SOME NORM INEQUALITIES IN MUSIELAK–ORLICZ SPACES

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Abstract. Our aim in this paper is to establish various norm inequalities in Musielak–Orlicz spaces. We give a generalization of a result due to Cruz-Uribe, Fiorenza, Martell and Pérez and apply it to obtain norm inequalities for classical operators as well as an Olsen inequality in Musielak–Orlicz spaces.

1. Introduction

There has been a considerable amount of studies on the variable exponent Lebesgue spaces $L^{p(\cdot)}$; see [5, 7] etc. for exhaustive account of this direction of research. In those studies, various kinds of norm inequalities were discussed, including those which show the boundedness of important operators. Cruz-Uribe, Fiorenza, Martell and Pérez [6] gave a method to obtain $L^{p(\cdot)}$-norm inequalities from $L^{p_0(w)}$-norm inequalities with a constant exponent $p_0$ and weights $w$. In fact, they proved [6, Theorem 1.3]:

**Theorem A.** Let $\mathcal{F}$ be a family of ordered pairs $(f, g)$ of nonnegative measurable functions on $\mathbb{R}^N$. Suppose that

$$
\int_{\mathbb{R}^N} f(x)^{p_0} w(x) \, dx \leq C_0 \int_{\mathbb{R}^N} g(x)^{p_0} w(x) \, dx
$$

for some $p_0 > 0$, for all $(f, g) \in \mathcal{F}$ and for all $A_1$-weights $w$ with a constant $C_0$ depending only on $p_0$ and the $A_1$-constant of $w$. Let $p(\cdot)$ be a variable exponent such that

$$
1 \leq p^− = \text{ess inf}_{x \in \mathbb{R}^N} p(x) \leq p^+ = \text{ess sup}_{x \in \mathbb{R}^N} p(x) < \infty.
$$

If $p_0 < p^−$ and the Hardy–Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'(\mathbb{R}^N)}$, then there is a constant $C > 0$ such that

$$
\|f\|_{L^{p(\cdot)}} \leq C \|g\|_{L^{p(\cdot)}}
$$

for all $(f, g) \in \mathcal{F}$ with $g \in L^{p(\cdot)}(\mathbb{R}^N)$.

In the present paper, we call this theorem CFMP-theorem.

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Variable exponent Lebesgue spaces are special cases of Musielak–Orlicz spaces, which were first considered by Nakano as modulared function spaces in [25] and then developed by Musielak as generalized Orlicz spaces in [22]. Our main aim in this paper is to extend Theorem A to Musielak–Orlicz spaces $L^\Phi(\mathbb{R}^N)$ defined by a general function $\Phi(x,t)$ satisfying certain conditions (Theorem 5.2). See Section 2 for the definition of $\Phi$ and $L^\Phi(\mathbb{R}^N)$.

Many types of norm inequalities depend on the boundedness of the Hardy–Littlewood maximal operator $M$. The boundedness of $M$ on $L^\Phi(\mathbb{R}^N)$ was established in [19, Corollary 4.4]; we give its improvement in Section 3 of the present paper. The proof of Theorem A also depends on the boundedness of $M$ on the dual space of $L^p(\mathbb{R}^N)$.

In Section 4, we study properties of the complementary function of $\Phi$ and look for conditions on $\Phi$ that assure the boundedness of $M$ on the dual space of $L^\Phi(\mathbb{R}^N)$. We follow [6] for the proof of our generalization of Theorem A, Theorem 5.2, and applications of extrapolation theorems to obtain vector-valued inequalities in $L^\Phi(\mathbb{R}^N)$. As applications of Theorem 5.2, we prove $L^\Phi$-norm inequalities for classical operators such as sharp maximal operators and singular integral operators in Section 6. We shall also show the $L^\Phi$-version of Kerman–Sawyer inequality.

Using the vector-valued inequality, in Section 7 we shall establish a decomposition result for functions in Musielak–Orlicz spaces as an extension of [23] and [24] for the case of Lebesgue spaces with variable exponents and Orlicz spaces. See [4, 13, 14, 26, 27, 37] for related results. As an application of the decomposition result, we obtain an Olsen inequality in the final section. By an Olsen inequality, or a trace inequality, we mean an inequality of type

\begin{equation}
\|g \cdot I_\alpha f\|_X \leq C\|g\|_Y \cdot \|f\|_Z
\end{equation}

for some Banach function spaces $X$, $Y$ and $Z$, where $I_\alpha f$ is the Riesz potential (of order $\alpha$) of $f$. There is a vast amount of literatures on Olsen inequalities [11, 12, 28, 29, 30, 32, 33, 34, 36]. We shall show that (1.2) holds with $X = Z = L^\Phi(\mathbb{R}^N)$ and a certain Morrey space $Y$.

Throughout this paper, let $C$ denote various constants independent of the variables in question, and $C(a,b,\ldots)$ a constant that depends on $a,b,\ldots$.

2. Preliminaries

We consider a function

$\Phi(x,t) = t\phi(x,t): \mathbb{R}^N \times [0, \infty) \to [0, \infty)$

satisfying the following conditions $(\Phi1)$–$(\Phi3)$:

$(\Phi1)$ $\phi(\cdot,t)$ is measurable on $\mathbb{R}^N$ for each $t \geq 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbb{R}^N$;

$(\Phi2)$ there exists a constant $A_1 \geq 1$ such that

$A_1^{-1} \leq \phi(x,1) \leq A_1$ for all $x \in \mathbb{R}^N$;

$(\Phi3)$ $\phi(x, \cdot)$ is uniformly almost increasing on $(0, \infty)$, namely there exists a constant $A_2 \geq 1$ such that

$\phi(x,t) \leq A_2 \phi(x,at)$ for all $x \in \mathbb{R}^N$ whenever $t > 0$ and $a > 1$. 

Let $\tilde{\phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\Phi(x, t) = \int_0^t \tilde{\phi}(x, r) \, dr$$

for $x \in \mathbb{R}^N$ and $t \geq 0$. Then $\tilde{\phi}(x, \cdot)$ is continuous nondecreasing, $\Phi(x, \cdot)$ is convex and

$$\Phi(x, t/2) \leq \Phi(x, t) \leq A_2 \Phi(x, t)$$

for all $x \in \mathbb{R}^N$ and $t \geq 0$. Given $\Phi(x, t)$ as above, the associated Musielak–Orlicz space

$$L^\Phi(\mathbb{R}^N) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^N): \int_{\mathbb{R}^N} \Phi(y, |f(y)|/\lambda) \, dy < \infty \text{ for some } \lambda > 0 \right\}$$

is a Banach space with respect to the norm (cf. [22])

$$\|f\|_\Phi = \|f\|_{L^\Phi(\mathbb{R}^N)} = \inf \left\{ \lambda > 0: \int_{\mathbb{R}^N} \Phi(y, |f(y)|/\lambda) \, dy \leq 1 \right\}.$$

We shall also consider the following conditions: Let $\varepsilon \geq 0$, $\nu > 0$ and $\omega > 0$.

($\Phi_3; \varepsilon$) $t \mapsto t^{-\varepsilon} \phi(x, t)$ is uniformly almost increasing on $(0, \infty)$, namely there exists a constant $A_{2, \varepsilon} \geq 1$ such that

$$\phi(x, t) \leq A_{2, \varepsilon} t^{-\varepsilon} \phi(x, at) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } t > 0 \text{ and } a > 1;$$

($\Phi_4$) $\phi(x, \cdot)$ satisfies the uniform doubling condition, namely there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0;$$

($\Phi_5; \nu$) For every $\gamma > 0$, there exists a constant $B_{\gamma, \nu} \geq 1$ such that

$$\Phi(x, t) \leq B_{\gamma, \nu} \Phi(y, t)$$

whenever $|x - y| \leq \gamma t^{-\nu}$ and $t \geq 1$;

($\Phi_6; \omega$) there exist a function $g$ on $\mathbb{R}^N$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbb{R}^N$, $g^\omega \in L^1(\mathbb{R}^N)$ and

$$B_\infty^{-1} \Phi(x, t) \leq \Phi(x', t) \leq B_\infty \Phi(x, t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

**Example 2.1.** Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on $\mathbb{R}^N$ such that

(P1) $1 \leq p^- = \inf_{x \in \mathbb{R}^N} p(x) \leq \sup_{x \in \mathbb{R}^N} p(x) = p^+ < \infty$

and

(Q1) $-\infty < q_j^- = \inf_{x \in \mathbb{R}^N} q_j(x) \leq \sup_{x \in \mathbb{R}^N} q_j(x) = q_j^+ < \infty$

for all $j = 1, \ldots, k$. Set $L_c(t) = \log(c + t)$ for $c \geq e$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\tilde{\Phi}(x, t) = t^{p(x)} \prod_{j=1}^k (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x, t)$ satisfies ($\Phi_1$), ($\Phi_2$) and ($\Phi_4$). It satisfies ($\Phi_3$) if there is a constant $K \geq 0$ such that $K(p(x) - 1) + q_j(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $j = 1, \ldots, k$; in particular if $p^- > 1$ or $q_j^- \geq 0$ for all $j = 1, \ldots, k$. If $p^- > 1$, then $\Phi(x, t)$ satisfies ($\Phi_3; \varepsilon$) for $0 < \varepsilon < p^- - 1$. 


Moreover, we see that $\Phi(x, t)$ satisfies (\Phi5; \nu) for every $\nu > 0$ if (P2) $p(\cdot)$ is log-Hölder continuous, namely

\[ |p(x) - p(y)| \leq \frac{C_p}{L(x, y)} \]

with a constant $C_p \geq 0$ and (Q2) $q_j(\cdot)$ is $(j + 1)$-log-Hölder continuous, namely

\[ |q_j(x) - q_j(y)| \leq \frac{C_{q_j}}{L^{j+1}(x, y)} \]

with constants $C_{q_j} \geq 0$, $j = 1, \ldots, k$.

Finally, we see that $\Phi(x, t)$ satisfies (\Phi6; $\omega$) for every $\omega > 0$ with $g(x) = 1/(1 + |x|)^{(N+1)/\omega}$ if $p(\cdot)$ is log-Hölder continuous at $\infty$, namely if it satisfies

\[ (P3) \ |p(x) - p(x')| \leq \frac{C_{p, \infty}}{L(|x'|)} \quad \text{whenever} \ |x'| \geq |x| \quad \text{with a constant} \ C_{p, \infty} \geq 0. \]

Note that $(\Phi3; 0) = (\Phi3)$. If $\Phi(x, t)$ satisfies $(\Phi3; \epsilon)$, then it satisfies $(\Phi3; \epsilon')$ for $0 \leq \epsilon' \leq \epsilon$. If $\Phi(x, t)$ satisfies $(\Phi3; \epsilon)$, then

\[ \bar{\phi}(x, t) \leq A_{2, \epsilon}^{-\epsilon} \bar{\phi}(x, at) \quad \text{for all} \ x \in \mathbb{R}^N \quad \text{whenever} \ t > 0 \text{ and } a > 1 \]

and

\[ \bar{\Phi}(x, t) \leq A_{2, \epsilon}^{-1-\epsilon} \bar{\Phi}(x, at) \quad \text{for all} \ x \in \mathbb{R}^N \quad \text{whenever} \ t > 0 \text{ and } a > 1. \]

If $\Phi(x, t)$ satisfies $(\Phi5; \nu)$, then it satisfies $(\Phi5; \nu')$ for all $\nu' \geq \nu$; if $\Phi(x, t)$ satisfies $(\Phi6; \omega)$, then it satisfies $(\Phi6; \omega')$ for all $\omega' \geq \omega$.

