UNIVERSAL TEICHMÜLLER SPACES
AND F(p, q, s) SPACE

Xiaogao Feng, Shengjin Huo and Shuan Tang*

China West Normal University, College of Mathematics and Information
Nanchong, 637002, P. R. China; and
Soochow University, Department of Mathematics
Suzhou, 215006, P. R. China; fengxiaogao603@163.com
Tianjin Polytechnic University, Department of Mathematics
Tianjin, 300387, P. R. China; sjhuo@pku.edu.cn
Guizhou Normal University, School of Mathematics Sciences
Guiyang 550001, P. R. China; tsaflyhigher@163.com

Abstract. In this paper, we introduce the F(p, s)-Teichmüller space and investigate its Schwarzian derivative model and pre-logarithmic derivative model. In particular, we prove that the pre-logarithmic derivative model is a disconnected subset of Besov type space F(p, s) and the Bers projection is holomorphic.

1. Introduction

Let \( \Delta = \{ z : |z| < 1 \} \) be the unit disk in the complex plane \( \mathbb{C} \), \( \Delta^* = \mathbb{C} \setminus \overline{\Delta} \) be the outside of the unit disk and \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \) be the unit circle. Let \( \alpha > 0 \), the Bloch-type space \( B^\alpha \) consists of all holomorphic functions \( f \) on \( \Delta \) such that
\[
\| f \|_B = \sup_{z \in \Delta} (1 - |z|^2)^\alpha |f'(z)| < \infty,
\]
and the subspace \( B^\alpha_0 \) consists of all functions \( f \in B^\alpha \) such that
\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.
\]

We denote by BMO(\( S^1 \)) the space of all integrable functions on \( S^1 \) such that
\[
\| u \|_{BMO} = \sup_I \frac{1}{|I|} \int_I |u - u_I| d\theta < \infty,
\]
where \( I \) is any arc on \( S^1 \), \(|I|\) denotes the Lebesgue measure of \( I \), and
\[
\frac{1}{|I|} \int_I u d\theta
\]
is the average of \( u \) over \( I \). A holomorphic function \( f \) on \( \Delta \) belongs to BMOA(\( \Delta \)) if and only if it is a Poisson integral of some function which belongs to BMO(\( S^1 \)).

For any \( a \in \Delta \), set \( \varphi_a(z) = \frac{z - a}{1 - \overline{a}z} \), \( z \in \Delta \). For \( p > 1 \), \( q > -2 \) and \( s \geq 0 \), the space \( F(p, q, s) \) consists of all holomorphic functions \( f \) on the unit disk \( \Delta \) with the

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*The first and the corresponding author.
following finite norm
\[
\|f\|_{F_{p,q,s}}^p = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi(z)|^2)^s \, dx \, dy < \infty,
\]
and \( F_0(p, q, s) \) consists of all functions \( f \in F(p, q, s) \) with
\[
\lim_{|a| \to 1} \iint_{\Delta} |f'(z)|^p (1 - |z|^2)^q (1 - |\varphi(z)|^2)^s \, dx \, dy = 0.
\]
The space \( F(p, q, s) \) was introduced by Zhao [23]. It is well known that \( F(p, q, s) \) is trivial if \( q + s \leq -1 \). For \( p \geq 1 \), \( F(p, q, s) \) is a Banach space contained in the Bloch-type space \( B^\alpha \) and \( F_0(p, q, s) \subset B^\alpha_0 \) with \( \alpha = \frac{2 + 2}{p} \). It is also known that \( F(2, 0, 1) \) is the BMOA space (see [8]) and \( F(2, 0, s) \) is the \( Q_s \) space (see [19, 20]). In this paper, we shall mainly concentrate on the Besov type space \( F(p, s) = F(p, p - 2, s) \) and \( F_0(p, s) = F_0(p, p - 2, s) \).

Let \( \Omega = \Delta \) or \( \Omega = \Delta^* \). Given an arc \( I \) of the unit circle \( S^1 \), the Carleson box is defined by
\[
S_{\Omega}(I) = \begin{cases} 
{\{ z \in \Delta : 1 - |I| \leq |z| < 1, z/|z| \in I \},} & \Omega = \Delta, \\
{\{ z \in \Delta^* : 1 \leq |z| < 1 + |I|, z/|z| \in I \},} & \Omega = \Delta^*.
\end{cases}
\]
Let \( s > 0 \). A positive measure \( \lambda \) on \( \Omega \) is called an \( s \)-Carleson measure if
\[
\|\lambda\|_{C,s} = \sup_{I \subset S^1} \lambda(S_{\Omega}(I)) / |I|^s < \infty,
\]
and a compact \( s \)-Carleson measure if
\[
\lim_{|I| \to 0} \lambda(S_{\Omega}(I)) / |I|^s = 0.
\]
We denote by \( CM_s(\Omega) \) the set of all \( s \)-Carleson measures on \( \Omega \) and \( CM_{s,\Omega} \) the set of all compact \( s \)-Carleson measures on \( \Omega \). 1-Carleson measure is the classical Carleson measure. By [19], we know that a positive measure \( \lambda \) on \( \Delta \) is an \( s \)-Carleson measure if and only if
\[
\sup_{a \in \Delta} \iint_{\Delta} \left( \frac{1 - |a|^2}{1 - \overline{a}z |^2} \right)^s d\lambda(z) < \infty,
\]
and is a compact \( s \)-Carleson measure if and only if
\[
\lim_{|a| \to 1} \iint_{\Delta} \left( \frac{1 - |a|^2}{1 - \overline{a}z |^2} \right)^s d\lambda(z) = 0.
\]
Let \( f \) be a quasiconformal mapping of the complex plane \( \mathbb{C} \) onto itself. Then \( f \) is a homeomorphism with locally integral distributional derivatives, and satisfies the Beltrami equation \( f_{\bar{z}} = \mu f_z \) with \( \|\mu\|_{\infty} = \sup_{z \in \mathbb{C}} |\mu(z)| < 1 \). Here we use the notations
\[
f_{\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f, \quad f_z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) f.
\]
This function \( \mu \) is called the complex dilatation of \( f \). The measurable Riemann mapping theorem (see [1]) says that for each measurable function \( \mu \) on the complex plane \( \mathbb{C} \) with \( \|\mu\|_{\infty} < 1 \), there is a quasiconformal mapping \( f \) on \( \mathbb{C} \) with complex dilatation \( \mu \) and \( f \) is unique up to a Möbius transformation of \( \mathbb{C} \) onto itself.

