\mathcal{G}^{(p)} \colon (0,\infty)^3 \to \mathbf{R}$ *is an analytic function,* $s_j \in \{0,\theta_j\}$ *for* $j = 1,\ldots,K$,

$$\begin{aligned}
g^p(u,v,w;s_1,\ldots,s_{K-2}) &= \mathcal{G}^{(p)}\left(\frac{|\triangle_{0,w}\gamma|}{|w|}, \frac{|\triangle_{0,v}\gamma|}{|v|}, \frac{|\triangle_{v,w}\gamma|}{|v-w|}\right)\Gamma(u,v,w,s_1,s_2) \\
&\cdot \left(\bigotimes_{i=3}^{K_1} \gamma'(u+s_iv)\right) \otimes \left(\bigotimes_{j=K_1}^{K_2} \gamma'(u+s_iw)\right) \otimes \left(\bigotimes_{j=K_2}^{K-2} \gamma'(u+v+s_i(w-v))\right),
\end{aligned}$$

and $\Gamma(u,v,w,s_1,s_2)$ *is a term of one of the four types*

$$\frac{(\gamma'(u+s_1w)-\gamma'(u+s_1v))\otimes(\gamma'(u+s_2w)-\gamma'(u+s_2v))}{|v|^{p-2}|w|^{p-2}|v-w|^p},$$
$$\frac{|\gamma'(u+s_1w)-\gamma'(u+s_2w)|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p},$$
$$\frac{|\gamma'(u+s_1v)-\gamma'(u+s_2v)|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p},$$
$$\frac{|\gamma'(u+v+s_1(w-v))-\gamma'(u+v+s_2(w-v))|^2}{|v|^{p-2}|w|^{p-2}|v-w|^p}.$$

Proof. Using

$$\begin{aligned}&\frac{|\triangle_{w,0}\gamma\wedge\triangle_{v,0}\gamma|^2}{|\triangle_{w,0}\gamma|^{p+2}|\triangle_{v,0}\gamma|^p|\triangle_{v,w}\gamma|^p}\cdot\langle\triangle_{w,0}\gamma,\triangle_{w,0}h\rangle\\&=\mathcal{G}^{(p)}\left(\frac{|\triangle_{0,w}\gamma|}{|w|},\frac{|\triangle_{0,v}\gamma|}{|v|},\frac{|\triangle_{v,w}\gamma|}{|v-w|}\right)\frac{|\triangle_{w,0}\gamma\wedge\triangle_{v,0}\gamma|^2}{|w|^{p+2}|v|^p|v-w|^p}\cdot\langle\triangle_{w,0}\gamma,\triangle_{w,0}h\rangle\end{aligned}$$

where $\mathcal{G}^{(p)}(z_1,z_2,z_2)=z_1^{-p-2}z_2^{-p}z_3^{-p}$ together with

$$\begin{aligned}\frac{\triangle_{w,0}\gamma}{w}\wedge\frac{\triangle_{v,0}\gamma}{v}&=\left(\frac{\triangle_{w,0}\gamma}{w}-\frac{\triangle_{v,0}\gamma}{v}\right)\wedge\frac{\triangle_{v,0}\gamma}{v}\\&=\int_0^1\int_0^1(\gamma'(u+s_1w)-\gamma'(u+s_1v))\wedge\gamma'(u+s_3v)\,\mathrm{d}s_1\,\mathrm{d}s_3\end{aligned}$$

we see that the first term of R^1 is of type 1. Similarly, one gets that all the terms of R^1 are of type 1.

For the term R^2 we use

$$\begin{aligned}\left\langle P^{\perp}_{\triangle_{0,v}\gamma}(\triangle_{0,w}\gamma),\triangle_{0,w}h\right\rangle&=\frac{1}{|\triangle_{0,v}\gamma|^2}\langle(\triangle_{0,v}\gamma\wedge\triangle_{0,w}\gamma),(\triangle_{0,v}\gamma\wedge\triangle_{0,w}h)\rangle\\&\overset{(4.1)}{=}\frac{v^2w^2}{|\triangle_{0,v}\gamma|^2}\int\cdots\int_{[0,1]^4}\gamma'(u+s_1v)\otimes\gamma'(u+s_2w)\otimes\gamma'(u+s_3v)\\&\qquad\otimes h'(u+s_4w)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,\mathrm{d}s_4\end{aligned}$$

