SOBOLEV AND TRUDINGER TYPE INEQUALITIES ON GRAND MUSIELAK–ORLICZ–MORREY SPACES

Fumi-Yuki Maeda, Yoshihiro Mizuta, Takao Ohno and Tetsu Shimomura

Abstract. Our aim in this paper is to establish generalizations of Sobolev’s inequality and Trudinger’s inequality for general potentials of functions in grand Musielak–Orlicz–Morrey spaces.

1. Introduction

Grand Lebesgue spaces were introduced in [15] for the study of Jacobian. They play important roles also in the theory of partial differential equations (see [10], [16] and [29], etc.). The generalized grand Lebesgue spaces appeared in [12], where the existence and uniqueness of the non-homogeneous N-harmonic equations were studied. The boundedness of the maximal operator on the grand Lebesgue spaces was studied in [9]. For variable exponent Lebesgue spaces, see [6] and [7]. In [21] and [17], grand Morrey spaces and generalized grand Morrey spaces were introduced. For Morrey spaces, we refer to [24] and [27]. Further, grand Morrey spaces of variable exponent were considered in [11].

On the other hand, the classical Sobolev’s inequality for Riesz potentials of $L^p$-functions (see, e.g. [2, Theorem 3.1.4 (b)]) has been extended to various function spaces. For Morrey spaces, Sobolev’s inequality was studied in [1], [27], [5], [25], etc., for Morrey spaces of variable exponent in [3], [13], [14], [22], [23], etc., for grand Morrey spaces in [21] and [17], and also for grand Morrey spaces of variable exponent in [11]. Recently, Sobolev’s inequality has been extended by the authors [19] to an inequality for general potentials of functions in Musielak–Orlicz–Morrey spaces.

The classical Trudinger’s inequality for Riesz potentials of $L^p$-functions (see, e.g. [2, Theorem 3.1.4 (c)]) has been also extended to function spaces as above; see [22], [23] for Morrey spaces of variable exponent, [11] for grand Morrey spaces of variable exponent and [20] for Musielak–Orlicz–Morrey spaces.

In this paper, we define (generalized) grand Musielak–Orlicz–Morrey space on a bounded open set in $\mathbb{R}^N$ and give a Sobolev type inequality as well as a Trudinger type inequality for general potentials of functions in such spaces.
Throughout this paper, let $C$ denote various constants independent of the variables in question. The symbols $g \lesssim h$ and $g \sim h$ means that $g \leq Ch$ and $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$ respectively.

2. Preliminaries

Let $G$ be a bounded open set in $\mathbb{R}^N$ and let $d_G$ denote the diameter of $G$. We consider a function

$$
\Phi(x, t) = t\phi(x, t): G \times [0, \infty) \to [0, \infty)
$$

satisfying the following conditions (Φ1)–(Φ4):

(Φ1) $\phi(\cdot, t)$ is measurable on $G$ for each $t \geq 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \geq 1$ such that

$$
A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in G;
$$

(Φ3) there exists a constant $\varepsilon_0 > 0$ such that $t \mapsto t^{-\varepsilon_0}\phi(x, t)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$$
t^{-\varepsilon_0}\phi(x, t) \leq A_2 s^{-\varepsilon_0}\phi(x, s)
$$

for all $x \in G$ whenever $0 < t < s$;

(Φ4) there exists a constant $A_3 \geq 1$ such that

$$
\phi(x, 2t) \leq A_3\phi(x, t) \quad \text{for all } x \in G \text{ and } t > 0.
$$

Note that (Φ3) implies that

$$
t^{-\varepsilon}\phi(x, t) \leq A_2 s^{-\varepsilon}\phi(x, s)
$$

for all $x \in G$ and $0 < \varepsilon \leq \varepsilon_0$ whenever $0 < t < s$. Also note that (Φ2), (Φ3) and (Φ4) imply

$$
0 < \inf_{x \in G} \phi(x, t) \leq \sup_{x \in G} \phi(x, t) < \infty
$$

for each $t > 0$ and there exists $\omega > 1$ such that

$$(A_1A_2)^{-1}t^{1+\varepsilon_0} \leq \Phi(x, t) \leq A_1A_2A_3t^\omega
$$

for $t \geq 1$; in fact we can take $\omega \geq 1 + \log A_3/\log 2$.

We shall also consider the following condition:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$
\phi(x, t) \leq B_\gamma \phi(y, t)
$$

whenever $|x - y| \leq \gamma t^{-1/N}$ and $t \geq 1$.

Let $\tilde{\phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$
\overline{\Phi}(x, t) = \int_0^t \tilde{\phi}(x, r)\,dr.
$$

Then $\overline{\Phi}(x, \cdot)$ is convex and

$$
\frac{1}{2A_3}\Phi(x, t) \leq \overline{\Phi}(x, t) \leq A_2\Phi(x, t)
$$

for all $x \in G$ and $t \geq 0$. 
Example 2.1. Let $p(\cdot)$ and $q_j(\cdot), j = 1, \ldots, k$, be measurable functions on $G$ such that

$$1 < p^- := \inf_{x \in G} p(x) \leq \sup_{x \in G} p(x) =: p^+ < \infty$$

and

$$-\infty < q_j^- := \inf_{x \in G} q_j(x) \leq \sup_{x \in G} q_j(x) =: q_j^+ < \infty, \quad j = 1, \ldots k.$$ 

Set $L(t) := \log(e + t), L^{(1)}(t) = L(t)$ and $L^{(j)}(t) = L(L^{j-1}(t)), j = 2, \ldots$. Then,

$$\Phi_p(\cdot), \{q_j(\cdot)\}(x, t) = t^{p(x)} \prod_{j=1}^k (L^{(j)}(t))^{q_j(x)}$$

satisfies (Φ1), (Φ2), (Φ3) with $0 < \varepsilon_0 < p^--1$ and (Φ4). (2.1) holds for any $\omega > p^+$. $\Phi_p(\cdot), \{q_j(\cdot)\}(x, t)$ satisfies (Φ5) if $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{L(1/|x - y|)} \quad (x \neq y)$$

and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \leq \frac{C_{q_j}}{L^{(j+1)}(1/|x - y|)} \quad (x \neq y)$$

for $j = 1, \ldots, k$ (cf. [19, Example 2.1]).

We also consider a function $\kappa(x, r): G \times (0, d_G) \to (0, \infty)$ satisfying the following conditions:

$(\kappa 1)$ $\kappa(x, \cdot)$ is continuous on $(0, d_G)$ for each $x \in G$ and satisfies the uniform doubling condition: there is a constant $Q_1 \geq 1$ such that

$$Q_1^{-1} \kappa(x, r) \leq \kappa(x, r') \leq Q_1 \kappa(x, r)$$

for all $x \in G$ whenever $0 < r \leq r' \leq 2r < d_G$;

$(\kappa 2)$ $r \mapsto r^{-\delta} \kappa(x, r)$ is uniformly almost increasing for some $\delta > 0$, namely there is a constant $Q_2 > 0$ such that

$$r^{-\delta} \kappa(x, r) \leq Q_2 s^{-\delta} \kappa(x, s)$$

for all $x \in G$ whenever $0 < r < s < d_G$;

$(\kappa 3)$ there is a constant $Q_3 \geq 1$ such that

$$Q_3^{-1} \min(1, r^N) \leq \kappa(x, r) \leq Q_3$$

for all $x \in G$ and $0 < r < d_G$.

Example 2.2. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on $G$ such that $\nu^- := \inf_{x \in G} \nu(x) > 0$, $\nu^+ := \sup_{x \in G} \nu(x) \leq N$ and $-c(N - \nu(x)) \leq \beta(x) \leq c$ for all $x \in G$ and some constant $c > 0$. Then $\kappa(x, r) = r^{\nu(x)}(\log(e + 1/r))^{\beta(x)}$ satisfies $(\kappa 1), (\kappa 2)$ and $(\kappa 3)$; we can take any $0 < \delta < \nu^-$ for $(\kappa 2)$.

Given $\Phi(x, t)$ and $\kappa(x, r)$, we define the Musielak–Orlicz–Morrey space $L^{\Phi, \kappa}(G)$ by

$$L^{\Phi, \kappa}(G) = \left\{ f \in L^1_{loc}(G); \sup_{x \in G, 0 < r < d_G} \frac{\kappa(x, r)}{|B(x, r)|} \int_{B(x, r) \cap G} \Phi(y, |f(y)|) \, dy < \infty \right\}.$$
Then there exists for all \( x \) independent of \( \sigma \)

\[
\Phi(x, t) := \Phi(t^\gamma) = \int_0^t \Phi(y, |f(y)|) dy < \infty
\]

with the norm

\[
\|f\|_{\Phi; G} = \inf \left\{ \lambda > 0; \int_G \Phi(y, |f(y)|)/\lambda \, dy \leq 1 \right\}
\]

(cf. [26]).

In case \( \kappa(x, r) = r^N \), \( L^{\Phi, \kappa}(G) \) is the Musielak–Orlicz space

\[
L^\Phi(G) = \left\{ f \in L^{1}_{\text{loc}}(G); \int_G \Phi(y, |f(y)|) \, dy < \infty \right\}
\]

with the norm

\[
\|f\|_{\Phi; G} = \inf \left\{ \lambda > 0; \int_G \Phi(y, |f(y)|)/\lambda \, dy \leq 1 \right\}.
\]

**Remark 2.3.** The Musielak–Orlicz spaces \( L^\Phi(G) \) include

- Orlicz spaces defined by Young functions satisfying the doubling condition;
- variable exponent Lebesgue spaces.

The Musielak–Orlicz–Morrey spaces \( L^{\Phi, \kappa}(G) \) include Morrey spaces as well as variable exponent Morrey spaces.

### 3. Grand Musielak–Orlicz–Morrey space

For \( \varepsilon \geq 0 \), set \( \Phi_\varepsilon(x, t) := t^{-\varepsilon} \Phi(x, t) = t^{1-\varepsilon} \phi(x, t) \). Then, \( \Phi_\varepsilon(x, t) \) satisfies \((\Phi1), (\Phi2)\) with the same \( A_1 \) and \((\Phi4)\) with the same \( A_3 \). If \( \Phi(x, t) \) satisfies \((\Phi5)\), then so does \( \Phi_\varepsilon(x, t) \) with the same \( \{B_\gamma\}_{\gamma > 0} \).

