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ELECTROSTATIC SKELETONS

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Dedicated to the memory of Herbert Stahl.

Abstract. Let P be the equilibrium potential of a compact set K in $\mathbf{R}^n$. An electrostatic skeleton of K is a positive measure μ such that the closed support S of μ has connected complement and empty interior, and the Newtonian (or logarithmic, when $n = 2$) potential of μ is equal to P near infinity. We prove the existence and uniqueness of an electrostatic skeleton for any simplex.

1. Introduction

Let $K \subset \mathbf{R}^n$ be a compact set which is regular for the Dirichlet problem. The equilibrium measure for K is the unique probability measure $\nu = \nu_K$ which maximizes the energy

$$I(\mu) = \iint \log|z-w| d\mu(z)\, d\mu(w) \quad \text{for } n = 2,$$

or

$$I(\mu) = -\iint |z-w|^{2-n}\, d\mu(z) d\mu(w) \quad \text{for } n > 2,$$

among all Borel probability measures μ on K. We denote the equilibrium potential by

$$P_\nu(z) = \begin{cases} \displaystyle\int \log|z-w|\, d\nu(w) & \text{for } n = 2, \\ \displaystyle -\int |z-w|^{2-n}\, d\nu(w) & \text{for } n > 2. \end{cases}$$

The equilibrium potential P_ν has the property that $P_\nu \geq I(\nu)$ on $\mathbf{R}^n$, and, $P_\nu = I(\nu)$ on K (cf. [2, 8]).

Definition 1.1. Let $K \subset \mathbf{R}^n$ be a compact set. A positive measure μ with closed support $S \subset K$ is called an *electrostatic skeleton* of K if S has empty interior, $\mathbf{R}^n \backslash S$ is connected, and the potential of μ matches the equilibrium potential outside K, i.e.,

$$P_\nu = P_\mu \quad \text{in } \mathbf{R}^n \backslash K.$$

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For example, an ellipsoid has an electrostatic skeleton supported on its $(n-1)$-dimensional focal ellipsoid [5]. For $n > 2$, if

$$F(x) = \sum_{i=k}^{d} \frac{a_k}{|x - x_k|^{n-2}}, \quad a_k > 0,$$

then the discrete measure with atoms of mass a_k at each point x_k is an electrostatic skeleton for the level sets $\{x \colon F(x) \geq L\}$, $L > 0$. The analogous examples for $n = 2$ are polynomial lemniscates which have skeletons supported on finite sets.

An electrostatic skeleton of a special region bounded by two circles in $\mathbf{R}^2$ appears in [3]. It is clear that there are sets K which do not have an electrostatic skeleton. For example, a Jordan curve in $\mathbf{R}^2$ whose boundary is nowhere analytic. If u is the equilibrium potential of such a K, then any level set $\{z \colon u(z) < c\}$ also does not have an electrostatic skeleton.

Motivated by his study of the asymptotic behavior of zeros of Bergman (area orthogonal) polynomials, Saff proposed the problem on existence of skeletons in 2003 and mentioned it at several conferences (see the reference to Saff in the recent paper [6]). In particular, Saff asked whether every convex polygon has an electrostatic skeleton. An analogous question with potentials of the volume or surface area measure instead of the equilibrium potential was considered by Gustafsson [1].

In this note, we prove the existence and uniqueness of electrostatic skeletons for simplices. Our proof of existence relies on results from [7, 4] on the convexity of the level hypersurfaces of the equilibrium potential of a convex set.

2. Existence and uniqueness for simplices

In the proof of existence and uniqueness we will work with a modified potential defined by $u_K := P_\nu - I(\nu)$. So u_K is a positive harmonic function in $\mathbf{R}^n \setminus K$ which satisfies

$$u_K(z) = \log|z| + O(1), \quad z \to \infty, \quad \text{for } n = 2,$$

and

$$u_K(x) = \text{constant} + O(1/|x|^{n-2}), \quad x \to \infty, \quad \text{for } n > 2,$$

and whose boundary values on K are zero. Thus for $n = 2$, u_K is the same as the Green function of $\mathbf{R}^2 \backslash K$ with pole at infinity.