together with the fact that for $w\in\mathbf{R}$, $u\in\mathbf{R}/\mathbf{Z}$

$$\begin{aligned}\frac{1}{|\triangle_{w,0}\gamma|^\alpha}-\frac{1}{|w|^\alpha}&=2\mathcal{G}^{(\alpha)}\left(\frac{|\triangle_{w,0}\gamma|}{|w|}\right)\frac{1-\frac{|\triangle_{w,0}\gamma|^2}{|w|^2}}{|w|^\alpha}\\&=\mathcal{G}^{(\alpha)}\left(\frac{|\triangle_{w,0}\gamma|}{|w|}\right)\iint\limits_{[0,1]^2}\frac{|\gamma'(u+s_1w)-\gamma'(u+s_2w)|^2}{|w|^\alpha}\,\mathrm{d}s_1\,\mathrm{d}s_2\end{aligned}\tag{4.3}$$

where $\mathcal{G}^{(\alpha)}(z) = \frac{1}{2} \cdot \frac{1-z^\alpha}{1-z^2} \cdot z^{-\alpha}$ is analytic on $(0, \infty)$ for $\alpha > 0$. Both equations together with

$$\begin{aligned}
&\frac{1}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |\triangle_{w,0}\gamma|^p} - \frac{1}{|v-w|^p \, |v|^{p-2} \, |w|^p} \\
&= \left(\frac{1}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |\triangle_{w,0}\gamma|^p} - \frac{1}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |w|^p} \right) \\
&+ \left(\frac{1}{|\triangle_{v,w}\gamma|^p \, |\triangle_{v,0}\gamma|^{p-2} \, |w|^p} - \frac{1}{|\triangle_{v,w}\gamma|^p \, |v|^{p-2} \, |w|^p} \right) \\
&+ \left(\frac{1}{|\triangle_{v,w}\gamma|^p \, |v|^{p-2} \, |w|^p} - \frac{1}{|v-w|^p \, |v|^{p-2} \, |w|^p} \right)
\end{aligned}$$

show that the first term of R_2 is the sum of terms of type 2 to 4. Similarly for the second term in R_2.

Let us turn to the last term, R_3. Again, we restrict to the first term, the second is parallel. We obtain

$$\begin{aligned}
&\frac{\left\langle P^\top_{\triangle_{v,0}\gamma}\left(\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}\right), \frac{\triangle_{w,0}h}{w}\right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} = \frac{\left\langle \frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{v,0}\gamma}{|\triangle_{v,0}\gamma|}\right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \left\langle \frac{\triangle_{v,0}\gamma}{|\triangle_{v,0}\gamma|}, \frac{\triangle_{w,0}h}{w}\right\rangle \\
&= \left|\frac{\triangle_{v,0}\gamma}{v}\right|^{-2} \frac{\left\langle \frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{v,0}\gamma}{v}\right\rangle}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \left\langle \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0}h}{w}\right\rangle \\
&= \left|\frac{\triangle_{v,0}\gamma}{v}\right|^{-2} \frac{\left(\left\langle \frac{\triangle_{w,0}\gamma}{w}, \frac{\triangle_{v,0}\gamma}{v}\right\rangle - 1\right)}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \left\langle \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0}h}{w}\right\rangle \\
&= -\tfrac{1}{2}\left|\frac{\triangle_{v,0}\gamma}{v}\right|^{-2} \frac{\left|\frac{\triangle_{w,0}\gamma}{w} - \frac{\triangle_{v,0}\gamma}{v}\right|^2}{|v-w|^p \, |v|^{p-2} \, |w|^{p-2}} \left\langle \frac{\triangle_{v,0}\gamma}{v}, \frac{\triangle_{w,0}h}{w}\right\rangle,
\end{aligned}$$

which gives rise to type 4. □

Our next task is to show that R^p is in fact a lower-order term. More precisely, we have

Proposition 4.3. (Regularity of the remainder term) *If $\gamma \in W^{(3p-4)/2+\sigma,2}_{\mathrm{ia}}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ for some $\sigma \geq 0$, then $R^p(\gamma, \cdot) \in \left(W^{3/2-\sigma+\varepsilon,2}\right)^*$ for any $\varepsilon > 0$.*

This statement together with Proposition 4.1 immediately leads to the proof of the regularity theorem which is deferred to the end of this section.