A homeomorphism \( h \) is said to be quasisymmetric if there is some \( M(h) > 0 \) such that \( |h(I^*)| \leq M(h)|h(I)| \) for any interval \( I \subset S^1 \) with \( |I| \leq \pi \), where \( I^* \) is the interval with the same center as \( I \) but with double length. Denote by
QS(S¹) the group of quasisymmetric homeomorphisms of the unit circle S¹. It is well known that a sense preserving self-homeomorphism h is quasisymmetric if and only if it can be extended to a quasiconformal self-homeomorphism of the unit disk Δ (see [3]). Douady and Earle [7] also gave a quasiconformal extension of h to the unit disk which is conformally invariant. Let Möb(S¹) be the group of all Möbius transformations of Δ onto itself. The universal Teichmüller space is the right coset space T = QS(S¹)/Möb(S¹).

Let M(Δ*) denote the unit ball of the Banach space $L^∞(Δ^*)$ of all bounded measurable functions on Δ*. For any $μ ∈ M(Δ^*)$, there exists a unique quasiconformal mapping $f_μ$ of C whose complex dilatation is equal to $μ$ in Δ* and is zero in Δ. We normalize $f_μ$ by

$$f_μ(0) = f_μ′(0) - 1 = f_μ''(0) = 0.$$  

We say that two Beltrami coefficients $μ_1$ and $μ_2$ in $M(Δ^*)$ are Teichmüller equivalent, which is denoted by $μ_1 ∼ μ_2$, if $f_{μ_1}(Δ) = f_{μ_2}(Δ)$. The universal Teichmüller space T can be described as $T = M(Δ^*)/∼$. We denote by $[μ]$ the equivalent class containing $μ ∈ M(Δ^*)$.

Let $S_Q$ be the class of all univalent analytic functions f in the unit disk Δ with the normalized condition $f(0) = f′(0) - 1 = 0$ that can be extended to a quasiconformal mapping in the whole plane. Set $T(1) = \{log f′ : f \text{ belongs to } S_Q\}$. It is known that $T(1)$ is an alternative model called pre-logarithmic derivative model of the universal Teichmüller space. $T(1)$ is a disconnected subset of the Bloch space $B^1$. Furthermore, $T_b = \{log f′ ∈ T(1): f(Δ) \text{ is bounded}\}$ and $T_δ = \{log f′ ∈ T(1): f(e^{iθ}) = ∞, \theta ∈ [0, 2π]\}$, are the connected components of $T(1)$ (see [24]).

A quasisymmetric homeomorphism h is called strongly quasisymmetric homeomorphism if for each $ε > 0$, there is a constant $δ > 0$ such that $|E| ≤ δ|I|$ implies that $|h(E)| ≤ ε|h(I)|$, where $I ∈ S¹$ is an interval and $E ⊂ I$ is a measurable subset. In other words, h is absolutely continuous and log h′ ∈ BMOA(S¹). It is equivalent to say that there exists a quasiconformal extension of h to Δ such that its complex dilatation $μ(z)$ satisfies

$$\frac{|μ(z)|^2}{1 - |z|^2} dx dy ∈ CM_1(Δ)$$

(see [2]). The Teichmüller space called BMO-Teichmüller space with respect to the strongly quasisymmetric homeomorphisms has been much studied in recently years (see [6], [16]). In particular, Astala and Zinsmeister [2] proved that the pre-logarithmic derivative model $BMOA ∩ T(1)$ of the BMO-Teichmüller space is disconnected open subset of $BMOA$.

Recently, Wulan and Ye [18] introduced the $Q_K$-Teichmüller space and showed that its pre-logarithmic derivative model is also disconnected subset of the $Q_K$ space.

We denote by $N(p, s)$ the space of all holomorphic functions f on Δ with the following finite norm

$$||f||_{N_p,s}^p = \sup_{α ∈ Δ} \int_Δ \int_Δ |f(z)|p (1 - |z|^2)^{s+2p-2} \frac{(1 - |α|^2)^s}{|1 - αz|^2s} dx dy.$$  

We say an analytic function f belongs to $N_0(p, s)$ if $f ∈ N(p, s)$ and

$$\lim_{|α| → 1} \int_Δ \int_Δ |f(z)|p (1 - |z|^2)^{s+2p-2} \frac{(1 - |α|^2)^s}{|1 - αz|^2s} dx dy = 0.$$  


Zboroska [25] obtained a characterization of the relationship between the pre-logarithmic derivative \( \log f' \) in space \( F(p,s) \) and the Schwarzian derivative \( S_f \) in space \( N(p,s) \).

**Theorem A.** [25] Let \( f \) be conformal on \( \Delta \), \( 0 \leq s < \infty \) and \( 1 \leq p < \infty \). Then \( \log f' \in F(p,s) \) if and only if \( S_f \in N(p,s) \), while \( \log f' \in F_0(p,s) \) if and only if \( S_f \in N_0(p,s) \).

It should be pointed out that the \( F(2,1) \) case was proved by Astala and Zinsmeister in [2] and the \( F(2,s) \) case was proved by Pau and Peláez in [13].

In this paper, we introduce the \( F(p,s) \)-Teichmüller space and investigate its Schwarzian derivative model and pre-logarithmic derivative model. In what follows, we always assume that \( p \geq 2 \) and \( s > 0 \). Denote by \( M_{p,s}(\Omega) \) the Banach space of all essentially bounded measurable functions \( \mu \) each of which induces an \( s \)-Carleson measure \( \lambda_{\mu}(z) := \frac{|\mu(z)|^p}{|1-\bar{z}z|^2} \) \( dx \, dy \in CM_s(\Omega) \). The norm of \( \mu \in M_{p,s}(\Omega) \) is defined as

\[
\| \mu \|_s = \| \mu \|_\infty + \| \lambda_{\mu} \|_{C,s}^{1/p},
\]

where \( \| \lambda_{\mu} \|_{C,s} \) is the \( s \)-Carleson norm of \( \lambda_{\mu} \) on \( \Omega \). \( M_{p,s,0}(\Omega) \) is the subspace of \( M_{p,s}(\Omega) \) which consists of all elements \( \mu \) such that \( \lambda_{\mu}(z) \in CM_{s,0}(\Omega) \). Set \( M_{p,s}^1(\Omega) = M_{p,s}(\Omega) \cap M(\Omega) \) and \( M_{p,s,0}^1(\Omega) = M_{p,s,0}(\Omega) \cap M(\Omega) \), where \( M(\Omega) \) denotes the unit ball of the Banach space \( L^\infty(\Omega) \) of all bounded measurable functions on \( \Omega \). We define the \( F(p,s) \)-Teichmüller space \( T_{F(p,s)} \) as \( T_{F(p,s)} = M_{p,s}^1(\Delta^*)/\sim \) and the \( F_0(p,s) \)-Teichmüller space \( T_{F_0(p,s)} \) as \( T_{F_0(p,s)} = M_{p,s,0}^1(\Delta^*)/\sim \).