The following example shows that if $0 < \nu' < \nu$ and $0 < \omega' < \omega$, then there exists $\Phi(x, t)$ satisfying $(\Phi j)$, $j = 1, 2, 3, 4$ such that it satisfies $(\Phi5; \nu)$ and $(\Phi6; \omega)$, while it does not satisfy $(\Phi5; \nu')$ nor $(\Phi6; \omega')$.

**Example 2.2.** For $p \geq 1$, $q > 0$ and $r > 0$, set

\[ \Phi(x, t) = \begin{cases} t^p \max(1, t^q \min(1, |x|)) & \text{if} \ t \geq 1, \\ t^p \max(t, \min(1/2, |x|^{-N/r})) & \text{if} \ t < 1. \end{cases} \]

This $\Phi(x, t)$ satisfies $(\Phi j)$, $j = 1, 2, 3, 4$; it satisfies $(\Phi3; p - 1)$. We shall show:

(a) $\Phi(x, t)$ satisfies $(\Phi5; \nu)$ if and only if $\nu \geq q$;

(b) $\Phi(x, t)$ satisfies $(\Phi6; \omega)$ if $\omega > r$ but does not satisfy $(\Phi6; \omega)$ if $\omega < r$.

**Proof of (a).** Let $t \geq 1$ and $|x - y| \leq \gamma t^{\nu}$. If $\nu \geq q$, then

\[ \min(1, |x|) \leq \min(1, |y|) + |x - y| \leq \min(1, |y|) + \gamma t^{\nu} \leq \min(1, |y|) + \gamma t^q, \]

so that

\[ \max(1, t^q \min(1, |x|)) \leq \max(1, t^q \min(1, |y|)) + \gamma, \]

which implies

\[ \Phi(x, t) \leq t^p \max(1, t^q \min(1, |y|)) + \gamma t^p \leq (1 + \gamma) \Phi(y, t). \]

Hence $\Phi(x, t)$ satisfies $(\Phi5; \nu)$ if $\nu \geq q$.

Next, suppose $\nu < q$. Let $e_1 = (1, 0, \ldots, 0)$. Since $\Phi(0, t) = t^p$ and $\Phi(t^{-\nu} e_1, t) = t^p \max(1, t^{-\nu}) = t^{p + q - \nu}$,

\[ \frac{\Phi(t^{-\nu} e_1, t)}{\Phi(0, t)} \to \infty \quad (t \to \infty). \]

This shows that $\Phi(x, t)$ does not satisfy $(\Phi5; \nu)$. 
Proof of (b). First, let $\omega > r$. Take

$$g_r(x) = \min\left(1/2, |x|^{-N/r}\right) \quad (x \in \mathbb{R}^N).$$

Then $0 < g_r(x) \leq 1/2$ for all $x \in \mathbb{R}^N$ and $g_r^\omega \in L^1(\mathbb{R}^N)$. If $g_r(x) \leq t < 1$ and $|x'| \geq |x|$, then $g_r(x') \leq g_r(x) \leq t$, so that $\Phi(x, t) = \Phi(x', t) = t^{p+1}$. Hence $\Phi(x, t)$ satisfies $(\Phi_6; \omega)$ if $\omega > r$.

Next, assume that $\omega < r$ and suppose that there exists a function $g$ on $\mathbb{R}^N$ such that $0 \leq g(x) < 1$ for all $x \in \mathbb{R}^N$ and

$$\Phi(x, t) \leq B\Phi(x', t) \quad \text{whenever } |x'| \geq |x| \text{ and } g(x) \leq t < 1$$

with a constant $B \geq 1$. We claim that there exists $R > 1$ such that

$$g(x) \geq |x|^{-N/\omega} \quad \text{for } |x| \geq R.$$

Suppose on the contrary that there exists a sequence $\{x_n\}$ such that $|x_n| \to \infty$ and $g(x_n) < |x_n|^{-N/\omega}$ for all $n$. We may assume $|x_n| \geq 2^n/N$. Then

$$\Phi(x_n, |x_n|^{-N/\omega}) = |x_n|^{-pN/\omega} \max(|x_n|^{-N/\omega}, |x_n|^{-N/r}) = |x_n|^{-pN/\omega + 1/\omega})N.$$

If we take $x_n' \in \mathbb{R}^N$ such that $|x_n'| = |x_n|^{p/\omega} > |x_n|$, then

$$\Phi(x_n', |x_n|^{-N/\omega}) = |x_n|^{-pN/\omega + 1} \max(|x_n|^{-N/\omega}, |x_n'|^{-N/\omega}) = |x_n|^{-pN/\omega + 1/\omega})N.$$

Hence,

$$\frac{\Phi(x_n, |x_n|^{-N/\omega})}{\Phi(x_n', |x_n|^{-N/\omega})} = |x_n|^{(1/\omega - 1/r)N} \to \infty \quad (n \to \infty),$$

which contradicts (2.2). Thus, (2.3) holds, and hence $g^\omega \notin L^1(\mathbb{R}^N)$, which means that $\Phi(x, t)$ does not satisfy $(\Phi_6; \omega)$ if $\omega < r$.

### 3. Boundedness of the maximal operator

For $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, its maximal function $Mf$ is defined by

$$Mf(x) = \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.$$

As the boundedness of the maximal operator $M$ on $L^p(\mathbb{R}^N)$, we give the following theorem, which is an improvement of [19, Corollary 4.4] by relaxing assumptions on $\Phi(x, t)$ in [19]. In fact, we shall show our result by assuming $(\Phi_5; \nu)$ and $(\Phi_6; \omega)$ below instead of $(\Phi_5)$ and $(\Phi_6)$ in [19]. Further, the result is proved without $(\Phi_4)$ which is assumed in [19].

**Theorem 3.1.** Suppose that $\Phi(x, t)$ satisfies $(\Phi_3; \varepsilon)$, $(\Phi_5; \nu)$ and $(\Phi_6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < (1 + \varepsilon)/N$ and $\omega \leq 1 + \varepsilon$. Then the maximal operator $M$ is bounded from $L^p(\mathbb{R}^N)$ into itself, namely

$$\|Mf\|_p \leq C_M\|f\|_p$$

for all $f \in L^p(\mathbb{R}^N)$.

We prove this theorem by modifying the proof of [19, Theorem 4.1].

For a nonnegative $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$, let

$$I(f; x, r) = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy$$

and

$$J(f; x, r) = \frac{1}{|B(x, r)|} \int_{B(x, r)} \Phi(y, f(y)) \, dy.$$
Lemma 3.2. Suppose \( \Phi(x, t) \) satisfies (\( \Phi_3; \varepsilon_1 \)) and (\( \Phi_5; \nu \)) for \( \varepsilon_1 > 0 \) and \( \nu > 0 \) satisfying \( \nu \leq (1 + \varepsilon_1)/N \). Then, given \( L \geq 1 \), there exist constants \( C_1 = C(L) \geq 2 \) and \( C_2 > 0 \) such that
\[
\Phi(x, I(f; x, r)/C_1) \leq C_2 J(f; x, r)
\]
for all \( x \in \mathbb{R}^N, r > 0 \) and for all nonnegative \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \) such that \( f(y) \geq 1 \) or \( f(y) = 0 \) for each \( y \in \mathbb{R}^N \) and
\[
(3.1) \quad \int_{\mathbb{R}^N} \Phi(y, f(y)) \, dy \leq L.
\]

Proof. Given \( f \) as in the statement of the lemma, \( x \in \mathbb{R}^N \) and \( r > 0 \), set \( I = I(f; x, r) \) and \( J = J(f; x, r) \). Note that (3.1) implies \( J \leq L |B(0, 1)|^{-1} r^{-N} \). By (\( \Phi_2 \)) and (\( \Phi_3 \)), \( \Phi(y, f(y)) \geq (A_1 A_2)^{-1} f(y) \), since \( f(y) \geq 1 \) or \( f(y) = 0 \). Hence \( I \leq A_1 A_2 J \). Thus, if \( J \leq 1 \), then
\[
\Phi(x, I/C_1) \leq A_2 J \Phi(x, 1) \leq A_1 A_2 J
\]
whenever \( C_1 \geq A_1 A_2 \).

Next, suppose \( J > 1 \). Since \( \Phi(x, t) \to \infty \) as \( t \to \infty \), there exists \( K \geq 1 \) such that
\[
\Phi(x, K) = \Phi(x, 1) J
\]
by (\( \Phi_1 \)) and the mean value theorem. With this \( K \), we have
\[
\int_{B(x, r)} f(y) \, dy \leq K |B(x, r)| + A_2 \int_{B(x, r)} f(y) \frac{\phi(y, f(y))}{\phi(y, K)} \, dy.
\]
Since \( K > 1 \), by (\( \Phi_3; \varepsilon_1 \)) we have
\[
\Phi(x, 1) J = \Phi(x, K) \geq A_2^{-1} K^{1+\varepsilon_1} \Phi(x, 1),
\]
so that \( J \geq A_2^{-1} K^{1+\varepsilon_1} \), which implies
\[
K^{1+\varepsilon_1} \leq A_2 \varepsilon_1 J \leq A_2 \varepsilon_1 L |B(0, 1)|^{-1} r^{-N},
\]
or \( r \leq \gamma K^{-(1+\varepsilon_1)/N} \) with \( \gamma = (A_2 \varepsilon_1 L |B(0, 1)|^{-1})^{1/N} \). Thus, if \( |y-x| \leq r \), then \( |y-x| \leq \gamma K^{-(1+\varepsilon_1)/N} \leq \gamma K^{-\nu} \). Hence, by (\( \Phi_5; \nu \)) there is \( \beta > 0 \), independent of \( f, x, r, \) such that
\[
\phi(x, K) \leq \beta \phi(y, K) \quad \text{for all } y \in B(x, r).
\]
Thus, we have
\[
\int_{B(x, r)} f(y) \, dy \leq K |B(x, r)| + \frac{A_2 \beta}{\phi(x, K)} \int_{B(x, r)} f(y) \phi(y, f(y)) \, dy
\]
\[
= K |B(x, r)| + A_2 \beta |B(x, r)| \frac{J}{\phi(x, K)}
\]
\[
= K |B(x, r)| \left( 1 + \frac{A_2 \beta}{\phi(x, 1)} \right) \leq K |B(x, r)| (1 + A_1 A_2 \beta).
\]
Therefore
\[
I \leq (1 + A_1 A_2 \beta) K,
\]
so that by (\( \Phi_2 \)) and (\( \Phi_3 \))
\[
\Phi(x, I/C_1) \leq A_2 \Phi(x, K) \leq A_1 A_2 J
\]
whenever \( C_1 \geq 1 + A_1 A_2 \beta \). \( \square \)

The next lemma can be shown in the same way as [19, Lemma 3.2]; note that the value of \( \omega \) is irrelevant in this lemma.
Lemma 3.3. Suppose \( \Phi(x, t) \) satisfies (\( \Phi6; \omega \)) for some \( \omega > 0 \). Then there exists a constant \( C_3 > 0 \) such that
\[
\Phi(x, f(x, r)/2) \leq C_3 \{ J(f; x, r) + \Phi(x, g(x)) \}
\]
for all \( x \in \mathbb{R}^N, r > 0 \) and for all nonnegative \( f \in L^{1}_{\text{loc}}(\mathbb{R}^N) \) such that \( g(y) \leq f(y) \leq 1 \) or \( f(y) = 0 \) for each \( y \in \mathbb{R}^N \), where \( g \) is the function appearing in (\( \Phi6; \omega \)).