To prove Proposition 4.3, we first note that, by partial integration, the terms of $R^{(p)}(\gamma, h)$ may be transformed into

$$\begin{aligned}
\int \cdots \int_{[0,1]^K} \iint_D \int_{\mathbf{R}/\mathbf{Z}} &\left((-\Delta)^{\tilde{\sigma}/2} g^p(\cdot, v, w; s_1, \ldots, s_{K-2})\right)(u) \\
&\otimes \left((-\Delta)^{-\tilde{\sigma}/2} h'\right)(u + s_{K-1}v + s_K w) \,\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w \,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_K
\end{aligned}$$

$$\leq \int \cdots \int_{[0,1]^K} \iint_D \|g^p(\cdot, v, w; \dots)\|_{W^{\tilde\sigma,1}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{K-2} \left\|(-\Delta)^{-\tilde\sigma/2} h'\right\|_{L^\infty(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)}$$

$$\leq \int \cdots \int_{[0,1]^K} \iint_D \|g^p(\dots)\|_{W^{\tilde\sigma,1}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)} \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{K-2} \|h\|_{W^{3/2+\varepsilon/2-\tilde\sigma,2}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)},$$

where $\tilde\sigma \in \mathbf{R}$, $\varepsilon > 0$ can be chosen arbitrarily, and $(-\Delta)^{\tilde\sigma/2}$ denotes the fractional Laplacian. We let $\tilde\sigma := 0$ if $\sigma = 0$ and $\tilde\sigma := \sigma - \frac{\varepsilon}{2}$ otherwise. Now the claim directly follows from the succeeding auxiliary result.

Lemma 4.4. (Regularity of the remainder integrand) *Let* $\gamma \in W^{(3p-4)/2+\sigma,2}_{\mathrm{ia}}$.

- *If* $\sigma = 0$ *then* $g^p \in L^1(\mathbf{R}/\mathbf{Z} \times D, \mathbf{R}^n)$ *and*
- *if* $\sigma > 0$ *then* $((v,w) \mapsto g^p(\cdot, v, w; \dots)) \in L^1(D, W^{\tilde\sigma,1}(\mathbf{R}/\mathbf{Z},\mathbf{R}^n))$ *for any* $\tilde\sigma < \sigma$,

then respective norms are bounded independently of $s_1, \dots, s_K$.

Proof. Recall that the argument of $\mathcal{G}^{(p)}$ is compact and bounded away from zero. Using arc-length parametrization, we immediately obtain for the first type

$$\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} |g^p(u,v,w)| \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}u \leq C \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{|\gamma'(u+s_1w) - \gamma'(u+s_1v)|^2}{|w|^{p-2}|v|^{p-2}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$
$$= C \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{|\gamma'(u+s_1(w-v)) - \gamma'(u)|^2}{|w|^{p-2}|v|^{p-2}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$
$$\overset{(1.3)}{\leq} C \|\gamma\|^2_{W^{(3p-4)/2,2}}.$$

For a term of the second type we get

$$\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} |g^p(u,v,w)| \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}u \leq C \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{|\gamma'(u+s_1w) - \gamma'(u+s_2w)|^2}{|w|^{p-2}|v|^{p-2}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$
$$= C \int_{\mathbf{R}/\mathbf{Z}} \int_0^{2/3} \frac{|\gamma'(u+s_1w) - \gamma'(u+s_2w)|^2}{|w|^{p-2}} \left(\int_0^{2/3-w} \frac{1}{v^{p-2}(v+w)^p} \,\mathrm{d}v \right) \mathrm{d}w\,\mathrm{d}u$$
$$\leq C \int_{\mathbf{R}/\mathbf{Z}} \int_0^{2/3} \frac{|\gamma'(u+s_1w) - \gamma'(u+s_2w)|^2}{|w|^{3p-5}} \left(\int_0^{\infty} \frac{1}{t^{p-2}(1+t)^p} \,\mathrm{d}t \right) \mathrm{d}w\,\mathrm{d}u$$
$$\leq C \,[\gamma]_{W^{(3p-4)/2,2}}$$

and of course the same estimate is true for a term of the third kind. For a term of type four, we get along the same lines

$$\iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} |g^p(u,v,w)| \,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}u$$
$$\leq C \iiint\limits_{\mathbf{R}/\mathbf{Z}\times D} \frac{|\gamma'(u+v+s_1(w-v) - \gamma'(u+v+s_2(w-v))|^2}{|w|^{p-2}|v|^{p-2}|w-v|^p} \,\mathrm{d}w\,\mathrm{d}v\,\mathrm{d}u$$

