FALCONER DISTANCE PROBLEM, ADDITIVE ENERGY AND CARTESIAN PRODUCTS

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Abstract. A celebrated result due to Wolff says if $E$ is a compact subset of $\mathbb{R}^2$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x - y|: x, y \in E\}$ is positive if the Hausdorff dimension of $E$ is greater than $\frac{3}{2}$. In this paper we improve the $\frac{3}{2}$ barrier by a small exponent for Cartesian products. In higher dimensions, also in the context of Cartesian products, we reduce Erdogan’s $d^2 + \frac{1}{3}$ exponent to $d^2 - \frac{1}{d-1}$. The proof uses a combination of Fourier analysis and additive combinatorics.

1. Introduction

The Falconer distance conjecture [3] says that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{d}{2}$, then the Lebesgue measure of the distance set $\Delta(E) = \{|x - y|: x, y \in E\}$ is positive.

The best known results are due to Wolff [7] in two dimensions and Erdogan [2] in higher dimensions. They proved that the Lebesgue measure of $\Delta(E)$ is positive if the Hausdorff dimension of $E$ is greater than $\frac{d}{2} + \frac{1}{3}$. This was accomplished by showing that if $s \in \left(\frac{d}{2}, \frac{d+2}{3}\right)$ is the Hausdorff dimension of $E$ and $\mu$ is a Frostman measure on $E$ which has finite $(s - \epsilon)$-energy $I_{s-\epsilon}(\mu)$ for all $\epsilon > 0$, then for all $\epsilon > 0$,

$$\int_{S^{d-1}} |\hat{\mu}(t\omega)|^2 \, d\omega \leq Ct^{-\frac{d+2-2s}{d-1} + \epsilon}. \quad (1.1)$$

In particular, in the two-dimensional case, which is the focus of this paper, the estimate takes the form

$$\int_{S^1} |\hat{\mu}(t\omega)|^2 \, d\omega \leq Ct^{-\frac{d}{2} + \epsilon}. \quad (1.2)$$

This estimate is then plugged into the Mattila integral,

$$\mathcal{M}(\mu) = \int_1^\infty \left( \int_{S^{d-1}} |\hat{\mu}(t\omega)|^2 \, d\omega \right)^2 \, t^{d-1} \, dt, \quad (1.3)$$

the most effective tool developed so far for the study of the Falconer distance problem.

Mattila proved in [5] that if $E$ is a compact set of Hausdorff dimension $s > \frac{d}{2}$ and $\mu$ is Borel measure supported on $E$ such that $\mathcal{M}(\mu) < \infty$, then the Lebesgue measure of $\Delta(E)$ is positive. For discussion about other versions of Mattila integrals, see [4].

Our results are the following.
Theorem 1.1. Let $E = A \times B$, where $A$ and $B$ are compact subsets of $\mathbb{R}$ with positive $s_A, s_B$-dimensional Hausdorff measure, respectively. If $s_A + s_B + \max(s_A, s_B) > 2$, the Lebesgue measure of $\Delta(E)$ is positive. In particular, if $\dim_H(E) = \dim_H(A) + \dim_H(B)$ and $\dim_H(A) \neq \dim_H(B)$, $\dim_H(E) > \frac{4}{3} - \frac{|\dim_H(A) - \dim_H(B)|}{3}$ implies $\Delta(E)$ has positive Lebesgue measure.

To state the result in the case $\dim_H(A) = \dim_H(B)$, we need the following definition.

Definition 1.2. Let $A$ be a compact subset of $\mathbb{R}^d$ of Hausdorff dimension $s_A$. We say $A$ is Ahlfors–David regular if there exists a Radon measure $\nu_A$ on $A$ and a constant $0 < C_{\nu_A} < \infty$ such that

$$C_{\nu_A}^{-1} r^{s_A} < \nu_A(B(x, r)) < C_{\nu_A} r^{s_A}, \ \forall x \in A, \ 0 < r < 1.$$  

Theorem 1.3. Suppose $E = A \times B$, $s_A = s_B = \alpha$ and $A$ is Ahlfors–David regular with $\nu_A$, $C_{\nu_A}$ such that (1.4) holds. Then there exists $\delta = \delta(C_{\nu_A}) > 0$ such that whenever $\alpha > \frac{4}{3} - \delta$, the Lebesgue measure of $\Delta(E)$ is positive.

We also obtain an improvement of Erdogan’s $\frac{d}{2} + \frac{1}{3}$ exponent in higher dimension for Cartesian products.

