A NEW APPROACH TO THE CORONA THEOREM FOR DOMAINS BOUNDED BY A $C^{1+\alpha}$ CURVE

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Abstract. We prove the corona theorem for domains whose boundary lies in a $C^{1+\alpha}$ curve. For that, we transfer $H^\infty$ on the complement of the curve onto a Denjoy domain and use the results from Garnett and Jones.

Introduction

Let $\Gamma$ be an unbounded $C^{1+\alpha}$ curve analytic at $\infty$, $E$ a compact subset of this curve with positive length and set $\Omega = C^* \setminus E$. Let us denote the space of bounded analytic functions on $\Omega$ by $H^\infty(\Omega)$. The corona theorem for this type of domains was already proved by Moore in [7]. The purpose of this paper is to present a new approach to this result.

**Theorem 1.** Let $f_1, f_2, \ldots, f_n \in H^\infty(\Omega)$ so that $\delta \leq \max |f_k(\omega)| \leq 1$, for all $\omega \in \Omega$ and some $\delta > 0$. Then, there exist $g_1, g_2, \ldots, g_n \in H^\infty(\Omega)$ such that $f_1g_1 + f_2g_2 + \ldots + f_ng_n = 1$ on $\Omega$.

The functions $\{f_k\}_{k=1}^n$ and $\{g_k\}_{k=1}^n$ are called corona data and corona solutions respectively, and $\delta$ and $n$ are the corona constants. When $\Gamma$ is the real line, the domain $\Omega$ is called a Denjoy domain. In this case, the theorem was proved by Garnett and Jones [5].

The first corona problem for simply connected domains was solved by Carleson in 1962 [1]. Since then, the result has been extended to some classes of infinitely connected domains, in particular to domains whose boundary lies in a Lipschitz graph and satisfies a thickness condition [8] or complements of Cantor sets [6].

For our approach, we will apply the following result proved in [2] which allows us to transfer the problem in $\Omega$ to a Denjoy domain.

**Theorem 2.** Let $\Gamma$ be an unbounded $C^{1+\alpha}$ curve analytic at $\infty$, and let $\rho$ denote a conformal map of $\mathbb{R}^2$ onto any of the regions bounded by $\Gamma$. Then, given a function $g \in L^\infty(\Gamma)$, the Cauchy integral $C_\Gamma(g) \in L^\infty(\mathbb{C})$ if and only if $C_R(f) \in L^\infty(\mathbb{C})$, where $f$ denotes the pullback of $g$ under the conformal mapping $\rho$.

This transfer is possible thanks to the existence of a quasiconformal extension of $\rho$ whose complex dilatation, $\mu$, verifies that $|\mu|^2/|y|^{1+\varepsilon} \, dx \, dy$ is a Carleson measure.

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relative to $\mathbf{R}$ for some $\varepsilon = \varepsilon(\alpha) > 0$. In fact, the existence of such an extension characterizes $C^{1+\alpha}$ curves [2, Theorem 1].

The paper is structured as follows: In section 1, we review some definitions and basic facts. The proof of Theorem 1 is presented in section 2.

1. Preliminaries

Let us denote complex variables by $z = x + iy$ and $\omega = \xi + i\eta$. $B_r(z)$ will denote the ball centered at $z$ and radius $r$ and $C$ will represent a positive constant that could be different throughout an inequality. Also, we shall write $\bar{\partial} = \partial/\partial \bar{z} = 1/2(\partial_x + i\partial_y)$ and $\partial = \partial/\partial z = 1/2(\partial_x - i\partial_y)$. For a square $Q$, we will denote by $\alpha Q$, $\alpha > 0$, the dilation of this square by a scale factor $\alpha$ and by $l(Q)$ its length.