Proof of Theorem 3.1. Choose \( p_0 \in (1, 1 + \varepsilon) \) such that \( p_0 \leq (1 + \varepsilon)/(N\nu) \) and consider the function
\[
\Phi_0(x, t) = \Phi(x, t)^{1/p_0}.
\]
Then \( \Phi_0(x, t) \) satisfies the conditions (\( \Phi1 \)), (\( \Phi2 \)), (\( \Phi5; \nu \)) and (\( \Phi6; \omega \)) with the same \( g \). Since
\[
\Phi_0(x, t) = t\phi_0(x, t) \quad \text{with} \quad \phi_0(x, t) = [t^{1-p_0} \phi(x, t)]^{1/p_0},
\]
condition (\( \Phi3; \varepsilon \)) implies that \( \Phi_0(x, t) \) satisfies (\( \Phi3; \varepsilon_1 \)) with \( \varepsilon_1 = (1 + \varepsilon)/p_0 - 1 > 0 \). Note that
\[
\frac{1 + \varepsilon_1}{N} = \frac{1 + \varepsilon}{p_0N} \geq \nu.
\]
Let \( f \geq 0 \) and \( \|f\|_{\Phi} \leq 1/2 \). Let \( f_1 = f\chi_{\{x : f(x) \geq 1\}}, f_2 = f\chi_{\{x : g(x) \leq f(x) < 1\}} \) with \( g \) in (\( \Phi6; \omega \)) and \( f_3 = f - f_1 - f_2 \), where \( \chi_E \) is the characteristic function of \( E \). Since \( \Phi(x, t) \geq (A_1 A_2)^{-1} \) for \( t \geq 1 \) by (\( \Phi3 \)),
\[
\Phi_0(x, t) \leq (A_1 A_2)^{1-1/p_0} \Phi(x, t) \leq (A_1 A_2)^{1-1/p_0} \Phi(x, 2t)
\]
if \( t \geq 1 \). Hence
\[
\int_{\mathbb{R}^N} \Phi_0(y, f_1(y)) \, dy \leq (A_1 A_2)^{1-1/p_0}.
\]
In view of (3.2), we can apply Lemma 3.2 to \( \Phi_0 \), \( f_1 \) and \( L = (A_1 A_2)^{1-1/p_0} \), and we have
\[
\Phi_0(x, Mf_1(x)/C_1) \leq C_2 M\Phi_0(\cdot, f_1(\cdot))(x),
\]
so that
\[
\Phi(x, Mf_1(x)/C_1) \leq C_2^{p_0} \left[ M\Phi_0(\cdot, f(\cdot))(x) \right]^{p_0}
\]
for all \( x \in \mathbb{R}^N \) with a constant \( C > 0 \) independent of \( f \).

Next, applying Lemma 3.3 to \( \Phi_0 \) and \( f_2 \), we have
\[
\Phi_0(x, Mf_2(x)/2) \leq C \left[ M\Phi_0(\cdot, f_2(\cdot))(x) + \Phi_0(x, g(x)) \right].
\]
Noting that \( \Phi_0(x, g(x)) \leq C g(x)^{1+\varepsilon}/p_0 \) by (\( \Phi3; \varepsilon \)), we have
\[
\Phi(x, Mf_2(x)/2) \leq C \left\{ \left[ M\Phi_0(\cdot, f(\cdot))(x) \right]^{p_0} + g(x)^{1+\varepsilon} \right\}
\]
for all \( x \in \mathbb{R}^N \) with a constant \( C > 0 \) independent of \( f \).

Since \( 0 \leq f_3 \leq g \leq 1, 0 \leq Mf_3 \leq Mg \leq 1 \). Hence we have
\[
\Phi(x, Mf_3(x)) \leq A_2 \Phi_0(x, Mg(x)) \leq (Mg(x))^{1+\varepsilon}
\]
for all \( x \in \mathbb{R}^N \) with a constant \( C > 0 \) independent of \( f \).

Combining (3.3), (3.4) and (3.5), and noting that \( g(x) \leq Mg(x) \) for a.e. \( x \in \mathbb{R}^N \), we obtain
\[
\Phi(x, Mf(x)/(C_1 + 3)) \leq C \left\{ \left[ M\Phi_0(\cdot, f(\cdot))(x) \right]^{p_0} + [Mg(x)]^{1+\varepsilon} \right\}
\]
for a.e. \( x \in \mathbb{R}^N \) with a constant \( C > 0 \) independent of \( f \).
Since $M$ is bounded on $L^{p_0}(\mathbb{R}^N)$ and on $L^{1+\varepsilon}(\mathbb{R}^N)$, there exists a constant $C_4 \geq 1$ such that
\[
\int_{\mathbb{R}^N} \Phi(x, Mf(x)/(C_1 + 3)) \, dx
\]
\[
\leq C \left\{ \int_{\mathbb{R}^N} [M\Phi_0(\cdot, f(\cdot))(y)]^{p_0} \, dy + \int_{\mathbb{R}^N} [Mg(y)]^{1+\varepsilon} \, dy \right\}
\]
\[
\leq C \left\{ \int_{\mathbb{R}^N} \Phi_0(x, f(x))^{p_0} \, dx + \int_{\mathbb{R}^N} g(x)^{1+\varepsilon} \, dx \right\}
\]
\[
\leq C \left\{ \int_{\mathbb{R}^N} \Phi(x, f(x)) \, dx + \int_{\mathbb{R}^N} g(x)^{\varepsilon} \, dx \right\} \leq C_4,
\]
so that
\[
\int_{\mathbb{R}^N} \Phi(y, Mf(y)/(A_2(C_1 + 3)C_4)) \, dy \leq 1.
\]
This completes the proof of the theorem. \hfill \Box

4. Properties of the complementary function

Hereafter, we assume that $\Phi(x, t)$ further satisfies

$(\Phi^3^*)$ $\lim_{t \to \infty} \phi(x, t) = \infty$ and $\lim_{t \to 0^+} \phi(x, t) = 0$ for every $x \in \mathbb{R}^N$.

Note that this condition implies the same condition with $\tilde{\phi}$ in place of $\phi$. Also, note that if $\Phi(x, t)$ satisfies $(\Phi^3; \varepsilon)$ for some $\varepsilon > 0$, then it satisfies $(\Phi^3^*)$.

Under this assumption, we consider the complementary function $\Phi^*(x, s)$ of $\Phi(x, t)$: set
\[
\phi^*(x, s) = \sup \{ t \geq 0 : \tilde{\phi}(x, t) \leq s \}
\]
and
\[
\Phi^*(x, s) = \int_0^s \phi^*(x, r) \, dr
\]
for $x \in \mathbb{R}^N$ and $s \geq 0$. Note that $\Phi^*(x, \cdot)$ is nonnegative, convex and $\Phi^*(x, 0) = 0$; $\Phi^*(x, t)$ satisfies $(\Phi 1)$ and $(\Phi 3)$.

Furthermore, we have

Proposition 4.1. (1) If $\Phi(x, t)$ satisfies $(\Phi 3; \varepsilon)$ for some $\varepsilon > 0$, then $\Phi^*(x, t)$ satisfies $(\Phi 2)$ and $(\Phi 4)$.

(2) Define $\varepsilon^* = (\log 2)/(\log A_3)$ where $A_3 > 1$ is a constant appearing in $(\Phi 4)$. If $\Phi(x, t)$ satisfies $(\Phi 4)$, then $\Phi^*(x, t)$ satisfies $(\Phi 3; \varepsilon^*)$.

Proof. (1) Suppose $\Phi(x, t)$ satisfies $(\Phi 3; \varepsilon)$ for some $\varepsilon > 0$. Then $\tilde{\phi}(x, t) \to \infty$ as $t \to \infty$ and $\tilde{\phi}(x, t) \to 0$ as $t \to 0$, both uniformly in $x \in \mathbb{R}^N$. It then follows that
\[
0 < \inf_{x \in \mathbb{R}^N} \phi^*(x, 1) \leq \sup_{x \in \mathbb{R}^N} \phi^*(x, 1) < \infty
\]
and
\[
0 < \inf_{x \in \mathbb{R}^N} \Phi^*(x, 1) \leq \sup_{x \in \mathbb{R}^N} \Phi^*(x, 1) < \infty,
\]
which is $(\Phi 2)$ for $\Phi^*(x, t)$.

Since
\[
\tilde{\phi}(x, (2A_2\varepsilon)^{-1/\varepsilon}t) \leq A_2\varepsilon(2A_2\varepsilon)^{-1}\tilde{\phi}(x, t) = \frac{1}{2} \phi(x, t)
\]
by (Φ3; ε),
\[
\phi^*(x, 2s) = \sup \{ t \geq 0 : \bar{\phi}(x, t) \leq 2s \}
\leq \sup \{ t \geq 0 : \bar{\phi}(x, (2A_2)\cdot 1/\varepsilon t) \leq s \} = (2A_2\cdot 1/\varepsilon) \phi^*(x, s),
\]
which implies
\[
\Phi^*(x, 2s) \leq 2A_3^* \Phi^*(x, s),
\]
where \( A_3^* = (2A_2)\cdot 1/\varepsilon \). Thus \( \Phi^*(x, t) \) satisfies (Φ4).

(2) First, we show
\[
(4.1) \quad \bar{\phi}(x, t) \leq s \Rightarrow \bar{\phi}(x, A_3^{-\varepsilon} a \cdot \varepsilon t) \leq as
\]
for \( t \geq 0 \) and \( a > 1 \). Let \( \bar{\phi}(x, t) \leq s \). If \( 1 < a \leq A_3 \), then
\[
\bar{\phi}(x, A_3^{-\varepsilon} a \cdot \varepsilon t) \leq \bar{\phi}(x, t) \leq s \leq as.
\]
If \( a \geq A_3 \), then using (Φ4) we have
\[
\bar{\phi}(x, A_3^{-\varepsilon} a \cdot \varepsilon t) \leq A_3(A_3^{-\varepsilon} a \cdot \varepsilon) \phi^*(x, t) = a \phi^*(x, t) \leq as.
\]
Thus (4.1) holds.

Now (4.1) implies
\[
\phi^*(x, s) \leq A_3^* a^{-\varepsilon} \phi^*(x, as)
\]
for \( s > 0 \) and \( a > 1 \), which in turn implies
\[
\Phi^*(x, s) \leq A_3^* a^{-1-\varepsilon} \Phi^*(x, as)
\]
whenever \( s > 0 \) and \( a > 1 \). This means that \( \Phi^*(x, t) \) satisfies (Φ3; ε*).

\[\square\]

**Lemma 4.2.** Suppose \( \Phi(x, t) \) satisfies (Φ3; ε) for some \( \varepsilon > 0 \). Define
\[
\eta(x, t) = \begin{cases} \Phi(x, t)/t & (t > 0), \\ 0 & (t = 0), \end{cases} \quad \eta^*(x, s) = \begin{cases} \Phi^*(x, s)/s & (s > 0), \\ 0 & (s = 0). \end{cases}
\]
for \( x \in \mathbb{R}^N, t \geq 0 \) and \( s \geq 0 \). Then there is a constant \( A_4 \geq 1 \) such that
\[
A_4^{-1} t \leq \eta^*(x, \eta(x, t)) \leq A_4 t
\]
for \( x \in \mathbb{R}^N \) and \( t \geq 0 \).

