VALUE DISTRIBUTION OF THE SEQUENCES OF THE DERIVATIVES OF ITERATED POLYNOMIALS

Yûsuke Okuyama
Kyoto Institute of Technology, Division of Mathematics
Sakyo-ku, Kyoto 606-8585, Japan; okuyama@kit.ac.jp

Abstract. We establish the equidistribution of the sequence of the averaged pullbacks of a Dirac measure at any value in $\mathbb{C} \setminus \{0\}$ under the derivatives of the iterations of a polynomials $f \in \mathbb{C}[z]$ of degree more than one towards the $f$-equilibrium (or canonical) measure $\mu_f$ on $\mathbb{P}^1$. We also show that for every $C^2$ test function on $\mathbb{P}^1$, the convergence is exponentially fast up to a polar subset of exceptional values in $\mathbb{C}$. A parameter space analog of the latter quantitative result for the monic and centered unicritical polynomials family is also established.

1. Introduction

Let $f \in \mathbb{C}[z]$ be a polynomial of degree $d > 1$. Let $\mu_f$ be the $f$-equilibrium (or canonical) measure on $\mathbb{P}^1$, which coincides with the harmonic measure $\mu_{K(f)}$ on the filled-in Julia set $K(f)$ of $f$ with respect to $\infty$. The exceptional set $E(f) := \{a \in \mathbb{P}^1 : \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty\}$ of $f$ contains $\infty$ and $\#E(f) \leq 2$. Brolin [2, Theorem 16.1] studied the value distribution of the sequence $(f^n : \mathbb{P}^1 \to \mathbb{P}^1)$ of the iterations of $f$, and established

\begin{equation}
\left\{ a \in \mathbb{P}^1 : \lim_{n \to \infty} \frac{(f^n)'(a)\delta_a}{d^n} = \mu_f \text{ weakly on } \mathbb{P}^1 \right\} = \mathbb{P}^1 \setminus E(f),
\end{equation}

which is more precise than the classical inclusion $\partial K(f) \subset \bigcup_{n \in \mathbb{N}} f^{-n}(a)$ for every $a \in \mathbb{P}^1 \setminus E(f)$. Here for every $h \in \mathbb{C}(z)$ of degree $> 0$ and every Radon measure $\nu$ on $\mathbb{P}^1$, the pullback $h^*\nu$ of $\nu$ under $h$ is a Radon measure on $\mathbb{P}^1$ so that for every $a \in \mathbb{P}^1$, when $\nu = \delta_a$, $h^*\delta_a = \sum_{w \in h^{-1}(a)} (\deg_h w) \delta_w$ on $\mathbb{P}^1$. Pursuing the analogy between the roles played by $E(f)$ in (1.1) and by the set of Valiron exceptional values in $\mathbb{P}^1$ of a transcendental meromorphic function on $\mathbb{C}$, Sodin [20], Russakovskii–Sodin [19], and Russakovskii–Shiffman [18] (see also [7], [15]) studied the value distribution of a sequence of rational maps between projective spaces from the viewpoint of Nevanlinna theory, in a quantitative way (cf. [22, Chapter V, §2]). Gauthier and Vigny [10, 1. in Theorem A] studied the value distribution of the sequence $((f^n)': \mathbb{P}^1 \to \mathbb{P}^1)$ of the derivatives of iterations of a polynomial $f \in \mathbb{C}[z]$ of degree $> 1$ (cf. [23]) possibly with a polar subset of exceptional values in $\mathbb{C} \setminus \{0\}$, in terms of dynamics of the tangent map $F(z, w) := (f(z), f'(z)w)$ on the tangent bundle $\mathbb{T} \mathbb{C}$. The aim of this article is to improve their result in two ways.

The first improvement of [10, 1. in Theorem A] is qualitative, but with no exceptional values.

https://doi.org/10.5186/aasfm.2017.4233

2010 Mathematics Subject Classification: Primary 37F10; Secondary 30D35, 32H50.

Key words: Value distribution, equidistribution, quantitative equidistribution, derivative, iterated polynomials, monic and centered unicritical polynomials family, complex dynamics, Nevanlinna theory.
Theorem 1. Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \). Then for every \( a \in \mathbb{C} \setminus \{0\} \),
\[
\lim_{n \to \infty} \frac{((f^n))^{*} \delta_a}{d^n - 1} = \mu_f
\]
weakly on \( \mathbb{P}^1 \).

In Theorem 1, the values \( a = 0, \infty \) are excluded since it is clear that for every \( n \in \mathbb{N} \), \( ((f^n))^{*}(\delta_n)/(d^n - 1) = \delta_n(\neq \mu_f) \), and it immediately follows from (1.1) and the chain rule that \( \lim_{n \to \infty}((f^n))^{*}\delta_n)/(d^n - 1) = \mu_f \) weakly on \( \mathbb{P}^1 \) if and only if \( E(f) = \{\infty\} \). In Gauthier–Vigny [10, 2, and 3, in Theorem A], they also established a result similar to Theorem 1 under the assumption that \( f \) has no Siegel disks (or the assumption that \( f \) is hyperbolic). Our proof of Theorem 1 is independent of their argument even in those cases.

The second improvement of [10, 1, in Theorem A] is quantitative, but with an at most polar subset of exceptional values in \( \mathbb{C} \).

Theorem 2. Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and suppose that \( E(f) = \{\infty\} \). Then for every \( \eta > \sup_{z \in \mathbb{C}}: \) superattracting periodic point of \( f \) \( \lim \sup_{n \to \infty} (\deg_z(f^n))^{1/n} \) there is a polar subset \( E = E_{f, \eta} \) in \( \mathbb{C} \) such that for every \( a \in \mathbb{C} \setminus \{0\} \) and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \),
\[
\int_{\mathbb{P}^1} \phi \cdot \frac{((f^n))^{*} \delta_a}{d^n - 1} = o((\eta/d)^n)
\]
as \( n \to \infty \).