Theorem 1. *Any simplex $K \subset \mathbf{R}^n$ has a unique electrostatic skeleton.*

Proof. First we prove existence. Let L_j, $j = 1, \ldots, n+1$, be the open $(n-1)$-faces of K, and ℓ_j the reflection in L_j. Then the harmonic function $u = u_K$ extends across each L_j by reflection:

$$u_j = -u \circ \ell_j, \quad j = 1, \ldots, n+1.$$

As K is convex, each of these functions u_j is a negative harmonic function defined throughout the whole interior of K. The boundary values of u_j on ∂K are zero on the closure $\overline{L_j}$ and strictly negative on $K \backslash \overline{L_j}$. It follows that the u_j are pairwise distinct negative harmonic functions in int K. So there is no open set $V \subset K$ where $u_j(x) = u_k(x)$ for $x \in V$ and $j \neq k$.

Next, define a function w by

$$w(x) = \begin{cases} u(x), & x \in \mathbf{R}^n \backslash K, \\ \max\{u_1(x), \ldots, u_{n+1}(x)\}, & x \in K. \end{cases} \tag{1}$$

Clearly, w is subharmonic in $\mathbf{R}^n$ and, harmonic in $\mathbf{R}^n \backslash K$. We claim that the Riesz measure μ of w is a skeleton. First we show that the potential of μ matches the equilibrium potential outside K. Notice that the function $h = w - P_\mu$ is harmonic in all of $\mathbf{R}^n$. Since $w = u$ outside K, it follows that h has logarithmic growth (when $n = 2$) or bounded growth (when $n > 2$) at infinity. In either case, this implies that h is a constant. In particular, $P_\nu - P_\mu$ is a constant. That this constant is zero follows because each of the terms behaves like $\log|z| + o(1)$ near infinity (correspondingly $-|x|^{2-n} + o(1)$ for $n > 2$). Therefore, $P_\mu = P_\nu$ outside K.

So it remains to prove that $\mathbf{R}^n \backslash S[w]$ is connected, where $S[w]$ is the support of the Riesz measure μ. The support can be explicitly described as:

$$S[w] = \{x \in K \colon u_i(x) = u_j(x), \text{ for some } i \neq j\}. \tag{2}$$

Suppose that $\mathbf{R}^n \backslash S[w]$ has a bounded component D, then $D \subset K$. To every component D of int $K \backslash S[w]$ corresponds a number $k(D)$ such that $w(x) = u_k(x)$, $x \in D$. Suppose without loss of generality that $k(D) = 1$, that is $u_1(x) > u_j(x)$, $z \in D$, $j \in \{2, 3, \ldots, n+1\}$. Let a be a point in D. Let R be the straight line segment connecting a point $b \in L_1$ with the vertex opposite to L_1 which passes through a. Let $R_2, R_3, \ldots, R_{n+1}$ be reflections of R with respect to the other faces $L_2, L_3, \ldots, L_{n+1}$. These segments all lie outside K. Introduce the direction on $R, R_2, R_3, \ldots, R_{n+1}$ from their common vertex. See Figure 1. We claim that our original equilibrium potential u is strictly increasing on these segments.

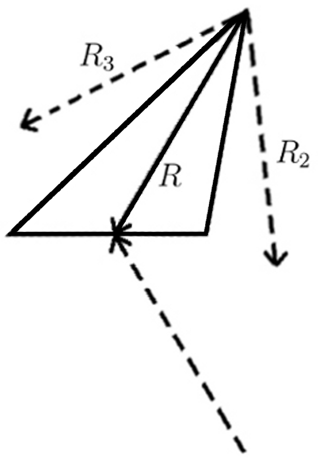

Figure 1. Construction for the proof of Theorem 1.

This follows from the fact that level hypersurfaces of the equilibrium potential of a convex set are convex (for $n = 2$, see [7], and for $n > 2$, [4]), and a straight line can cross a convex hypersurface at most twice; if the straight line segment begins inside a convex hypersurface and ends outside, then it crosses the boundary only once. Therefore, our segments $R_2, R_3, \ldots, R_{n+1}$ cross each level surface at most once, so the function u is increasing on $R_2, R_3, \ldots, R_{n+1}$. Therefore, the functions $u_j = -u \circ \ell_j$ with $j \in \{2, 3, \ldots, n+1\}$ are decreasing on R, while u_1 is increasing on R. So as we have $u_1(x) > \max\{u_2(x), u_3(x), \ldots, u_{n+1}(x)\}$ for $x = a$ and $x = b$ we conclude that this inequality holds on the whole segment $[a, b] \subset R$ giving a contradiction, which proves that S_1 does not divide $\mathbf{R}^n$. This proves the existence of a skeleton for simplices. Uniqueness follows from the proposition below. $\square$