If \( 0 \leq \varepsilon < \varepsilon_0 \), then \( \Phi_\varepsilon(x, t) \) satisfies \((\Phi3)\) with \( \varepsilon_0 \) replaced by \( \varepsilon_0 - \varepsilon \) and the same \( A_2 \). It follows that

\[
\frac{1}{2A_3} \Phi_\varepsilon(x, t) \leq \Phi(x, t) \leq A_2 \Phi_\varepsilon(x, t)
\]

for all \( x \in G, t \geq 0 \) and \( 0 \leq \varepsilon \leq \varepsilon_0 \). By \((\Phi3)\), we see that for \( 0 \leq \varepsilon \leq \varepsilon_0 \)

\[
\Phi_\varepsilon(x, at) \begin{cases} \leq A_2 a \Phi_\varepsilon(x, t) & \text{if } 0 \leq a \leq 1, \\ \geq A_2^{-1} a \Phi_\varepsilon(x, t) & \text{if } a \geq 1. \end{cases}
\]

Let

\[
\tilde{\sigma} = \sup \{ \sigma \geq 0; r^{N-\sigma} \kappa(x, r)^{-1} \text{ is bounded on } G \times (0, \min(1, d_G))] \}
\]

By \((\kappa2)\), \( 0 \leq \tilde{\sigma} \leq N \). If \( \tilde{\sigma} = 0 \), then let \( \sigma_0 = 0 \); otherwise fix any \( \sigma_0 \in (0, \tilde{\sigma}) \). We also take \( \delta_0 \) such that \( 0 < \delta_0 < \delta \) for \( \delta \) in \((\kappa2)\).

For \( -\delta_0 \leq \sigma \leq \sigma_0 \), set

\[
\kappa_\sigma(x, r) = r^{\sigma} \kappa(x, r)
\]

for \( x \in G \) and \( 0 < r < d_G \). Then \( \kappa_\sigma(x, r) \) satisfies \((\kappa1), (\kappa2)\) and \((\kappa3)\) with constants independent of \( \sigma \).

**Lemma 3.1.** For \( 0 \leq \varepsilon \leq \varepsilon_0 \), let

\[
\Phi_\varepsilon^{-1}(x, s) = \sup \{ t > 0 : \Phi_\varepsilon(x, t) < s \} \quad (x \in G, \ s > 0).
\]

Then there exists \( r_0 \in (0, \min(1, d_G)) \) such that \( \kappa_\sigma(x, r) \leq 1 \) and

\[
\Phi_\varepsilon^{-1}(x, \kappa_\sigma(x, r)^{-1}) \geq 1
\]

for all \( x \in G, 0 < r \leq r_0, -\delta_0 \leq \sigma \leq \sigma_0 \) and \( 0 < \varepsilon \leq \varepsilon_0 \).
Proof. By ($\kappa2$) and ($\kappa3$),
$$\kappa_\sigma(x, r) \leq Q_2Q_3 \min(1, d_G)^{-\delta}r^{\delta+\sigma} \leq Q_2Q_3 \min(1, d_G)^{-\delta}r^{\delta-\delta_0}$$
for $x \in G$, $0 < r < \min(1, d_G)$ and $-\delta_0 \leq \sigma \leq \sigma_0$. Hence, there is $r' \in (0, \min(1, d_G))$ such that $\kappa_\sigma(x, r') \leq 1$ for $x \in G$, $0 < r \leq r'$ and $-\delta_0 \leq \sigma \leq \sigma_0$. By (2.1), we see that
$$\Phi_\varepsilon^{-1}(x, \kappa_\sigma(x, r)) \geq C^{-1}\kappa_\sigma(x, r)^{-1/\omega} \geq C^{-1}r^{-\omega(-\delta_0)/\omega}$$
whenever $x \in G$, $0 < r \leq r'$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $0 < \varepsilon \leq \varepsilon_0$ with constants $C$, $C' > 0$ independent of $x, r, \sigma, \varepsilon$. Hence the assertion of the lemma holds if we take $r_0 \in (0, r']$ satisfying $r_0^{-(-\delta_0)/\omega} \geq C'$.

**Proposition 3.2.** Assume that $\Phi(x, t)$ satisfies (Φ5). If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_0$, $-\delta_0 \leq \sigma_j \leq \sigma_0$, $j = 1, 2$ and
$$\sigma_1 + \frac{\delta - \delta_0}{\omega}\varepsilon_1 \leq \sigma_2 + \frac{\delta - \delta_0}{\omega}\varepsilon_2,$$
then $L^{\Phi_\varepsilon, \kappa_\sigma_1}(G) \subset L^{\Phi_\varepsilon, \kappa_\sigma_2}(G)$ and
$$\|f\|_{\Phi_\varepsilon, \kappa_\sigma_2; G} \leq C\|f\|_{\Phi_\varepsilon, \kappa_\sigma_1; G}$$
for all $f \in L^{\Phi_\varepsilon, \kappa_\sigma_1}(G)$ with $C > 0$ independent of $\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2$. In particular,
$$L^{\Phi_\varepsilon, \kappa_\sigma}(G) \subset L^{\Phi_\varepsilon, \kappa_\sigma}(G)$$
if $0 \leq \varepsilon \leq \varepsilon_0$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $\sigma + ((\delta - \delta_0)/\omega)\varepsilon \geq 0$.

**Proof.** Let $\|f\|_{\Phi_\varepsilon, \kappa_\sigma_1; G} \leq 1$. Then
$$\frac{\kappa_\sigma_1(x, r)}{|B(x, r)|} \int_{B(x, r)} \Phi_\varepsilon_1(y, |f(y)|) \, dy \leq 1$$
for $x \in G$ and $0 < r < d_G$.
For $x \in G$ and $0 < r < d_G$, let
$$k(x, r) = \Phi_\varepsilon_1^{-1}(x, \kappa_\sigma_1(x, r))^{-1}$$
and
$$I(x, r) = \int_{B(x, r)} \Phi_\varepsilon_2(y, |f(y)|) \, dy.$$ We write $I(x, r) = I_1(x, r) + I_2(x, r)$, where
$$I_1(x, r) = \int_{B(x, r) \cap \{|y, |f(y)| \leq k(x, r)|\}} \Phi_\varepsilon_2(y, |f(y)|) \, dy$$
and
$$I_2(x, r) = \int_{B(x, r) \cap \{|y, |f(y)| > k(x, r)|\}} \Phi_\varepsilon_2(y, |f(y)|) \, dy.$$ If $|f(y)| \leq k(x, r)$, then
$$\Phi_\varepsilon_2(y, |f(y)|) \leq A_2\Phi_\varepsilon_2(y, k(x, r)) = A_2k(x, r)^{\varepsilon_2}\Phi_\varepsilon_1(y, k(x, r)).$$ Let $r_0 \in (0, \min(1, d_G))$ be the number given in Lemma 3.1. Then, (3.2) implies
$$k(x, r) \leq C\kappa_\sigma_1(x, r)^{-1} \leq Cr^{-N}$$
for $0 < r \leq r_0$ with constants independent of $x$, $\sigma_1$, $\varepsilon_1$. Hence, by (F5), there is a constant $B > 0$ independent of $x$, $\sigma_1$, $\varepsilon_1$, such that

$$
\Phi_{\varepsilon_1}(y, k(x, r)) \leq B\Phi_{\varepsilon_1}(x, k(x, r))
$$

whenever $|x - y| < r \leq r_0$. Therefore,

$$
I_1(x, r) \leq C|B(x, r)|k(x, r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(x, k(x, r)) = C|B(x, r)|k(x, r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x, r)^{-1}
$$

for $0 < r \leq r_0$.

On the other hand, if $|f(y)| > k(x, r)$, then

$$
\Phi_{\varepsilon_2}(y, |f(y)|) = |f(y)|^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y, |f(y)|) \leq k(x, r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y, |f(y)|),
$$

so that

$$
I_2(x, r) \leq k(x, r)^{\varepsilon_1-\varepsilon_2}\int_{B(x, r)} \Phi_{\varepsilon_1}(y, |f(y)|) dy \leq |B(x, r)|k(x, r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x, r)^{-1}
$$

for $0 < r \leq r_0$.

Therefore,

$$
I(x, r) \leq C|B(x, r)|k(x, r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x, r)^{-1},
$$

which implies

$$
\frac{\kappa_{\sigma_2}(x, r)}{|B(x, r)|}\int_{B(x, r)} \Phi_{\varepsilon_2}(y, |f(y)|) dy \leq C r^{\sigma_2-\sigma_1}k(x, r)^{\varepsilon_1-\varepsilon_2}
$$

for $0 < r \leq r_0$. Since $k(x, r)^{-1} \leq C r^{(\delta-\delta_0)/\omega}$ and $\sigma_2 - \sigma_1 + ((\delta - \delta_0)/\omega)(\varepsilon_2 - \varepsilon_1) \geq 0$ by assumption,

$$
\frac{\kappa_{\sigma_2}(x, r)}{|B(x, r)|}\int_{B(x, r)} \Phi_{\varepsilon_2}(y, |f(y)|) dy \leq C r^{\sigma_2-\sigma_1+((\delta-\delta_0)/\omega)(\varepsilon_2-\varepsilon_1)} \leq C
$$

for $0 < r \leq r_0$ with positive constants $C$’s independent of $x$, $\sigma_j$, $\varepsilon_j$ ($j = 1, 2$).

In case $r_0 < r < d_G$, we see

$$
I(x, r) \leq A_2 \int_{B(x, r)} \Phi_{\varepsilon_2}(y, 1) dy + \int_{B(x, r)} \Phi_{\varepsilon_1}(y, |f(y)|) dy
$$

$$
\leq A_1 A_2 |B(x, r)| + |B(x, r)|\kappa_{\sigma_1}(x, r)^{-1},
$$

so that

$$
\frac{\kappa_{\sigma_2}(x, r)}{|B(x, r)|}\int_{B(x, r)} \Phi_{\varepsilon_2}(y, |f(y)|) dy \leq A_1 A_2 \kappa_{\sigma_2}(x, r) + r^{\sigma_2-\sigma_1} \leq C
$$

with $C$ independent of $r$, $x$, $\sigma_1\sigma_2$.

Therefore, $\|f\|_{\Phi_{\varepsilon_2,\kappa_{\sigma_2}}; G} \leq C$ with $C > 0$ independent of $\varepsilon_1$, $\varepsilon_2$, $\sigma_1$, $\sigma_2$. 

Let $\eta(\varepsilon)$ be an increasing positive function on $(0, \infty)$ such that $\eta(0+) = 0$. Let $\xi(\varepsilon)$ be a function on $(0, \varepsilon_1]$ with some $\varepsilon_1 \in (0, \varepsilon_0/2]$ such that $-\delta_0 \leq \xi(\varepsilon) \leq \sigma_0$ for $0 < \varepsilon \leq \varepsilon_1$, $\xi(0+) = 0$ and $\varepsilon \mapsto \xi(\varepsilon) + ((\delta - \delta_0)/\omega)\varepsilon$ is non-decreasing; in particular, $\xi(\varepsilon) + ((\delta - \delta_0)/\omega)\varepsilon \geq 0$ for $0 < \varepsilon \leq \varepsilon_1$.