$$= C \int_{(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)} \int_0^{2/3} \frac{|\gamma'(u + s_1(w+v)) - \gamma'(u + s_2(w+v))|^2}{|w|^{p-2}}$$
$$\cdot \left(\int_0^{2/3-w} \frac{1}{(v+w)^p v^{p-2}} \,\mathrm{d}v \right) \mathrm{d}w \,\mathrm{d}u$$
$$\leq C \int_{(\mathbf{R}/\mathbf{Z},\mathbf{R}^n)} \int_0^{2/3} \frac{|\gamma'(u + s_1\tilde{w}) - \gamma'(u + s_2\tilde{w})|^2}{|\tilde{w}|^{3p-5}} \left(\int_0^{\infty} \frac{1}{(t+1)^p t^{p-2}} \,\mathrm{d}t \right) \mathrm{d}w \,\mathrm{d}u$$
$$\leq C \left[\gamma\right]_{W^{(3p-4)/2,2}} .$$

We prove the second claim only for terms of the first type, as the arguments for all other terms follow the same line of argments. We will derive a suitable bound on $\|g^p(\cdot, w)\|_{W^{\tilde{\sigma},r}}$ for some $r > 1$. To this end, we choose $q_1, \ldots, q_{K-2}$, which will be determined more precisely later on, such that

$$\sum_{i=1}^{K-2} \frac{1}{q_i} = \frac{1}{r}.$$

The product rule, Lemma A.1, then leads to

$$\|g^p(\cdot, w)\|_{W^{\tilde{\sigma},r}} \leq C \left\|\mathcal{G}^{(p)}\right\|_{W^{\tilde{\sigma},q_1}} \frac{\|\gamma'(\cdot + s_1 w) - \gamma'(\cdot + s_1 v)\|^2_{W^{\tilde{\sigma},2q_2}}}{|v|^{p-2} |w|^{p-2} |w-v|^p} \prod_{i=3}^{K-2} \|\gamma'\|_{W^{\tilde{\sigma},q_i}} .$$
$$= C \left\|\mathcal{G}^{(p)}\right\|_{W^{\tilde{\sigma},q_1}} \frac{\|\gamma'(\cdot + s_1(w-v)) - \gamma'(\cdot)\|^2_{W^{\tilde{\sigma},2q_2}}}{|v|^{p-2} |w|^{p-2} |w-v|^p} \prod_{i=3}^{K-2} \|\gamma'\|_{W^{\tilde{\sigma},q_i}} .$$

For the second factor, we now choose $q_2 > r$ so small that $W^{\sigma,2}$ embeds into $W^{\tilde{\sigma},2q_2}$. To this end, we set $\frac{1}{r} := 1 - (\sigma - \tilde{\sigma})$ and $\frac{1}{q_2} := 1 - 2(\sigma - \tilde{\sigma})$. and $q_i := \frac{K-3}{\sigma - \tilde{\sigma}}$ for $i = 1, 3, 4, \ldots, K-2$.

Then for the first factor we apply the chain rule, Lemma A.2. Recall that $\mathcal{G}^{(p)}$ is analytic and its argument is bounded below away from zero and above by 1. We infer

$$\left\|\mathcal{G}^{(p)}\right\|_{W^{\tilde{\sigma},q_1}} \leq C \left\|\gamma'\right\|_{W^{\tilde{\sigma},q_1}} .$$

The Sobolev embedding gives

$$\|\gamma'\|_{W^{\tilde{\sigma},q_i}} \leq C \|\gamma\|_{W^{(3p-4)/2+\sigma,2}} \leq C \quad \text{for } i = 1, 3, 4, \ldots, K-2.$$

Summarizing this leads to

$$\|g^p(\cdot, v, w)\|_{W^{\tilde{\sigma},r}} \leq C \frac{\|\gamma'(\cdot + s_1(w-v)) - \gamma'(\cdot)\|^2_{W^{\sigma,2}}}{|v|^{p-2} |w|^{p-2} |w-v|^p}$$

and finally

$$\iint_D \|g^p(\cdot, w)\|_{W^{\tilde{\sigma},r}} \,\mathrm{d}v \,\mathrm{d}w \leq C \iint_D \frac{\|\gamma'(\cdot + s_1(w-v)) - \gamma'(\cdot)\|^2_{W^{\sigma,2}}}{|v|^{p-2} |w|^{p-2} |w-v|^p} \,\mathrm{d}v \,\mathrm{d}w$$
$$\overset{(1.3)}{\leq} C \|\gamma\|_{W^{(3p-4)/2,2}} . \qquad \square$$