Proposition 2. *Let K be a convex polytope in $\mathbf{R}^n$. Suppose that K admits an electrostatic skeleton. Let w be defined as in (1) and $S[w]$ as defined in (2). If $S[w]$ does not divide $\mathbf{R}^n$, then the electrostatic skeleton is unique, and its closed support is $S[w]$.*

Proof. By assumption, the electrostatic skeleton exists. Let S be its support, and v the potential of the skeleton. Clearly $S \subset K$. By assumption, $S[w]$ does not divide $\mathbf{R}^n$. Then the complement $\operatorname{int} K \backslash S[w]$ consists of N components, D_j, where N is the number of $(n-1)$-faces of ∂K, such that D_j contains L_j.

As S is closed and has empty interior, the set $G = \operatorname{int} K \backslash S$ is open and non-empty. Let p be a point in G. As S does not divide space, there is a curve γ in $\mathbf{R}^n \backslash S$ starting from p and ending outside K, and v has an analytic continuation on this curve. Consider the point q where γ leaves K for the first time, and let $q \in L_k$ (it is clear that q belong to a face of dimension $n-2$ of K). As u_k is the immediate analytic continuation of u to the interior of K, we conclude that $v(x) = u_k(x)$ in a neighborhood of p. Therefore

$$v(x) \in \{u_1(x), \ldots, u_N(x)\} \quad \text{for every} \quad x \in K, \tag{3}$$

In particular, v is continuous. Recall that $u_k(x) > u_j(x)$ for $x \in D_k$ and all $j \neq k$. As v is continuous, it follows from (3) that $v(x) = u_k(x)$ for $x \in D_k$. So $v = w$ by continuity, and $S = S[w]$. □

3. Remarks

1. It is easy to show that a non-convex (Jordan) polygon in $\mathbf{R}^2$ cannot have a skeleton by considering the behavior near a corner with interior angle greater than π.

2. Now we give an example of a circular triangle which does not have an electrostatic skeleton. Let U be the unit disc, and $D \subset \Delta = \overline{\mathbf{C}} \backslash \overline{U}$ the hyperbolic triangle with vertices at $1, \exp(\pm 2\pi i/3)$. Then repeated reflections of D in the sides tile Δ. A more familiar picture of this tiling is obtained by changing the variable to $1/z$. Therefore the equilibrium potential of $K = \mathbf{C} \backslash D$ has an analytic continuation to $\Delta \backslash E$, where E is a discrete set of logarithmic singularities. As these singularities are dense on the unit circle, K cannot have an electrostatic skeleton.

3. When the equilibrium potential in the definition of skeleton is replaced by the volume measure on K, the resulting notion is called a "mother body". Gustafsson [1] proved existence and uniqueness of the mother body for every convex polytope in $\mathbf{R}^n$.

4. In general, the electrostatic skeleton of a Jordan region K does not have to be unique. This can be seen by starting from an example of non-uniqueness for the mother body problem that was constructed by Zidarov [9] (and it is also reproduced in [1]). For simplicity we only discuss this example for $n = 2$. Zidarov constructed two distinct probability measures whose supports are trees consisting of finitely many straight segments, and whose potentials coincide in a neighborhood of ∞. These potentials are equal in a neighborhood of ∞ with the potential of the area measure of a non-convex polygon. The level sets of these potentials $\{z \colon u(z) \leq c\}$ with sufficiently large c are Jordan regions with non-unique skeletons.

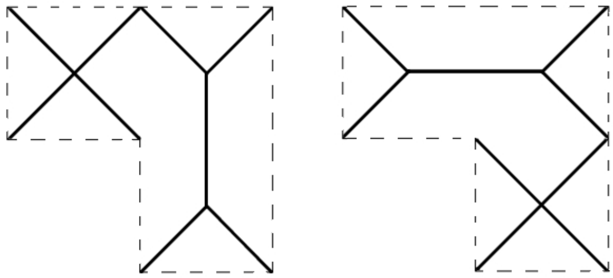

Figure 2. Two trees supporting positive measures whose potentials coincide near infinity.

5. For any convex polytope, define $S[w]$ as in Theorem 1. We conjecture that $S[w]$ does not divide $\mathbf{R}^n$. This would imply the existence of an electrostatic skeleton as in the proof of Theorem 1, and it would also imply uniqueness by Proposition 2.

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