The proof of Theorem 2 is based on Russakovskii–Shiffman [18] mentioned above, and on an improvement of it for the sequence of the iterations of a rational function of degree \( > 1 \) by Drasin and the author [6] (see also [4] and [21] in higher dimensions).

Remark 1.1. Under the assumption \( E(f) = \{\infty\} \) in Theorem 2, we have \( \sup_{z \in \mathbb{C}}: \) superattracting periodic point of \( f \) \( \lim \sup_{n \to \infty} (\deg_z(f^n))^{1/n} \in \{1, 2, \ldots, d - 1\} \), and \( = 1 \) if and only if there is no superattracting cycles of \( f \) in \( \mathbb{C} \). Here we adopt the convention \( \sup_{\emptyset} = 1 \). In the case that \( E(f) \neq \{\infty\} \), we point out the following better estimate than that in Theorem 2
\[
\int_{\mathbb{P}^1} \phi \cdot \frac{((f^n))^{*} \delta_a}{d^n - 1} = O(nd^{-n}) \quad \text{as} \quad n \to \infty
\]
for every \( a \in \mathbb{C} \setminus \{0\} \) and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), with no exceptional values; indeed, we can assume that \( f(z) = z^d \) without loss of generality (see Remark 3.1), and then \( f^n(z) = z^{dn} \) for every \( n \in \mathbb{N} \) and \( \mu_f \) is the normalized Lebesgue measure \( m_{\partial \mathbb{D}} \) on the unit circle \( \partial \mathbb{D} = \partial K(f) \). For every \( a = re^{i\theta} (r > 0, \theta \in \mathbb{R}) \), every \( C^1 \)-test function \( \phi \) on \( \mathbb{P}^1 \), and every \( n \in \mathbb{N} \), we have
\[
\left| \int_{\mathbb{P}^1} \phi \cdot ((f^n))^{*} \delta_a - \frac{1}{d^n - 1} \right| \leq \|\phi\|_{C^1} \cdot (e^{(\log(d^n))/d^n} - 1) \leq \|\phi\|_{C^1} \cdot Cnd^{-n}
\]
for some \( C > 0 \) independent of both \( \phi \) and \( n \), and if \( \phi \) is \( C^2 \), then by the midpoint method in numerically computing definite integrals, we also have
\[
\left| \int_{\mathbb{P}^1} \phi \cdot \frac{1}{d^n - 1} \right| \leq \|\phi\|_{C^2} \cdot C'd^{-n}
\]
for some \( C' > 0 \) independent of both \( \phi \) and \( n \).

Finally, let us focus on the (monic and centered) unicritical polynomials family
\[
f : \mathbb{C} \times \mathbb{P}^1 \ni (\lambda, z) \mapsto z^d + \lambda =: f_{\lambda}(z) \in \mathbb{P}^1
\]
of degree \( d > 1 \). The parameter space analog of Theorem 1 for the sequence \((f^n_{\lambda})^{*}(\lambda)\) in \( \mathbb{C}[\lambda] \) of the derivative of \( f^n_{\lambda} \) at its unique critical value \( z = \lambda \) in \( \mathbb{C} \) is also obtained by Gauthier–Vigny [10, Theorem 3.7]. We will also establish a parameter space analog of Theorem 2.
Theorem 3. Let $f$ be the monic and centered unicritical polynomials family of degree $d > 1$ defined as in (1.2). Then for every $\eta > 1$, there is a polar subset $E = E_{f, \eta}$ in $C$ such that for every $a \in C \setminus E$ and every $C^2$-test function $\phi$ on $P^1$, 
\[ \int_{P^1} \phi(\lambda) \left( \frac{(f^n)'(\lambda)^*\delta_a}{d^n - 1} - \mu_{C_d} \right)(\lambda) = O((\eta/d)^n) \]
as $n \to \infty$. Here $C_d$ is the connectedness locus of the family $f$ in the parameter space $C$ and $\mu_{C_d}$ is the harmonic measure on $C_d$ with pole $\infty$.

The proof of Theorem 3 is based on Russakovskii–Shiffman [18] mentioned above, and on a quantitative equidistribution of superattracting parameters by Gauthier–Vigny [9].

In Section 2, we recall a background from complex dynamics. In Sections 3, 4, and 5, we show Theorems 1, 2, and 3, respectively.

Notation 1.2. We adopt the convention $N = Z_{>0}$. For every $a \in C$ and every $r > 0$, set $D(a, r) := \{z \in C : |z - a| < r\}$. Let $\delta_z$ be the Dirac measure on $P^1$ at each $z \in P^1$. Let $[z, w]$ be the chordal metric on $P^1$ normalized as $[z, \infty] = 1/\sqrt{1 + |z|^2}$ on $P^1$ (following the notation in Nevanlinna’s and Tsuji’s books [14, 22]). Let $\omega$ be the Fubini–Study area element on $P^1$ normalized as $\omega(P^1) = 1$. The Laplacian $dd^c$ on $P^1$ is normalized as $dd^c(-\log|\cdot, \infty|) = \omega - \delta_\infty$ on $P^1$.