Theorem 1.4. Suppose that $E$ is a compact subset of $\mathbb{R}^d$ of the form $A_1 \times A_2 \times \cdots \times A_d$, where $A_j \subset \mathbb{R}$ has positive $s_j$-dimensional Hausdorff measure for all $1 \leq j \leq d$. Suppose that $\sum_{j=1}^d s_j > \frac{d^2}{2d-1}$. Then the Lebesgue measure of $\Delta(E)$ is positive.

1.1. Outline of the proof of Theorems 1.1, 1.3 and 1.4. Our argument consists of three basic steps.

- We first establish Theorem 1.1 which is accomplished using the imbalance inherent in the structure of the Mattila integral.
- The improvement of the $\frac{d}{2} + \frac{1}{3}$ exponent for Cartesian products in higher dimensions (Theorem 1.4) is accomplished in the same way regardless of whether the Hausdorff dimension of the fibers is the same.
- In the case when $\dim_H(A) = \dim_H(B)$, we use a recent result due to Dyatlov and Zahl [1] to show that when $A$ is Ahlfors–David regular, the additive energy of $A$ at scale $t^{-1}$,

$$\nu_A^{-1} \{(a_1, a_2, a_3, a_4) : |(a_1 + a_2) - (a_3 + a_4)| \leq t^{-1}\},$$

where $\nu_A$ is a Frostman measure on $A$, satisfies a better than trivial estimate, namely $C t^{-\dim_H(A) - \delta}$ for some $\delta > 0$, and then show that this leads to a slightly better exponent than $\frac{4}{3}$.

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2. Proof of Theorem 1.1

We shall repeatedly use the following simple estimate.

Lemma 2.1. (Solid Average) Suppose that $\nu$ is a compactly supported Borel measure on $\mathbb{R}^d$ such that $\nu(B(x, r)) \leq C r^\alpha$ for all $x \in \mathbb{R}^d$. Then for any bounded rectangle $R$,

$$\int_R |\hat{\nu}(tu)|^2 \, du \leq C_R t^{-\alpha}.$$
To prove the lemma observe that the left hand side is

\[ \leq \int |\widehat{\nu}(u)|^2 \hat{\psi}(u) \, du, \]

where \( \psi \) is a suitably chosen smooth compactly supported function. This expression equals

\[ \iint e^{2\pi i (x-y) \cdot tu} \hat{\psi}(u) \, du \, d\nu(x) \, d\nu(y) = \iint \psi(t(x-y)) \, d\nu(x) \, d\nu(y) \leq C R^{k-\alpha} \]

by assumption. This completes the proof of the lemma.

We now parameterize the upper semi-circle \( S^+_1 \) in the form

\[ \big\{(u, \sqrt{1-u^2}): -1 \leq u \leq 1\big\}. \]

The argument shall be carried out for this parameterization as the proof for the lower semi-circle is identical.

Let \( d\mu(x) = d\nu_A(x_1) \, d\nu_B(x_2) \), where \( \nu_A, \nu_B \) are Frostman probability measures on \( A \) and \( B \), respectively such that

\[ \nu_A(B(x, r)) \leq C r^{s_A}, \quad \nu_B(B(x, r)) \leq C r^{s_B}. \]

Assume without loss of generality that \( s_A \geq s_B \). Also assume \( E \) is not a point mass, which implies that either

\[ \exists a \in \mathbb{R}, \mu(\{(x_1, x_2): x_1 > a\}), \mu(\{(x_1, x_2): x_1 < a\}) > 0, \]

or

\[ \exists b \in \mathbb{R}, \mu(\{(x_1, x_2): x_2 > b\}), \mu(\{(x_1, x_2): x_2 < b\}) > 0. \]

Without loss of generality, we may assume \( \mu(\{(x_1, x_2): x_2 > b\}), \mu(\{(x_1, x_2): x_2 < b\}) > 0 \) for some \( b \in \mathbb{R} \). It follows that

(2.1) \[ \iint \frac{|x_2 - y_2|}{|x - y|} \, d\mu(x) \, d\mu(y) > 0. \]

Let \( \omega = (\cos(\theta), \sin(\theta)) \). Consider the modified Mattila integral

(2.2) \[ \int \left( \int_{S^1} |\widehat{\mu}(t\omega)|^2 \, |\sin(\theta)| \, d\omega \right)^2 \, t \, dt. \]

**Lemma 2.2.** Suppose (2.1) holds. Then the finiteness of the integral (2.2) implies that the Lebesgue measure of the distance set is positive.