Given $\Phi(x, t)$, $\kappa(x, r)$, $\eta(\varepsilon)$ and $\xi(\varepsilon)$, the associated (generalized) grand Musielak–Orlicz–Morrey space is defined by (cf. [17] for generalized grand Morrey space)

$$
\mathcal{G}_{\Phi, \kappa, \eta, \xi}(G) = \left\{ f \in \bigcap_{0 < \varepsilon \leq \varepsilon_1} L^{\Phi_{\varepsilon, \eta(\xi(\varepsilon))} \kappa_{\sigma_1}(\xi(\varepsilon))}(G); \|f\|_{\Phi_{\varepsilon, \eta(\xi(\varepsilon))} \kappa_{\sigma_1}(\xi(\varepsilon))} < \infty \right\},
$$
where
\[ \|f\|_{\Phi,\kappa;\xi;G} = \sup_{0<\epsilon\leq\epsilon_0} \eta(\epsilon) \|f\|_{\Phi,\kappa;\xi;G}. \]

\( \tilde{L}_{\eta,\xi}^{\Phi,\kappa}(G) \) is a Banach space with the norm \( \|f\|_{\Phi,\kappa;\eta;\xi;G} \). Note that, in view of Proposition 3.2, this space is determined independent of the choice of \( \epsilon_1 \).

In case \( \xi(\epsilon) \equiv 0 \), the symbol \( \xi \) may be omitted. If \( \kappa(x,r) = r^N \) and \( \xi(\epsilon) \equiv 0 \), then the symbol \( \kappa \) will be also omitted; namely
\[ \tilde{L}_{\eta}^{\Phi}(G) = \left\{ f \in \bigcap_{0<\epsilon\leq\epsilon_0} L^{\Phi,\kappa}(G); \|f\|_{\Phi,\eta;G} := \sup_{0<\epsilon\leq\epsilon_0} \eta(\epsilon) \|f\|_{\Phi,\kappa;G} < \infty \right\}. \]
This space may be called a grand Musielak–Orlicz space.

**Remark 3.3.** The grand Musielak–Orlicz space \( \tilde{L}_{\eta}^{\Phi}(G) \) include the following spaces:
- generalized grand Lebesgue spaces introduced in [4];
- grand Orlicz spaces introduced in [18] where \( \Phi(x,t) = \Phi(t) \) satisfying
  \[ \sup_{0<\epsilon\leq\epsilon_0} \eta(\epsilon) \int_1^\infty t^{-N-\epsilon} \Phi(t) \frac{dt}{t} < \infty \]
(see also [8]).

The (generalized) grand Musielak–Orlicz–Morrey space \( \tilde{L}_{\eta,\xi}^{\Phi,\kappa}(G) \) include also the following spaces:
- grand Morrey spaces introduced in [21] where \( \xi(\epsilon) \equiv 0 \);
- grand grand Morrey spaces introduced in [28] and generalized grand Morrey spaces introduced in [17] where \( \xi(\epsilon) \) is an increasing positive function on \((0,\infty)\).

4. Boundedness of the maximal operator

Hereafter, we shall always assume that \( \Phi(x,t) \) satisfies (\( \Phi_5 \)). For a nonnegative \( f \in L^1_{\text{loc}}(G), x \in G, 0 < r < d_G \) and \( \epsilon > 0 \), set
\[ I(f;x,r) := \frac{1}{|B(x,r)|} \int_{B(x,r) \cap G} f(y) \, dy \]
and
\[ J_\epsilon(f;x,r) := \frac{1}{|B(x,r)|} \int_{B(x,r) \cap G} \Phi_\epsilon(y, f(y)) \, dy. \]
We show a Jensen type inequality for functions in \( L^{\Phi,\kappa,\sigma}(G) \).

**Lemma 4.1.** There exists a constant \( C > 0 \) (independent of \( \epsilon \) and \( \sigma \)) such that
\[ \Phi_\epsilon(x, I(f;x,r)) \leq CJ_\epsilon(f;x,r) \]
for all \( x \in G, 0 < r < d_G, 0 < \epsilon \leq \epsilon_0 \) and for all nonnegative \( f \in L^1_{\text{loc}}(G) \) such that \( f(y) \geq 1 \) or \( f(y) = 0 \) for each \( y \in G \) and \( \|f\|_{\Phi,\kappa;G} \leq 1 \) with \( -\delta_0 \leq \sigma \leq \delta_0 \).

**Proof.** Let \( f \) be as in the statement of the lemma and let \( I = I(f;x,r) \) and \( J_\epsilon = J_\epsilon(f;x,r) \) for \( x \in G, 0 < r < d_G \) and \( 0 < \epsilon \leq \epsilon_0 \). Note that \( \|f\|_{\Phi,\kappa;G} \leq 1 \) implies \( J_\epsilon \leq 2A_3\kappa_\sigma(x,r)^{-1} \) by (3.1).
By (Φ2) and (3.2), \( \Phi_\varepsilon(y, f(y)) \geq (A_1A_2)^{-1}f(y) \), since \( f(y) \geq 1 \) or \( f(y) = 0 \). Hence \( I \leq A_1A_2J_\varepsilon \). Thus, if \( J_\varepsilon \leq 1 \), then

\[
\Phi_\varepsilon(x, I) \leq (A_1A_2J_\varepsilon)A_2\phi(x, A_1A_2) \leq CJ_\varepsilon.
\]

Next, suppose \( J_\varepsilon > 1 \). Since \( \Phi_\varepsilon(x, t) \to \infty \) as \( t \to \infty \), there exists \( K_\varepsilon \geq 1 \) such that

\[
\Phi_\varepsilon(x, K_\varepsilon) = \Phi_\varepsilon(x, 1)J_\varepsilon.
\]

Then \( K_\varepsilon \leq A_2J_\varepsilon \) by (3.2). With this \( K_\varepsilon \), we have

\[
\int_{B(x,r) \cap G} f(y) \, dy \leq K_\varepsilon |B(x,r)| + A_2 \int_{B(x,r) \cap G} f(y) \frac{f(y)^{-\varepsilon}\phi(y, f(y))}{K_\varepsilon^{-\varepsilon}\phi(y, K_\varepsilon)} \, dy.
\]

Since \( \kappa_\varepsilon(x,r)J_\varepsilon \leq 2A_3 \),

\[
1 \leq K_\varepsilon \leq A_2J_\varepsilon \leq 2A_2A_3\kappa_\varepsilon(x,r)^{-1} \leq C\varepsilon^{-N}
\]

with a constant \( C > 0 \) independent of \( \varepsilon \) and \( \sigma \). Hence, by (Φ5) there is \( \beta \geq 1 \), independent of \( f, x, r, \varepsilon \) and \( \sigma \) such that

\[
\phi(x, K_\varepsilon) \leq \beta\phi(y, K_\varepsilon)
\]

for all \( y \in B(x,r) \). Thus, we have

\[
\int_{B(x,r) \cap G} f(y) \, dy \leq K_\varepsilon |B(x,r)| + A_2\beta \int_{B(x,r) \cap G} \frac{\Phi_\varepsilon(y, f(y))}{K_\varepsilon^{-\varepsilon}\phi(x, K_\varepsilon)} \, dy
\]

\[
= K_\varepsilon |B(x,r)| + A_2\beta |B(x,r)|\frac{J_\varepsilon}{K_\varepsilon^{-\varepsilon}\phi(x, K_\varepsilon)}.
\]

Since

\[
K_\varepsilon^{-\varepsilon}\phi(x, K_\varepsilon) = K_\varepsilon^{-1}\Phi_\varepsilon(x, K_\varepsilon) = K_\varepsilon^{-1}J_\varepsilon\Phi_\varepsilon(x, 1) \geq A_1^{-1}K_\varepsilon^{-1}J_\varepsilon,
\]

it follows that

\[
I \leq (1 + A_1A_2\beta)K_\varepsilon,
\]

so that by (Φ2), (Φ3) and (Φ4)

\[
\Phi_\varepsilon(x, I) \leq C\Phi_\varepsilon(x, K_\varepsilon) \leq CJ_\varepsilon
\]

with constants \( C > 0 \) independent of \( f, x, r, \varepsilon \) and \( \sigma \) as required.

For a locally integrable function \( f \) on \( G \), the Hardy–Littlewood maximal function \( Mf \) is defined by

\[
Mf(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r) \cap G} |f(y)| \, dy.
\]

The following lemma can be proved in a way similar to the proof of [25, Theorem 1]:

**Lemma 4.2.** Let \( p_0 > 1 \) and \( -\delta_0 \leq \sigma \leq \sigma_0 \). Then there exists a constant \( C > 0 \) independent of \( \sigma \) for which the following holds: If \( f \) is a measurable function such that

\[
\int_{B(x,r) \cap G} |f(y)|^{p_0} \, dy \leq |B(x,r)|\kappa_\sigma(x,r)^{-1}
\]

for all \( x \in G \) and \( 0 < r < d_G \), then

\[
\int_{B(x,r) \cap G} [Mf(y)]^{p_0} \, dy \leq C|B(x,r)|\kappa_\sigma(x,r)^{-1}
\]
for all $x \in G$ and $0 < r < d_G$.

**Lemma 4.3.** There is a constant $C > 0$ (independent of $\varepsilon$ and $\sigma$) such that

$$\|Mf\|_{\Phi_\varepsilon, \kappa_\sigma, G} \leq C\|f\|_{\Phi_\varepsilon, \kappa_\sigma, G}$$

for all $f \in L^{\Phi_\varepsilon, \kappa_\sigma}(G)$ whenever $0 < \varepsilon \leq \varepsilon_0/2$ and $-\delta_0 \leq \sigma \leq \sigma_0$.

**Proof.** Set $p_0 = 1 + \varepsilon_0/2$ and consider the function

$$\tilde{\Phi}(x, t) = \Phi(x, t)^{1/p_0}.$$

Then $\tilde{\Phi}(x, t)$ also satisfies all the conditions $(\Phi_j)$, $j = 1, 2, \ldots, 5$ with $\varepsilon_0$ replaced by $\varepsilon_0' = \varepsilon_0/(2 + \varepsilon_0)$. In fact, it trivially satisfies $(\Phi_j)$ for $j = 1, 2, 4, 5$. Since

$$t^{-\varepsilon_0}t^{-1}\tilde{\Phi}(x, t) = \left[t^{-\varepsilon_0}\phi(x, t)\right]^{1/p_0},$$

condition $(\Phi3)$ implies that $\tilde{\Phi}(x, t)$ satisfies $(\Phi3)$ with $\varepsilon_0$ replaced by $\varepsilon_0'$.

Let $0 < \varepsilon \leq \varepsilon_0/2$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $f \geq 0$ and $\|f\|_{\Phi_\varepsilon, \kappa_\sigma, G} \leq 1$. Let $f_1 = f\chi_{\{x: f(x) \geq 1\}}$ and $f_2 = f - f_1$, where $\chi_E$ is the characteristic function of $E$.

Since $\tilde{\Phi}_\varepsilon(x, t) \geq 1/(A_1A_2)$ for $t \geq 1$, we see that

$$\tilde{\Phi}_\varepsilon/p_0(x, t) = \tilde{\Phi}_\varepsilon(x, t)^{1/p_0} \leq (A_1A_2)^{1-1/p_0}\tilde{\Phi}_\varepsilon(x, t)$$

if $t \geq 1$, so that

$$\int_{B(x, r) \cap G} \tilde{\Phi}_\varepsilon/p_0(y, f_1(y)) \, dy \leq 2(A_1A_2)^{1-1/p_0}A_3|B(x, r)|\kappa_\sigma(x, r)^{-1}$$

for every $x \in G$ and $0 < r < d_G$. Hence $\|f_1\|_{\Phi_{\varepsilon/p_0, \kappa_\sigma, G}} \leq c_0$ with $c_0 > 0$ independent of $\varepsilon$ and $\sigma$.