2. Background

2.1. Dynamics of rational functions. Let $f \in C(z)$ be of degree $d > 1$. Let $C(f)$ be the critical set of $f$. The Julia and Fatou sets of $f$ are defined by $J(f) := \{z \in P^1 : f E \}$ be the monic and centered unicritical polynomials family of degree $d > 1$ defined as in (1.2). Then for every $\eta > 1$, there is a polar subset $E = E_{f, \eta}$ in $C$ such that for every $a \in C \setminus E$ and every $C^2$-test function $\phi$ on $P^1$, 
\[ \int_{P^1} \phi(\lambda) \left( \frac{(f^n)'(\lambda)^*\delta_a}{d^n - 1} - \mu_{C_d} \right)(\lambda) = O((\eta/d)^n) \]
as $n \to \infty$. Here $C_d$ is the connectedness locus of the family $f$ in the parameter space $C$ and $\mu_{C_d}$ is the harmonic measure on $C_d$ with pole $\infty$.
By a standard telescope argument, there exists the locally uniform limit
\[ g_f := \lim_{n \to \infty} \frac{-\log|f^n(\cdot)|}{d^n} \]
on \mathbb{C}. Setting \( g_f(\infty) := +\infty \), we have \( g_f \circ f = d \cdot g_f \) on \( \mathbb{P}^1 \), and for every \( n \in \mathbb{N} \), we also have \( g_f^n = g_f \) on \( \mathbb{P}^1 \). The restriction of \( g_f \) to \( I_\infty(f) \) coincides with the Green function on \( I_\infty(f) \) with pole \( \infty \), and the measure
\[ \mu_{K(f)} := dd^c g_f + \delta_\infty \]
coincides with the harmonic measure on \( K(f) \) with pole \( \infty \). In particular, \( \text{supp} \mu_{K(f)} \subset \partial K(f) \), and in fact \( \mu_{K(f)} = \mu_f \) on \( \mathbb{P}^1 \). The function \( z \mapsto g_f(z) - \log|z| \) extends harmonically to an open neighborhood of \( \infty \) in \( I_\infty(f) \) so the function \( z \mapsto -\log[z, \infty] - g_f(z) \) extends continuously to \( \mathbb{P}^1 \).

The following is substantially shown in Buff [3, the proof of Theorem 4].

**Theorem 2.1.** (Buff) Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and let \( z_0 \in \mathbb{C} \). If \( g_f(z_0) \geq \max_{c \in C(f) \cap \infty} g_f(c) \), then \( |f'(z_0)| \leq d^2 \cdot e^{d(d-1)g_f(z_0)} \), and the equality never holds if \( (C(f) \cap \infty) \cap I_\infty(f) \neq \emptyset \).

For more details on polynomial dynamics and potential theory, see Brolin [2, Chapter III], and also Ransford’s book [17].

## 3. Proof of Theorem 1

Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \). For every \( a \in \mathbb{C} \) and every \( n \in \mathbb{N} \), the functions \( \log |(f^n)'(a)|/(d^n - 1) - g_f \) and \( \log \max \{1, |(f^n)'|\}/(d^n - 1) - g_f \) extend continuously to \( \mathbb{P}^1 \). Set \( a_d = a_d(f) := \lim_{n \to \infty} f(z)/z^d \in \mathbb{C} \setminus \{0\} \).

**Remark 3.1.** Since the question is affine invariant, we could assume \( |a_d| = 1 \) without loss of generality, by replacing \( f \) with \( c^{-1} \circ f \circ c \) for such \( c \in \mathbb{C} \setminus \{0\} \) that \( c^{d-1} = a_d^{-1} \) if necessary (for every \( c \in \mathbb{C} \setminus \{0\} \), \( z \mapsto c \cdot z \) is also denoted by \( c \)). In this article, we would not normalize \( f \) as \( |a_d| = 1 \) in order to make it explicit which computations would be independent of such a normalization.

**Lemma 3.2.** On \( I_\infty(f) \setminus \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(C(f) \cap \mathbb{C}) \),
\[ \lim_{n \to \infty} \frac{\log |(f^n)'|}{d^n - 1} = 0 \]
locally uniformly.

**Proof.** For every \( n \in \mathbb{N} \) and every \( z \in \mathbb{C} \), by a direct calculation, we have
\[ \frac{\log |(f^n)'(z)|}{d^n - 1} - \frac{\log |d^n \cdot a_d^{(d^n - 1)/(d-1)}|}{d^n - 1} = \frac{1}{d^n - 1} \int |z - u|(dd^c \log |(f^n)'|)(u) \]
\[ = \frac{1}{d^n - 1} \int \sum_{j=0}^{n-1} \left( \int |z - |d((f^j)' \cdot \delta_u)| \right)(dd^c \log |f'(w)|)(w) \]
\[ = \frac{1}{d^n - 1} \int \sum_{j=0}^{n-1} \left( \log |f^j(z)| - \log |a_d^{(d^{j+1} - 1)/(d-1)}| \right)(dd^c \log |f'(w)|)(w) \]
\[ = \frac{1}{d^n - 1} \int \sum_{j=0}^{n-1} \left( \log |f^j(z)| - \log |a_d^{(d^{j+1} - 1)/(d-1)}| \right)(dd^c \log |f'(w)|)(w) \]
\[ = \frac{1}{d^n - 1} \int \sum_{j=0}^{n-1} \left( \log |f^j(z), w| - \log |f^j(z), \infty| - \log |w, \infty| \right)(dd^c \log |f'(w)|)(w) \]
\[ - \log |a_d| \cdot \sum_{j=0}^{n-1} \left. \right|_{w=z^d} \cdot w^{-j}. \]
Then noting that \( g_f \circ f = d \cdot g_f \) on \( \mathbb{P}^1 \), for every \( n \in \mathbb{N} \) and every \( z \in \mathbb{P}^1 \), we have
\[
\frac{\log|(f^n)'(z)|}{d^n - 1} - g_f(z) = \frac{1}{d^n - 1} \int_{\mathbb{C}} \left( \sum_{j=0}^{n-1} \log[f^j(z), w] \right) (dd^c \log |f'|)(w)
\]
\[
+ \frac{d-1}{d^n - 1} \sum_{j=0}^{n-1} \left( - \log[f^j(z), \infty] - g_f(f^j(z)) \right)
\]
\[
+ \left( - \int_{\mathbb{C}} \log[w, \infty](dd^c \log |f'|)(w) + \log d + \log |a_d| \right) \frac{n}{d^n - 1},
\]
which with \( \sup_{z \in \mathbb{P}^1} \left( - \log[z, \infty] - g_f(z) \right) < \infty \) completes the proof. \( \square \)