**Proof.** First, we note that
\[
(4.2) \quad t \leq \phi^*(x, \bar{\phi}(x, t)) \leq A_2^{1/\varepsilon} t
\]
for all \( x \in \mathbb{R}^N \) and \( t > 0 \). In fact, the first inequality is obvious from the definition of \( \phi^* \). Suppose \( \bar{\phi}(x, at) \leq \bar{\phi}(x, t) \) with \( a \geq 1 \). Then, by (Φ3; ε),
\[
\bar{\phi}(x, t) \geq \bar{\phi}(x, at) \geq A_2^{-1} a \cdot \varepsilon \bar{\phi}(x, t),
\]
so that \( a \leq A_2^{1/\varepsilon} \). This shows the second inequality of (4.2).

Since \( \phi^*(x, \cdot) \) and \( \bar{\phi}(x, \cdot) \) are non-decreasing, so are \( \eta^*(x, \cdot) \) and \( \eta(x, \cdot) \); and \( \eta^*(x, s) \leq \phi^*(x, s) \) as well as \( \eta(x, t) \leq \bar{\phi}(x, t) \). Hence, by (4.2), we have
\[
\eta^*(x, \eta(x, t)) \leq \phi^*(x, \bar{\phi}(x, t)) \leq A_2^{1/\varepsilon} t.
\]

On the other hand,
\[
\eta(x, t) \geq \frac{1}{t} \int_{t/2}^t \bar{\phi}(x, r) dr \geq \frac{1}{2} \bar{\phi}(x, t/2)
\]
and, similarly, $\eta^*(x, s) \geq (1/2)\phi^*(x, s/2)$. Hence
$$
\eta^*(x, \eta(x,t)) \geq \eta^* \left( x, \frac{1}{2}\phi(x,t/2) \right) \geq \frac{1}{2} \phi^* \left( x, \frac{1}{4}\phi(x,t/2) \right).
$$
Thus, by Proposition 4.1 (1) and (4.2), we have
$$
\eta^*(x, \eta(x,t)) \geq \frac{1}{2(2A_3^3)^2}\phi^*(x, \tilde{\phi}(x,t/2)) \geq \frac{1}{2(4(A_3^3)^2)}t. \quad \square
$$

**Proposition 4.3.** Suppose $\Phi(x,t)$ satisfies $(\Phi3; \varepsilon)$ and $(\Phi4)$ for some $\varepsilon > 0$.

1. If $\Phi(x,t)$ satisfies $(\Phi5; \nu)$, then $\Phi^*(x,s)$ satisfies $(\Phi5; \nu/\varepsilon)$.
2. If $\Phi(x,t)$ satisfies $(\Phi6; \omega)$, then $\Phi^*(x,s)$ satisfies $(\Phi6; \omega/\varepsilon)$.

**Proof.** (1) Let $\gamma' > 0$ and $|x - y| \leq \gamma't^{-\nu/\varepsilon}$. First, we consider the case $t \geq A_1A_2$. Since $\eta(x,1) \leq A_1A_2$, there is $s \geq 1$ such that $t = \eta(x,s)$. Since
$$
\eta(x,s) \geq A_2^{-1}s^\varepsilon \eta(x,1) \geq (2A_1A_2\varepsilon A_3)^{-1}s^\varepsilon,
$$
we have
$$
|x - y| \leq \gamma'(2A_1A_2\varepsilon A_3)^{\nu/\varepsilon}s^{-\nu}.
$$
Hence, by $(\Phi5; \nu)$ and (2.1),
$$
\tilde{B}^{-1}\eta(x,s) \leq \eta(y,s) \leq \tilde{B}\eta(x,s)
$$
with $\tilde{B} = 2A_2A_3B_{\gamma,\varepsilon}$, $\gamma = \gamma'(2A_1A_2\varepsilon A_3)^{\nu/\varepsilon}$. By Proposition 4.1 (1), there is a constant $\tilde{B}' \geq 1$ such that $\eta^*(z, \tilde{B}t) \leq \tilde{B}'\eta^*(z,t)$ for all $z \in \mathbb{R}^N$ and $t > 0$. Then, using Lemma 4.2 twice, we have
$$
\eta^*(y,t) = \eta^*(y, \eta(x,s)) \leq \eta^*(y, \tilde{B}\eta(y,s)) 
\leq \tilde{B}'\eta^*(y, \eta(y,s)) \leq A_1\tilde{B}'s \leq A_4\tilde{B}'\eta^*(x,t)
$$
and
$$
\eta^*(x,t) = \eta^*(x, \eta(x,s)) \leq A_4s \leq A_4^2\eta^*(y, \eta(y,s)) 
\leq A_4^2\eta^*(y, \tilde{B}\eta(y,s)) \leq A_4^2\tilde{B}'\eta^*(y, \eta(y,s)) = A_4^2\tilde{B}'\eta^*(y,t).
$$
Thus
$$
\eta^*(x,t) \geq \eta^*(y,t) \leq \tilde{B}''\eta^*(x,t) \quad \text{if } t \geq A_1A_2
$$
with $\tilde{B}'' = A_4^2\tilde{B}'$.

Next, let $C_1 = \inf_{z \in \mathbb{R}^N} \eta^*(z,1)$ and $C_2 = \sup_{z \in \mathbb{R}^N} \eta^*(z, A_1A_2)$. Then $C_1 > 0$ and $C_2 < \infty$ by Proposition 4.1 (1). Then
$$
C_1C_2^{-1}\eta^*(x,t) \leq \eta^*(y,t) \leq C_1^{-1}C_2\eta^*(x,t) \quad \text{for } 1 \leq t \leq A_1A_2.
$$
Now, (4.3) and (4.4) show that $\Phi^*(x,t)$ satisfies $(\Phi5; \nu/\varepsilon)$.

(2) Let $g(x)$ be the function appearing in $(\Phi6; \omega)$ for $\Phi(x,t)$. Set
$$
g^*(x) = \min \left( \frac{1}{2A_1A_3}, \eta(x, g(x)) \right).
$$
Then, $0 \leq g^*(x) \leq 1/2 < 1$ and
$$
g^*(x) \leq \eta(x, g(x)) \leq A_1A_2A_2\varepsilon g(x)^{\varepsilon},
$$
which implies $(g^*)^{\nu/\varepsilon} \in L^1(\mathbb{R}^N)$. We want to show that there exists a constant $B''_\infty \geq 1$ such that
$$
(B''_\infty)^{-1}\eta^*(x,t) \leq \eta^*(x',t) \leq B''_\infty\eta^*(x,t)
$$
whenever $|x'| \geq |x|$ and $g^*(x) \leq t \leq 1$. 
First, suppose \( g^*(x) < t < 1/(2A_1A_3) \). Then \( g^*(x) = \eta(x, g(x)) \). Take \( s > 0 \) such that \( \eta(x, s) = t \). Then \( \eta(x, g(x)) < \eta(x, s) < \eta(x, 1) \), which implies \( g(x) < s < 1 \). Thus, by \((\Phi_6; \omega)\) and \((2.1)\)
\[
\bar{B}_t^{-1} \eta(x, s) \leq \eta(x', s) \leq \bar{B}_t \eta(x, s)
\]
or,
\[
\bar{B}_t^{-1} \leq \eta(x', s) \leq \bar{B}_t
\]
whenever \( |x'| \geq |x| \), where \( \bar{B}_t = 2A_2A_3 B_\infty \). Again by Proposition 4.1 (1), there is a constant \( B^* \geq 1 \) such that \( \eta^*(z, \bar{B}_t) \leq B^* \eta^*(z, t) \) for all \( z \in \mathbb{R}^N \) and \( t > 0 \). Then, by Lemma 4.2, we have
\[
\eta^*(x, t) = \eta^*(x, \eta(x, s)) \leq A_4 s \leq A_4^2 \eta^*(x', \eta(x', s)) \leq A_4^2 \eta^*(x', \bar{B}_t) \leq A_4^2 B^* \eta^*(x', t)
\]
and
\[
\eta^*(x', t) \leq \eta^*(x', \bar{B}_t \eta(x', s)) \leq B^* \eta^*(x', \eta(x', s)) \leq A_4 B^* s \leq A_4^2 B^* \eta^*(x, \eta(x, s)) = A_4^2 B^* \eta^*(x, t).
\]
Thus, we have shown that \((4.5)\) holds for \( g^*(x) < t < 1/(2A_1A_3) \) and \( |x'| \geq |x| \) with \( B'_\infty = A_2^2 B^* \). By continuity, this holds also for \( t = g^*(x) \).

Next, let \( C'_1 = \inf_{z \in \mathbb{R}^N} \eta^*(z, 1/(2A_1A_3)) \) and \( C'_2 = \sup_{z \in \mathbb{R}^N} \eta^*(z, 1) \). Then \( C'_1 > 0, C'_2 < \infty \) and
\[
C'_1(C'_2)^{-1} \eta^*(x, t) \leq \eta^*(x', t) \leq (C'_1)^{-1} C'_2 \eta^*(x, t)
\]
for \( 1/(2A_1A_3) \leq t \leq 1 \) and \( |x'| \geq |x| \), which shows \((4.5)\) for \( 1/(2A_1A_3) \leq t \leq 1 \) with \( B'_\infty = (C'_1)^{-1} C'_2 \).

**Proposition 4.4.**  
(1) For \( f \in L^\Phi(\mathbb{R}^N) \),
\[
\|f\|_\phi \leq \sup \left\{ \int_{\mathbb{R}^N} f(x)g(x) \, dx : g \in L^{\Phi^*}(\mathbb{R}^N), \|g\|_{\Phi^*} \leq 1 \right\} \leq 2\|f\|_\Phi.
\]

(2) If a measurable function \( f \) satisfies
\[
\sup \left\{ \int_{\mathbb{R}^N} |f(x)g(x)| \, dx : g \in L^{\Phi^*}(\mathbb{R}^N), \|g\|_{\Phi^*} \leq 1 \right\} < \infty,
\]
then \( f \in L^\Phi(\mathbb{R}^N) \).

**Proof.**  
(1) This assertion is proved in [22, Theorem 13.11].

(2) Given a measurable function \( f \), set \( E_n = \{ x \in \mathbb{R}^N : |x| \leq n, |f(x)| \leq n \} \) and \( f_n = f \chi_{E_n} \) for \( n \in \mathbb{N} \). Then \( f_n \in L^\Phi(\mathbb{R}^N) \) and by \((4.6)\), we have
\[
\|f_n\|_\Phi \leq \sup \left\{ \int_{\mathbb{R}^N} |f(x)g(x)| \, dx : g \in L^{\Phi^*}(\mathbb{R}^N), \|g\|_{\Phi^*} \leq 1 \right\} < \infty.
\]
By the monotone convergence theorem, we conclude that \( f \in L^\Phi(\mathbb{R}^N) \).