**Proof.** To prove this lemma, one simply replaces the distance measure in the derivation of the Mattila integral in [7], given by the relation

\[ \int f(t) \, d\nu^*_0(t) = \iint f(|x - y|) \, d\mu(x) \, d\mu(y) \]

by a slightly modified distance measure given by

\[ \int f(t) \, d\nu_0(t) = \iint f(|x - y|) \frac{|x_2 - y_2|}{|x - y|} \, d\mu(x) \, d\mu(y). \]

As in [7], define

\[ d\nu(t) = e^{i\frac{\pi}{4}} t^{-\frac{1}{2}} d\nu_0(t) + e^{-i\frac{\pi}{4}} |t|^{-\frac{1}{2}} d\nu_0(-t) \]
and it follows that
\begin{equation}
\hat{\nu}(t) = \int |x - y|^{\frac{1}{2}} \cos(2\pi(t|x - y| - \frac{1}{8})) \frac{|x_2 - y_2|}{|x - y|} d\mu(x) d\mu(y).
\end{equation}

On the other hand,
\begin{equation}
\int |\hat{\mu}(\omega)|^2 \sin \theta \, d\theta = \int \left( \int e^{2\pi i(x - y) \cdot (\omega)} |\sin \theta| \, d\theta \right) d\mu(x) d\mu(y).
\end{equation}

Let $\theta_{x-y}$ be the angle between the vector $x - y$ and the $x$-axis. Then $|\sin \theta_{x-y}| = \frac{|x_2 - y_2|}{|x - y|}$. We may assume $s$, the Hausdorff dimension of $E$, is not greater than $\frac{3}{2}$. By stationary phase (see, e.g. [8] for details),
\begin{equation}
\int e^{2\pi i(x - y) \cdot (\omega)} |\sin \theta| \, d\theta
= 2(|t||x - y|)^{-\frac{1}{2}} \cos(2\pi(t|x - y| - \frac{1}{8}) |\sin \theta_{x-y}| + O((t|x - y|)^{-\frac{3}{2}})
= 2(|t||x - y|)^{-\frac{1}{2}} \cos(2\pi(t|x - y| - \frac{1}{8}) \frac{|x_2 - y_2|}{|x - y|} + O((t|x - y|)^{-s+\epsilon}).
\end{equation}

Putting (2.3), (2.4), (2.5) together, one can see
\begin{equation}
||\hat{\nu}||^2_2 = \int_{|t| \leq 1} |\hat{\nu}(t)|^2 \, dt + \int_{|t| \geq 1} |\hat{\nu}(t)|^2 \, dt
\leq 1 + \int_1^\infty \left( \int |\hat{\mu}(\omega)|^2 |\sin \theta| \, d\theta \right)^2 \, dt + CI_{s-\epsilon}(\mu),
\end{equation}
which proves the lemma since $I_{s-\epsilon}(\mu) < \infty$. 

We now proceed with the estimation of (2.2). It follows that
\begin{equation}
\int_{S^1} |\hat{\mu}(\omega)|^2 |\sin(\theta)| \, d\omega
\end{equation}
is bounded by the sum of two terms of the form
\begin{equation}
\int_{-1}^1 |\hat{\nu}_A(tu)|^2 \left| \hat{\nu}_B \left( \pm t \sqrt{1 - u^2} \right) \right|^2 \, du \leq \int_{-1}^1 |\hat{\nu}_A(tu)|^2 \, du \leq C t^{-s_A}
\end{equation}
by Lemma 2.1. Plugging (2.6) into the modified Mattila integral (2.2) we see that
\begin{equation}
\mathcal{M}(\mu) \leq C \int \int_{S^1} |\hat{\nu}(\omega)|^2 \cdot t^{-s_A} \, d\omega \, dt = C \int |\hat{\nu}(\xi)|^2 |\xi|^{-s_A} \, d\xi
= C' \int \int |x - y|^{-2 + s_A} \, d\mu(x) \, d\mu(y)
\end{equation}
and this energy integral (see e.g. [8] or [6]) is finite if
\begin{equation}
s_A + s_B > 2 - s_A,
\end{equation}
as desired.
3. Proof of Theorem 1.3