**Lemma 3.3.** There is \( C = C_f > 0 \) such that for every \( n \in \mathbb{N} \) and every \( z \in \mathbb{P}^1 \),
\[
\frac{\log \max \{1, |(f^n)'(z)|\}}{d^n - 1} - g_f(z) \leq \frac{Cn}{d^n - 1}.
\]

**Proof.** Set \( C = C_f := (d-1) \cdot \sup_{z \in \mathbb{P}^1} | - \log[z, \infty] - g_f(z) | \)
\[
+ (d-1) \cdot \sup_{w \in C(f) \cap \mathbb{C}} | \log[w, \infty] | + \log d + \log |a_d| \in \mathbb{R}_{>0}.
\]

Then for every \( n \in \mathbb{N} \) and every \( z \in \mathbb{C} \), from (3.2), we have \( |(f^n)'(z)| \leq e^{Cn} \cdot e^{(d^n-1)g_f(z)} \), which with \( g_f \geq 0 \) on \( \mathbb{P}^1 \) completes the proof. \( \square \)

We note that \( \max_{c \in \bigcup_{n \in \mathbb{N}, c \neq 0} f^{-n}(C(f) \cap \mathbb{C})} g_f(c) = \max_{c \in C(f) \cap \mathbb{C}} g_f(c) < \infty \) by \( g_f \circ f = d \cdot g_f \) on \( \mathbb{P}^1 \).

**Lemma 3.4.** For every \( a \in \mathbb{C} \setminus \{0\} \),
\[
\lim_{n \to \infty} \int_{\mathbb{P}^1} \left| \frac{\log|f^n)' - a|}{d^n - 1} - g_f \right| d\omega = 0.
\]

**Proof.** Fix \( a \in \mathbb{C} \setminus \{0\} \). The sequence \( \{ \log|f^n)' - a|/(d^n - 1) \} \) of subharmonic functions on \( \mathbb{C} \) is locally uniformly bounded from above on \( \mathbb{C} \); indeed, by the chain rule and \( \lim_{n \to \infty} |f^n)'(z)| = +\infty \), for every \( R > 0 \) so large that \( \{|z| = R\} \subset I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, c \neq 0} f^{-n}(C(f) \cap \mathbb{C}) \), we have \( \lim_{n \to \infty} \inf_{|z| = R} |(f^n)'(z)| = +\infty \), which with the maximum modulus principle yields \( \sup_{|z| \leq R} |(f^n)'(z)| - a| \leq \sup_{|z| \leq R} 2 |(f^n)'(z)| \) for every \( n \in \mathbb{N} \) large enough. Then by Lemma 3.3, we have \( \limsup_{n \to \infty} \sup_{|z| \leq R} |(f^n)' - a|/(d^n - 1) \leq \sup_{|z| = R} g_f(z) < \infty \). By Lemma 3.2 and \( g_f > 0 \) on \( I_\infty(f) \), for every compact subset \( C \) in \( I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, c \neq 0} f^{-n}(C(f) \cap \mathbb{C}) \), we also have \( 1/2 \leq |(f^n)' - a|/(d^n)' \leq 2 \) on \( C \) for every \( n \in \mathbb{N} \) large enough, so in particular
\[
\lim_{n \to \infty} \left( \frac{\log|f^n)' - a|}{d^n - 1} - g_f \right) = \lim_{n \to \infty} \left( \frac{\log|f^n)'|}{d^n - 1} - g_f \right) = 0
\]
locally uniformly on \( I_\infty(f) \setminus \bigcup_{n \in \mathbb{N}, c \neq 0} f^{-n}(C(f) \cap \mathbb{C}) \).

Let \( m_2 \) be the Lebesgue measure on \( \mathbb{C} \). By a compactness principle for a locally uniformly upper bounded sequence of subharmonic functions on a domain in \( \mathbb{R}^m \) which is not locally uniformly convergent to \( -\infty \) (see Azarin [1, Theorem 1.1.1], Hörmander’s book [11, Theorem 4.1.9(a)]), we can choose a sequence \( (n_j) \in \mathbb{N} \) tending to \( +\infty \) as \( j \to \infty \) such that the \( L^1_{\text{loc}}(\mathbb{C}, m_2) \)-limit \( \phi := \lim_{j \to \infty} (\log|f^{n_j})' - a|/(d^{n_j} - 1) \) exists and is subharmonic on \( \mathbb{C} \). Choosing a subsequence of \( (n_j) \) if
necessary, we have \( \phi = \lim_{j \to \infty} (\log |(f^{n_j})' - a|)/(d^{n_j} - 1) \) Lebesgue a.e. on \( C \). Then by (3.5), we have \( \phi \equiv g_f \) Lebesgue a.e. on \( C \setminus (K(f) \cup \bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(C(f) \cap C)) \), and in turn on \( C \setminus K(f) \) by the subharmonicity of \( \phi \) and the harmonicity of \( g_f \) there. Let us show that \( \phi = g_f \) Lebesgue a.e. on the whole \( C \), and then \( \lim_{n \to \infty} (\log |(f^n)' - a|)/(d^n - 1) = g_f \) in \( L_{\infty}(C, m_2) \), which with the locally uniform convergence (3.5) will complete the proof since \( \max_{w \in \mathbb{H}} \left| f^{-n}(C(f) \cap C) \right| g_f(w) < \infty \) and the Radon–Nikodym derivative \( d\omega/dm_2 \) is continuous so locally bounded on \( C \).

By \( \log(1/|w, \infty|) \) \( - \log \max\{1, |w|\} \) \( \leq \log \sqrt{N} \) on \( C \) and Lemma 3.3, for every \( n \in \mathbb{N} \), we have

\[
\frac{\log |(f^n)' - a|}{d^n - 1} - g_f = \frac{\log |(f^n)' - a|}{d^n - 1} + \left( \frac{\log(1/|(f^n)'|, \infty)}{d^n - 1} - g_f \right) + \frac{\log(1/|a, \infty|)}{d^n - 1} \leq \frac{C_f \cdot n}{d^n - 1} + \frac{\log \sqrt{N} + \log(1/|a, \infty|)}{d^n - 1}
\]
on \( C \), so \( \phi \leq g_f \) Lebesgue a.e. on \( C \) and in turn on \( C \) by the subharmonicity of \( \phi \) and the continuity of \( g_f \) on \( C \). Hence \( \phi - g_f \) is \( 0 \) and is upper semicontinuous on \( C \).