A Jordan curve $\Gamma$ is said to be of class $C^n$ $(n = 1, 2, \ldots)$ if it has a parametrization $\varphi(\tau) = f(e^{i\tau}), 0 \leq \tau \leq 2\pi$, that is $n$ times continuously differentiable and satisfies that $\varphi'(\tau) \neq 0, \forall \tau$. Furthermore, it is of class $C^{n+\alpha}$, for $0 < \alpha < 1$, if

$$|\varphi^{(n)}(\tau_1) - \varphi^{(n)}(\tau_2)| \leq C|\tau_1 - \tau_2|^\alpha.$$ 

Given a function $F$ on $\Gamma$ define its Cauchy integral $f(z) = C_\Gamma(F)(z)$ off $\Gamma$ by

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta, \quad \zeta \notin \Gamma.$$ 

We define the jump of $f = C\Gamma(F)$ across $\Gamma$ at a point $z$, $J(f)(z)$, as $f_+(z) - f_-(z)$, where $f_+$ and $f_-$ denote the boundary values of $f$. As the classical Plemelj formula states,

$$f_\pm(z) = \pm \frac{1}{2} F(z) + \frac{1}{2\pi i} \text{P.V.} \int_\Gamma \frac{F(\omega)}{\omega - z} d\omega, \quad z \in \Gamma.$$ 

Hence $f_+(z) - f_-(z) = F(z)$. Also, $f$ is holomorphic off $\Gamma$ so that $\bar{\partial} f = 0$ on $\mathbf{C}\setminus\Gamma$.

A positive measure $\lambda$ on $\mathbf{C}$ is called a Carleson measure relative to a given chord-arc curve $\Gamma$ if there exists a constant $C > 0$ such that $\lambda(B_R(z)) \leq CR$ for all $z \in \Gamma$ and $R > 0$. The smallest such $C$ is the norm of $\lambda$, $\|\lambda\|_C$. Furthermore, if

$$\lim_{r \to 0} \sup_{R < r} \frac{\lambda(B_z(R))}{R} = 0,$$

then we say that $\lambda$ is a vanishing Carleson measure or that it satisfies a $o(1)$-Carleson condition.

2. Proof of the Theorem

Let $\Omega_+$ and $\Omega_-$ be the two regions bounded by the $C^{1+\alpha}$ curve $\Gamma$ and $\rho$ be a conformal map from $\mathbf{R}^2_-$ onto $\Omega_-$. It was proved in [2] that $\rho$ extends to a global quasi-conformal map whose dilatation $\mu$ satisfies that $\nu = |\mu|^2/|y|^{1+\varepsilon} dx\,dy$ is a Carleson measure relative to $\mathbf{R}$ where $\varepsilon = \varepsilon(\alpha)$. In fact, for this extension, it holds that $|\partial \rho(z)| \simeq |\rho'(\bar{z})|$ if $0 < \text{Im}(z) < \varepsilon_0$ for some $\varepsilon_0 = \varepsilon_0(\alpha)$ small enough [2, Proof of Theorem 1].

Besides, since $\Gamma$ is analytic at $\infty$, we will assume that $\mu$ has compact support. We will keep the notation fixed for the rest of the proof, that is, $\rho$ is a quasi-conformal mapping associated to $\Gamma$, $\mu$ is its complex dilatation and $\varepsilon$ is such that $\nu$ is a Carleson measure.
Let \( E_0 = \rho^{-1}(E) \subset \mathbb{R} \) and \( \Omega_0 = \mathbb{C} \setminus E_0 \). Note that \( E_0 \) is closed and has positive length ([9], Theorem 6.8). Define the space
\[
H^\infty(\Omega_0, \mu) = \{ f \circ \rho : f \in H^\infty(\Omega) \}.
\]

Observe that if \( g = f \circ \rho \in H^\infty(\Omega_0, \mu) \), then \( \bar{\partial} f = 0 \) on \( \Omega \) translates into \( (\bar{\partial} - \mu \partial) g = 0 \) on \( \Omega_0 \), and as well, the jump of \( g \) across \( E_0 \) is given by \( j(g) = j(f) \circ \rho \). Also, as \( \Gamma \) is a \( C^{1+\alpha} \) curve, \( \lambda = |y| |\partial g|^2 \, dx \, dy \) is a Carleson measure relative to \( \mathbb{R} \) [2, Proof of Theorem 2].