It is noted that \( F(2,1) \)-Teichmüller space is the \( \text{BMO} \)-Teichmüller space and the limit case \( F(2,0) \)-Teichmüller space is the Weil–Petersson Teichmüller space (see [5]) which has been much investigated in recently years and has wide applications to various areas such as mathematical physics, differential equation and computer vision. The limit case \( F(p,0) \)-Teichmüller space is the \( p \)-integrable Teichmüller space (see [9, 17, 21]).

The pre-logarithmic derivative model \( \tilde{T}_{F(p,s)} \) of \( F(p,s) \)-Teichmüller space is the set of functions \( \log f' \), where \( f \) is conformal on \( \Delta \) and admits a quasiconformal extension to the whole plane \( \mathbb{C} \) such that its complex dilatation \( \mu \) satisfies

\[
\frac{|\mu(z)|^p}{(|z|^2-1)^{2-s}} \, dx \, dy \in CM_s(\Delta^*).
\]

In this paper, we shall prove

**Theorem 1.1.** Let \( p \geq 2 \) and \( 0 < s < 2 \). \( \tilde{T}_{F(p,s)} \) is a disconnected subset of the space \( F(p,s) \). Furthermore, \( \overline{T}_b = \{ \log f' \in \tilde{T}_{F(p,s)} : f(\Delta) \text{ is bounded} \} \) and \( \overline{T}_b = \{ \log f' \in \tilde{T}_{F(p,s)} : f(e^{i\theta}) = \infty \}, \theta \in [0,2\pi) \), are the connected components of \( \tilde{T}_{F(p,s)} \).

Let \( B_\infty(\Delta) \) denote the Banach space of all holomorphic functions on \( \Delta \) with norm

\[
\| \varphi \|_{B_\infty} = \sup_{z \in \Delta} |\varphi(z)| (1 - |z|^2)^2 < \infty.
\]

The Schwarzian derivative \( S_f \) of a conformal mapping \( f \) on \( \Delta \) is defined as

\[
S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.
\]
The Bers projection $\Phi: M(\Delta^*) \to B_\infty(\Delta)$ is defined by $\mu \mapsto S_{f_* \mu}$. The holomorphy of the Bers projection is important in the theory of Teichmüller space. It is known that $\Phi: M(\Delta^*) \to B_\infty(\Delta)$ is holomorphic and descends down to a mapping $B: T \to B_\infty(\Delta)$ known as the Bers embedding. Via the Bers embedding, $T$ carries a natural complex Banach manifold structure so that $B$ is a holomorphic split submersion. For the Bers projection $\Phi$ on $M^1_{p,s}(\Delta^*)$, we also obtain the following result.

**Theorem 1.2.** Let $p \geq 2$ and $0 < s < 2$. The Bers projection $\Phi: M^1_{p,s}(\Delta^*) \to N(p,s)$ is holomorphic.

Fix $z_0 \in \Delta^*$. For $\mu \in M^1_{p,s}(\Delta^*)$, let $g_{\mu}^{z_0}$ (abbreviated to be $g_{\mu}$) be the quasi-conformal mapping on the complex plane $C$ whose complex dilatation equals to $\mu$ in $\Delta^*$ and zero in $\Delta$, normalized by $g_{\mu}(0) = g_{\mu}'(0) - 1 = 0$, $g_{\mu}(z_0) = \infty$. Consider the pre-Bers projection mapping $L_{z_0}$ on $M^1_{p,s}(\Delta^*)$ by setting $L_{z_0}(\mu) = \log g_{\mu}'$. Then $\bigcup_{z_0 \in \Delta^*} L_{z_0}(M^1_{p,s}(\Delta^*)) = \overline{F(p,s)} \cap F(p,s)^0$, where $F(p,s)^0$ consists of all functions $\varphi \in F(p,s)$ with $\varphi(0) = 0$.

**Theorem 1.3.** Let $p \geq 2$ and $0 < s < 2$. For $z_0 \in \Delta^*$, the pre-Bers projection mapping $L_{z_0}: M^1_{p,s}(\Delta^*) \to F(p,s)^0$ is holomorphic.

Let $f$ be a conformal mapping on $\Delta$. The Grunsky kernel function is defined as

$$U(f, \zeta, z) = \frac{f'(\zeta)f''(z)}{(f'(-\zeta) - f''(z))^2} - \frac{1}{(z - \zeta)^2}, \quad (\zeta, z) \in \Delta \times \Delta.$$  \hspace{1cm} (11)

Let $h$ be a quasisymmetric homeomorphism on the unit circle $S^1$. Then a kernel function induced by $h$ is defined as

$$\phi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(1 - \zeta w)^2(1 - z h(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta. \hspace{1cm} (12)$$

These two kernel functions induce two functions,

$$U(f, z) = \left( \frac{1}{\pi} \int_{\Delta} |U(f, \zeta, z)|^2 d\zeta d\eta \right)^{\frac{1}{2}}, \quad z \in \Delta \hspace{1cm} (13)$$

and

$$\phi_h(z) = \left( \frac{1}{\pi} \int_{\Delta} |\phi_h(\zeta, z)|^2 d\zeta d\eta \right)^{\frac{1}{2}}, \quad z \in \Delta. \hspace{1cm} (14)$$

The functions $U(f, z)$ and $\phi_h(z)$ are important in Teichmüller theory (see [10], [15], [16]). They were used to characterize when a quasisymmetric homeomorphism is symmetric in [10] or belongs to the Weil–Petersson class in [15]. They were also used to study the BMO-Teichmüller theory in [16].

For any quasisymmetric homeomorphism $h$, there exists a unique pair of conformal mappings $f \in S_Q$ on $\Delta$ and $g$ on $\Delta^*$, such that $f(0) = f'(0) - 1 = 0$, $g(\infty) = \infty$ and $h = f^{-1} \circ g$ on $S^1$. We call this a normalized decomposition of $h$. Conversely, for each $f \in S_Q$, there exists a quasisymmetric homeomorphism $h$ with normalized decomposition $h = f^{-1} \circ g$ on $S^1$ (see [11]). We have the following result.

**Theorem 1.4.** Let $p \geq 2$, $0 < s < 2$ and $h$ be a sense-preserving quasisymmetric homeomorphism with normalized decomposition $h = f^{-1} \circ g$. Then the following statements are equivalent:

1. $\log f' \in F_0(p,s)$;
2. $|S_f(z)|^p(1 - |z|^2)^{2p+s-2} dx dy \in CM_{s,0}(\Delta)$;
(3) \( f \) can be extended to a quasiconformal mapping to the whole plane such that its complex dilatation \( \mu \) satisfies \( \frac{|\mu(z)|^p}{(1-|z|^2)^{p+1}} \) \( dz \) \( dy \) \( \in \) \( CM_{n,0}(\Delta^*) \).

(4) \( |U(f, z)|^p(1-|z|^2)^{p+1} \) \( dz \) \( dy \) \( \in \) \( CM_{n,0}(\Delta) \).