Now suppose to the contrary that the open subset \( \{z \in C : \phi(z) < g_f(z)\} \) in \( C \) is non-empty. Then by \( \phi \equiv g_f \) on \( C \setminus K(f) \), there is a bounded Fatou component \( U \) of \( f \) containing a component \( W \) of \( \{z \in C : \phi(z) < g_f(z)\} \). Since \( \phi \leq g_f = 0 \) on \( U \subset K(f) \), by the maximum principle for subharmonic functions, we in fact have \( U = W \).

Taking a subsequence of \( (n_j) \) if necessary, we can assume that \( (f^{n_j})|U \) is locally uniformly convergent to a holomorphic function \( g \) on \( U \) as \( j \to \infty \) without loss of generality. We claim that \( g' \equiv a \) on \( U \), so we can say \( g \in C[z] \); indeed, fixing a domain \( D \Subset U = W \), by a version of Hartogs’s lemma on subharmonic functions (see Hörmander’s book [11, Theorem 4.1.9(b)]) and the upper semicontinuity of \( \phi \), we have \( \limsup_{n \to \infty} \sup_{z \in D} (\log |(f^{n_j})' - a|)/(d^{n_j} - 1) \leq \sup \phi \phi < 0 \). Hence \( g' = (\lim_{j \to \infty} f^{n_j})' = \lim_{j \to \infty} (f^{n_j})' \equiv a \) on \( D \), so \( g' \equiv a \) on \( U \) by the identity theorem for holomorphic functions.

Hence, under the assumption that \( a \neq 0 \), the locally uniform limit \( g \) on \( U \) is non-constant. So by Hurwitz’s theorem and the classification of cyclic Fatou components, there is \( N \in \mathbb{N} \) such that \( V := f^{n_N}(U) = g(U) \supset g(D) \) is a Siegel disk of \( f \) and, setting \( p := \min\{n \in \mathbb{N} : f^n(V) = V\} \), that \( p(n_j - n_N) \) for every \( j \geq N \). We can fix a holomorphic injection \( h : V \to C \) such that for some \( \alpha \in \mathbb{R} \setminus \Omega \), setting \( \lambda := e^{2\pi \alpha} \), we have \( h \circ f^p = \lambda \cdot h \) on \( V \), so for every \( j \geq N \), \( h \circ f^{n_j} = \lambda^{(n_j-n_N)/p} \cdot (h \circ f^{n_N}) \) on \( U \). Then taking a subsequence of \( (n_j) \) if necessary, there also exists the limit

\[
\lambda_0 := \lim_{j \to \infty} \lambda^{(n_j-n_N)/p}
\]
in \( \partial D \), so that \( h \circ g = \lim_{j \to \infty} h \circ f^{n_j} = \lambda_0 \cdot (h \circ f^{n_N}) \) on \( U \). In particular,

\[
(h \circ f^{n_j} - h \circ g) = (\lambda^{(n_j-n_N)/p} - \lambda_0) \cdot (h \circ f^{n_N})
\]
on \( U \). Set \( w_0 := h^{-1}(0) \in V \), so that \( f^p(w_0) = w_0 \), and fix \( z_0 \in f^{-n_N}(w_0) \cap U \), so that \( f^{n_j}(z_0) = w_0 \) for every \( j \geq N \) and \( g(z_0) = \lim_{j \to \infty} f^{n_j}(z_0) = w_0 \).

We claim that

\[
\frac{\log |(f^{n_j})'(z_0) - a|}{d^{n_j} - 1} = \frac{\log |\lambda^{(n_j-n_N)/p} - \lambda_0|}{d^{n_j} - 1} + O(d^{-n_j})
\]
as \( j \to \infty \); for, by the chain rule applied to both sides in \((3.6)\) and \( h'(w_0) \neq 0 \) (and \( g'(z_0) = a \)), we have

\[
(f^{n_j})'(z_0) - a = (\lambda^{(n_j-n_N)/p} - \lambda_0) \cdot (f^{n_N})'(z_0),
\]

which also yields \((f^{n_N})'(z_0) \neq 0\) by \((f^{n_j})'(z_0) = (f^{n_j-n_N})'(w_0) \cdot (f^{n_N})'(z_0)\) and the assumption \(a \neq 0\). We also claim that

\[
\liminf_{j \to \infty} \frac{1}{d^{n_j}} \log |\lambda^{(n_j-n_N)/p} - \lambda_0| \geq 0
\]

(cf. [16, Proof of Theorem 3]); indeed, for every domain \( D \subseteq U \setminus f^{-n_N}(w_0) \), since \( h^{-1} \) is Lipschitz continuous on \( h(\bigcup_{n \in \mathbb{N}} (f^p)^n(D)) \cup g(D) \subseteq h(V) \) and \( \sup_D |h \circ f^{n_N}| > 0 \), from \((3.6)\), we observe that

\[
\left( \sup_D \frac{1}{d^{n_j}} \log |f^{n_j} - g| \right) = \frac{1}{d^{n_j}} \log |\lambda^{(n_j-n_N)/p} - \lambda_0| + O\left(d^{-n_j}\right)
\]

as \( j \to \infty \). On the other hand, for every domain \( \tilde{D} \) intersecting \( \partial U \) in \( \mathbb{C} \), fixing \( \tilde{z} \in \tilde{D} \cap I_\infty(f) \neq \emptyset \), we observe that

\[
\liminf_{j \to \infty} \frac{1}{d^{n_j}} \sup_{\tilde{D}} \log |f^{n_j} - g| \geq g_f(\tilde{z}) > 0.
\]