We improve the upper bound of (1.2) to prove the theorem. More precisely, under the assumptions of Theorem 1.3, there exists $\delta = \delta(C_{\nu_A}) > 0$ such that

$$\int_{S^1} |\hat{\mu}(t\omega)|^2 \, d\omega \lesssim t^{-\alpha - \delta},$$

where $\mu = \nu_A \times \nu_B$. First we deal with the case when $\theta$ is close to 0. We have

$$\int_0^\delta |\hat{\nu}_A(t \cos(\theta))|^2 |\hat{\nu}_B(t \sin(\theta))|^2 \, d\theta \leq \int |\hat{\nu}_B(tu)|^2 \hat{\psi}(u/\delta) \, du$$

with an appropriately chosen cut-off function $\psi$. This expression equals

$$\delta \int \hat{\psi}(\delta t(u - v)) \, d\nu_B(u) \, d\nu_B(v) \leq C\delta^{1-\alpha} \cdot t^{-\alpha}.$$

Choosing $\delta = t^{-\gamma_0}$, where $\gamma_0$ is a small positive number to be determined later, we see that the expression in (3.1) is

$$\leq C t^{-\gamma_0(1-\alpha)} \cdot t^{-\alpha}.$$

We can deal with the neighborhood near $\frac{\pi}{2}$ in the same way, so we omit this part of the calculation.

Now consider

$$\int_I |\hat{\nu}_A(t \cos(\theta))|^2 |\hat{\nu}_B(t \sin(\theta))|^2 \, d\theta,$$

where $I$ is an interval that excludes both $(0, t^{-\gamma_0})$ and a fixed neighborhood of $\frac{\pi}{2}$. By Cauchy–Schwartz, this expression (3.3) is bounded by

$$C \left( \int_I |\hat{\nu}_A(t \cos(\theta))|^4 \, d\theta \right)^{\frac{1}{2}} \cdot \left( \int_I |\hat{\nu}_B(t \sin(\theta))|^4 \, d\theta \right)^{\frac{1}{2}}.$$

Making the change of variables $u = \cos(\theta)$ and $v = \sin(\theta)$, respectively, we see that this expression is

$$\leq C t^{\gamma_0} \left( \int |\hat{\nu}_A(tu)|^4 \hat{\psi}(u) \, du \right)^{\frac{1}{2}} \cdot \left( \int |\hat{\nu}_B(tu)|^4 \hat{\psi}(u) \, du \right)^{\frac{1}{2}} = C t^{\gamma_0} \sqrt{I} \cdot \sqrt{II},$$

where $\psi$ is a smooth positive function whose Fourier transform has compact support.

Expanding each expression and changing the order of integration, we obtain

$$I = \int \int \int \psi(t(u_1 - u_2 + u_3 - u_4)) \, d\nu_A(u_1) \, d\nu_A(u_2) \, d\nu_A(u_3) \, d\nu_A(u_4)$$

$$\lesssim \nu_A \times \nu_A \times \nu_A \times \nu_A \{(u_1, u_2, u_3, u_4) \in A^4 : |(u_1 + u_2) - (u_3 + u_4)| < t^{-1}\}$$

and

$$II = \int \int \int \psi(t(u_1 - u_2 + u_3 - u_4)) \, d\nu_B(u_1) \, d\nu_B(u_2) \, d\nu_B(u_3) \, d\nu_B(u_4)$$

$$\lesssim \nu_B \times \nu_B \times \nu_B \times \nu_B \{(u_1, u_2, u_3, u_4) \in B^4 : |(u_1 + u_2) - (u_3 + u_4)| < t^{-1}\}.$$

Observe that we trivially have

$$I \lesssim t^{-\alpha}; \quad II \lesssim t^{-\alpha}.$$

It follows that

$$C t^{\frac{(\alpha - dim_B(A \times B))}{2}} \sqrt{I} \cdot \sqrt{II} \leq C t^{\gamma_0} \leq C t^{\gamma_0} \cdot t^{-\frac{dim_B(A \times B)}{2}},$$
which recovers Wolff’s $\frac{4}{3}$ exponent as $\gamma_0 \to 0$. Moreover, the only way this estimate does not beat $\frac{4}{3}$ is if

$$I, II \geq Ct^{-\alpha + \frac{3}{2}}$$

for a sequence of $t$’s going to infinity. The following theorem due to Dyatlov and Zahl ([1]) shows that this cannot happen for Ahlfors–David regular sets.