(5) \( |\phi_h(\zeta)|^p(1-|\zeta|^2)^{p+1} \) \( dz \) \( dy \) \( \in \) \( CM_{n,0}(\Delta) \).

In what follows, \( C(\cdot) \) will denote constant that depends only on the elements put in the bracket.

2. Bers projection and pre-Bers projection

In order to prove Theorem 1.2 and Theorem 1.3, we need some lemmas as follows.

**Lemma 2.1.** [22] Suppose that \( k > -1, r, t > 0, \) and \( r+t-k > 2 \). If \( t < k+2 < r \), then there exists a universal constant \( C > 0 \) such that for all \( z, \zeta \in \Delta \),

\[
\int \frac{(1-|w|^2)^k}{1-\overline{w}z} \left| 1-\overline{w}\zeta \right| r \left| 1-\overline{w}\zeta \right| t \ nu dv \leq C \left(1-\frac{|z|^2}{1-\overline{z}\zeta} \right)^{2+k-r} \left| 1-\overline{z}\zeta \right| r,
\]

where \( w = u + iv \).

**Lemma 2.1.** Let \( \alpha > 0, \beta > 0 \) and \( s < 1 + \alpha/2 \). For a positive measure \( \lambda \) on \( \Delta \), set

\[
\lambda_c(z) = \int \frac{(1-|w|^2)^{\alpha}(1-|w|^2)^{\beta}}{|1-\overline{w}|^{\alpha+\beta+2}} \lambda(w) \; du dv.
\]

If \( \lambda \in CM_s(\Delta) \), then \( \lambda_c \in CM_s(\Delta) \) and there exists a constant \( C' > 0 \) such that

\[
||\lambda_c||_{C,s} \leq C' ||\lambda||_{C,s},
\]

while \( \lambda_c \in CM_{s,0}(\Delta) \) if \( \lambda \in CM_{s,0}(\Delta) \).

**Proof.** Set \( k = \alpha, r = \alpha + \beta + 2, t = 2s \) and note that \( s < 1 + \alpha/2 \), it follows from Lemma 2.1 that there exists a universal constant \( C > 0 \) such that for any \( a, w \in \Delta \),

\[
\int \frac{(1-|w|^2)^{\alpha}}{|1-\overline{w}|^{\alpha+\beta+2}|1-\overline{z}|^{2s}} \; dx \; dy \leq C \frac{(1-|w|^2)^{-\beta}}{|1-\overline{w}|^{2s} |1-\overline{a}|^{2s} |1-\overline{z}|^{2s}}.
\]

By Lemma 4.1.1 in Xiao [19], there exist some constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
C_1 ||\lambda||_{C,s} \leq \sup_{a \in \Delta} \int \lambda_c(z) \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \; dx \; dy \leq C_2 ||\lambda||_{C,s}.
\]

Consequently, we conclude from (15) that

\[
\int \frac{\lambda_c(z)(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \; dx \; dy = \int \left( \int \frac{(1-|w|^2)^{\alpha}(1-|w|^2)^{\beta}}{|1-\overline{w}|^{\alpha+\beta+2}} \lambda(w) \; du \; dv \right) \frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \; dx \; dy
\]

\[
= \int \frac{(1-|a|^2)^s}{|1-\overline{w}|^{\alpha+\beta+2}} \lambda(w) \; du \; dv \int \frac{(1-|z|^2)^{\alpha}}{|1-\overline{w}|^{\alpha+\beta+2}|1-\overline{z}|^{2s}} \; dx \; dy
\]

\[
\leq C \int \lambda(w) \frac{(1-|a|^2)^s}{|1-\overline{a}|^{2s}} \; du \; dv.
\]
If \( \lambda \in CM_{s}(\Delta) \), then from (16) and (17), we deduce that \( \overline{\lambda} \in CM_{s}(\Delta) \) and there is a constant \( C' = \frac{CC}{C_{1}} \) such that \( \| \overline{\lambda} \|_{C,s} \leq C'' \| \lambda \|_{C,s} \). If \( \lambda \in CM_{s,0}(\Delta) \), then

\[
\lim_{|a| \to 1} \iint_{\Delta} \lambda(w) \left( \frac{1 - |a|^{2}}{1 - \overline{a}w} \right)^{s} \, du \, dv = 0.
\]

We deduce from (17) that

\[
\lim_{|a| \to 1} \iint_{\Delta} \overline{\lambda}(z) \left( \frac{1 - |a|^{2}}{1 - \overline{a}z} \right)^{s} \, dx \, dy = 0.
\]

Therefore \( \overline{\lambda} \in CM_{s,0}(\Delta) \). 

Let \( f \) be a conformal mapping on \( \Delta^{*} \). For any \( z \in \Delta^{*} \), set

\[
\beta_{z}(w) = \frac{1 + wz}{w + \overline{z}} \quad \text{and} \quad \gamma_{f}(w) = \frac{(|z|^{2} - 1)}{w - f(z)} f'(z).
\]

Then \( \beta_{z} \) is an automorphism of \( \Delta^{*} \) sending \( \infty \) to \( z \). We need a representation theorem of the Schwarzian derivative, which is proved by Astala and Zinsmeister [2].

**Lemma 2.3.** [2] Let \( f \) be a conformal mapping on \( \Delta^{*} \) and admits a quasiconformal extension to the whole plane, then for any \( z \in \Delta^{*} \) and \( w = u + iv \),

\[
S_{f}(z) = -\frac{3(|z|^{2} - 1)^{2}}{2\pi} \iint_{\Delta} \overline{\partial g(w)} \, du \, dv,
\]

where \( g = \gamma_{f} \circ f \circ \beta_{z} \).

We first show that the Bers projection is well defined.

**Proposition 2.4.** Let \( p \geq 2 \) and \( 0 < s < 2 \). If \( \mu \in M_{p,s}(\Delta^{*}) \), then \( \Phi(\mu) \in N(p,s) \).