**5. A generalization of a theorem of CFMP**

In this section, we give a generalization of a result due to Cruz-Uribe, Fiorenza, Martell and Pérez [6, Theorem 1.3]. Before we state the theorem, we prepare the following lemma, which is easily verified:
Lemma 5.1. For $\theta > 0$, set

$$\Phi_\theta(x,t) = \bar{\Phi}(x,t^{1/\theta}).$$

Then:

1. $\Phi_\theta(x,t)$ also satisfies (Φ1) and (Φ2);
2. if $\Phi(x,t)$ satisfies (Φ3; $\varepsilon$) and $\theta \leq 1 + \varepsilon$, then $\Phi_\theta(x,t)$ satisfies (Φ3; $(1+\varepsilon-\theta)/\theta$);
3. if $\Phi(x,t)$ satisfies (Φ4), then $\Phi_\theta(x,t)$ also satisfies (Φ4): for

$$\phi_\theta(x,t) = \begin{cases} \Phi_\theta(x,t)/t & (t > 0), \\ 0 & (t = 0), \end{cases}$$

we have

$$\phi_\theta(x,2t) \leq A_{3,\theta}\phi_\theta(x,t)$$

with $A_{3,\theta} = 2^{1/\theta-1}A_3^{j(\theta)}$, where $j(\theta)$ is the integer such that

$$j(\theta) - 1 < 1/\theta \leq j(\theta);$$

4. if $\Phi(x,t)$ satisfies (Φ5; $\nu$), then $\Phi_\theta(x,t)$ satisfies (Φ5; $\nu/\theta$); if $\Phi(x,t)$ satisfies (Φ6; $\omega$), then $\Phi_\theta(x,t)$ satisfies (Φ6; $\omega/\theta$).

Further, if $\theta \leq 1 + \varepsilon$, then

$$\|f\|_{\Phi_\theta} = \|\|f\|^{1/\theta}\|_{\Phi}$$

for $f \in L^{\Phi_\theta}(\mathbb{R}^N)$.

Theorem 5.2. Suppose $\Phi(x,t)$ satisfies (Φ3; $\varepsilon$), (Φ4), (Φ5; $\nu$) and (Φ6; $\omega$) for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ and let $0 < p_0 < 1 + \varepsilon$. Assume that

$$\nu < \frac{(1+\varepsilon-p_0)(1+\varepsilon^*(p_0))}{N} \quad \text{and} \quad \omega \leq (1+\varepsilon-p_0)(1+\varepsilon^*(p_0)),$$

where, defining $j(p_0)$ by (5.2), we write

$$\varepsilon^*(p_0) = \frac{\log 2}{\log A_{3,p_0}} = \frac{\log 2}{\log(2^{1/p_0-1}A_3^{j(p_0)})}.$$ 

Let $\mathcal{F}$ be a family of ordered pairs $(f,g)$ of nonnegative measurable functions on $\mathbb{R}^N$. If

$$\int_{\mathbb{R}^N} f(x)^{p_0}w(x) \, dx \leq C_0 \int_{\mathbb{R}^N} g(x)^{p_0}w(x) \, dx$$

for all $(f,g) \in \mathcal{F}$ and for all $A_1$-weights $w$ with a constant $C_0$ depending only on $p_0$ and the $A_1$-constant of $w$, then there is a constant $C > 0$ such that

$$\|f\|_{\Phi} \leq C\|g\|_{\Phi}$$

for all $(f,g) \in \mathcal{F}$ with $g \in L^{\Phi}(\mathbb{R}^N)$.

Proof. By Lemma 5.1, $\Phi_{p_0}(x,t)$ satisfies (Φ1), (Φ2), (Φ4) with constant $A_{3,p_0} = 2^{1/p_0-1}A_3^{j(p_0)}$, (Φ3; $(1+\varepsilon-p_0)/p_0$), (Φ5; $\nu/p_0$) and (Φ6; $\omega/p_0$). Let $\Psi(x,t) = \Phi_{p_0}^{-1}(x,t)$. Then, by Propositions 4.1 and 4.3, $\Psi(x,t)$ satisfies (Φ1), (Φ2), (Φ4), (Φ3; $\varepsilon^*(p_0)$), (Φ5; $\nu/(1+\varepsilon-p_0)$) and (Φ6; $\omega/(1+\varepsilon-p_0)$). Therefore, the maximal operator $M$ is bounded on $L^{\Psi}(\mathbb{R}^N)$ by (5.4) and Theorem 3.1, namely there is a constant $A > 0$ such that

$$\|Mh\|_{\Psi} \leq A\|h\|_{\Psi} \quad \text{for all} \ h \in L^{\Psi}(\mathbb{R}^N).$$
Let \( j = 0, 1, \ldots \). Denote by \( M^j \) the \( j \)-fold composition of \( M \), where it is understood that \( M^0 f = |f| \). Consider an operator \( T : L^\Psi (R^N) \to L^\Psi (R^N) \) defined by

\[
Th = \sum_{j=0}^{\infty} \frac{M^j h}{2^j A^2}, \quad h \in L^\Psi (R^N).
\]

Note that \( \|Th\|_\Psi \leq 2\|h\|_\Psi \).

Now, let \( h \in L^\Psi (R^N) \) and \( h \geq 0 \). Then \( Th \geq h \) and \( M(Th) \leq 2ATh \). A direct calculation shows that \( Th \) is an \( A_\Psi \)-weight with \( A_1 \)-constant less than or equal to \( 2A \) (see, e.g., [9, Lemma 5.1]). Therefore by our assumption there is a constant \( C_0 \) independent of \( h \) such that

\[
\int_{R^N} f(x)^p h(x) \, dx \leq \int_{R^N} f(x)^p Th(x) \, dx \leq C_0 \int_{R^N} g(x)^p Th(x) \, dx
\]

for all \((f, g) \in \mathcal{F}\).

Thus, if \((f, g) \in \mathcal{F}\) with \( g \in L^{\Phi}(R^N) \), then, by Proposition 4.4 (applied to \( \Phi_{p_0} \)), we have

\[
\int_{R^N} f(x)^p h(x) \, dx \leq 2C_0 \|g\|_{\Phi_{p_0}} \|Th\|_{\Psi} \leq 4C_0 \|g\|_{\Phi_{p_0}} \|h\|_{\Psi}
\]

for all \( h \in L^\Psi (R^N) \) with \( h \geq 0 \). Therefore, by Proposition 4.4 again, \( f^p \in L^{\Phi_{p_0}} (R^N) \) and \( \|f^p\|_{\Phi_{p_0}} \leq 4C_0 \|g\|_{\Phi_{p_0}} \). By (5.3), \( f \in L^{\Phi} (R^N) \) and

\[
\|f\|_{\Phi} \leq (4C_0)^{1/p_0} \|g\|_{\Phi}.
\]

\textbf{Remark 5.3.} If \( p_0 \geq 1 \), then \( \varepsilon^*(p_0) \geq \varepsilon^*(1) = \varepsilon^* \).

By using two types of extrapolation theorems as in [6], we obtain the following corollaries. Let \( \mathcal{F} \) be a family of ordered pairs \((f, g)\) of nonnegative measurable functions on \( R^N \).

\textbf{Corollary 5.4.} (cf. [6, Corollary 1.10]) Suppose \( \Phi(x, t) \) satisfies \((\Phi3; \varepsilon)\), \((\Phi4)\), \((\Phi5; \nu)\) and \((\Phi6; \omega)\) with \( \varepsilon > 0 \), \( \nu > 0 \) and \( \omega > 0 \) satisfying

\[
\nu < (1 + \varepsilon)/N
\]

and

\[
\omega < 1 + \varepsilon.
\]

Let \( 0 < p_0 < \infty \). If (5.6) holds for all \((f, g) \in \mathcal{F}\) and for all \( A_\Psi \)-weights \( w \) with a constant \( C_0 \) depending only on \( p_0 \) and the \( A_\Psi \)-constant of \( w \), then there is a constant \( C > 0 \) such that

\[
\|f\|_{\Phi} \leq C \|g\|_{\Phi}
\]

for all \((f, g) \in \mathcal{F}\) with \( g \in L^{\Phi}(R^N) \). Furthermore,

\[
\left\| \left( \sum_j (f_j)^q \right)^{1/q} \right\|_{\Phi} \leq C \left\| \left( \sum_j (g_j)^q \right)^{1/q} \right\|_{\Phi}
\]

for every \( 0 < q < \infty \) and \( \{(f_j, g_j)\} \subset \mathcal{F} \).

\textbf{Proof.} By an extrapolation theorem [6, Theorem 6.1], for every \( 0 < p < \infty \) and \( w \in A_\infty \),

\[
\int_{R^N} f(x)^p w(x) \, dx \leq C \int_{R^N} g(x)^p w(x) \, dx, \quad (f, g) \in \mathcal{F}
\]
and, for every $0 < p, q < \infty$ and $w \in A_\infty$,
\[
\int_{\mathbb{R}^N} \left( \sum_j f_j(x)^q \right)^{p/q} w(x) \, dx \leq C \int_{\mathbb{R}^N} \left( \sum_j g_j(x)^q \right)^{p/q} w(x) \, dx.
\]

Choosing $p_1 > 0$ satisfying $\nu < (1 + \varepsilon - p_1)/N$ and $\omega < 1 + \varepsilon - p_1$ and applying Theorem 5.2 with this $p_1$ in place of $p_0$, we obtain the first assertion. The second assertion can be derived by applying Theorem 5.2 to the family
\[
\mathcal{F}_q = \left\{ \left( \left( \sum_j (f_j)^q \right)^{1/q}, \left( \sum_j (g_j)^q \right)^{1/q} \right) : \{(f_j, g_j)\}_j \subset \mathcal{F} \right\}.
\]

**Corollary 5.5.** (cf. [6, Corollary 1.11]) Suppose $\Phi(x, t)$ satisfies $(\Phi_3; \varepsilon)$, $(\Phi_4)$, $(\Phi_5; \nu)$ and $(\Phi_6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying
\[
\nu < \varepsilon(1 + \varepsilon^*)/N
\]
and
\[
\omega < \varepsilon(1 + \varepsilon^*)
\]
for $\varepsilon^*$ given in Proposition 4.1. Let $\mathcal{F}$ be a family of ordered pairs $(f, g)$ of nonnegative measurable functions on $\mathbb{R}^N$. Let $1 < p_0 < \infty$. If (5.6) holds for all $(f, g) \in \mathcal{F}$ and for all $A_{p_0}$-weights $w$ with a constant $C_0$ depending only on $p_0$ and the $A_{p_0}$-constant of $w$, then there is a constant $C > 0$ such that
\[
\|f\|_{\Phi} \leq C\|g\|_{\Phi}
\]
for all $(f, g) \in \mathcal{F}$ with $g \in L_\Phi(\mathbb{R}^N)$. Furthermore,
\[
\left\| \left( \sum_j (f_j)^q \right)^{1/q} \right\|_{\Phi} \leq C \left\| \left( \sum_j (g_j)^q \right)^{1/q} \right\|_{\Phi}
\]
for every $1 < q < \infty$ and $\{(f_j, g_j)\}_j \subset \mathcal{F}$.