Now fix \( z_1 \in U \) and \( z' \in \partial U \) such that \( D(z_1, |z' - z_1|) \subseteq U \setminus f^{-n_N}(w_0) \). Then for every \( \epsilon \in (0, |z' - z_1|) \), using Cauchy’s estimate applied to \( f^{n_j} - g \in \mathbb{C}[z] \) around \( z_1 \), we have

\[
|f^{n_j} - g| \leq \sum_{k=0}^{d^{n_j}} \sup_{D(z_1, |z' - z_1| - \epsilon)} \left|f^{n_j} - g\right| \cdot |z_1|^k
\]

\[
\leq \left( \sup_{D(z_1, |z' - z_1| - \epsilon)} |f^{n_j} - g| \right) \cdot \sum_{k=0}^{d^{n_j}} \left( \frac{|z' - z_1| + \epsilon}{|z'_1 - z_1| - \epsilon} \right)^k
\]

on \( D(z', \epsilon) \), so since \( z' \in D(z', \epsilon) \cap \partial U \) and \( D(z_1, |z' - z_1| - \epsilon) \subseteq U \setminus f^{-n_N}(w_0) \), by \((**)\) and \((*)\), we have

\[
0 \leq \left( \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log \sup_{D(z', \epsilon)} \left|f^{n_j} - g\right| \right)
\]

\[
\leq \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log \sup_{D(z_1, |z' - z_1| - \epsilon)} \left|f^{n_j} - g\right| + \log \frac{|z' - z_1| + \epsilon}{|z'_1 - z_1| - \epsilon}
\]

\[
\leq \left( \liminf_{j \to \infty} \frac{1}{d^{n_j}} \log |\lambda^{(n_j-n_N)/p} - \lambda_0| + \log \frac{|z' - z_1| + \epsilon}{|z'_1 - z_1| - \epsilon} \right)
\]

This yields \((3.8)\) as \( \epsilon \to 0 \).

Once \((3.7)\) and \((3.8)\) are at our disposal, using a version of Hartogs’s lemma on subharmonic functions again, we have

\[
\phi(z_0) \geq \limsup_{j \to \infty} \frac{\log |(f^{n_j})'(z_0) - a|}{d^{n_j} - 1} \geq \liminf_{j \to \infty} \frac{\log |\lambda^{(n_j-n_N)/p} - \lambda_0|}{d^{n_j} - 1} \geq 0,
\]

which contradicts \( \phi < g_f = 0 \) on \( U = W \).
For every \( a \in \mathbb{C} \setminus \{0\} \) and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), by Lemma 3.4, we have
\[
\left| \int_{\mathbb{P}^1} \phi \frac{d}{d^n} \left( \frac{1}{d^n-1} \delta_a - \mu_f \right) \right| = \left| \int_{\mathbb{P}^1} \phi \frac{d}{d^n} \left( \frac{\log |(f^n)'(\cdot) - a|}{d^n-1} - g_f \right) \right| 
\leq \left( \sup_{\mathbb{P}^1} \frac{d}{d^n} \right) \cdot \int_{\mathbb{P}^1} \left| \frac{\log |(f^n)'(z) - a|}{d^n-1} - g_f \right| d\omega(z) \to 0 \quad \text{as} \ n \to \infty,
\]
where the Radon–Nikodym derivative \((d \phi)/d\omega\) on \( \mathbb{P}^1 \) is bounded on \( \mathbb{P}^1 \).

4. Proof of Theorem 2

Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and suppose that \( E(f) = \{\infty\} \). Then
\[
\sup_{z \in \mathbb{C}} \text{superattracting periodic point of } f \limsup_{n \to \infty} (\deg_z (f^n))^{1/n} = \sup_{c \in \mathbb{C} \cap \mathbb{C}} \text{superattracting periodic under } f \limsup_{n \to \infty} (\deg_c (f^n))^{1/n} \in \{1, 2, \ldots, d-1\}
\]
(recall the convention \( \sup_{\emptyset} = 1 \)). Set \( a_d := a_d(f) = \lim_{n \to \infty} f(z)/z^d \in \mathbb{C} \setminus \{0\} \). For every \( n \in \mathbb{N} \), the functions \((\log(1/[(f^n)', \infty])/(d^n-1) - g_f)\) and \((\log \max\{1, |(f^n)'|\})/(d^n-1) - g_f\) extend continuously to \( \mathbb{P}^1 \).

Lemma 4.1. For every \( \eta > \sup_{c \in \mathbb{C}(\mathbb{P}^1) \cap \mathbb{C}} \text{periodic under } f \limsup_{n \to \infty} (\deg_c (f^n))^{1/n} \),
\[
\int_{\mathbb{P}^1} \left| \frac{\log(1/[(f^n)', \infty])}{d^n-1} - g_f(z) \right| d\omega(z) = o(\eta/d^n)
\]
as \( n \to \infty \).

Proof. For every \( n \in \mathbb{N} \), from (3.2), we have
\[
\int_{\mathbb{P}^1} \left| \frac{\log |(f^n)'(z)|}{d^n-1} - g_f(z) \right| d\omega(z)
\leq \frac{1}{d^n-1} \int_{\mathbb{C}} \left( \sum_{j=0}^{n-1} \int_{\mathbb{P}^1} \log \frac{1}{|f^j(z), w|} d\omega(z) \right) \left( d^c \log |f'(w)| d\omega(z) \right) + C_f \cdot n
\]
where \( C_f > 0 \) is defined in (3.4). By [6, Theorem 2], for every \( \eta > \sup_{c \in \mathbb{C}(\mathbb{P}^1) \cap \mathbb{C}} \text{periodic under } f \limsup_{n \to \infty} (\deg_c (f^n))^{1/n} \) and every \( w \in \mathbb{C} = \mathbb{P}^1 \setminus E(f) \) under the assumption \( E(f) = \{\infty\} \), we have
\[
\int_{\mathbb{P}^1} \log \frac{1}{|f^n(z), w|} d\omega(z) = o(\eta^n)
\]
as \( n \to \infty \), which with Lemma 3.3 and \( 0 \leq \log(1/|w, \infty|) - \log \max\{1, |w|\} \leq \log \sqrt{2} \) on \( \mathbb{C} \) completes the proof.