Let $F_\varepsilon(x) = \Phi_\varepsilon(x, f(x))$. Then $\tilde{\Phi}_\varepsilon/p_0(x, f(x)) = F_\varepsilon(x)^{1/p_0}$. Applying Lemma 4.1 to $\tilde{\Phi}_\varepsilon/p_0$ and $f_1/c_0$, we have

$$\Phi_\varepsilon(x, Mf_1(x)) \leq \left[\tilde{\Phi}_\varepsilon/p_0(x, Mf_1(x))\right]^{p_0} \leq C[M(F_\varepsilon^{1/p_0}(x))]^{p_0}.$$

On the other hand, since $Mf_2 \leq 1$, we have by (\Phi2) and (\Phi3)

$$\Phi_\varepsilon(x, Mf_2(x)) \leq A_1A_2.$$

Thus, we obtain

$$\Phi_\varepsilon(x, Mf(x)) \leq C \left\{M(F_\varepsilon^{1/p_0}(x))^{p_0} + 1\right\}$$

for $x \in G$ with a constant $C > 0$ independent of $f$ and $\varepsilon$. Hence

$$\int_{B(x, r) \cap G} \Phi_\varepsilon(y, Mf(y)) \, dy \leq C \left\{\int_{B(x, r) \cap G} \left[M(F_\varepsilon^{1/p_0}(y))^{p_0} \, dy + |B(x, r)|\right]\right\}$$

for $x \in G$ and $0 < r < d_G$. Since $\|f\|_{\Phi_\varepsilon, \kappa_\sigma, G} \leq 1$ and $\Phi_\varepsilon(y, f(y)) = F_\varepsilon(y) = (F_\varepsilon^{1/p_0}(y))^{p_0}$, Lemma 4.2 implies

$$\int_{B(x, r) \cap G} \left[M(F_\varepsilon^{1/p_0}(y))^{p_0} \, dy \leq C|B(x, r)|\kappa_\sigma(x, r)^{-1}\right.$$
which shows
\[ \|Mf\|_{\Phi, \kappa, \eta, \xi, G} \leq C \|f\|_{\Phi, \kappa, \eta, \xi, G} \]
with a constant \( C > 0 \) independent of \( \varepsilon \) and \( \sigma \).

From this lemma we obtain the boundedness of the maximal operator on \( \tilde{L}_{\eta, \xi}^{\Phi, \kappa} (G) \).

**Theorem 4.4.** The maximal operator \( M \) is bounded from \( \tilde{L}_{\eta, \xi}^{\Phi, \kappa} (G) \) into itself; namely there exists a constant \( C > 0 \) such that
\[ \|Mf\|_{\Phi, \kappa, \eta, \xi, G} \leq C \|f\|_{\Phi, \kappa, \eta, \xi, G} \]
for all \( f \in \tilde{L}_{\eta, \xi}^{\Phi, \kappa} (G) \).

**Corollary 4.5.** If \( \Phi_{\rho(\cdot), \{q_j(\cdot)\}} (x, t) \) satisfies the conditions in Example 2.1, then the maximal operator \( M \) is bounded from \( \tilde{L}_{\eta}^{\Phi_{\rho(\cdot), \{q_j(\cdot)\}}} (G) \) into itself.

5. Sobolev type inequality

**Lemma 5.1.** [19, Lemma 5.1] Let \( F(x, t) \) be a positive function on \( G \times (0, \infty) \) satisfying the following conditions:

1. \( F(x, \cdot) \) is continuous on \( (0, \infty) \) for each \( x \in G \);
2. there exists a constant \( K_1 \geq 1 \) such that \( K_1^{-1} \leq F(x, 1) \leq K_1 \) for all \( x \in G \);
3. \( t \mapsto t^{-\varepsilon'} F(x, t) \) is uniformly almost increasing for \( \varepsilon' > 0 \); namely there exists a constant \( K_2 \geq 1 \) such that \( t^{-\varepsilon'} F(x, t) \leq K_2 s^{-\varepsilon'} F(x, s) \) for all \( x \in G \) whenever \( 0 < t < s \).

Set
\[ F^{-1}(x, s) = \sup\{t > 0 ; F(x, t) < s\} \]
for \( x \in G \) and \( s > 0 \). Then:

1. \( F^{-1}(x, \cdot) \) is non-decreasing;
2. \( F^{-1}(x, \lambda s) \leq (K_2 \lambda)^{1/\varepsilon'} F^{-1}(x, s) \) for all \( x \in G \), \( s > 0 \) and \( \lambda \geq 1 \);
3. \( F(x, F^{-1}(x, t)) = t \) for all \( x \in G \) and \( t > 0 \);
4. \( K_2^{-1/\varepsilon'} t \leq F^{-1}(x, F(x, t)) \leq K_2^{2/\varepsilon'} t \) for all \( x \in G \) and \( t > 0 \);
5. \( \min \left\{ 1, \left( \frac{s}{K_1 K_2} \right)^{1/\varepsilon'} \right\} \leq F^{-1}(x, s) \leq \max\{1, (K_1 K_2)^{1/\varepsilon'} \} \) for all \( x \in G \) and \( s > 0 \).

**Remark 5.2.** \( F(x, t) = \Phi_{\rho}(x, t) \) \( (0 < \varepsilon \leq \varepsilon_0) \) satisfies (F1), (F2) and (F3) with \( \varepsilon' = 1 \), \( K_1 = A_1 \) and \( K_2 = A_2 \) and \( F(x, t) = \kappa_{\varepsilon \rho}(x, t) \) \( (-\delta_0 \leq \sigma \leq \sigma_0) \) satisfies (F1), (F2) and (F3) with \( \varepsilon' = \delta = \delta_0 \), \( K_1 = Q_1 \) and \( K_2 = Q_2 \).

**Lemma 5.3.** There exists a constant \( C > 0 \) such that
\[ \frac{\eta(\varepsilon)}{|B(x, r)|} \int_{B(x, r) \cap G} f(y) \, dy \leq C \Phi_{-1}^{-1}(x, \kappa_{\varepsilon \rho}(x, r)^{-1}) \]
for all \( x \in G \), \( 0 < r < d_G \), \( 0 < \varepsilon \leq \varepsilon_1 \) and nonnegative functions \( f \) on \( G \) such that \( \|f\|_{\Phi, \kappa, \eta, \xi, G} \leq 1 \).
Proof. Let $f$ be a nonnegative function on $G$ such that $\|f\|_{\Phi, \kappa, \eta, \xi, G} \leq 1$. Then we have by (3.1)

$$
\kappa_{\xi}(x, r) \int_{B(x, r) \cap G} \Phi_{\xi}(y, \eta(y)) \, dy \leq 2A_3
$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \leq \varepsilon_1$. Fix $\varepsilon$ and let $f_1 = f \chi_{(x; \eta(y)f(x) \geq 1)}$ and $f_2 = f - f_1$. By Lemma 4.1,

$$
\Phi_{\varepsilon}(x, 1) \leq A_2 \Phi_{\varepsilon}(x, 1) \leq A_1 A_2,
$$

we have

$$
\Phi_{\varepsilon}(x, 1) \leq C_1 \kappa_{\xi}(x, r)^{-1}
$$

with a constant $C_1 \geq 1$ independent of $x$, $r$, $\varepsilon$. Hence, we find by Lemma 5.1 with $F = \Phi_{\varepsilon}$ and $\varepsilon' > 1$

$$
\frac{1}{B(x, r)} \int_{B(x, r) \cap G} \eta(y) f(y) \, dy \leq A_2 \Phi_{\varepsilon}(x, 1) \leq C_1 \kappa_{\xi}(x, r)^{-1}
$$

as required.

As a potential kernel, we consider a function

$$
J(x, r) : G \times (0, d_G) \to [0, \infty)
$$

satisfying the following conditions:

(J1) $J(\cdot, r)$ is measurable on $G$ for each $r \in (0, d_G)$;

(J2) $J(x, \cdot)$ is non-increasing on $(0, d_G)$ for each $x \in G$;

(J3) $\int_0^{d_G} J(x, r) r^{N-1} dr \leq J_0 < \infty$ for every $x \in G$.

Example 5.4. Let $\alpha(\cdot)$ be a measurable function on $G$ such that

$$
0 < \alpha^- := \inf_{x \in G} \alpha(x) \leq \sup_{x \in G} \alpha(x) =: \alpha^+ < N.
$$

Then, $J(x, r) = r^{\alpha(x)-N}$ satisfies (J1), (J2) and (J3).

For a nonnegative measurable function $f$ on $G$, its $J$-potential $Jf$ is defined by

$$
Jf(x) = \int_G J(x, |x-y|) f(y) \, dy \quad (x \in G).
$$

Set

$$
\overline{J}(x, r) = \frac{N}{r^N} \int_0^r J(x, \rho) \rho^{N-1} d\rho
$$

for $x \in G$ and $0 < r < d_G$. Then $J(x, r) \leq \overline{J}(x, r)$. Further, $\overline{J}(x, \cdot)$ is non-increasing and continuous on $(0, d_G)$ for each $x \in G$. Also, set

$$
Y_J(x, r) = r^N \overline{J}(x, r)
$$
for $x \in G$ and $0 < r < d_G$.

We consider the following condition:

$(\Phi_KJ)$ there exist constants $\delta' > 0$ and $A_1 \geq 1$ such that

\[ s^\delta Y_J(x, s) \Phi_\varepsilon(x, \kappa_\sigma(x, s)^{-1}) \leq A_1 t^\delta Y_J(x, t) \Phi_\varepsilon^{-1}(x, \kappa_\sigma(x, t)^{-1}) \]

for all $x \in G$ whenever $0 < t < s < d_G$, $0 < \varepsilon \leq \varepsilon_0/2$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $\sigma + ((\delta - \delta_0)/\omega) \varepsilon \geq 0$.

**Lemma 5.5.** Assume $(\Phi_KJ)$. Then there exists a constant $C > 0$ such that

\[ \int_{r}^{d_G} \rho^N \Phi_\varepsilon^{-1}(x, \kappa_\sigma(x, \rho)^{-1}) d(-J(x, \cdot))(\rho) \leq CY_J(x, r) \Phi_\varepsilon^{-1}(x, \kappa_\sigma(x, r)^{-1}) \]

for all $x \in G$, $0 < r \leq d_G/2$, $0 < \varepsilon \leq \varepsilon_1/2$ and $-\min(\delta_0, ((\delta - \delta_0)/\omega) \varepsilon) \leq \sigma \leq \sigma_0$.