**Proof.** By the extrapolation theorem [6, Theorem 6.2], for every $1 < p < \infty$ and $w \in A_p$,
\[
\int_{\mathbb{R}^N} f(x)^p w(x) \, dx \leq C \int_{\mathbb{R}^N} g(x)^p w(x) \, dx, \quad (f, g) \in \mathcal{F}
\]
and, for every $1 < p, q < \infty$ and $w \in A_p$,
\[
\int_{\mathbb{R}^N} \left( \sum_j f_j(x)^q \right)^{p/q} w(x) \, dx \leq C \int_{\mathbb{R}^N} \left( \sum_j g_j(x)^q \right)^{p/q} w(x) \, dx.
\]

Choosing $1 < p_1 < 1 + \varepsilon$ satisfying $\nu < (1 + \varepsilon - p_1)(1 + \varepsilon^*)/N$ and $\omega < (1 + \varepsilon - p_1)(1 + \varepsilon^*)$ and applying Theorem 5.2 with this $p_1$ in place of $p_0$, we obtain the first assertion. The second assertion follows from the same arguments as in the previous corollary. \qed

**Remark 5.6.** Assumptions (5.7) and (5.8) are weaker than (5.9) and (5.10), respectively. In fact, we see that
\[
(A_1 A_{2, \varepsilon})^{-1} \varepsilon^* t \leq \phi(x, t) \leq A_1 A_2 \varepsilon^* t^{1/\varepsilon^*}
\]
for $t \geq 1$, which implies $\varepsilon \varepsilon^* \leq 1$, so that $\varepsilon(1 + \varepsilon^*) \leq 1 + \varepsilon$.

From Corollary 5.5 with the pairs $(M_f, |f|)$, we obtain vector-valued inequalities for $M$ on $L_\Phi(\mathbb{R}^N)$. Recall that $\varepsilon^*(p_0)$ is defined by (5.5).
Corollary 5.7. Suppose that $\Phi(x,t)$ satisfies $(\Phi 3; \varepsilon)$, $(\Phi 4)$, $(\Phi 5; \nu)$ and $(\Phi 6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$. Let $q > 1$.

1. If $\nu < \varepsilon(1 + \varepsilon^*)/N$ and $\omega < \varepsilon(1 + \varepsilon^*)$, then

$$\left\| \left( \sum_{j=1}^{\infty} (Mf_j)^q \right)^{1/q} \right\|_{\Phi} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_{\Phi}$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ of measurable functions.

2. If $\nu < (1 + \varepsilon - 1/q)(1 + \varepsilon^*(1/q))/N$ and $\omega < (1 + \varepsilon - 1/q)(1 + \varepsilon^*(1/q))$, then

$$\left\| \sum_{j=1}^{\infty} (Mf_j)^q \right\|_{\Phi} \leq C \left\| \sum_{j=1}^{\infty} |f_j|^q \right\|_{\Phi}$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ of measurable functions.

Proof. (1) This is a direct consequence of Corollary 5.5 applied to the family $\mathcal{F} = \{(Mf, |f|): |f| \geq 1\}$; see [1].

(2) By Lemma 5.1, $\Phi_{1/q}(x,t) = \Phi(x,t')$ satisfies $(\Phi 3; \varepsilon')$, $(\Phi 4)$, $(\Phi 5; \nu')$ and $(\Phi 6; \omega')$ with $\varepsilon' = q(1 + \varepsilon) - 1$, $\nu' = q \nu$ and $\omega' = q \omega$.

By assumption,

$$\varepsilon' > 0, \quad 0 < \nu' < \varepsilon'(1 + \varepsilon^*(1/q))/N, \quad 0 < \omega' < \varepsilon'(1 + \varepsilon^*(1/q)).$$

Hence, by Corollary 5.5, we have

$$\left\| \left( \sum_{j=1}^{\infty} (Mf_j)^q \right)^{1/q} \right\|_{\Phi_{1/q}} \leq C \left\| \left( \sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_{\Phi_{1/q}},$$

which implies the required inequality in view of Lemma 5.1. \qed

6. Some applications of CFMP-theorem

6.1. Sharp maximal function. For $f \in L^1_{loc}(\mathbb{R}^N)$, the sharp maximal function $M^#f$ is defined by

$$M^#f(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, dy,$$

where $f_B = (1/|B|) \int_B f(x) \, dx$ for a ball $B$.

Since $0 \leq M^#f \leq 2Mf$, by Theorem 3.1, we have

Proposition 6.1. Suppose $\Phi(x,t)$ satisfies $(\Phi 3; \varepsilon)$, $(\Phi 5; \nu)$ and $(\Phi 6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < (1 + \varepsilon)/N$ and $\omega \leq 1 + \varepsilon$. Then

$$\|M^#f\|_{\Phi} \leq 2CM\|f\|_{\Phi}$$

for all $f \in L^\Phi(\mathbb{R}^N)$.

The following inequality is known (cf. [16]): for $0 < p < \infty$,

$$\int_{\mathbb{R}^N} [Mf(x)]^p w(x) \, dx \leq C \int_{\mathbb{R}^N} [M^#f(x)]^p w(x) \, dx$$

for all $f \in L^\infty(\mathbb{R}^N)$ (= the space of $L^\infty$-functions with compact support) and $w \in A_\infty$.

Thus, applying Corollary 5.4 to $\mathcal{F} = \{(Mf, M^#f): f \in L^\infty(\mathbb{R}^N)\}$, we have
Proposition 6.2. Suppose \( \Phi(x, t) \) satisfies (\( \Phi_3; \varepsilon \)), (\( \Phi_4 \)), (\( \Phi_5; \nu \)) and (\( \Phi_6; \omega \)) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1+\varepsilon)/N \) and \( \omega < 1+\varepsilon \). Then
\[
\|Mf\|_\Phi \leq C\|M^\#f\|_\Phi
\]
for all \( f \in L^\infty_c(\mathbb{R}^N) \).

In view of Propositions 6.1 and 6.2, we can state:

Corollary 6.3. Suppose \( \Phi(x, t) \) satisfies (\( \Phi_3; \varepsilon \)), (\( \Phi_4 \)), (\( \Phi_5; \nu \)) and (\( \Phi_6; \omega \)) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1+\varepsilon)/N \) and \( \omega < 1+\varepsilon \). Then
\[
C^{-1}\|f\|_\Phi \leq \|M^\#f\|_\Phi \leq C\|f\|_\Phi
\]
for \( f \in L^\Phi(\mathbb{R}^N) \).

6.2. Singular integral operators. We consider a singular integral operator \( T \) associated to a standard kernel \( k(x, y) \) (see, e.g., [7, Section 6.3]). By \( C^\infty_0(\mathbb{R}^N) \) we denote the set of all compactly supported \( C^\infty \)-functions in \( \mathbb{R}^N \).

Recall the following result due to Alvarez and Pérez [3]:

Lemma 6.4. Let \( T \) be a singular integral operator associated to a standard kernel and suppose \( T \) extends to a bounded operator from \( L^1(\mathbb{R}^N) \) to \( wL^1(\mathbb{R}^N) \). Then, for \( 0 < \theta < 1 \) there exists a constant \( C(\theta) > 0 \) such that
\[
M^\#(|Tf|^\theta)(x) \leq C(\theta)|Mf(x)|^\theta
\]
for all \( f \in C^\infty_0(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \).

Theorem 6.5. Let \( T \) be a singular integral operator associated to a standard kernel and suppose \( T \) extends to a bounded operator from \( L^1(\mathbb{R}^N) \) to \( wL^1(\mathbb{R}^N) \). Suppose \( \Phi(x, t) \) satisfies (\( \Phi_3; \varepsilon \)), (\( \Phi_4 \)), (\( \Phi_5; \nu \)) and (\( \Phi_6; \omega \)) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1+\varepsilon)/N \) and \( \omega < 1+\varepsilon \). Then \( T \), defined initially on \( C^\infty_0(\mathbb{R}^N) \), can be extended to a bounded operator from \( L^\Phi(\mathbb{R}^N) \) into itself.

Proof. Let \( 0 < \theta < 1 \). By Lemma 5.1, \( \Phi_\theta(x, t) = \Phi(x, t^{1/\theta}) \) satisfies (\( \Phi_3; \varepsilon' \)) with \( \varepsilon' = (1+\varepsilon-\theta)/\theta \), (\( \Phi_5; \nu/\theta N \)) and (\( \Phi_6; \omega/\theta \)). Note that \( (1+\varepsilon)/\theta = 1+\varepsilon' \).

Let \( f \in C^\infty_0(\mathbb{R}^N) \). Then, using Proposition 6.2, the above lemma and then Theorem 3.1, we obtain
\[
|Tf|_\Phi = \left( \|Tf\|_{\Phi_\theta} \right)^{1/\theta} \leq C\|M^\#(|Tf|^\theta)\|_{\Phi_\theta}^{1/\theta} = C\|M^\#(|Tf|^\theta)\|_{\Phi}^{1/\theta} \leq C\|f\|_{\Phi}.
\]
Since \( C^\infty_0(\mathbb{R}^N) \) is dense in \( L^\Phi(\mathbb{R}^N) \) (cf. [20]), we obtain the required assertion. \( \square \)

Remark 6.6. If \( K \) is a locally integrable function on \( \mathbb{R}^N \setminus \{0\} \) such that its Fourier transform is bounded and
\[
|K(x)| \leq \frac{C}{|x|^N}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{N+1}}, \quad x \neq 0.
\]
Then, for the singular integral operator \( T_K \) defined by \( T_K f = K * f \) and for \( 1 < p < \infty \), there exists a constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^N} |T_K f(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^N} |f(x)|^p w(x) \, dx
\]
for all \( w \in A_p \) and \( f \in L^p(\mathbb{R}^N; w) \) (see, e.g., [9, Theorem 3.1, p. 411]). Therefore, by Corollary 5.5, we have
Proposition 6.7. Suppose \( \Phi(x, t) \) satisfies \( (\Phi 3; \varepsilon), (\Phi 4), (\Phi 5; \nu) \) and \( (\Phi 6; \omega) \) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < \varepsilon (1 + \varepsilon^*)/N \) and \( \omega < \varepsilon (1 + \varepsilon^*) \). Then

\[
\| T_K f \|_\Phi \leq C \| f \|_\Phi
\]

for all \( f \in L^\Phi(\mathbb{R}^N) \).

6.3. Kerman–Sawyer inequality. In this subsection, let \( k(r) \) be a non-negative nonincreasing lower semi-continuous function on \((0, \infty)\) such that

\[
\int_0^1 k(r)r^{N-1}dr < \infty
\]

and there is \( R_0 > 0 \) such that \( k(r) \) is positive and satisfies the doubling condition on \((0, R_0)\), i.e., \( k(r) \leq C_0 k(2r) \) for \( 0 < r < R_0/2 \).

Set \( k(0) = \liminf_{r \to 0^+} k(r) \). \( k(0) \) may be \( \infty \). With an abuse of notation, we write \( k(x) = k(|x|) \) for \( x \in \mathbb{R}^N \). Let \( k^*(r) = r^{-N} \int_0^r k(t) t^{N-1} dt \). The \( k \)-maximal function of a non-negative measure \( \mu \) is defined by

\[
M_k \mu(x) = \sup_{r>0} k^*(r) \mu(B(x, r)).
\]

Kerman and Sawyer [17, Theorem 2.2] showed that

\[
C^{-1} \| M_k \mu \|_p \leq \| k \ast \mu \|_p \leq C \| M_k \mu \|_p
\]

for \( 1 < p < \infty \). The left inequality follows from

\[
M_k \mu(x) \leq CM(k \ast \mu)(x) \quad \text{for all } x \in \mathbb{R}^N,
\]

which was proved in [2, Lemma 4.3.1] and [17, Theorem 2.2(a)], and the boundedness of the maximal operator \( M \). This inequality, together with our Theorem 3.1, gives

Proposition 6.8. Suppose \( \Phi(x, t) \) satisfies \( (\Phi 3; \varepsilon), (\Phi 5; \nu) \) and \( (\Phi 6; \omega) \) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1 + \varepsilon)/N \) and \( \omega \leq 1 + \varepsilon \). Then

\[
\| M_k \mu \|_\Phi \leq C \| k \ast \mu \|_\Phi.
\]

As a weighted version of the right inequality in (6.1), we have

Lemma 6.9. Let \( 1 < p < \infty \) and \( w \in A_p \). Then there exists a constant \( C > 0 \) such that

\[
\int_{\mathbb{R}^N} [(k \ast \mu)(x)]^p w(x) dx \leq C \int_{\mathbb{R}^N} \mu w(x) dx
\]

for all nonnegative measures \( \mu \) on \( \mathbb{R}^N \).