**Definition 3.1.** [1, Dyatlov and Zahl] Let $X \subset [0, 1]^d$ and $\nu$ be an outer measure on $X$ with $0 < \nu(X) < \infty$. For $r > 0$, define the (scale $r$) additive energy by

$$E(X, \nu, r) = \nu \times \nu \times \nu \{(u_1, u_2, u_3, u_4) \in X^4 : |(u_1 + u_2) - (u_3 + u_4)| < r\}.$$

**Theorem 3.2.** [1, Dyatlov and Zahl] Let $X \subset [0, 1]$ be an Ahlfors–David regular set of Hausdorff dimension $\alpha$ and $\nu$ be a measure on $X$ such that for some constant $0 < C_\nu < \infty$,

$$C_\nu^{-1}r^\alpha < \nu(B(x, r)) < C_\nu r^\alpha, \forall x \in X, \ 0 < r < 1.$$

Then

$$E(X, \nu, r) \leq \tilde{C} r^{\alpha + \beta_\nu}$$

for some $\beta_\nu > 0$ and some $\tilde{C} > 0$. In particular, we can choose

$$\beta_\nu = \alpha e^{-\exp[K(1+\log C_\nu)^{1/2}(1-\alpha)^{-1/2}]}$$

where $K$ is an absolute constant; $\tilde{C}$ depends only on $\alpha$ and $C_\nu$.

From Theorem 3.2, Definition 1.2 and the trivial estimate (3.6) of $II$, it follows that

$$I \lesssim t^{-\beta_{\nu_A}}, II \lesssim t^{-\alpha},$$

where $\beta_{\nu_A}$ is defined in Theorem 3.2. All implicit constants are finite, independent on $t$. Together with the estimate near $\theta = 0$ (3.2), we can bound (1.2) by

$$Ct^{-\gamma_0(1-\alpha)-\alpha} + Ct^{\frac{\mu}{\alpha}}t^{-\alpha-\gamma},$$

where $\gamma = \gamma(C_{\nu_A}) > 0$. Let $\gamma_0 > 0$ be a small enough, we get

$$\int_{S^1} |\hat{\mu}(t\xi)|^2 \, d\xi \lesssim t^{-\alpha-\delta}$$

for some $\delta = \delta(C_{\nu_A}) > 0$.

4. Proof of Theorem 1.4

Let $\nu_j$ denote the restriction of the $s_j$-dimensional Hausdorff measure to $A_j$ and assume without loss of generality that $s_1 \geq s_2 \geq \cdots \geq s_d$. Parameterize the upper half-sphere in the form

$$\left\{ \left(u_1, u_2, \ldots, u_{d-1}, \sqrt{1 - u_1^2 - \cdots - u_{d-1}^2} \right) : -1 \leq u_j \leq 1 \right\}.$$

Let $\mu$ denote the product measure on $E$, $\theta_\omega$ be the angle between the vector $\omega \in S^{d-1}$ and the hyperplane $\{x_d = 0\}$. Without loss of generality, we may assume

$$\mu(\{(x_1, \ldots, x_d) : x_d > a\}) \mu(\{(x_1, \ldots, x_d) : x_d < a\}) > 0$$

for some $a \in \mathbb{R}$. An argument identical to the one in the proof of Lemma 2.2 shows the finiteness of

$$\int_{1}^{\infty} \left( \int_{S^{d-1}} |\hat{\mu}(t\omega)|^2 |\sin \theta_\omega| \, d\omega \right)^2 t^{d-1} \, dt$$
implies that the distance set has positive Lebesgue measure. It follows that
\[ \int_{S^{d-1}} |\hat{\mu}(t\omega)|^2 |\sin \theta_{\omega}| \, d\omega \]
is bounded by two terms of the form
\[ \int_{-1}^{1} \cdots \int_{-1}^{1} |\hat{\nu}_1(tu_1)|^2 \cdots |\hat{\nu}_{d-1}(tu_{d-1})|^2 \cdot |\hat{\nu}_d (\pm t\sqrt{1 - |u|^2})|^2 \, du_1 \cdots du_{d-1} \]
\[ \leq \int_{-1}^{1} \cdots \int_{-1}^{1} |\hat{\nu}_1(tu_1)|^2 \cdots |\hat{\nu}_{d-1}(tu_{d-1})|^2 \, du_1 \cdots du_{d-1}. \]
By Lemma 2.1, this quantity is
\[ \leq C t^{-s_1 - \cdots - s_{d-1}} \leq C t^{-\frac{d}{d+1}}, \]
where \( s = \sum_{j=1}^{d} s_j \).

Plugging this estimate into the Mattila integral (4.1) we obtain
\[ C \int \int |\hat{\mu}(t\omega)|^2 t^{d-1} \cdot t^{-\frac{d}{d+1}} \, d\omega \, dt = C \int |\hat{\mu}(\xi)|^2 |\xi|^{-d} t^{-\frac{d}{d+1}} \, d\xi \]
and this integral is finite if
\[ d - s < \frac{s (d - 1)}{d}, \]
which is the case if
\[ s > \frac{d^2}{2d - 1}, \]
as desired.

References


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