Proof of Theorem 4. We start with the Euler–Lagrange equation

$$\delta \mathrm{intM}^{(p,q)}(\gamma, h) + \lambda \left\langle \gamma', h' \right\rangle_{L^2} = 0 \tag{4.4}$$

for any $h \in C^\infty(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$ where $\lambda \in \mathbf{R}$ is a Lagrange parameter stemming from the side condition (fixed length). Using (4.2) this reads

$$12Q^{(p)}(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} + 12R^{(p)}(\gamma, h) = 0. \tag{4.5}$$

Since first variation of the length functional satisfies

$$\langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbf{Z}} |2\pi k|^2 \left\langle \hat{\gamma}_k, \hat{h}_k \right\rangle_{\mathbf{C}^d},$$

we get using Proposition 4.1 that there is a $\tilde{c} > 0$ such that

$$12Q^{(p)}(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbf{Z}} \tilde{\varrho}_k \left\langle \hat{\gamma}_k, \hat{h}_k \right\rangle_{\mathbf{C}^d} \tag{4.6}$$

where

$$\tilde{\varrho}_k = \tilde{c}\, |k|^{3p-4} + o\left(|k|^{p-1}\right) \quad \text{as } |k| \nearrow \infty.$$

Assuming that $\gamma \in W_{\mathrm{ia}}^{(3p-4)/2+\sigma,2}$ for some $\sigma \geq 0$, we infer

$$12Q^{(p)}(\gamma, \cdot) + \lambda \langle \gamma', \cdot' \rangle_{L^2} \in \left(W^{3/2-\sigma+\varepsilon,2}\right)^*$$

applying Proposition 4.3 to (4.5). Equation (4.6) implies

$$\left(\tilde{\varrho}_k\, |k|^{-3/2+\sigma-\varepsilon}\, \hat{\gamma}_k\right)_{k \in \mathbf{Z}} \in \ell^2.$$

Recalling that $\tilde{\varrho}_k\, |k|^{-3p+4}$ converges to a positive constant as $|k| \nearrow \infty$, we are led to

$$\gamma \in W^{\frac{3p-4}{2} + \sigma + \frac{3p-7}{2} - \varepsilon}.$$

Choosing $\varepsilon := \frac{3p-7}{4} > 0$, we gain a positive amount of regularity that does not depend on σ. So by a simple iteration we get $\gamma \in W^{s,2}$ for all $s \geq 0$. □

Appendix A. Product and chain rule

As in [5], we make use of the following results which we briefly state for the readers' convenience.

Lemma A.1. (Product rule) *Let $q_1, \ldots, q_k \in (1, \infty)$ with $\sum_{i=1}^k \frac{1}{q_k} = \frac{1}{r} \in (1, \infty)$ and $s > 0$. Then, for $f_i \in W^{s,q_i}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$, $i = 1, \ldots, k$,*

$$\left\| \prod_{i=1}^k f_i \right\|_{W^{s,r}} \leq C_{k,s} \prod_{i=1}^k \|f_i\|_{W^{s,q_i}}.$$

We also refer to Runst and Sickel [37, Lem. 5.3.7/1 (i)]. For the following statement, one mainly has to treat $\left\|(D^k\psi) \circ f\right\|_{W^{\sigma,p}}$ for $k \in \mathbf{N} \cup \{0\}$ and $\sigma \in (0, 1)$ which is e.g. covered by [37, Thm. 5.3.6/1 (i)].

Lemma A.2. (Chain rule) *Let $f \in W^{s,p}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^n)$, $s > 0$, $p \in (1, \infty)$. If $\psi \in C^\infty(\mathbf{R})$ is globally Lipschitz continuous and ψ and all its derivatives vanish at 0 then $\psi \circ f \in W^{s,p}$ and*

$$\|\psi \circ f\|_{W^{s,p}} \leq C \|\psi\|_{C^k} \|f\|_{W^{s,p}}$$

where k is the smallest integer greater than or equal to s.

Appendix B. Equivalence of fractional seminorms

We give a straightforward proof of the equivalence of two seminorms on the Sobolev–Slobodeckiĭ spaces we used in this article.