Lemma 4.2. For every \( \eta > 1 \), the Valiron exceptional set
\[
E_V((f^n)'), \eta) := \left\{ a \in \mathbb{P}^1 : \limsup_{n \to \infty} \frac{1}{\eta^n} \int_{\mathbb{P}^1} \log \frac{1}{|f^n)'(z), a|} d\omega(z) > 0 \right\}
\]
of the sequence \((f^n)'\) of the derivatives of the iterations of \( f \) with respect to the sequence \((\eta^n)\) in \( \mathbb{R}_{>0} \) is a polar subset in \( \mathbb{P}^1 \).
Proof. This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence \(((f^n)')\) in \(C[z]\) since \(\sum_{n\in\mathbb{N}} 1/\eta^n < \infty\) for every \(\eta > 1\). 

For every \(\eta > \sup_{c \in C(f)/C_c} \text{periodic under } f \lim_{n \to \infty} (\deg((f^n)))^{1/n}\), every \(a \in C \setminus E_V(((f^n)'), (\eta^n))\), and every \(C^2\)-test function \(\phi\) on \(\mathbb{P}_1\), by Lemmas 4.1 and 4.2, we have

\[
\left| \int_{\mathbb{P}_1} \phi \left( \frac{((f^n)')^d a}{d^n - 1} - \mu_f \right) \right| = \left| \int_{\mathbb{P}_1} \phi \left( \frac{\log([f^n]''(a))}{d^n - 1} + \frac{\log(1/([f^n]'(\infty)) - g_f)}{d^n - 1} \right) \right|
\]

\[
\leq \left( \sup_{\mathbb{P}_1} \frac{d\omega}{d\phi} \right) \left( \frac{1}{d^n - 1} \int_{\mathbb{P}_1} \frac{1}{(f^n)(z)} d\omega(z) \right)
\]

\[
+ \left( \int_{\mathbb{P}_1} \frac{\log(1/((f^n)'(\infty))}{d^n - 1} \right) - g_f \right) d\omega(z)
\]

\[
= o((\eta/d)^n) \quad \text{as } n \to \infty,
\]

where the Radon–Nikodym derivative \((dd^c\phi)/d\omega\) on \(\mathbb{P}_1\) is bounded on \(\mathbb{P}_1\). 

5. Proof of Theorem 3

Let \(f: \mathbb{C} \times \mathbb{P}_1 \ni (\lambda, z) \mapsto z^d + \lambda =: f_\lambda(z) \in \mathbb{P}_1\) be the monic and centered unicritical polynomials family of degree \(d > 1\). For every \(n \in \mathbb{N}\), \((f^n_\lambda(\lambda), (f^n_\lambda)'(\lambda)) \in C[\lambda]\) are of degree \(d^n\), \(d^n - 1\), respectively.

5.1. Background on the family \(f\). Recall the definitions in Subsection 2.2.

The following constructions are due to Douady–Hubbard [5] and Siobony.

For every \(\lambda \in \mathbb{C}\), \(f_\lambda'(z) = d \cdot z^{d-1}\), so \(C(f_\lambda) \cap \mathbb{C} = \{0\}\) and \(f_\lambda(0) = \lambda\). The connectedness locus \(C_d := \{\lambda \in \mathbb{C}: \lambda \in K(f_\lambda)\}\) of the family \(f\) is a compact subset in \(\mathbb{C}\), and \(H_\infty = H_{d, \infty} := \mathbb{P}_1 \setminus C_d\) is a simply connected domain containing \(\infty\) in \(\mathbb{P}_1\). Moreover, the locally uniform limit

\[
g_{H_\infty}(\lambda) := g_{f_\lambda}(\lambda) = d \cdot g_{f_\lambda}(0) = \lim_{n \to \infty} \frac{-\log([f^n_\lambda(\lambda)], \infty]}{d^n}
\]

exists on \(\mathbb{C}\). Setting \(g_{H_\infty}(\infty) := +\infty\), the restriction of \(g_{H_\infty}\) to \(H_\infty\) coincides with the Green function on \(H_\infty\) with pole \(\infty\), and the measure

\[
\mu_{C_d} := dd^c\mu_{H_\infty} + \delta_\infty \quad \text{on } \mathbb{P}_1
\]

coincides with the harmonic measure on \(C_d\) with pole \(\infty\). In particular, \(z \mapsto g_{H_\infty}(z) - \log |z|\) extends harmonically to an open neighborhood of \(\infty\) in \(H_\infty\), and \(\text{supp} \mu_{C_d} \subset \partial C_d\) (in fact, the equality holds).

5.2. Proof of Theorem 3. For every \(n \in \mathbb{N}\), \(\lambda \mapsto (\log |(f^n_\lambda)'(\lambda)|)/(d^n - 1) - g_{H_\infty}(\lambda)\) and \(\lambda \mapsto (\log \max\{1, |(f^n_\lambda)'(\lambda)|\})/(d^n - 1) - g_{H_\infty}(\lambda)\) on \(\mathbb{C}\) extend continuously to \(\mathbb{P}_1\).