Before proving the corona theorem, we need some preliminary lemmas.

**Lemma 2.1.** If \( g \in H^\infty(\Omega_0, \mu) \), then \( \tau = |\mu||\partial g| \, dx \, dy \) is a vanishing Carleson measure.

**Proof.** For any \( s \in \mathbb{R}, r > 0 \):
\[
\int_{B_r(s)} \frac{|\mu(z)|^2}{|y|} \, dx \, dy = \int_{B_r(s)} \left( \frac{|\mu(z)|^2}{|y|^{1+\varepsilon}} \right) \, dx \, dy \lesssim \|\nu\|_C r^{1+\varepsilon}.
\]
Therefore,
\[
\int_{B_r(s)} |\mu(z)||\partial g(z)| \, dx \, dy \leq \left( \int_{B_r(s)} \frac{|\mu(z)|^2}{|y|} \, dx \, dy \right)^{1/2} \left( \int_{B_r(s)} |\partial g(z)|^2 \, |y| \, dx \, dy \right)^{1/2}
\]
(1)
\[
\lesssim \|\nu\|^{1/2} C \|\lambda\|^{1/2} r^{1+\varepsilon/2},
\]
and \( \tau = |\mu||\partial g| \, dx \, dy \) is a vanishing Carleson measure relative to \( \mathbb{R} \). \( \square \)

**Lemma 2.2.** There exists \( \varepsilon_0 > 0 \) such that if \( g \in H^\infty(\Omega_0, \mu) \) and \( z \in \Omega_0 \) with \( 0 < |\text{Im}(z)| < \varepsilon_0 \), then \( |y||\partial g(z)| < C \), where \( C = C(\|g\|_\infty, \|\mu\|_\infty) \) and \( \varepsilon_0 = \varepsilon_0(\alpha) \).

**Proof.** Let \( f \in H^\infty(\Omega) \) such that \( g = f \circ \rho \). Then, \( \delta_\mu(\omega)|f'(\omega)| \leq C \), \( \forall \omega \in \mathbb{C} \setminus \Gamma \) and \( C = C(\|f\|_\infty) \).

Let \( z \in \mathbb{R}^2 \) and \( \omega = \rho(z) \). Since \( \rho \) is conformal on \( \mathbb{R}^2 \), by Koebe’s distortion theorem,
\[
|y||\partial g(z)| = |y||f'(\rho(z))||\rho'(z)| \approx \delta_\rho(\omega)|f'(\omega)| \leq C.
\]
If \( z \in \mathbb{R}^2_+ \), as we mentioned before, we can choose \( \varepsilon_0 \) so that, if \( 0 < |\text{Im}(z)| < \varepsilon_0 \) then, \( |\partial \rho(z)| \approx |\rho'(z)| \). Hence, as above,
\[
|y||\partial g(z)| = |y||\partial \rho(z)||f'(\rho(z))| \approx \delta_\rho(\rho(z))|f'(\rho(z))|.
\]
By the distortion theorem for quasiconformal mappings \( \delta_\rho(\rho(z)) \approx \delta_\rho(z) \) with comparison constants depending on \( \|\mu\|_\infty \), which concludes the proof. \( \square \)

Before stating the next lemma, we will review some facts already developed in [2] which follow Semmes’s approach in [10]. Let \( g \in H^\infty(\Omega_0, \mu) \), then \( g = f \circ \rho \) for some \( f \in H^\infty(\Omega) \). Consider now the jump of \( g, j(g) \), and set \( \tilde{g} = C_\mathbb{R}(j(g)) \). If we define \( G = g - \tilde{g} \), then \( \partial G = \mu \partial g \) on \( \Omega_0 \) and since \( G \) has no jump across \( E_0 \), we can consider that this equation holds on all \( \mathbb{C} \) in the sense of distributions. We can then apply Cauchy’s formula to obtain
\[
G(z_0) = \frac{1}{\pi i} \int_{\mathbb{C}} \frac{\tilde{g}(z)}{z - z_0} \, dx \, dy = \frac{1}{\pi i} \int_{\mathbb{C}} \frac{\mu(z)\partial g(z)}{z - z_0} \, dx \, dy, \quad \text{for all} \ z_0 \in \mathbb{C}.
\]