Lemma B.1. *For $s \in (0,1)$, $p \in [1,\infty)$ the seminorms*

$$[f]_{W^{1+s,p}} := \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/2}^{1/2} \frac{|f'(u+w) - f'(u)|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p}, \tag{0.11}$$

$$[\![f]\!]_{W^{1+s,p}} := \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - 2f(u) + f(u-w)|^p}{|w|^{1+(1+s)p}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \tag{0.12}$$

are equivalent on $W^{1,p}$.

Proof. We first prove the equivalence of the two norms for smooth f. The fundamental theorem of calculus and the triangle inequality tell us

$$\begin{aligned}
[\![f]\!]_{W^{1+s,p}} &= \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - 2f(u) + f(u-w)|^p}{|w|^{1+(s+1)p}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \\
&= \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|\int_0^1 f'(u+\tau w) - f'(u-\tau w) \,\mathrm{d}\tau|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \\
&\le \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|\int_0^1 f'(u+\tau w) - f'(u) \,\mathrm{d}\tau|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \\
&\quad + \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|\int_0^1 f'(u) - f'(u-\tau w) \,\mathrm{d}\tau|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \\
&= 2 \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \int_0^1 \frac{|f'(u+\tau w) - f'(u)|^p}{|w|^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p}.
\end{aligned}$$

Using Fubini's theorem and substituting $\tilde{w} = \tau w$, we can estimate this further by

$$2 \left(\int_{\mathbf{R}/\mathbf{Z}} \int_0^1 \tau^{sp} \int_{-\tau/4}^{\tau/4} \frac{|f'(u+\tilde{w}) - f'(u)|^p}{|\tilde{w}|^{1+sp}} \,\mathrm{d}\tilde{w} \,\mathrm{d}\tau \,\mathrm{d}u \right)^{1/p} \le \frac{2}{1+sp} [f]_{W^{1+s,p}}.$$

Hence,

$$[\![f]\!]_{W^{1+s,p}} \le \frac{2}{1+sp} [f]_{W^{1+s,p}}.$$

To get an estimate in the other direction, we calculate for $\varepsilon > 0$

$$\begin{aligned}
[\![f]\!]_{W^{1+s,p}} &= \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|f(u+w) - 2f(u) + f(u-w)|^p}{|w|^{1+(s+1)p}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p} \\
&= \left(\int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|\int_0^1 f'(u+\tau w) - f'(u-\tau w) \,\mathrm{d}\tau|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\geq \left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u+\tau w)-f'(u-\tau w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&\quad -\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{0}^{1-\varepsilon} f'(u+\tau w)-f'(u-\tau w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&=: I_1 - I_2.
\end{aligned}$$

Substituting $\tilde{\tau}=\frac{\tau}{1-\varepsilon}$ and $\tilde{w}=(1-\varepsilon)w$, we get

$$\begin{aligned}
I_2 &= \left(\int_{\mathbf{R}/\mathbf{Z}}(1-\varepsilon)^{(1+s)p}\int_{-(1-\varepsilon)/4}^{(1-\varepsilon)/4}\frac{|\int_{0}^{1} f'(u+\tilde{\tau}\tilde{w})-f'(u-\tilde{\tau}\tilde{w})\,\mathrm{d}\tilde{\tau}|^p}{|\tilde{w}|^{1+sp}}\,\mathrm{d}\tilde{w}\,\mathrm{d}u\right)^{1/p}\\
&\leq (1-\varepsilon)^{1+s}[\![f]\!]_{W^{1+s,p}}.
\end{aligned}$$

For I_1 we observe

$$\begin{aligned}
I_1 &\geq \left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u+w)-f'(u-w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&\quad -\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u+\tau w)-f'(u+w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&\quad -\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u-\tau w)-f'(u-w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&= \varepsilon\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|f'(u+w)-f'(u-w)|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&\quad -2\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u+\tau w)-f'(u+w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}.
\end{aligned}$$

To bound the first integral from below, we calculate

$$\begin{aligned}
[f]_{W^{1+s,p}} &= \left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/2}^{1/2}\frac{|f'(u+w)-f'(u)|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}\\
&= 2^{-s}\left(\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|f'(u+w)-f'(u-w)|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/p}.
\end{aligned}$$