**Proof.** Let \( \mu \in M_{p,s}(\Delta^{*}) \) and \( f_{\mu} \) be the normalized quasiconformal mapping \( f_{\mu} \) of \( \mathbb{C} \) whose complex dilatation is \( \mu \) in \( \Delta^{*} \) and is zero in \( \Delta \). Set \( \hat{f}(\zeta) = s \circ f_{\mu} \circ s(\zeta) \), where \( s(\zeta) = 1/\zeta \). Then \( \hat{f} \) is a quasiconformal mapping of the whole plane \( \mathbb{C} \) whose complex dilatation \( \mu_{\hat{f}}(\zeta) \) satisfies \( |\mu_{\hat{f}}(\zeta)| = |\mu(\zeta)| \) in \( \Delta \) and is zero in \( \Delta^{*} \). By a change of variable, we conclude that

\[
\lambda_{\mu_{\hat{f}}} = \frac{|\mu_{\hat{f}}(\zeta)|^{p}}{(1 - |\zeta|^{2})^{2-s}} d\xi \, d\eta \in CM_{s}(\Delta) \quad \text{and} \quad \| \lambda_{\mu_{\hat{f}}} \|_{C,s} = \| \lambda_{\mu} \|_{C,s}.
\]

Let \( g = \gamma_{f} \circ \hat{f} \circ \beta_{z} \), where \( \gamma_{f} \) and \( \beta_{z} \) are defined in (20). The area theorem of univalent functions yields

\[
\iint_{\Delta} J_{g}(\zeta) d\xi \, d\eta \leq \pi,
\]

where \( J_{g} \) is the Jacobian determinant of \( g \). Noting that \( \mu_{g} = \mu_{f \circ \beta_{z}} \) and \( \| \mu_{g} \|_{\infty} = \| \mu \|_{\infty} \), by (21), (23) and Hölder’s inequality, we get

\[
|S_{\hat{f}}(z)|^{p}(|z|^{2} - 1)^{2p} = \left( \frac{3}{2\pi} \right)^{p} \iint_{\Delta} (\mu_{\hat{f} \circ \beta_{z}} \partial g)(\zeta) d\xi \, d\eta \left| \frac{p}{p} \right|^{p} \leq \left( \frac{3}{2\pi} \right)^{p} \left( 1 - \| \mu_{g} \|_{\infty} \right)^{2} \iint_{\Delta} |\mu_{\hat{f} \circ \beta_{z}}(\zeta)|^{p} d\xi \, d\eta \right.
\]

\[
= C_{1}(\| \mu \|_{\infty}) \iint_{\Delta} \frac{|\mu_{\hat{f}}(w)|^{p}(|z|^{2} - 1)^{2}}{|w - z|^{4}} \, du \, dv,
\]
where $C_1(||\mu||_\infty) = (\frac{\pi}{2})^p \frac{2^{1-p}}{(1-||\mu||_\infty^2)^{\frac{p}{2}}}$. Consequently, for $b \in \Delta^*$, set $a = 1/b \in \Delta$, by (22), (24) and Lemma 2.2, we have

$$
\int \int_{\Delta^*} |S_f(z)|^p (|z|^2 - 1)^{2p+s-2} \left(\frac{|b|^2 - 1}{1 - wz}\right)^s dx dy
\leq C_1(||\mu||_\infty) \int \int_{\Delta} |\mu_f(w)|^p (1 - |w|^2)^{2-s} \left(\frac{1}{1 - w^2}\right)^s du dv \int \int_{\Delta} (1 - |w|^2)^{-s} \left(\frac{1}{1 - \alpha z}\right)^{2s} dx dy
\leq C_2(||\mu||_\infty) \int \int_{\Delta} |\mu_f(w)|^p (1 - |w|^2)^{2-s} \left(\frac{1}{1 - \alpha z}\right)^{2s} du dv
\leq C_3(||\mu||_\infty) \lambda\mu ||_{\mathbb{C}, s} = C_3(||\mu||_\infty) \lambda\mu ||_{\mathbb{C}, s}.
$$

(25)

Noting that $|S_f(z)| = |S_{f_\nu}(\frac{1}{z})| = \frac{1}{|z|^s}$, we get from (25) that

$$
||S_{f_\nu}||_{p,s}^p = \sup_{a \in \Delta^*} \int \int_{\Delta^*} |S_f(z)|^p (|z|^2 - 1)^{2p+s-2} \left(\frac{|b|^2 - 1}{1 - wz}\right)^s dx dy < \infty.
$$

Which implies that $\Phi(\mu) = S_{f_\nu} \in N(p, s)$ if $\mu \in M_{p,s}^1(\Delta^*)$. The proof follows. \(\square\)

Before proving Theorem 1.2, we first recall some basic facts about the infinite dimensional holomorphy (see [11, p. 206], [12, p. 86-87]). Let $E$ and $F$ be two complex Banach space and $U$ an open subset in $E$, a mapping $f: U \rightarrow F$ is holomorphic if and only if it is continuous (locally boundedness is also enough) and the complex Gateaux derivative $d_x(\lambda)$ defined as

$$
d_x(\lambda) = \lim_{t \rightarrow 0} \frac{f(x + t\lambda) - f(x)}{t}
$$

exists for each $(x, \lambda) \in U \times E$.

Let $F^*$ denote the dual space of $F$ in the usual sense. For a subset $A$ of $F^*$, we define $A^\perp = \{y \in F: y^*(y) = 0, y^* \in A\}$. A subset $A$ is called total if $A^\perp = \{0\}$.

**Proposition 2.5.** [11, 12] $f: U \rightarrow F$ is holomorphic if and only if it satisfies one of the following conditions.

(i) The mapping $f$ is local bounded and for every $(x, \lambda) \in U \times E$, the mapping $t \mapsto f(x + t\lambda)$ is holomorphic from an open neighborhood of zero in the complex plane $\mathbb{C}$ to $F$.

(ii) The mapping $f$ is continuous and there exists a total subset $A$ of $F^*$ such that for every $y^* \in A$, the function $y^*(f): U \rightarrow \mathbb{C}$ is holomorphic.

We are now in a position to prove Theorem 1.2. Our proof is based on the proof of Theorem 3 in Cui [5].

**Proof of Theorem 1.2.** We first show that mapping $\Phi: M_{p,s}^1(\Delta^*) \rightarrow N(p, s)$ is continuous. Let $\widehat{\mu} \in M_{p,s}^1(\Delta^*)$, $\widehat{\nu} \in M_{p,s}^1(\Delta^*)$. It is sufficient to show that there is a constant $C(||\widehat{\mu}||_\infty, ||\widehat{\nu}||_\infty)$ such that

$$
||\Phi(\widehat{\mu}) - \Phi(\widehat{\nu})||_{N_{p,s}} \leq C(||\widehat{\mu}||_\infty, ||\widehat{\nu}||_\infty) ||\widehat{\mu} - \widehat{\nu}||_s.
$$

(27)