This lemma is essentially proved in [15, Section 5] and [18, Proposition 1’]. By using the method given in the proof of [35, Theorem 3.1.2] and modifying the proof of [2, Part II, Theorem 4.3.1] to the weighted case, we can prove this lemma.

By this lemma and Corollary 5.5, we have

Theorem 6.10. Suppose \( \Phi(x, t) \) satisfies \( (\Phi 3; \varepsilon), (\Phi 4), (\Phi 5; \nu) \) and \( (\Phi 6; \omega) \) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < \varepsilon (1 + \varepsilon^*)/N \) and \( \omega < \varepsilon (1 + \varepsilon^*) \). Then there exists a constant \( C > 0 \) such that

\[
\| k \ast \mu \|_\Phi \leq C \| M_k \mu \|_\Phi
\]

for all nonnegsive measures \( \mu \) on \( \mathbb{R}^N \) such that \( M_k \mu \in L^\Phi(\mathbb{R}^N) \).
7. Decomposition for Musielak–Orlicz spaces

In this section, we give a decomposition theorem for functions in \( L^\Phi(\mathbb{R}^N) \).

**Theorem 7.1.** Let \( d \in \mathbb{N} \cup \{0\} \). Suppose \( \Phi(x,t) \) satisfies \((\Phi_3;\varepsilon), (\Phi_4), (\Phi_5;\nu)\) and \((\Phi_6;\omega)\) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1 + \varepsilon)/N \) and \( \omega < 1 + \varepsilon \). Then every \( f \in L^\Phi(\mathbb{R}^N) \) has a decomposition

\[
f = \sum_{j=1}^{\infty} \lambda_j a_j,
\]

with \( a_j \in L^\infty(\mathbb{R}^N) \) and \( \lambda_j \in [0,\infty), j = 1,2, \ldots, \) such that

\[
|a_j| \leq \chi_{Q_j} \quad \text{for a cube } Q_j \text{ for each } j,
\]

\[
\int_{\mathbb{R}^N} a_j(x)x^\beta \, dx = 0 \quad \text{for all multi-indices } \beta \text{ with } |\beta| \leq d, \quad \text{for each } j
\]

and

\[
\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_\Phi \leq C\|f\|_\Phi
\]

with a constant \( C > 0 \) independent of \( f \).

To prove this theorem, we introduce the grand maximal operator which is originally used in the definition of the Hardy space \( H^p(\mathbb{R}^N) \) with \( 0 < p < \infty \). The grand maximal operator \( \mathcal{M} \) is defined by

\[
(7.1) \quad \mathcal{M}f(x) = \sup\{|t^{-N}\varphi(t^{-1}\cdot) * f(x)| : t > 0, \varphi \in \mathcal{F}_L\}
\]

for \( f \in \mathcal{S}'(\mathbb{R}^N) \) and \( x \in \mathbb{R}^N \), where \( L \) is a fixed large integer and

\[
\mathcal{F}_L = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^N) : \sup_{x \in \mathbb{R}^N} (1 + |x|)^L |\partial^\beta \varphi(x)| \leq 1 \text{ for all } \beta \text{ with } |\beta| \leq L \right\}.
\]

Here and below, we suppose \( L \geq N + 1 \).

**Lemma 7.2.** Suppose that \( \Phi \) satisfies \((\Phi_3;\varepsilon), (\Phi_5;\nu)\) and \((\Phi_6;\omega)\) for \( \varepsilon > 0, \nu > 0 \) and \( \omega > 0 \) satisfying \( \nu < (1 + \varepsilon)/N \) and \( \omega < 1 + \varepsilon \). If \( f \in L^\Phi(\mathbb{R}^N) \), then \( \mathcal{M}f \in L^\Phi(\mathbb{R}^N) \) and

\[
(7.2) \quad \|\mathcal{M}f\|_\Phi \leq C\|f\|_\Phi.
\]

**Proof.** In view of Theorem 3.1, it is enough to show

\[
(7.3) \quad \mathcal{M}f(x) \leq C Mf(x)
\]

for \( f \in \mathcal{S}'(\mathbb{R}^N) \cap L^1_{\text{loc}}(\mathbb{R}^N) \). Let \( h_t(x) = t^{-N}(1 + |x|/t)^{-L} \) for \( t > 0 \). Then,

\[
|h_t * f(x)| \leq \|h_t\|_1 Mf(x) = \|h_t\|_1 Mf(x)
\]

(cf., e.g., [8, Proposition 2.7] or [10, Theorem 2.1.10]). Since \( |\varphi| \leq h_1 \) if \( \varphi \in \mathcal{F}_L \),

\[
|t^{-N}\varphi(t^{-1}\cdot) * f(x)| \leq h_t * |f|(x) \leq \|h_1\|_1 Mf(x)
\]

for all \( \varphi \in \mathcal{F}_L, t > 0 \) and \( x \in \mathbb{R}^N \), which yields (7.3) with \( C = \|h_1\|_1 < \infty \). \( \square \)

We shall use the following lemma. We refer to [31, Chap. III, §2] for the proof.

**Lemma 7.3.** Let \( f \in \mathcal{S}'(\mathbb{R}^N) \cap L^1_{\text{loc}}(\mathbb{R}^N) \), \( d \in \mathbb{N} \cup \{0\} \) and \( r > 0 \). Set \( \mathcal{O} = \{ y \in \mathbb{R}^N : \mathcal{M}f(y) > r \} \) and consider a collection of cubes \( \{Q_k^*\} \) which has the bounded intersection property and for which \( \mathcal{O} = \bigcup_k Q_k^* \). (Such a collection can be obtained...
via the Whitney decomposition of $\mathcal{O}$.) Then, $f$ is expressed as $f = g + \sum b_k$ with $b_k$, $g \in S'(\mathbb{R}^N) \cap L^1_{\text{loc}}(\mathbb{R}^N)$ such that $b_k$ is supported in $Q_k^*$ and

$$\int_{\mathbb{R}^N} b_k(x)x^\beta\,dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \leq d$$

for each $k$. Furthermore,

$$|g(x)| \leq Cr \quad \text{for all } x \in \mathbb{R}^N$$

and

$$\mathcal{M}b_k \leq C \left( \mathcal{M}f \cdot \chi_{Q_k^*} + r \cdot \frac{\ell_k^{N+d+1}}{|x_k|^{N+d+1}} \chi_{\mathbb{R}^N \setminus Q_k^*} \right)$$

with constants $C$ depending only on $N$, where $x_k$ and $\ell_k$ denote the center and the side-length of $Q_k^*$, respectively.

**Remark 7.4.** We have the following pointwise estimate from [10, Example 2.1.8]:

$$\left( \frac{\ell_k}{\ell_k + |x - x_k|} \right)^N \leq CM\chi_{Q_k^*}(x)$$

with $C > 0$ independent of $k$, so that (7.5) implies

$$\mathcal{M}b_k \leq C \left( \mathcal{M}f \cdot \chi_{Q_k^*} + r(M\chi_{Q_k^*})^{(N+d+1)/N} \right).$$

**Proof of Theorem 7.1.** Choose $d_1 \geq d$ ($d_1 \in \mathbb{N}$) such that

$$\nu \leq \frac{1 + \varepsilon - 1/q}{N} \quad \text{and} \quad \omega \leq 1 + \varepsilon - 1/q$$

for $q = (N + d_1 - 1)/N$. For each $j \in \mathbb{Z}$, let

$$\mathcal{O}_j = \{ x \in \mathbb{R}^N : \mathcal{M}f(x) > 2^j \}.$$ 

By the previous lemma and remark, we find collections of cubes $\{Q_{j,k}^*\}_{k \in K_j}$ having the bounded intersection property such that $\bigcup_{k \in K_j} Q_{j,k}^* = \mathcal{O}_j$; and we have a decomposition

$$f = g_j + b_j, \quad b_j = \sum_{k \in K_j} b_{j,k}$$

with $b_{j,k}$, $g_j \in S'(\mathbb{R}^N) \cap L^1_{\text{loc}}(\mathbb{R}^N)$ such that $b_{j,k}$ supported in $Q_{j,k}^*$,

$$\int_{\mathbb{R}^N} b_{j,k}(x)x^\beta\,dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \leq d_1,$$

$$|g_j(x)| \leq C2^j$$

and

$$\mathcal{M}b_{j,k} \leq C \left( \mathcal{M}f \cdot \chi_{Q_{j,k}^*} + 2^j(M\chi_{Q_{j,k}^*})^{1/q} \right).$$

Since $g_j \to 0$ uniformly as $j \to -\infty$, $g_j \to 0$ in $S'$ as $j \to -\infty$. On the other hand, by (7.7) we have

$$\|b_j\|_\Phi \leq \left\| \sum_k \mathcal{M}b_{j,k} \right\|_\Phi \leq C \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} + \sum_k 2^j(M\chi_{Q_{j,k}^*})^q \right\|_\Phi$$

$$\leq C \left\| \mathcal{M}f \cdot \chi_{\mathcal{O}_j} \right\|_\Phi + C \left\| 2^j \sum_k (M\chi_{Q_{j,k}^*})^q \right\|_\Phi.$$
Now, by Corollary 5.7 (2), we see
\[ \left\| 2^j \sum_k (M \chi_{Q_j,k})^q \right\|_{\Phi} \leq C \left\| 2^j \sum_k \chi_{Q_j,k}^* \right\|_{\Phi} \leq C \left\| M f \cdot \chi_{\mathcal{O}_j} \right\|_{\Phi}. \]
Hence,
\[ \|b_j\|_{\Phi} \leq C \left\| M f \cdot \chi_{\mathcal{O}_j} \right\|_{\Phi} \to 0 \]
as \( j \to \infty \), which implies that \( b_j \to 0 \) in the sense of distributions as \( j \to \infty \).

Therefore
\[ (7.8) \quad f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j), \]
with the sum converging in the sense of distributions. Going through the same arguments as in [31, pp. 108–109], we have functions \( \{A_{j,k}\}_{k \in K_j} \) such that
\[ (7.9) \quad g_{j+1} - g_j = \sum_{k \in K_j} A_{j,k}, \]
\[ |A_{j,k}| \leq C_0 2^j \chi_{Q^*_{j,k}} \]
for some universal constant \( C_0 \) and \( \int_{\mathbb{R}^N} A_{j,k}(x)x^\beta \, dx = 0 \) for all \( \beta \) with \( |\beta| \leq d \).