**Lemma 2.3.** Assume that \( \text{supp}(\mu) \subset Q \) for some \( Q \) centered at a real point with length \( R \leq \varepsilon_0/4 \). Let \( g \in H^\infty(\Omega_0, \mu) \) and \( \tilde{g} \in H^\infty(\Omega_0) \) so that \( j(g) = j(\tilde{g}) \) and set \( G = g - \tilde{g} \). Then for all \( z \in \mathbb{C}, |G(z)| \leq CR^{\varepsilon/(2+\varepsilon)} \), where \( C = C(\|g\|_\infty, \|\nu\|_C) \).
Therefore, we obtain by lemma 2.2:

\[ |G(z_0)| \lesssim \int_{Q_0} \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy = \int_Q \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy \]

(2)

where \( Q_0 \) is the square centered at \( z_0 \) and length \( l(Q_0) = |y_0| \). To bound the first integral in (2), set \( p = 2 + \varepsilon \) and \( q = (2 + \varepsilon)/(1 + \varepsilon) \). Then

\[ \int_{Q_0} \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy \leq \left( \int_{Q_0} |\mu(z)\partial g(z)|^{2+\varepsilon} \, dx \, dy \right)^{\frac{1}{2+\varepsilon}} \left( \int_{Q_0} |z - z_0|^{-\frac{2+\varepsilon}{1+\varepsilon}} \, dx \, dy \right)^{\frac{1+\varepsilon}{2+\varepsilon}}. \]

(3)

As \( \nu = |\mu|^2/|y|^{1+\varepsilon} \) is a Carleson measure relative to \( R \) and \( |y| \geq |y_0|/2 \) for \( z \in Q_0 \), we obtain by lemma 2.2:

\[ \int_{Q_0} |\mu(z)\partial g(z)|^{2+\varepsilon} \, dx \, dy \lesssim \int_{Q_0} \frac{|\mu(z)|^{2+\varepsilon}}{|y|^{2+\varepsilon}} \, dx \, dy \]

\[ \lesssim \frac{2}{|y_0|} \int_{2Q_0} \frac{|\mu(z)|^2}{|y|^{1+\varepsilon}} \, dx \, dy \leq 4\|\nu\|_C. \]

(4)

Let us now consider \( B_0 = B_r(z_0) \) so that \( r \simeq |y_0| \) and \( Q_0 \subset B_0 \). By changing variables to polar coordinates,

\[ \int_{Q_0} |z - z_0|^{-\frac{2+\varepsilon}{1+\varepsilon}} \, dx \, dy \leq \int_{B_0} |z - z_0|^{-\frac{2+\varepsilon}{1+\varepsilon}} \, dx \, dy \leq C(\varepsilon) r^{\frac{\varepsilon}{1+\varepsilon}} \simeq C(\varepsilon)|y_0|^\frac{\varepsilon}{1+\varepsilon}. \]

(5)

Therefore, by (3), (4) and (5)

\[ \int_{Q_0} \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy \lesssim C(\|\nu\|_C, \varepsilon)|y_0|^\varepsilon \lesssim C(\|\nu\|_C, \varepsilon)R^{\varepsilon}. \]

(6)

To bound the second integral in (2), consider an open cover of \( Q \setminus Q_0 \) with squares, \( Q_i \), centered at \( z_0 \) and length \( l(Q_i) = 2^i|y_i| \), \( i \geq 1 \). Note that it is sufficient a cover with \( M \) squares such that \( M \lesssim \log_2(R/|y_0|) \). Then, by (1)

\[ \int_{Q \setminus Q_0} \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy \lesssim \sum_{i=1}^M 2^{-i} \int_{Q \setminus Q_{i-1}} |\mu(z)\partial g(z)| \, dx \, dy \]

\[ \lesssim \sum_{i=1}^M 2^{-i} (2^i|y_i|)^{1+\varepsilon/2} \lesssim |y_0|^{\varepsilon/2} (2^i)^{M} \lesssim R^{\varepsilon/2}. \]

(7)

Therefore, by (2), (6) and (7), \( |G(z_0)| \lesssim C(\|\nu\|_C, \|\lambda\|_C, \varepsilon)R^{\varepsilon/(2+\varepsilon)} \).