Set $\widehat{f}_1 = f_{\widehat{\mu}}$, $\widehat{f}_2 = f_{\widehat{\nu}}$ and $f_i(\zeta) = s \circ \widehat{f}_i \circ s(\zeta)$, $i = 1, 2$, where $s(\zeta) = 1/\zeta$. Then $f_1$ is a quasiconformal mapping of $\mathbb{C}$ whose complex dilatation is equal to $\mu(\zeta) = \widehat{\mu}(s(\zeta)) \frac{2|G|}{s(\zeta)}$ in $\Delta$ and is zero in $\Delta^*$, while $f_2$ is a quasiconformal mapping of $\mathbb{C}$ whose complex dilatation is equal to $\nu(\zeta) = \widehat{\nu}(s(\zeta)) \frac{2|G|}{s(\zeta)}$ in $\Delta$ and is zero in $\Delta^*$. Thus the
correspondence between \(\mu\) and \(\hat{\mu}\) is one-to-one and \(\|\hat{\mu}\|\infty = \|\mu\|\infty, \|\nu\|\infty = \|\nu\|\infty\). By a change of variable, we conclude that

\[
\|\lambda_{\mu_f}\|_{C,s} = \|\lambda_{\mu}\|_{C,s}, \quad \|\lambda_{\nu_f}\|_{C,s} = \|\lambda_{\nu}\|_{C,s}
\]

and

\[
\|\Phi(\hat{\mu}) - \Phi(\hat{\nu})\|_p^{\Phi} = \sup_{b \in \Delta^*} \int_{\Delta} |S_{f_1}(z) - S_{f_2}(z)|^p \left( \frac{|z|^2 - 1}{1 - \frac{1}{b^2}} \right)^{s-2} \frac{\|h\|^2 - 1}{|1 - \frac{1}{b^2}|^2} \, dx \, dy.
\]

Let \(g^\mu = \gamma_{f_1} \circ f_1 \circ \beta_z\) and \(g^\nu = \gamma_{f_2} \circ f_2 \circ \beta_z\). By Lemma 2.3, we have

\[
S_{f_1}(z) - S_{f_2}(z) = \frac{-3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} \overline{\partial}(g^\mu - g^\nu)(w) \, du \, dv.
\]

Set \(\mu_{\beta_z}(w) = \mu(\beta_z(w))\frac{\nu(\beta_z(w))}{\nu(\beta_z)}\), \(\nu_{\beta_z}(w) = \nu(\beta_z(w))\frac{\nu(\beta_z)}{\nu(\beta_z)}\). Let \(H\) be the Beuring–Ahlfors operator defined as

\[
H(\phi)(\zeta) = -\frac{1}{\pi} \int_{C} \frac{\phi(z)}{(\zeta - z)^2} \, dx \, dy,
\]

the integral is understood in the sense of Cauchy principal value. The representation theorem of quasiconformal mapping says that \(\overline{\partial}g^\mu = \mu_{\beta_z}(I + H\overline{\partial}g^\mu)\), \(\overline{\partial}g^\nu = \nu_{\beta_z}(I + H\overline{\partial}g^\nu)\) (see [1, Chapter V]). Consequently, we conclude that

\[
\overline{\partial}(g^\mu - g^\nu) = \mu_{\beta_z}(I + H\overline{\partial}g^\mu) - \nu_{\beta_z}(I + H\overline{\partial}g^\nu)
\]

\[
= \mu_{\beta_z} - \nu_{\beta_z} + \mu_{\beta_z}H\overline{\partial}g^\mu - \mu_{\beta_z}H\overline{\partial}g^\nu + \mu_{\beta_z}H\overline{\partial}g^\nu - \nu_{\beta_z}H\overline{\partial}g^\nu
\]

\[
= (\mu_{\beta_z} - \nu_{\beta_z})(H\overline{\partial}g^\nu + I) + \mu_{\beta_z}H\overline{\partial}(g^\mu - g^\nu).
\]

Since \(\overline{\partial}g^\nu = I + H\overline{\partial}g^\nu\), we have

\[
\overline{\partial}(g^\mu - g^\nu) = (I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})(I + H\overline{\partial}g^\nu) = (I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu.
\]

Thus it follows from (30) that

\[
S_{f_2}(z) - S_{f_1}(z) = -\frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} (I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(w) \, du \, dv.
\]

Since \((I - \mu_{\beta_z}H)^{-1} = I + \mu_{\beta_z}H(I - \mu_{\beta_z}H)^{-1}\), we have

\[
S_{f_2}(z) - S_{f_1}(z)
\]

\[
= -\frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} (I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(\zeta) \, d\xi \, d\eta
\]

\[
= -\frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} (\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(\zeta) \, d\xi \, d\eta
\]

\[
- \frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} \mu_{\beta_z}H(I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(\zeta) \, d\xi \, d\eta
\]

\[
= L_1 + L_2,
\]

where

\[
L_1 = -\frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} (\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(\zeta) \, d\xi \, d\eta
\]

and

\[
L_2 = -\frac{3(|z|^2 - 1)^{-2}}{2\pi} \int_{\Delta} \mu_{\beta_z}H(I - \mu_{\beta_z}H)^{-1}(\mu_{\beta_z} - \nu_{\beta_z})\overline{\partial}g^\nu(\zeta) \, d\xi \, d\eta.
\]
By using the method similar to (24), we get

$$\left| L_1 \right|^p (|z|^2 - 1)^{2p} \leq C(\|\mu\|_\infty) \iint_{\Delta} \frac{|\mu(w) - \nu(w)|^p (|z|^2 - 1)^2}{|w - z|^4} \, du \, dv.$$  

Consequently, similar to (25), by Lemma 2.2 and a change of variable, we obtain

$$\sup_{b \in \Delta^*} \left| \int_{\Delta} |L_1(z)|^p (|z|^2 - 1)^{2p+s-2} \left( \frac{|b|^2 - 1|^s}{|1 - \overline{b}z|^2s} \right) \, dx \, dy \right| \leq C(\|\mu\|_\infty) \sup_{a \in \Delta} \iint_{\Delta} \frac{|\mu(w) - \nu(w)|^p (1 - |a|^2)^s}{(1 - |w|^2)^{2s}} \, du \, dv. \tag{34}$$

We now estimate $L_2$. It is noted that when $\|\mu\|_\infty < 1$, the operator $I - \mu H$ is invertible on $L^2(\Delta)$ and the norm of its inverse $(I - \mu H)^{-1}$ is less than $1/(1 - \|\mu\|_\infty)$. Thus we have

$$\left| L_2 \right|^p (|z|^2 - 1)^{4} = \left( \frac{3}{2\pi} \right)^2 \iint_{\Delta^*} \mu_{\beta} H (I - \mu_{\beta} H)^{-1} ((\mu_{\beta} - \nu_{\beta}) \partial g^\nu)(\zeta) \, d\xi \, d\eta \right|^2 \leq \frac{9}{4\pi^2 (1 - \|\mu\|_\infty)^2} \iint_{\Delta} \left| \mu_{\beta} \right|^2 \, d\xi \, d\eta \iint_{\Delta} \left| (\mu_{\beta} - \nu_{\beta}) \partial g^\nu(\zeta) \right|^2 \, d\xi \, d\eta \leq \frac{9}{4\pi (1 - \|\mu\|_\infty)^2 (1 - \|\nu\|_\infty^2)} \iint_{\Delta} \left| \mu_{\beta} \right|^2 \, d\xi \, d\eta = \frac{9}{4\pi (1 - \|\mu\|_\infty)^2 (1 - \|\nu\|_\infty^2)} \iint_{\Delta} \left| \mu(\zeta) \right|^2 \left| (|z|^2 - 1)^2 \right| \, d\xi \, d\eta. \tag{35}$$