**Proof.** We follow the proof of [19, Lemma 6.2], noting that the constants are independent of $\varepsilon$ and $\sigma$. \hfill \Box

**Lemma 5.6.** Assume $(\Phi_KJ)$. Then there exists a constant $C > 0$ such that

\[ \eta(\varepsilon) \int_{G \setminus B(x, r)} J(x, |x - y|) f(y) dy \leq CY_J(x, r) \Phi_\varepsilon^{-1}(x, \kappa_\xi(\varepsilon)(x, r)^{-1}) \]

for all $x \in G$, $0 < r \leq d_G/2$, $0 < \varepsilon \leq \varepsilon_1$ and $f \geq 0$ satisfying $\| f \|_{\Phi, \kappa; \eta; \xi; G} \leq 1$.

**Proof.** By the integration by parts, we have

\[ \int_{G \setminus B(x, r)} J(x, |x - y|) f(y) dy \]

\[ \leq J(x, d_G - 0) \int_{G} f(y) dy + \int_{r}^{d_G} \left( \int_{B(x, \rho) \cap G} f(y) dy \right) d(-J(x, \cdot))(\rho), \]

where $J(x, d_G - 0) = \lim_{\rho \to d_G - 0} J(x, \rho)$. Hence, by Lemma 5.3, we have

\[ \eta(\varepsilon) \int_{G \setminus B(x, r)} J(x, |x - y|) f(y) dy \leq C \left\{ Y_J(x, d_G) \Phi_\varepsilon^{-1}(x, \kappa_\xi(\varepsilon)(x, d_G)^{-1}) \right. \]

\[ + \left. \int_{r}^{d_G} |B(x, \rho)| \Phi_\varepsilon^{-1}(x, \kappa_\xi(\varepsilon)(x, \rho)^{-1}) d(-J(x, \cdot))(\rho) \right\}. \]

Hence by $(\Phi_KJ)$ and the previous lemma we obtain the required result. \hfill \Box

**Lemma 5.7.** Assume $(\Phi_KJ)$. Then there exists a constant $C > 0$ such that

\[ \eta(\varepsilon) J f(x) \leq C \left\{ \eta(\varepsilon) M f(x) Y_J \left( x, \kappa_\xi^{-1}(\varepsilon)(x, \Phi_\varepsilon(x, \xi(\varepsilon) M f(x))^{-1}) \right) \right\} + 1 \]

for all $x \in G$, $0 < \varepsilon \leq \varepsilon_1$ and $f \geq 0$ satisfying $\| f \|_{\Phi, \kappa; \eta; \xi; G} \leq 1$.

**Proof.** Let $f$ be a nonnegative function on $G$ such that $\| f \|_{\Phi, \kappa; \eta; \xi; G} \leq 1$. For $0 < r \leq d_G/2$, we write

\[ J f(x) = \int_{B(x, r) \cap G} J(x, |x - y|) f(y) dy + \int_{G \setminus B(x, r)} J(x, |x - y|) f(y) dy \]

\[ = J_1(x) + J_2(x). \]

First note that

\[ J_1(x) \leq CY_J(x, r) M f(x) \]
(see, e.g., [30, p. 63, (16)]). By Lemma 5.6, we have
\[
\eta(\varepsilon)J_2(x) \leq CY_f(x, r)\Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x, r)^{-1}).
\]
Hence
\[
\eta(\varepsilon)J_f(x) \leq CY_f(x, r) \left\{ \eta(\varepsilon)Mf(x) + \Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x, r)^{-1}) \right\}
\]
for \( x \in G, \ 0 < r \leq d_G/2 \) and \( 0 < \varepsilon \leq \varepsilon_0 \).

We consider two cases.

Case 1: \( d_G/2 < \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x))^{-1}) \). In this case, let \( r = d_G/2 \). Since
\[
\Phi_\varepsilon(x, \eta(\varepsilon)Mf(x)) \leq Q_2\kappa_{\xi(\varepsilon)}(x, d_G/2)^{-1} \leq Q_2Q_3 \max\{1, (d_G/2)^{-N}\},
\]
it follows that \( \eta(\varepsilon)Mf(x) \leq C_1 \) with a constant \( C_1 > 0 \) independent of \( x \) and \( \varepsilon \).

Also,
\[
\Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x, r)^{-1}) = \Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x, d_G/2)^{-1}) \leq C_2
\]
with a constant \( C_2 > 0 \) independent of \( x \) and \( \varepsilon \). Hence, by (5.1) and (J3),
\[
\eta(\varepsilon)J_f(x) \leq C
\]
with a constant \( C > 0 \) independent of \( x \) and \( \varepsilon \).

Case 2: \( d_G/2 \geq \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x))^{-1}) \). In this case, take
\[
r = \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x))^{-1}).
\]
Then \( \kappa_{\xi(\varepsilon)}(x, r)^{-1} = \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x)) \), so that by Lemma 5.1(4)
\[
\Phi_\varepsilon^{-1}(x, \kappa_{\xi(\varepsilon)}(x, r)^{-1}) \leq C\eta(\varepsilon)Mf(x)
\]
with a constant \( C > 0 \) independent of \( x \) and \( \varepsilon \). Hence, by (5.1)
\[
\eta(\varepsilon)J_f(x) \leq CY_f(x, r)\eta(\varepsilon)Mf(x)
\]
\[
= C\eta(\varepsilon)Mf(x)Y_f\left(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x, \eta(\varepsilon)Mf(x))^{-1})\right)
\]
with a constant \( C > 0 \) independent of \( x \) and \( \varepsilon \). \( \square \)

The following theorem gives a Sobolev type inequality for potentials \( Jf \) of \( f \in \tilde{L}_{\eta, \Phi}(G) \). Example 5.9 below shows that this theorem includes known Sobolev type inequalities as special cases.

**Theorem 5.8.** Assume \((\Phi, \kappa, J)\). Suppose a function
\[
\Psi(x, t) : G \times [0, \infty) \to [0, \infty)
\]
satisfies \((\Phi 1) – (\Phi 4)\) with \( \varepsilon_0 \) replaced by some \( \varepsilon_0' \) in \((\Phi 3)\) and
\((\Psi \Phi)\) there exist a constant \( A' \geq 1 \) and a strictly increasing continuous function \( \zeta(\varepsilon) \)
on \([0, \varepsilon_1]\) such that \( \zeta(0) = 0 \), \( \varepsilon \mapsto \xi(\varepsilon) + ((\delta - \delta_0)/\omega^*)\zeta(\varepsilon) \) is non-decreasing with \( \omega^* > 1 \) such that \( \Psi(x, t) \leq C\omega^* \) for \( t \geq 1 \), and
\[
\Psi_{\xi(\varepsilon)}\left(x, tY_f\left(x, \kappa^{-1}_{\xi(\varepsilon)}(x, \Phi_\varepsilon(x, t)^{-1})\right)\right) \leq A\Phi_\varepsilon(x, t)
\]
for all \( x \in G, \ t \geq 1 \) and \( 0 < \varepsilon \leq \varepsilon_1 \).
Then there exists a constant \( C > 0 \) such that
\[
\|Jf\|_{\Psi, \kappa, \eta, \xi, \zeta^{-1}, \xi_0, \zeta_0^{-1}; G} \leq C\|f\|_{\Phi, \kappa, \eta, \xi; G}
\]
for all \( f \in \tilde{L}_{\eta, \Phi}(G) \).
Proof. Let \( f \) be a nonnegative function on \( G \) such that \( \|f\|_{\Phi,\kappa;\eta;G} \leq 1 \). Choose \( \varepsilon'_1 \in (0, \varepsilon] \) such that \( \zeta(\varepsilon') \leq \varepsilon'_0 \). Let \( x \in G, 0 < r < d_G \) and \( 0 < \varepsilon \leq \varepsilon'_1 \). By Lemma 5.7 and (\( \Psi,\Phi \)) we have
\[
\Psi_{\zeta(\varepsilon)}(x, \eta(\varepsilon) Jf(x)) \\
\leq C \left\{ \Psi_{\zeta(\varepsilon)} \left(x, \eta(\varepsilon) Mf(x) Y_J \left(x, \kappa_{\zeta(\varepsilon)}^{-1} \left(x, \Phi_e(x, \eta(\varepsilon) Mf(x))^{-1} \right) \right) \right\} + 1 \\
\leq C \left\{ \Phi_e(x, \eta(\varepsilon) Mf(x)) + 1 \right\}.
\]
Note that \( \|f\|_{\Phi,\kappa;\eta;G} \leq 1 \) implies \( \|Mf\|_{\Phi,\kappa;\eta;G} \leq C \) by Theorem 4.4. Hence there is a constant \( C'_1 > 0 \) such that
\[
\frac{\kappa_{\zeta(\varepsilon)}(x, r)}{|B(x, r)|} \int_{B(x, r) \cap G} \Phi_e(y, \eta(\varepsilon) Mf(y)) \, dy \leq C'_1
\]
for all \( x \in G, 0 < r < d_G \) and \( 0 < \varepsilon \leq \varepsilon'_1 \). Therefore, there is another constant \( C'_2 > 0 \) such that
\[
\frac{\kappa_{\zeta(\varepsilon)}(x, r)}{|B(x, r)|} \int_{B(x, r) \cap G} \Psi_{\zeta(\varepsilon)}(y, \eta(\varepsilon) Jf(y)) \, dy \leq C'_2
\]
for all \( x \in G, 0 < r < d_G \) and \( 0 < \varepsilon \leq \varepsilon'_1 \), so that
\[
\frac{\kappa_{\zeta(\varepsilon)}(x, r)}{|B(x, r)|} \int_{B(x, r) \cap G} \Psi_{\zeta(\varepsilon)}(y, \eta(\varepsilon) Jf(y)) \, dy \leq C'_2
\]
for all \( x \in G, 0 < r < d_G \) and \( 0 < \varepsilon' \leq \zeta(\varepsilon'_1) \), which implies the required result. \( \square \)

Example 5.9. Let \( \Phi(x, t) = \Phi_{p(t), q(t)}(x, t) \) be as in Example 2.1, \( \kappa(x, r) = r^{\nu(x)}(\log(e + 1/r))^{\beta(x)} \) be as in Example 2.2 and \( J(x, r) = r^{\alpha(x) - N} \) be as in Example 5.4.