Lemma 5.1. For every \(n \in \mathbb{N}\) and every \(\lambda \in \mathbb{C}\),

\[
\left(3.3'\right) \quad \frac{\log \max\{1, |(f^n_\lambda)'(\lambda)|\}}{d^n - 1} - g_{H_\infty}(\lambda) \leq \frac{n \log(d^2)}{d^n - 1}.
\]

Proof. For every \(n \in \mathbb{N}\) and every \(\lambda \in \mathbb{C}\), by \(g_{f_\lambda} = g_{f_\lambda}\) on \(\mathbb{P}_1\) and \(g_{f_\lambda} \circ f_\lambda = d \cdot g_{f_\lambda}\) on \(\mathbb{P}_1\), we have \(g_{f_\lambda}(\lambda) = g_{f_\lambda}(\lambda) = d \cdot g_{f_\lambda}(0) \geq g_{f_\lambda}(0) = \max_{c \in C(f_\lambda) \cap \mathbb{C}} g_{f_\lambda}(c) = \max_{c \in C(f_\lambda) \cap \mathbb{C}} g_{f_\lambda}(c)\), so by Theorem 2.1, we have \(|(f^n_\lambda)'(\lambda)| \leq (d^n)^2 e^{(d^n - 1)g_{f_\lambda}(\lambda)} = (d^n)^2 e^{(d^n - 1)g_{f_\lambda}(\lambda)} (d^n)^2 e^{(d^n - 1)g_{H_\infty}(\lambda)}\).

This with \(g_{H_\infty}(\lambda) \geq 0\) completes the proof. \(\square\)
Lemma 5.2.

\[ \int_{\mathbb{P}^1} \left| \log \left( \frac{1}{(f^n_\lambda)'(\lambda), \infty} \right) - g_{H_\infty}(\lambda) \right| d\omega(\lambda) = O(n^2 d^{-n}) \]

as \( n \to \infty \).

Proof. For every \( n \in \mathbb{N} \), by the third equality in (3.1) for \( f_\lambda \) evaluated at \( z = \lambda \), we have

\[ \frac{\log \left| (f^n_\lambda)'(\lambda) \right|}{d^n - 1} - \frac{n \log d}{d^n - 1} = \frac{d - 1}{d^n - 1} \sum_{j=0}^{n-1} \log |f_j^\lambda(\lambda)| = \frac{d - 1}{d^n - 1} \sum_{j=0}^{n-1} \log |f_{j+1}^\lambda(0)|, \]

so that

\[ \int_{\mathbb{P}^1} \left| \log \frac{|(f^n_\lambda)'(\lambda)|}{d^n - 1} - g_{H_\infty}(\lambda) \right| d\omega(\lambda) \leq \frac{d - 1}{d^n - 1} \sum_{j=0}^{n-1} \int_{\mathbb{P}^1} \left| \log |f_{j+1}^\lambda(0)| - d^j \cdot g_{H_\infty}(\lambda) \right| d\omega(\lambda) + \frac{n \log d}{d^n - 1} \]

(4.1’)

\[ = O(n^2 d^{-n}) \quad \text{as} \quad n \to \infty \]

since by Gauthier–Vigny [9, §4.3, Proof of Theorem A], we have

\[ \int_{\mathbb{P}^1} \left| \log |f^{n+1}_\lambda(0)| - d^n \cdot g_{H_\infty}(\lambda) \right| d\omega(\lambda) = O(n) \]

as \( n \to \infty \). This with Lemma 5.1 and \( 0 \leq \log(1/|w, \infty|) - \log \max \{1, |w|\} \leq \log \sqrt{2} \) on \( \mathbb{C} \) completes the proof. \( \square \)

Lemma 5.3. For every \( \eta > 1 \), the Valiron exceptional set

\[ E_V(((f^n_\lambda)'(\lambda)), (\eta^n)) := \left\{ a \in \mathbb{P}^1 : \limsup_{n \to \infty} \frac{1}{\eta^n} \int_{\mathbb{P}^1} \log \left| \frac{1}{(f^n_\lambda)'(\lambda), a} \right| d\omega(\lambda) > 0 \right\} \]

of the sequence \(((f^n_\lambda)'(\lambda))\) in \( \mathbb{C}[\lambda] \) with respect to the sequence \((\eta^n)\) in \( \mathbb{R}_{>0} \) is a polar subset in \( \mathbb{P}^1 \).

Proof. This is an application of Russakovskii–Shiffman [18, Proposition 6.2] to the sequence \(((f^n_\lambda)'(\lambda))\) in \( \mathbb{C}[\lambda] \) since \( \sum_{n \in \mathbb{N}} 1/\eta^n < \infty \) for every \( \eta > 1 \). \( \square \)

For every \( \eta > 1 \), every \( a \in \mathbb{C} \setminus E_V(((f^n_\lambda)'(\lambda)), (\eta^n)) \), and every \( C^2 \)-test function \( \phi \) on \( \mathbb{P}^1 \), by Lemmas 5.2 and 5.3, we have

\[ \left| \int_{\mathbb{P}^1} \phi(\lambda) d \left( \frac{(f^n_\lambda)'(\lambda))^* \delta_a}{d^n - 1} - \mu_C \right)(\lambda) \right| \]

\[ = \left| \int_{\mathbb{P}^1} \phi(\lambda) \, dd^c \left( \log \left| \frac{1}{(f^n_\lambda)'(\lambda), a} \right| d\omega(\lambda) - g_{H_\infty}(\lambda) \right) \right| \]

\[ \leq \left( \sup_{\mathbb{P}^1} \left| \frac{dd^c \phi}{d\omega} \right| \right) \left( \frac{1}{d^n - 1} \int_{\mathbb{P}^1} \log \left| \frac{1}{(f^n_\lambda)'(\lambda), a} \right| d\omega(\lambda) \right) + \int_{\mathbb{P}^1} \left| \log \left| \frac{1}{(f^n_\lambda)'(\lambda), \infty} \right| - g_{H_\infty}(\lambda) \right| d\omega(\lambda) \]

\[ = o((\eta/d)^n) \quad \text{as} \quad n \to \infty, \]

where the Radon–Nikodym derivative \((dd^c \phi)/d\omega\) on \( \mathbb{P}^1 \) is bounded on \( \mathbb{P}^1 \). \( \square \)
Acknowledgement. The author thanks the referee for a very careful scrutiny and invaluable comments. This research was partially supported by JSPS Grant-in-Aid for Scientific Research (C), 15K04924.

References


Received 26 July 2016 • Accepted 23 September 2016