Noting that $p \geq 2$, by Hölder’s inequality, we get

$$\left| L_2 \right|^p \leq \left( \frac{9}{4\pi (1 - \|\mu\|_\infty)^2 (1 - \|\nu\|_\infty^2)} \right) \iint_{\Delta} \left| \mu(\zeta) \right|^p \left| \frac{1}{|\zeta - z|^4} \right|^2 \, d\xi \, d\eta \left( \iint_{\Delta} \left| \frac{1}{|\zeta - z|^4} \right|^2 \, d\xi \, d\eta \right)^{\frac{2p-2}{2}} \tag{36} \leq C_1(\|\mu\|_\infty, \|\nu\|_\infty) \|\mu - \nu\|_\infty^p \iint_{\Delta} \left| \frac{\mu(\zeta)}{|\zeta - z|^4} \right|^p \left| (|z|^2 - 1)^2 \right| \, d\xi \, d\eta,$$

Similar to $L_1$, we can deduce that

$$\sup_{b \in \Delta^*} \left| \int_{\Delta} |L_2(z)|^p (|z|^2 - 1)^{2p+s-2} \left( \frac{|b|^2 - 1|^s}{|1 - \overline{b}z|^2s} \right) \, dx \, dy \right| \leq C_2(\|\mu\|_\infty, \|\nu\|_\infty) \|\mu - \nu\|_\infty^p \sup_{a \in \Delta} \iint_{\Delta} \frac{|\mu(w)|^p (1 - |a|^2)^s}{(1 - |w|^2)^{2s}} \, du \, dv. \tag{37}$$

Combining (28), (29), (32), (34) and (37), we deduce that (27) holds and thus the mapping $\Phi: M_{p,s}(\Delta^*) \to N(p, s)$ is continuous.

We now prove that the Bers projection $\Phi: M_{p,s}(\Delta^*) \to N(p, s)$ is holomorphic. For each $z \in \Delta$, we define a continuous linear functional $l_z$ on the Banach space $N(p, s)$ by $l_z(\varphi) = \varphi(z)$ for $\varphi \in N(p, s)$. Then the set $A = \{l_z: z \in \Delta\}$ is a total subset of the dual space of $N(p, s)$. Now for each $z \in \Delta$, each pair $(\mu, \nu) \in M_{p,s}(\Delta^*) \times M_{p,s}(\Delta^*)$ and small $t$ in the complex plane, by the well known holomorphic dependence of quasiconformal mappings on parameters (see [11, Theorem 3.1 in Chapter II], [1, Chapter V]), we conclude that $l_z(\Phi(\mu + t\nu)) = S_{p, s} \Phi(\mu + t\nu)(z)$ is a holomorphic function of $t$. From Proposition 2.5, the Bers projection $\Phi: M_{p,s}(\Delta^*) \to N(p, s)$ is holomorphic. This completes the proof.

Checking the proof of Theorem 1.2, we can show the following
Theorem 2.6. Let \( p \geq 2 \) and \( 0 < s < 2 \). The Bers projection \( \Phi: M_{p,s,0}^{1}(\Delta^{*}) \rightarrow N_{0}(p,s) \) is holomorphic.

We now prove Theorem 1.3.

Proof of Theorem 1.3. It follows from Theorem A and Proposition 2.4 that the mapping \( L_{z_{0}}: M_{p,s}^{1}(\Delta^{*}) \rightarrow F(p,s)^{0} \) is well defined. We can prove \( L_{z_{0}}: M_{p,s}^{1}(\Delta^{*}) \rightarrow F(p,s)^{0} \) is holomorphic by the same reasoning as the proof of the holomorphy of \( \Phi: M_{p,s}^{1}(\Delta^{*}) \rightarrow N(p,s) \). Thus it is enough to show that \( L_{z_{0}}: M_{p,s}^{1}(\Delta^{*}) \rightarrow F(p,s)^{0} \) is continuous. For \( \mu, \nu \in M_{p,s}^{1}(\Delta^{*}) \), it follows from the proof of [11, Theorem 3.1 in Chapter II] that

\[
\sup_{z \in \Delta} (1 - |z|^{2}) \left| \frac{g_{\mu}''}{g_{\mu}'} - \frac{g_{\nu}''}{g_{\nu}'} \right| \leq C(\|\mu\|_{\infty} - \|\nu\|_{\infty}).
\]

By Theorem 1.2, we conclude that

\[
\|S_{g_{\mu}}(z) - S_{g_{\nu}}(z)\|_{N_{p,s}} \leq C_{1}(\|\mu\|_{\infty}, \|\nu\|_{\infty})\|\mu - \nu\|_{s}.
\]

It follows from Chapter 4 in [14] that there is a constant \( C_{2} > 0 \) which is independent of \( \mu \) and \( \nu \) such that

\[
\int_{\Delta} \left| \frac{g_{\mu}''}{g_{\mu}'} - \frac{g_{\nu}''}{g_{\nu}'} \right| (1 - |z|^{2})^{p-2}(1 - |a|^{2})^{s} \frac{dx dy}{|1 - \overline{a}z|^{2s}}
\]

\[
\leq C_{2} \left\| \frac{g_{\mu}''(0)}{g_{\mu}'}(0) - \frac{g_{\nu}''(0)}{g_{\nu}'}(0) \right\|^{p} + C_{2} \int_{\Delta} \left( \left( \frac{g_{\mu}''}{g_{\mu}'} \right)' - \left( \frac{g_{\nu}''}{g_{\nu}'} \right)' \right)^{p} \frac{(1 - |z|^{2})^{2p-2}(1 - |a|^{2})^{s}}{|1 - \overline{a}z|^{2s}} dx dy.
\]

By the definition of the Schwarzian derivative, we get

\[
\left( \frac{g_{\mu}''}{g_{\mu}'} \right)' - \left( \frac{g_{\nu}''}{g_{\nu}'} \right)' \leq 2p |S_{g_{\mu}} - S_{g_{\nu}}|^{p} + 2p \left( \frac{\left( \frac{g_{\mu}''}{g_{\mu}'} \right)^{2}}{\frac{g_{\mu}''}{g_{\mu}'} + \frac{g_{\nu}''}{g_{\nu}'} - \frac{g_{\nu}''}{g_{\nu}'} \frac{g_{\mu}''}{g_{\mu}'} \right)^{p} + 2p \left( \frac{g_{\mu}''}{g_{\mu}'} + \frac{g_{\nu}''}{g_{\nu}'} \right)^{p} \right)
\]