Note that \( \sigma_0 = 0 \) if \( \nu^+ := \sup_{x \in G} \nu(x) = N \) and \( 0 < \sigma_0 < N - \nu^+ \) if \( \nu^+ < N \). We may take \( 0 < \delta_0 < \delta < \nu^- \) and \( \omega > \nu^+ \). Then,
\[
\sigma + \frac{\delta - \delta_0}{\omega} \varepsilon < \sigma + \frac{\nu^-}{\nu^+} \varepsilon \leq \sigma + \frac{\nu(x)}{p(x)} \varepsilon.
\]
Hence, if \( \sigma + ((\delta - \delta_0)/\omega) \varepsilon \geq 0 \), then
\[
(5.2) \quad \frac{\nu(x) + \sigma}{p(x) - \varepsilon} \geq \frac{\nu(x)}{p(x)}.
\]
Since
\[
Y_J(x, r) \Phi^{-1}_e(x, \kappa_\sigma(x, r)) \sim r^{\alpha(x) - (\nu(x) + \sigma)/(p(x) - \varepsilon)} \left[ Q(x, 1/r) (\log(e + 1/r))^{\beta(x)} \right]^{-1/(p(x) - \varepsilon)},
\]
where \( Q(x, t) = \prod_{j=1}^k (L_j(t))^{Q_j(x)} \), we see that condition (\( \Phi,\kappa,J \)) holds if
\[
\inf_{x \in G} \left( \frac{\nu(x)}{p(x)} - \alpha(x) \right) > 0.
\]
Set
\[
\Psi(x, t) = \left[ \Phi_{p(t), q(t)}(x, t) \right]^{\nu(x)/(p(x) - \varepsilon)} \left[ \frac{\nu(x)}{p(x)} \right]^{(\nu(x)/(p(x) - \varepsilon))},
\]
where \( 1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x) \). We see
\[
t Y_J(x, \kappa^{-1}_\sigma(x, \Phi_e(x, t)^{-1})) \sim t^{\beta(x)/(p^*(x) + \varepsilon \alpha(x)/(\nu(x) + \sigma))} \left[ Q(x, t) (\log(e + t))^{\beta(x)} \right]^{-\alpha(x)/(\nu(x) + \sigma)},
\]
where \(1/p^*_\alpha(x) = 1/p(x) - \alpha(x)/(\nu(x) + \sigma)\). Hence
\[
\Psi \left( x, tY_f(x, \kappa_\sigma^{-1}(x, \Phi_\alpha(x, t)^{-1})) \right) \\
\sim t^p(x)p^*(x)/p^*_\alpha(x) \alpha(x) \log(e + t) - p^*(x)\alpha(x)/(\nu(x) + \sigma) \\
\cdot Q(x, t)p^*(x)/p(x) \log(e + t) \sigma(p^*(x) - p(x))/\nu(x) + \epsilon(p^*(x)\alpha(x)/(\nu(x) + \sigma) + 1) \\
= \Phi_\alpha(x, t) \sigma(p^*(x) - p(x))/\nu(x) + 2p^*(x)/p(x) \log(e + t)\alpha(x)/(\nu(x) + \sigma) \\
\cdot Q(x, t)^\sigma(p^*(x)/p(x) - p(x))/\nu(x) + \sigma(p^*(x)\alpha(x)/(\nu(x) + \sigma)).
\]

Here, note that \(\xi(\epsilon) + (\nu^-/p^+)\epsilon \geq 0\) implies \(\nu(x) + \xi(\epsilon) > \nu(x)/2\) if \(0 < \epsilon \leq 1/2\). Let \(0 < \epsilon \leq \min(1/2, \epsilon_1)\). Let \(\theta = (\delta - \delta_0)/\omega\). Since
\[
\frac{\xi(\epsilon)}{\nu(x) + \xi(\epsilon)} \leq \frac{\xi(\epsilon) + \theta \epsilon}{\nu(x)} \quad \text{and} \quad \frac{p^*(x)\alpha(x)}{\nu(x) + \xi(\epsilon)} + 1 \leq 2\frac{p^*(x)}{p(x)},
\]
\[
\Psi \left( x, tY_f(x, \kappa_\xi^{-1}(x, \Phi_\xi(x, t)^{-1})) \right) \\
\lesssim \Phi_\xi(x, t)^{\xi(\epsilon)} \sigma(p^*(x) - p(x))/\nu(x) + 2p^*(x)/p(x) \log(e + t)\nu(x) \alpha(x)/(\nu(x) + \sigma) \\
\cdot Q(x, t)^\sigma(p^*(x)/p(x) - p(x))/\nu(x) + \sigma(p^*(x)\alpha(x)/(\nu(x) + \sigma)) \log(e + t)) \alpha(x)/(\nu(x) + \sigma)
\]
for \(t \geq 1\) with a constant \(m_1 \geq 0\). In view of (5.2), we also see that
\[
tY_f(x, \kappa_\xi^{-1}(x, \Phi_\xi(x, t)^{-1})) \gtrsim p(x)/p^*(x) \log(e + t)^{-m_2}
\]
with a constant \(m_2 \geq 0\), which implies
\[
\Psi \left( x, tY_f(x, \kappa_\xi^{-1}(x, \Phi_\xi(x, t)^{-1})) \right) \\
\lesssim \Phi_\xi(x, t)^{\xi(\epsilon)} \sigma(p^*(x)/p(x) - p(x))/\nu(x) + 2p^*(x)/p(x) \log(e + t)\nu(x) \alpha(x)/(\nu(x) + \sigma) \\
\cdot Q(x, t)^\sigma(p^*(x)/p(x) - p(x))/\nu(x) + \sigma(p^*(x)\alpha(x)/(\nu(x) + \sigma)) \log(e + t)) \alpha(x)/(\nu(x) + \sigma)
\]
for \(t \geq 1\).

Now, let \(\zeta(\epsilon) = a\epsilon + b(\xi(\epsilon) + \theta \epsilon)\ (a, b > 0)\). If \(a > 2\sup_{x \in G}(p^*(x)/p(x))^2\), then
\[
\sup_{x \in G, t \geq 1} \left\{ t^p(x)/p^*(x) \log(e + t)^{-m_2} \right\}^{-a} t^{2p^*(x)/p(x)} < \infty
\]
and if \(b > \sup_{x \in G} p^*(x)(p^*(x) - p(x))/p(x)\nu(x)\), then
\[
\sup_{x \in G, t \geq 1} \left\{ t^p(x)/p^*(x) \log(e + t)^{-m_2} \right\}^{-b} t^{(p^*(x) - p(x))/\nu(x) \log(e + t)} \nu(x) < \infty,
\]
so that \(\Psi(x, t)\) satisfies condition \((\Psi \Phi)\) with \(\zeta(\epsilon) = a\epsilon + b(\xi(\epsilon) + \theta \epsilon)\) \((0 < \epsilon \leq \min(1/2, \epsilon_1))\).

6. Trudinger type inequality

In this section, we consider Trudinger type inequality on \(\tilde{L}_{\eta, \xi}(G)\).

**Lemma 6.1.** Let \(t_1, t_2 > 0\). If
\[
\Phi(x, t_1) \leq K \Phi(x, t_2)
\]
for some \(x \in G\) with \(K \geq A_2^{-1}\), then \(t_1 \leq A_2 K t_2\).
\textbf{Proof.} Assume \( t_1 > A_2Kt_2 \). Note that \( t_1 > t_2 \). Using (\( \Phi 3 \)), we have
\[
\Phi(x, t_1) = t_1 \phi(x, t_1) > Kt_2 \phi(x, t_2) = K \Phi(x, t_2),
\]
which contradicts the assumption. \( \square \)

In this section, we assume:
(\( \Xi \)) \( \xi(\varepsilon) \leq a \varepsilon \) for \( 0 < \varepsilon \leq \varepsilon_1 \) with some \( a \geq 0 \).

Recall that \( \xi(\varepsilon) \geq -((\delta - \delta_0)/\omega)\varepsilon \) by assumption. Let
\[
\varepsilon(r) = (\log(e + 1/r))^{-1}
\]
for \( r > 0 \) and let \( r_1 \leq (0, \min(1, d_G)) \) be such that \( \varepsilon(r) \leq \varepsilon_1 \) for \( 0 < r \leq r_1 \).

\textbf{Lemma 6.2.} There exists a constant \( C \geq 1 \) such that
\[
C^{-1} \Phi^{-1}(x, \kappa(x, r)^{-1}) \leq \Phi_{\varepsilon(r)}^{-1}(x, \kappa_\xi(\varepsilon(r)) (x, r)^{-1}) \leq C \Phi^{-1}(x, \kappa(x, r)^{-1})
\]
for all \( x \in G \) and \( 0 < r \leq r_1 \).

\textbf{Proof.} Fix \( x \in G \) and set
\[
t_0(r) = \Phi^{-1}(x, \kappa(x, r)^{-1}) \quad \text{and} \quad t(r) = \Phi_{\varepsilon(r)}^{-1}(x, \kappa_\xi(\varepsilon(r)) (x, r)^{-1})
\]
for \( 0 < r \leq r_1 \). Then
\[
\Phi(x, t_0(r)) = \kappa(x, r)^{-1} = r^{\xi(\varepsilon(r))} \kappa_\xi(\varepsilon(r)) (x, r)^{-1}
\]
(6.1)
\[
= r^{\xi(\varepsilon(r))} \Phi_{\varepsilon(r)}(x, t(r)) = r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)} \Phi(x, t(r)).
\]
Thus, in view of Lemma 6.1, it is enough to show that there exists a constant \( K \geq 1 \) independent of \( x \) such that
\[
K^{-1} \leq r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)} \leq K
\]
for all \( 0 < r \leq r_1 \).

Note that
\[
e^{-a} \leq r^{a \varepsilon(r)} \leq r^{\xi(\varepsilon(r))} \leq r^{-(\delta - \delta_0)/\omega}\varepsilon(r) \leq e^{(\delta - \delta_0)/\omega}
\]
(6.3)
for \( 0 < r \leq r_1 \) and that
\[
Q_3^{-1} \leq \kappa(x, r)^{-1} \leq Q_3 \left( 1 + \frac{1}{r} \right)^N
\]
by (\( \kappa 3 \)).

If \( t(r) \leq 1 \), then by (6.1) and (6.3)
\[
Q_3^{-1} \leq \kappa(x, r)^{-1} = r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)} \Phi(x, t(r)) \leq e^{(\delta - \delta_0)/\omega} t(r)^{1-\varepsilon(d_G)} A_1 A_2 t(r)^{1-\varepsilon(d_G)},
\]
so that \( t(r) \geq C_1^{-1} \) with a constant \( C_1 \geq 1 \) independent of \( x \). Thus
\[
C_1^{-\varepsilon(d_G)} \leq t(r)^{\varepsilon(r)} \leq 1
\]
if \( t(r) \leq 1 \).

If \( t(r) \geq 1 \), then by (6.1) and (6.3) again
\[
Q_3 \left( 1 + \frac{1}{r} \right)^N \geq \kappa(x, r)^{-1} \geq e^{-a} t(r)^{1-\varepsilon(r)} \phi(x, t(r)) \geq e^{-a} (A_1 A_2)^{-1} t(r)^{1-\varepsilon(d_G)},
\]
so that \( t(r) \leq C_2[(1 + 1/r)^N]^{1/(1-ε(d_G))} \) with \( C_2 \geq 1 \) independent of \( x \). Since \( (1 + 1/r)^{ε(r)} \) is bounded for \( r > 0 \), it follows that

\[
1 \leq t(r)^{ε(r)} \leq C_2^{ε(\epsilon)} \left[ \left( 1 + \frac{1}{r} \right)^N \right]^{ε(r)/(1-ε(d_G))} \leq C_3
\]

if \( t(r) \geq 1 \), with a constant \( C_3 \geq 1 \) independent of \( x \).

Therefore, (6.2) holds with \( K = \max\{e^{(6-80)/\omega} C_1^{ε(d_G)}, e^3C_3\} \). □

**Lemma 6.3.** There exists a constant \( C > 0 \) such that

\[
(6.4) \quad \frac{1}{|B(x, r)|} \int_{B(x, r) \cap G} f(y) \, dy \leq C \Phi^{-1}(x, \kappa(x, r)^{-1}) \eta \left( (\log(e + 1/r))^{-1} \right)^{-1}
\]

for all \( x \in G, 0 < r < d_G \) and nonnegative \( f \in \widetilde{L}^{\Phi_{x, x}}(G) \) with \( \|f\|_{\Phi_{x, \eta; \xi; \xi} G} \leq 1 \).