The second integral can be estimated further

$$\begin{aligned}
&\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\frac{|\int_{1-\varepsilon}^{1} f'(u+\tau w)-f'(u+w)\,\mathrm{d}\tau|^p}{|w|^{1+sp}}\,\mathrm{d}w\,\mathrm{d}u\\
&\leq \varepsilon^{p-1}\int_{\mathbf{R}/\mathbf{Z}}\int_{-1/4}^{1/4}\int_{1-\varepsilon}^{1}\frac{|f'(u+\tau w)-f'(u+w)|^p}{|w|^{1+sp}}\,\mathrm{d}\tau\,\mathrm{d}w\,\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^{p-1} \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \int_{1-\varepsilon}^{1} \frac{|f'(u+(\tau-1)w)-f'(u)|^p}{|w|^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}u \\
&= \varepsilon^{p-1} \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \int_{0}^{\varepsilon} \frac{|f'(u-\tau w)-f'(u)|^p}{|w|^{1+sp}} \,\mathrm{d}\tau \,\mathrm{d}w \,\mathrm{d}u \\
&\leq \frac{\varepsilon^{sp+p}}{sp+1} \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/4}^{1/4} \frac{|f'(u+w)-f'(u)|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u.
\end{aligned}
$$

So we finally arrive at

$$
[\![f]\!]_{W^{1+s,p}} \geq 2^s \varepsilon \,[f]_{W^{1+s,p}} - 2\frac{\varepsilon^{s+1}}{\sqrt[p]{sp+1}}\,[f]_{W^{1+s,p}} - (1-\varepsilon)^{1+s}[\![f]\!]_{W^{1+s,p}}.
$$

For $\varepsilon = 2^{1-2/s}$ this leads to

$$
[\![f]\!]_{W^{1+s,p}} \geq 2^{-1-2/s}\,[f]_{W^{1+s,p}}\,.
$$

To get the statement for $f \in W^{1,p}$ we use a standard mollifier $\phi \in C^\infty(\mathbf{R})$ with $\phi \geq 0$, bounded support and

$$
\int_{\mathbf{R}} \phi \,\mathrm{d}x = 1.
$$

We set $\phi_\varepsilon(x) := \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$ and

$$
f_\varepsilon = f * \phi_\varepsilon.
$$

Then f_ε converges to f in $W^{1,p}$ and we can hence chose a sequence $\varepsilon_k \to 0$ such that $f_k := f_{\varepsilon_k}$ converge pointwise almost everywhere to f and f_k' to f'.

Using Hölder's inequality, we see that

$$
\begin{aligned}
[f_\varepsilon]^p_{W^{s+1,p}} &= \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/2}^{1/2} \frac{|\int_{\mathbf{R}}(f'(u+w-z)-f'(u-z))\phi_\varepsilon(z)\,\mathrm{d}z|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \\
&\leq \left(\int_{\mathbf{R}} \phi_\varepsilon(\tilde z)\,\mathrm{d}\tilde z\right)^{p-1} \int_{\mathbf{R}} \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/2}^{1/2} \frac{|(f'(u+w-z)-f'(u-z))|^p\phi_\varepsilon(z)}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \,\mathrm{d}z \\
&= \int_{\mathbf{R}} \phi_\varepsilon(z) \int_{\mathbf{R}/\mathbf{Z}} \int_{-1/2}^{1/2} \frac{|(f'(u+w)-f'(u))|^p}{|w|^{1+sp}} \,\mathrm{d}w \,\mathrm{d}u \,\mathrm{d}z \\
&= [f]^p_{W^{s+1,p}}
\end{aligned}
$$

for all $\varepsilon > 0$. Similarly,

$$
[\![f_\varepsilon]\!]_{W^{s+1,p}} \leq [\![f]\!]_{W^{s+1,p}}.
$$

Hence Fatou's lemma tells us that

$$
[\![f]\!]_{W^{1+s,p}} \leq \liminf_{k\to\infty} [\![f_k]\!]_{W^{1+s,p}} \leq C \liminf_{k\to\infty}\,[f_k]_{W^{1+s,p}} \leq C\,[f]_{W^{1+s,p}}
$$

and

$$
[f]_{W^{1+s,p}} \leq \liminf_{k\to\infty}\,[f_k]_{W^{1+s,p}} \leq C \liminf_{k\to\infty} [\![f_k]\!]_{W^{1+s,p}} \leq C[\![f]\!]_{W^{1+s,p}}.
$$

This completes the proof of the lemma. □

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