For \( z_0 \in (2Q \cap R) \), let \( Q^i \) be the square centered at \( z_0 \) and length \( l(Q^i) = 2^{2-i}R \), \( i \geq 0 \). Since \( \partial G = \mu \partial g \) and \( \text{supp}(\mu) \subset Q \), by (1)

\[ |G(z_0)| \lesssim \int_Q \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy = \sum_{i \geq 0} \int_{Q \setminus Q^{i+1}} |\mu(z)\partial g(z)| \, dx \, dy \]

\[ \lesssim \frac{1}{R} \sum_{i \geq 0} 2^i \int_{Q_i} |\mu(z)\partial g(z)| \, dx \, dy \lesssim \frac{1}{R} \sum_{i \geq 0} 2^i l(Q^i)^{1+\varepsilon/2} \lesssim R^{\varepsilon/2}. \]

Therefore, \( |G(z_0)| \leq CR^{\varepsilon/2} \) for \( C = C(\|\mu\|_C, \|\lambda\|_C, \varepsilon) \).
Finally, let \( z_0 \in \mathbb{C} \setminus 2Q \). Then, by (1)
\[
|G(z_0)| \lesssim \int_Q \frac{|\mu(z)\partial g(z)|}{|z - z_0|} \, dx \, dy \leq \frac{1}{R} \int_Q |\mu(z)\partial g(z)| \, dx \, dy
\]

\[
\leq C(\|\nu\|C, \|\lambda\|C) R^{-1/2}.
\]

We now prove Theorem 1:

**Theorem 1.** Let \( f_1, f_2, \ldots, f_n \in H^\infty(\Omega) \) so that \( \delta \leq \max_j |f_j(\omega)| \leq 1 \), for all \( \omega \in \Omega \) and some \( \delta > 0 \). Then, there exist \( g_1, g_2, \ldots, g_n \in H^\infty(\Omega) \) such that \( f_1g_1 + f_2g_2 + \ldots + f_ng_n = 1 \) on \( \Omega \).

**Proof.** Garnel and Gamelin [3] showed that it is sufficient to prove it locally, that is, for \( \zeta \in \Gamma \) there exists a neighborhood of \( \zeta \) on which it is true and such that the size of this neighborhood is determined by \( \delta, n \) and other parameters concerning \( \Gamma \) (see also [4, p. 358]).

We can then assume that \( \mu(z) = 0 \) outside a square \( Q \) centered at a real point with length \( R \), for a small enough \( R = R(n, \delta, \Gamma) \) to be determined later. To see this, consider the solution \( \tilde{\rho} \) of the Beltrami equation \( \partial \tilde{\rho} = \mu \partial \tilde{\rho} \) for \( z \in Q \), \( \partial \tilde{\rho} = 0 \) otherwise. Then, \( \rho = F \circ \tilde{\rho} \) where \( F \) is an univalent function in the region \( \tilde{\rho}(Q) \), and therefore it will be enough to prove the corona theorem for the domain \( \Omega = C \setminus \tilde{\rho}(E_0) \).

Since the dilatation coefficient \( \tilde{\mu} = \mu \chi_Q \) obviously satisfies that \( |\tilde{\mu}|^2/|y|^{1+\epsilon} \, dx \, dy \) is a Carleson measure, we know that \( \tilde{\Gamma} = \tilde{\rho}(\mathbb{R}) \) is also a \( C^{1+\delta} \) curve for \( \tilde{\alpha} = \tilde{\alpha}(\alpha, \|\mu\|_\infty) \) ([2], Theorem 1) and therefore all the previous lemmas apply if we replace \( \Gamma, \mu \) and \( \rho \) by the corresponding \( \tilde{\Gamma}, \tilde{\mu} \) and \( \tilde{\rho} \). To avoid excessive use of notation, we will drop the tilde notation.