**Proof.** Let \( f \) be a nonnegative measurable function on \( G \) such that \( \|f\|_{\Phi_{x, \eta; \xi; \xi} G} \leq 1 \).

If \( 0 < r \leq r_1 \), then by Lemma 5.3

\[
\frac{1}{|B(x, r)|} \int_{B(x, r) \cap G} f(y) \, dy \leq C \Phi^{-1}_{\epsilon(r)}(x, \kappa_{\epsilon(r)}(x, r)^{-1}) \eta(\epsilon(r))^{-1}
\]

for all \( x \in G \). Hence, using the above lemma we obtain (6.4).

In case \( r_1 < r < d_G \), note that

\[
\Phi^{-1}_{\epsilon(r_1)}(x, \kappa_{\epsilon(r_1)}(x, r_1)^{-1}) \leq C \Phi^{-1}(x, \kappa(x, r)^{-1})
\]

by (\( \kappa_3 \)) and Lemma 5.1(5). Hence, by Lemma 5.3 with \( \epsilon = \epsilon(r_1) \), we obtain (6.4) in this case, too. □

In this section, we also assume that

(J3') \( J(x, r) \leq C_J r^{-s} \) for \( x \in G \) and \( 0 < r \leq d_G \) with constants \( 0 \leq \varsigma < N \) and \( C_J > 0 \);

(J4) there is \( r_0 \in (0, d_G) \) such that

\[
\inf_{x \in G} J(x, r_0) > 0 \quad \text{and} \quad \inf_{x \in G} \frac{\overline{J}(x, r_0)}{\overline{J}(x, d_G)} > 1.
\]

Here note that (J3') implies (J3).

**Example 6.4.** Let \( \alpha(\cdot) \) and \( J(x, r) \) be as in Example 5.4. Then, \( J(x, r) \) satisfies (J3') and (J4) (with \( \varsigma = N - \alpha^- \)). In particular, it satisfies (J4) with any \( r_0 \in (0, d_G) \).

We consider the function

\[
\Gamma(x, s) = \begin{cases}
\int_{x/r_0}^{d_G} \rho^N \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} d(\overline{J}(x, \cdot)) (\rho) & \text{if } s \geq \frac{1}{r_0}, \\
\Gamma(x, 1/r_0) r_0 s & \text{if } 0 \leq s \leq 1/r_0
\end{cases}
\]

for every \( x \in G \), where \( r_0 \) is the number given in (J4). \( \Gamma(x, \cdot) \) is strictly increasing and continuous for each \( x \in G \).

**Lemma 6.5.** There exist positive constants \( C' \) and \( C'' \) such that

(a) \( \Gamma(x, s) \leq C's^\eta \left( (\log(e + s))^{-1} \right)^{-1} \) for all \( x \in G \) and \( s \geq 1/r_0 \) with \( \varsigma \) in condition (J3').
(b) $\Gamma(x, 1/r_0) \geq C'' > 0$ for all $x \in G$.

Proof. First note from $(k3)$ and Lemma 5.1(5) that

$$C^{-1} \leq \Phi^{-1}(x, \kappa(x, r)^{-1}) \leq Cr^{-N}.$$  

By (6.5) and (J3'),

$$\Gamma(x, s) \leq C\eta \left( (\log(e + s))^{-1} \right)^{-1} \int_{1/s}^{d_G} d(-\overline{J}(x, \cdot))(\rho)$$

$$\leq C\eta \left( (\log(e + s))^{-1} \right)^{-1} \overline{J}(x, 1/s) \leq C'ss' \eta \left( (\log(e + s))^{-1} \right)^{-1}$$

for all $x \in G$ and $s \geq 1/r_0$; and

$$\Gamma(x, 1/r_0) \geq C^{-1} \int_{r_0}^{d_G} \rho^N d(-\overline{J}(x, \cdot))(\rho) \geq C^{-1}r_0^N \int_{r_0}^{d_G} d(-\overline{J}(x, \cdot))(\rho)$$

$$= C^{-1}r_0^N (\overline{J}(x, r_0) - \overline{J}(x, d_G)) \geq C'' > 0,$$

where we used (J4) to obtain the inequalities in the last line. \hfill $\square$

Lemma 6.6. There exists a constant $C > 0$ such that

$$\int_{G \setminus B(x, \delta)} J(x, |x - y|) f(y) \, dy \leq CT \left( x, \frac{1}{\delta} \right)$$

for all $x \in G$, $0 < \delta \leq r_0$ and nonnegative $f \in \tilde{L}^{\Phi, \kappa}(G)$ with $\|f\|_{\Phi, \kappa; \eta; \xi; G} \leq 1$.

Proof. By integration by parts, Lemma 6.3, (6.5), (J3') and Lemma 6.5(b), we have

$$\int_{G \setminus B(x, \delta)} J(x, |x - y|) f(y) \, dy \leq \int_{G \setminus B(x, \delta)} \overline{J}(x, |x - y|) f(y) \, dy$$

$$\leq C \left\{ d_G^N \overline{J}(x, d_G) \Phi^{-1}(x, \kappa(x, d_G)^{-1}) \eta \left( (\log(e + 1/d_G))^{-1} \right)^{-1}$$

$$+ \int_{\delta}^{d_G} \rho^N \Phi^{-1}(x, \kappa(x, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} d(-\overline{J}(x, \cdot))(\rho) \right\}$$

$$\leq C \{ \Gamma(x, 1/r_0) + \Gamma(x, 1/\delta) \} \leq CT \left( x, \frac{1}{\delta} \right). \hfill \square$$

Lemma 6.7. Let $0 < \lambda < N$ and define

$$I_{\lambda} f(x) = \int_{G} |x - y|^\lambda \tilde{f}(y) \, dy$$

for a nonnegative measurable function $f$ on $G$ and

$$\omega_{\lambda}(z, r) = \frac{1}{1 + \int_{r}^{d_G} \rho^\lambda \Phi^{-1}(z, \kappa(z, \rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{dp}{\rho}}$$

for $z \in G$. Then there exists a constant $C_{I, \lambda} > 0$ such that

$$\frac{\omega_{\lambda}(z, r)}{|B(z, r)|} \int_{B(z, r) \cap G} I_{\lambda} f(x) \, dx \leq C_{I, \lambda}$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \tilde{L}^{\Phi, \kappa}(G)$ with $\|f\|_{\Phi, \kappa; \eta; \xi; G} \leq 1$. 

Proof. Let $z \in G$. Let $f(x) = 0$ for $x \in \mathbb{R}^N \setminus G$ and write
\[
I_\lambda f(x) = \int_{B(z,2r)} |x-y|^{\lambda-N} f(y) \, dy + \int_{G \setminus B(z,2r)} |x-y|^{\lambda-N} f(y) \, dy
\]
for $x \in G$. By Fubini’s theorem,
\[
\int_{B(z,r) \cap G} I_1(x) \, dx = \int_{B(z,2r)} \left( \int_{B(z,r) \cap G} |x-y|^{\lambda-N} \, dx \right) f(y) \, dy
\]
\[
\leq \int_{B(z,2r)} \left( \int_{B(y,3r)} |x-y|^{\lambda-N} \, dx \right) f(y) \, dy
\]
\[
\leq C \int_{B(z,2r)} \left( \int_0^{3r} t^{\lambda} \frac{dt}{t} \right) f(y) \, dy \leq \frac{C}{\lambda} r^\lambda \int_{B(z,2r)} f(y) \, dy.
\]
Now, by Lemma 6.3, (κ2) and Lemma 5.1(2), we have
\[
r^\lambda \int_{B(z,2r)} f(y) \, dy \leq C r^\lambda |B(z,2r)| \Phi^{-1}(z, \kappa(z,2r)^{-1}) \eta \left( (\log(e + 1/(2r)))^{-1} \right)^{-1}
\]
\[
\leq C |B(z,r)| \int_r^{2r} \rho^{\lambda} \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}
\]
if $0 < r < d_G/2$ and, by Lemma 6.3 and (6.5), we have
\[
r^\lambda \int_{B(z,2r)} f(y) \, dy = r^\lambda \int_{B(z,d_G)} f(y) \, dy
\]
\[
\leq Cd_G^\lambda |B(z,d_G)| \Phi^{-1}(z, \kappa(z,d_G)^{-1}) \eta \left( (\log(e + 1/d_G))^{-1} \right)^{-1} \leq C |B(z,r)|
\]
if $d_G/2 \leq r < d_G$. Therefore
\[
\int_{B(z,r) \cap G} I_1(x) \, dx \leq \frac{C |B(z,r)|}{\lambda \omega_\lambda(z,r)}
\]
for all $0 < r < d_G$.

For $I_2$, first note that $I_2(x) = 0$ if $x \in G$ and $r \geq d_G/2$. Let $0 < r < d_G/2$. Since
\[
I_2(x) \leq C \int_{G \setminus B(z,2r)} |z-y|^{\lambda-N} f(y) \, dy \quad \text{for} \quad x \in B(z,r) \cap G,
\]
by integration by parts and Lemma 6.3, we have
\[
I_2(x) \leq C \left\{ d_G^{\lambda} \Phi^{-1}(z, \kappa(z,d_G)^{-1}) \eta \left( (\log(e + 1/d_G))^{-1} \right)^{-1}
\]
\[
+ \int_{2r}^{d_G} \rho^{\lambda} \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta \left( (\log(e + 1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho} \right\}
\]
\[
\leq \frac{C}{\omega_\lambda(z,r)}
\]
for all $x \in B(z,r) \cap G$. Hence
\[
\int_{B(z,r) \cap G} I_2(x) \, dx \leq \frac{C |B(z,r)|}{\omega_\lambda(z,r)}.
\]
Thus this lemma is proved.\qed
For every $a > 1$, there exists $b > 0$ such that

$$\Gamma(x, as) \leq b\Gamma(x, s)$$

for all $x \in G$ and $s > 0$.

By $(\Gamma_{\log})$, together with Lemma 6.5, we see that $\Gamma(x, s)$ satisfies the uniform doubling condition in $s$:

**Lemma 6.8.** [20, Lemma 4.2] For every $a > 1$, there exists $b > 0$ such that $\Gamma(x, as) \leq b\Gamma(x, s)$ for all $x \in G$ and $s > 0$.

Now we consider the following condition (J5):

(J5) there exists $0 < \lambda < N - \varsigma$ such that $r \mapsto r^{N-\lambda}J(x, r)$ is uniformly almost increasing on $(0, d_G)$ for $\varsigma$ in condition (J3').