Let us set
\[ a_{j,k} = \frac{A_{j,k}}{C_0 2^j}, \quad \lambda_{j,k} = C_0 2^j, \quad \text{for } k \in K_j, \ j \in \mathbb{Z}. \]
Then
\[ |a_{j,k}| \leq \chi_{Q^*_{j,k}}, \quad \int_{\mathbb{R}^N} a_{j,k}(x)x^\beta \, dx = 0 \quad \text{for all } \beta \text{ with } |\beta| \leq d \]
and
\[ f = \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} \lambda_{j,k} a_{j,k}, \]
where the summation is convergent in the sense of distributions.

What remains to show is the estimate
\[ (7.10) \quad \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} \lambda_{j,k} \chi_{Q^*_{j,k}} \right\|_{\Phi} \leq C \left\| f \right\|_{\Phi}. \]
By the bounded intersection property, \( \sum_{k \in K_j} \chi_{Q^*_{j,k}} \leq C \chi_{\mathcal{O}_j} \), and by the definition of \( \mathcal{O}_j \), \( \sum_{j \in \mathbb{Z}} 2^j \chi_{\mathcal{O}_j} \leq 2M f \). Hence
\[ \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} \lambda_{j,k} \chi_{Q^*_{j,k}} \right\|_{\Phi} = C_0 \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} 2^j \chi_{Q^*_{j,k}} \right\|_{\Phi} \leq C \left\| M f \right\|_{\Phi}. \]
Required inequality (7.10) now follows from Lemma 7.2. \( \square \)
8. An application to Olsen inequality

For $0 < q \leq p < \infty$, recall that the Morrey space $\mathcal{M}_q^p(\mathbb{R}^N)$ is defined by
$$\mathcal{M}_q^p(\mathbb{R}^N) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^N) : \| f \|_{\mathcal{M}_q^p} < \infty \},$$
where
$$\| f \|_{\mathcal{M}_q^p} = \sup_{Q: \text{cube}} |Q|^2^{-\frac{1}{p}} \left( \int_Q |f(y)|^q \, dy \right)^{\frac{1}{q}}.$$

For $0 < \alpha < N$, we define the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $\mathbb{R}^N$ by
$$I_\alpha f(x) = \int_{\mathbb{R}^N} |x - y|^{\alpha - N} f(y) \, dy.$$

Here it is natural to assume that
$$\int_{\mathbb{R}^N} (1 + |y|)^{\alpha - N} |f(y)| \, dy < \infty$$
(see [21, Theorem 1.1, Chapter 2]), which is a necessary and sufficient condition for the integral defining $I_\alpha f(x)$ to converge for almost all $x \in \mathbb{R}^N$.

**Theorem 8.1.** Suppose $\Phi(x,t)$ satisfies $(\Phi_3; \varepsilon)$, $(\Phi_4)$, $(\Phi_5; \nu)$ and $(\Phi_6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < \varepsilon(1 + \varepsilon^*)/N$ and $\omega < \varepsilon(1 + \varepsilon^*)$ for $\varepsilon^*$ given in Proposition 4.1. Let
$$0 < \alpha < \frac{N\varepsilon^*}{1 + \varepsilon^*} \quad \text{and} \quad 1 + \frac{1}{\varepsilon^*} < u \leq \frac{N}{\alpha},$$
Then
$$\| g \cdot I_\alpha f \|_{\Phi} \leq C \| g \|_{\mathcal{M}_q^p} \| f \|_{\Phi}$$
for all $f \in L^q(\mathbb{R}^N)$ satisfying (8.1) and $g \in \mathcal{M}_u^{N/\alpha}(\mathbb{R}^N)$.

To prove Theorem 8.1, we need the following lemmas:

**Lemma 8.2.** [14, Lemma 4.2] Let $d \in \mathbb{N} \cup \{0\}$. Suppose $h$ is an $L^\infty$-function supported on a cube $Q$. Assume in addition that $\int_{\mathbb{R}^N} x^\beta h(x) \, dx = 0$ for all $\beta$ with $|\beta| \leq d$. Then,
$$\| I_\alpha h(x) \| \leq C_{\alpha,d} \| h \|_{\infty} \ell(Q)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(N+d+1-\alpha)}} \chi_{2^kQ}(x) \quad (x \in \mathbb{R}^N),$$
where $\ell(Q)$ denotes the side length of $Q$.

**Lemma 8.3.** Suppose $\Phi$ satisfies $(\Phi_3; \varepsilon)$, $(\Phi_4)$, $(\Phi_5; \nu)$ and $(\Phi_6; \omega)$ for $\varepsilon > 0$, $\nu > 0$ and $\omega > 0$ satisfying $\nu < \varepsilon(1 + \varepsilon^*)/N$ and $\omega \leq \varepsilon(1 + \varepsilon^*)$. Let $\{Q_j\}_{j=1}^\infty$ be a sequence of cubes, $\{a_j\}_{j=1}^\infty$ be a sequence of non-negative functions in $L^\infty(\mathbb{R}^N)$ for $u > 1 + 1/\varepsilon^*$ and let $\{\lambda_j\}_{j=1}^\infty$ be a sequence of non-negative numbers. If $\text{supp } a_j \subset Q_j$, $\|a_j\| \leq |Q_j|^{1/u}$ for each $j$ and $\left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\Phi} < \infty$, then $\sum_{j=1}^\infty \lambda_j a_j \in L^p(\mathbb{R}^N)$ and
$$\left\| \sum_{j=1}^\infty \lambda_j a_j \right\|_{\Phi} \leq C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{\Phi}.$$

**Proof of Lemma 8.3.** Consider $g \in L^q(\mathbb{R}^N)$ with $\|g\|_{\Phi^*} \leq 1$ and set
$$\Lambda(g) = \int_{\mathbb{R}^N} |g(x)| \sum_{j=1}^\infty \lambda_j a_j(x) \, dx.$$
By the Hölder inequality, we obtain

\[
|\Lambda(g)| \leq \sum_{j=1}^{\infty} \lambda_j \left( \int_{Q_j} |g(x)|^{u'} \, dx \right)^{1/u'} \|a_j\|_u \leq C \sum_{j=1}^{\infty} \lambda_j |Q_j| \left( \inf_{Q_j} M[|g|^{u'}] \right)^{1/u'}
\]

\[
\leq C \int_{\mathbb{R}^n} \left( \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(x) \right) \left( M[|g|^{u'}](x) \right)^{1/u'} \, dx
\]

\[
\leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\| \left\| \left( M[|g|^{u'}] \right)^{1/u'} \right\|_{\Phi,u'},
\]

where \(1/u + 1/u' = 1\). By Propositions 4.1 and 4.3, \(\Phi^*(x, t)\) satisfies \((\Phi3; \varepsilon^*)\), \((\Phi5; \nu/\varepsilon)\) and \((\Phi6; \omega/\varepsilon)\). Hence, by Lemma 5.1, \(\Phi^*_{\varepsilon}(x, t) = \Phi^*(x, t^{1/u'})\) satisfies \((\Phi3; (1 + \varepsilon^*/u' - 1), (\Phi5; \nu/(\varepsilon u'))\) and \((\Phi6; \omega/(\varepsilon u'))\). Note that \(u' < 1 + \varepsilon^*\). By our assumption \(\nu/(\varepsilon u') < (1 + \varepsilon^*/N)\) and \(\omega/(\varepsilon u') \leq 1 + \varepsilon^*\). Thus, by Theorem 3.1, the maximal operator \(M\) is bounded on \(L^{\Phi^*_u}(\mathbb{R}^N)\). Hence

\[
\left\| \left( M[|g|^{u'}] \right)^{1/u'} \right\|_{\Phi^*_u} = \left\| M[|g|^{u'}] \right\|_{\Phi^*_u}^{1/u'} \leq C \left\| |g|^{u'} \right\|_{\Phi^*_u} = C \|g\|_{\Phi^*} \leq C,
\]

which implies

\[
|\Lambda(g)| \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\Phi}
\]

for all \(g \in L^{\Phi^*_u}(\mathbb{R}^N)\) with \(\|g\|_{\Phi^*} \leq 1\). Now the required conclusion follows from Proposition 4.4. 

\[ \square \]

**Proof of Theorem 8.1.** First note that \(\nu < (1 + \varepsilon)/N\) and \(\omega < 1 + \varepsilon\) by Remark 5.6. Choose \(d \in \mathbb{N}\) so large that

\[
(8.4) \quad \nu \leq 1 - \frac{N}{N + d} \qquad \text{and} \quad \omega \leq 1 + \varepsilon - \frac{N}{N + d}.
\]

Let \(f \in L^{\Phi}(\mathbb{R}^N)\) satisfy (8.1). We decompose \(f\) according to Theorem 7.1 with \(d\) chosen as above; \(f = \sum_{j=1}^{\infty} \lambda_j a_j\), where \(a_j \in L^{\infty}(\mathbb{R}^N)\) and \(\lambda_j \in [0, \infty), j = 1, 2, \ldots\), satisfying the conditions in Theorem 7.1 for cubes \(\{Q_j\}_{j=1}^{\infty}\).

By Lemma 8.2,

\[
|I_a f| \leq C \sum_{k=1}^{\infty} \frac{1}{2k(N + d + 1)} \ell(2^k Q_j)^\alpha \chi_{2^k Q_j},
\]

so that

\[
|g \cdot I_a f| \leq |g| \sum_{j=1}^{\infty} \lambda_j |I_a a_j| \leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j}{2k(N + d + 1)} \left( |g| \ell(2^k Q_j)^\alpha \chi_{2^k Q_j} \right).
\]

Let

\[
a_{j, k}(x) = \frac{\ell(2^k Q_j)^\alpha}{\|g\|_{\mathcal{M}_u^{N/\alpha}}} |g(x)| \chi_{2^k Q_j}(x).
\]

Since

\[
|g \cdot \chi_{2^k Q_j}|_u \leq |2^k Q_j|^{-\alpha/N + 1/u} \|g\|_{\mathcal{M}_u^{N/\alpha}} = \ell(2^k Q_j)^{-\alpha} |2^k Q_j|^{1/u} \|g\|_{\mathcal{M}_u^{N/\alpha}},
\]
we see that \(|a_{j,k}| \leq |2^k Q_j|^{1/u}\). Hence, by Lemma 8.3, we have
\[
\|g \cdot I_\alpha f\|_\Phi \leq C\|g\|_{\mathcal{M}^{N/\alpha}_u} \left\| \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j}{2^{k(N+d+1)}} \cdot \chi_{2^k Q_j} \right\|_\Phi.
\]
Observe that \(\chi_{2^k Q_j} \leq 2^{KN} M \chi_{Q_j}\). Hence
\[
\|g \cdot I_\alpha f\|_\Phi \leq C\|g\|_{\mathcal{M}^{N/\alpha}_u} \left\| \sum_{j=1}^\infty \sum_{k=1}^\infty \frac{\lambda_j}{2^{k(N+d+1)}} 2^k(N+d) \cdot (M \chi_{Q_j})^{(N+d)/N} \right\|_\Phi.
\]
By Corollary 5.7 (2) and (8.4), we can remove the maximal operator \(M\) and we obtain
\[
\|g \cdot I_\alpha f\|_\Phi \leq C\|g\|_{\mathcal{M}^{N/\alpha}_u} \left\| \sum_{j=1}^\infty \lambda_j (M \chi_{Q_j})^{(N+d)/N} \right\|_\Phi \leq C\|g\|_{\mathcal{M}^{N/\alpha}_u} \|f\|_\Phi,
\]
which is the required inequality. \(\square\)

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