Let \( f_k^* = f_k \circ \rho \) on \( \Omega_0 \). Then, the jump of \( f_k^* \) across \( E_0 \) is indeed the pullback of \( j(f_k) \) under the mapping \( \rho \), that is, \( j(f_k^*) = j(f_k) \circ \rho \). Note that \( f_1^*, \ldots, f_n^* \in H^\infty(\Omega_0, \mu) \).

Set \( \tilde{f}_k = C_\mathbb{R}(j(f_k^*)) \). By Theorem 2, \( \tilde{f}_k \in H^\infty(\Omega_0) \). First, we want to show that \( \tilde{f}_k \) are corona data in \( \Omega_0 \). So, let \( G_k = f_k^* - \tilde{f}_k \) and \( z_0 \in \Omega_0 \). Then, there exists \( 1 \leq j \leq n \) such that \( \delta \leq |f_j^*(z_0)| \leq |G_j(z_0)| + |\tilde{f}_j(z_0)| \). By lemma 2.3, \( |G_j(z_0)| \leq R^{1/(2+\epsilon)} \leq \delta/2 \) for a sufficiently small \( R \) and therefore \( \delta/2 \leq |\tilde{f}_j(z_0)| \).

According to Garnett and Jones’ theorem for Denjoy domains [5], there exist \( \tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n \in H^\infty(\Omega_0) \) such that \( \tilde{f}_1 \tilde{h}_1 + \ldots + \tilde{f}_n \tilde{h}_n = 1 \) with \( ||\tilde{h}_k||_\infty \leq C(n, \delta) \).

Define \( H_k^* = j(\tilde{h}_k) \). Then, \( H_k^* \in L^\infty(\mathbb{R}) \) and \( \tilde{h}_k = C_\mathbb{R}(H_k^*) \). Set \( H_k = H_k^* \circ \rho^{-1} \) on \( \Gamma \) and define \( h_k = C_\Gamma(H_k) \). Although \( \{h_k\}_{k=1}^n \subset H^\infty(\Omega) \) by Theorem 2, they are not corona solutions as they do not verify that \( f_1h_1 + f_2h_2 + \ldots + f_nh_n = 1 \) on \( \Omega \). They clearly satisfy that \( \sum g_jf_j = 1 \). We just need to prove that \( f_1, f_2, \ldots, f_n \) are also bounded. For that, it is sufficient to show that \( \sum f_kh_k \) is close to 1.

Let us denote \( h_k^* = h_k \circ \rho \in H^\infty(\Omega_0, \mu) \). Note that \( j(h_k^*) = j(\tilde{h}_k) \). For any \( z \in \Omega_0 \) and by lemma 2.3:

\[
|\sum_{k=1}^n f_k(\rho(z))h_k(\rho(z)) - 1| = |\sum_{k=1}^n f_k^*(z)h_k^*(z) - \sum_{i=1}^n \tilde{f}_k(z)\tilde{h}_k(z)|
\]

\[
\leq \sum_{k=1}^n |f_k^*(z)||h_k^*(z) - \tilde{h}_k(z)| + \sum_{k=1}^n |\tilde{h}_k(z)||f_k(z) - \tilde{f}_k(z)|
\]
(8) \[ \lesssim nR^{\varepsilon/(2+\varepsilon)} + nC(n, \delta)R^{\varepsilon/(2+\varepsilon)} \leq 1/2 \]

for a sufficiently small \( R \).

As a final remark, this new approach encourages us to find solutions to the corona problem for domains bounded by other quasicircles. For that, one would need to find conditions on \( \mu \) so that we can transfer \( H^\infty \) on the complement of a curve onto the corresponding Denjoy domain.

References


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