**Example 6.9.** Let $J$ be as in Example 5.4. It satisfies (J5) with $0 < \lambda < \alpha^-$.  

**Theorem 6.10.** Assume that $\Gamma$ satisfies $(\Gamma_{\log})$ and $J$ satisfies (J5). For each $x \in G$, let $\gamma(x) = \sup_{s>0} \Gamma(x, s)$. Suppose $\Lambda(x, t): G \times [0, \infty) \to [0, \infty]$ satisfies the following conditions:

(A1) $\Lambda(\cdot, t)$ is measurable on $G$ for each $t \in [0, \infty)$; $\Lambda(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in G$;

(A2) there is a constant $A'_1 \geq 1$ such that $\Lambda(x, t) \leq \Lambda(x, A'_1 s)$ for all $x \in G$ whenever $0 < t < s$;

(A3) $\Lambda(x, \Gamma(x, s)/A'_2) \leq A'_2 s$ for all $x \in G$ and $s > 0$ with constants $A'_2, A'_3 \geq 1$ independent of $x$.

Then, for $\lambda$ given in (J5), there exists a constant $C^* > 0$ such that $Jf(x)/C^* \leq \gamma(x)$ for a.e. $x \in G$ and

$$\frac{\omega_\lambda(z, r)}{|B(z, r)|} \int_{B(z, r) \cap G} \Lambda \left( x, \frac{Jf(x)}{C^*} \right) \, dx \leq 1$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \tilde{L}_{\Phi, \kappa, \eta, \xi}(G)$ with $\|f\|_{\Phi, \kappa, \eta, \xi; G} \leq 1$.

By $(\Gamma_{\log})$ and (A3), the assertion of this theorem can be considered as exponential integrability of $Jf$; cf. Corollary 6.12 below.

**Proof.** Let $f$ be a nonnegative measurable function on $G$ such that $\|f\|_{\Phi, \kappa, \eta, \xi; G} \leq 1$. Fix $x \in G$. For $0 < \delta \leq r_0$, Lemma 6.6, (J5) and (J3') imply

$$Jf(x) \leq \int_{B(x, \delta)} J(x, |x-y|)f(y) \, dy + CT \left( x, \frac{1}{\delta} \right)$$

$$= \int_{B(x, \delta)} |x-y|^{N-\lambda} J(x, |x-y|) |x-y|^{\lambda-N} f(y) \, dy + CT \left( x, \frac{1}{\delta} \right)$$

$$\leq C \left\{ \delta^{N-\lambda} J(x, \delta) I_{\lambda} f(x) + \Gamma \left( x, \frac{1}{\delta} \right) \right\}$$

$$\leq C \left\{ \delta^{N-\varsigma-\lambda} I_{\lambda} f(x) + \Gamma \left( x, \frac{1}{\delta} \right) \right\}$$

with constants $C > 0$ independent of $x$. 

If $I_\lambda f(x) \leq 1/r_0$, then we take $\delta = r_0$. Then, by Lemma 6.5(b)

$$Jf(x) \leq CT \left( x, \frac{1}{r_0} \right).$$

By Lemma 6.8, there exists $C_1^* > 0$ independent of $x$ such that

(6.6) \hspace{1cm} Jf(x) \leq C_1^* \Gamma \left( x, \frac{1}{2A_3^*} \right) \quad \text{if } I_\lambda f(x) \leq 1/r_0.

Next, suppose $1/r_0 < I_\lambda f(x) < \infty$. Let $m = \sup_{s \geq 1/r_0, x \in G} \Gamma(s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define $\delta$ by

$$\delta^{N-\varsigma-\lambda} = \frac{r_0^{N-\varsigma-\lambda}}{m} \Gamma(x, I_\lambda f(x))(I_\lambda f(x))^{-1}.$$

Since $(x, I_\lambda f(x))(I_\lambda f(x))^{-1} \leq m$, $0 < \delta \leq r_0$. Then by Lemma 6.5(b)

$$\frac{1}{\delta} \leq CT(x, I_\lambda f(x))^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)} \leq CT(x, 1/r_0)^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)} \leq C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}.$$

Hence, using $(\Gamma_{\log})$ and Lemma 6.8, we obtain

$$\Gamma \left( x, \frac{1}{\delta} \right) = \Gamma(x, C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}) \leq CT(x, I_\lambda f(x)).$$

By Lemma 6.8 again, we see that there exists a constant $C_2^* > 0$ independent of $x$ such that

(6.7) \hspace{1cm} Jf(x) \leq C_2^* \Gamma \left( x, \frac{1}{2C_{I,\lambda}A_3^*} I_\lambda f(x) \right) \quad \text{if } 1/r_0 < I_\lambda f(x) < \infty,

where $C_{I,\lambda}$ is the constant given in Lemma 6.7.

Now, let $C^* = A_1^* A_2^* \max(C_1^*, C_2^*)$. Then, by (6.6) and (6.7),

$$\frac{Jf(x)}{C^*} \leq \frac{1}{A_1^* A_2^*} \max \left\{ \Gamma \left( x, \frac{1}{2A_3^*} \right), \Gamma \left( x, \frac{1}{2C_{I,\lambda}A_3^*} I_\lambda f(x) \right) \right\}$$

whenever $I_\lambda f(x) < \infty$. Since $I_\lambda f(x) < \infty$ for a.e. $x \in G$ by Lemma 6.7, $Jf(x)/C^* \leq \gamma(x)$ a.e. $x \in G$, and by (A2) and (A3), we have

$$\Lambda \left( x, \frac{Jf(x)}{C^*} \right) \leq \max \left\{ \Lambda \left( x, \Gamma \left( x, \frac{1}{2A_3^*} \right) /A_2^* \right), \Lambda \left( x, \Gamma \left( x, \frac{1}{2C_{I,\lambda}A_3^*} I_\lambda f(x) \right) /A_2^* \right) \right\} \leq \frac{1}{2} + \frac{1}{2C_{I,\lambda}} I_\lambda f(x)$$

for a.e. $x \in G$. Thus, noting that $\omega(x, r) \leq 1$ and using Lemma 6.7, we have

$$\frac{\omega(x, r)}{|B(x, r)|} \int_{B(x, r) \cap G} \Lambda \left( x, \frac{Jf(x)}{C^*} \right) \ dx \leq \frac{\omega(x, r)}{2C_{I,\lambda}} I_\lambda f(x) \ dx \leq \frac{1}{2} + \frac{1}{2} = 1$$

for all $z \in G$ and $0 < r < d_G$. \qed
Remark 6.11. If $\Gamma(x, s)$ is bounded, that is,
\[ \sup_{x \in G} \int_{0}^{d_G} \rho^{N} \Phi^{-1} (x, \kappa(x, \rho)^{-1}) \eta \left((\log(e + 1/\rho))^{-1}\right) \frac{d(-\mathcal{J}(x, \cdot))}{\rho} < \infty, \]
then by Lemma 6.6 we see that $J[f]$ is bounded for every $f \in \widetilde{L}_{\eta, \xi}^{\Phi, \kappa}(G)$. In particular, if $\omega_{N-\varsigma}(x, r)^{-1}$ is bounded, that is,
\[ \sup_{x \in G} \int_{0}^{d_G} \rho^{N-\varsigma} \Phi^{-1} (x, \kappa(x, \rho)^{-1}) \eta \left((\log(e + 1/\rho))^{-1}\right) \frac{d\rho}{\rho} < \infty, \]
then $\Gamma(x, s)$ is bounded by (J3'), and hence $J[f]$ is bounded for every $f \in \widetilde{L}_{\eta, \xi}^{\Phi, \kappa}(G)$.

If we further assume a continuity of the potential kernel $J$ like condition (J5) in our paper [20], then we can show a continuity of $Jf$ for $f \in \widetilde{L}_{\eta, \xi}^{\Phi, \kappa}(G)$, as in [20, Theorem 5.3].

Applying Theorem 6.10 to special $\Phi$, $\kappa$ and $J$, we obtain the following corollary:

Corollary 6.12. Let $\kappa(x, r)$ and $\alpha(x)$ be as in Examples 2.2 and 5.4 and let $p(x)$ and $q(x)$ be as in Examples 2.1. Set $\eta(t) = t^\theta$ for $\theta > 0$, $\Phi(x, t) = t^{p(x)}(\log(e + t))^{q(x)}$ and
\[ I_{\alpha(x)} f(x) = \int_{G} |x - y|^{-N} f(y) dy \]
for a nonnegative locally integrable function $f$ on $G$. Assume that
\[ \alpha(x) - \nu(x)/p(x) = 0 \quad \text{for all } x \in G. \]

(1) Suppose that
\[ \inf_{x \in G} (-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) > 0. \]

Then for $0 < \lambda < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that
\[ \frac{p^{\nu(z)/p(z) - \lambda}}{|B(z, r)|} \int_{B(z, r) \cap G} \exp \left( \frac{I_{\alpha(z)} f(x)}{C^*} \right) \frac{dx}{(p(x) + \theta p(x) - \beta(x) - q(x))^{p(x)/p(x) - \beta(x) - q(x)}} \leq C^{**} \]
for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}_{\eta, \xi}^{\Phi, \kappa}(G)$ with $\|f\|_{\Phi, \kappa; \eta, \xi; G} \leq 1$.

(2) If
\[ \sup_{x \in G} (-q(x)/p(x) - \beta(x)/p(x) + \theta + 1) \leq 0, \]

then for $0 < \lambda < \alpha^-$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that
\[ \frac{p^{\nu(z)/p(z) - \lambda}}{|B(z, r)|} \int_{B(z, r) \cap G} \exp \left( \frac{I_{\alpha(z)} f(x)}{C^*} \right) \frac{dx}{(p(x) + \theta p(x) - \beta(x) - q(x))^{p(x)/p(x) - \beta(x) - q(x)}} \leq C^{**} \]
for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}_{\eta, \xi}^{\Phi, \kappa}(G)$ with $\|f\|_{\Phi, \kappa; \eta, \xi; G} \leq 1$.

Proof. In the present situation, we see that
\[ \Gamma(x, s) \sim \begin{cases} (\log(e + s))^{-q(x)/p(x) - \beta(x)/p(x) + \theta + 1} & \text{in case (1),} \\
\log(\log(e + s)) & \text{in case (2)} \end{cases} \]
for all $x \in G$ and $s \geq 1/r_0 = 2/d_G$. Hence, we may take
\[ \Lambda(x, t) = \begin{cases} \text{exp}\left(\frac{p(x)(p(x) + \theta p(x) - q(x) - \beta(x))}{p(x) + \theta p(x) - \beta(x) - q(x)}\right) & \text{in case (1),} \\
\text{exp}(\text{exp} t) & \text{in case (2).} \end{cases} \]
On the other hand,
\[ \omega_N(z, r) \sim r^{\nu(z)/p(z)-\lambda'} (\log(e + 1/r))^{-q(x)/p(x)-\beta(x)/p(x)+\theta} \]
for all \( z \in G, 0 < s < d_G \) and \( 0 < \lambda' < \alpha^- \), so that
\[ r^{\nu(z)/p(z)-\lambda} \leq C \omega_N(z, r) \]
if \( 0 < \lambda < \lambda' < \alpha^- \). Thus, given \( 0 < \lambda < \alpha^- \), Theorem 6.10 implies the required results.

□

References


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