Taking \( z = 0 \) in (38), we get

\[
\left| \frac{g_{\mu}''(0)}{g_{\mu}'}(0) - \frac{g_{\nu}''(0)}{g_{\nu}'}(0) \right|^{p} \leq C_{p}(\|\mu\|_{\infty})\|\mu - \nu\|_{p}^{p}.
\]

It follows from (38), (39) and (41) that

\[
\int_{\Delta} \left( \frac{g_{\mu}''}{g_{\mu}'} \right)' - \left( \frac{g_{\nu}''}{g_{\nu}'} \right)' \frac{(1 - |z|^{2})^{2p-2}(1 - |a|^{2})^{s}}{|1 - \overline{a}z|^{2s}} dx dy
\]

\[
\leq 2p \int_{\Delta} \left| S_{g_{\mu}} - S_{g_{\nu}} \right|^{p} \frac{(1 - |z|^{2})^{2p-2}(1 - |a|^{2})^{s}}{|1 - \overline{a}z|^{2s}} dx dy
\]

\[
+ 2p \int_{\Delta} \left| g_{\mu}'' + g_{\nu}'' \right| \left( \frac{g_{\mu}''}{g_{\mu}'} - \frac{g_{\nu}''}{g_{\nu}'} \right)^{p} \frac{(1 - |z|^{2})^{2p-2}(1 - |a|^{2})^{s}}{|1 - \overline{a}z|^{2s}} dx dy
\]

\[
\leq 2p C_{p}(\|\mu\|_{\infty}, \|\nu\|_{\infty})\|\mu - \nu\|_{s}^{p}
\]

\[
+ 2p \sup_{z \in \Delta} (1 - |z|^{2})^{p} \left| \frac{g_{\mu}''}{g_{\mu}'} - \frac{g_{\nu}''}{g_{\nu}'} \right| \int_{\Delta} \left( \frac{g_{\mu}''}{g_{\mu}'} + \frac{g_{\nu}''}{g_{\nu}'} \right)^{p} \frac{(1 - |z|^{2})^{2p-2}(1 - |a|^{2})^{s}}{|1 - \overline{a}z|^{2s}} dx dy
\]

\[
\leq 2p C_{p}(\|\mu\|_{\infty}, \|\nu\|_{\infty})\|\mu - \nu\|_{s}^{p}
\]

\[
+ 4p C_{p}(\|\mu\|_{\infty}) \log g_{\mu}' F_{p,p-2,s} + \| \log g_{\nu}' F_{p,p-2,s} \|_{p} \| \mu - \nu \|_{p}^{p}.
\]
Combining (40), (42) and (43), we get
\[
\|L_{z_0}(\mu) - L_{z_0}(\nu)\|_{F(p,p-2,s)} \\
\leq C_3 \left(\|\mu\|_{\infty}, \|\nu\|_{\infty}, \|\log g'\|_{F(p,p-2,s)}, \|\log g''\|_{F(p,p-2,s)}\right) \|\mu - \nu\|_s.
\]
This completes the proof of Theorem 1.3. \qed

Similarly, we have the following

**Theorem 2.7.** Let \( p \geq 2 \) and \( 0 < s < 2 \). For \( z_0 \in \Delta^* \), the pre-Bers projection mapping \( L_{z_0}: M^{1}_{p,s,0}(\Delta^*) \rightarrow F_0(p,s)^0 \) is holomorphic.

### 3. Proofs of Theorem 1.1 and Theorem 1.4

In this section, we prove Theorem 1.1 and Theorem 1.4.

**Proof of Theorem 1.1.** Let \( f' \in \tilde{T}_{F(p,s)} \). Then \( f \) is a quasiconformal mapping of the complex plane \( \mathbb{C} \) whose complex dilatation \( \mu \) satisfies \( \mu = \frac{|\mu(z)|^p}{(|z|^2 - 1)^{p-2}} \) \( dx \, dy \in CM_s(\Delta^*) \) and equals to zero in \( \Delta \). Let \( f' \) be the quasiconformal mapping in \( \mathbb{C} \) with \( f^{-1}(\infty) = (f')^{-1}(\infty) \) and \( f\tilde{f}' = t\mu \partial f' \). Consider the path \( t \mapsto \log(f')^t, 0 \leq t \leq 1 \), in the space \( F(p,s) \). Set \( g = f' \) and \( h = f^s \). By Theorem 1.3, we conclude that
\[
\|\log g' - \log h\|_{F(p,p-2,s)} \leq C \left(\|\mu\|_s, \|\log g'\|_{F(p,p-2,s)}, \|\log h'\|_{F(p,p-2,s)}\right) |t - s|.
\]
This implies that the path \( t \mapsto \log(f')^t, 0 \leq t \leq 1 \), is continuous in the space \( F(p,s) \). Consequently, each \( f' \in \tilde{T}_{F(p,s)} \) can be connected by a continuous path to an element \( \log \varphi' \in F(p,s) \), where \( \varphi \) is a Möbius transformation of \( \mathbb{C} \). If \( \varphi(\Delta) \) is unbounded, then \( f(\zeta) = \varphi(\zeta) = \infty \) for some \( \zeta \in \partial \Delta \). Otherwise \( \varphi(\Delta) \) is bounded, we consider the path \( r \mapsto \log \varphi'_{r} \), where \( \varphi_r = \varphi(rz), 0 \leq r \leq 1 \). It is easy to see that this is a path which connects the point \( \log \varphi' \) to the point \( 0 \) in \( F(p,s) \). It turns out that \( \tilde{T}_b = \{ f' \in \tilde{T}_{F(p,s)} : f(\Delta) \text{ bounded} \} \) and \( \tilde{T}_\theta = \{ \log f' \in \tilde{T}_{F(p,s)} : \lim_{z \to e^{i\theta}} f(z) = \infty \}, 0 \leq \theta \leq 2\pi \), are connected. By [24], elements in different classes cannot be connected even in Bloch space. We conclude that \( \tilde{T}_b \) and \( \tilde{T}_\theta \) are the connected components of \( \tilde{T}_{F(p,s)} \). \qed

Before proving Theorem 1.4, we need a lemma which was proved by Shen and Wei in [16].

**Lemma 3.1.** Let \( h \) be a quasisymmetric homeomorphism on \( S^1 \) with normalized decomposition \( h = f^{-1} \circ g \) and \( \nu \) be the complex dilatation of a quasiconformal extension of \( h^{-1} \) to \( \Delta \). Then
\[
\frac{1}{36} (1 - |z|^2)^2 |S_f(z)|^2 \leq U^2(f, z) \leq \phi_h^2(\overline{z}) \leq \frac{1}{\pi} \int_{\Delta} \frac{|
u(w)|^2}{1 - |
u(w)|^2} \frac{1}{1 - \overline{w}z^2} du \, dv.
\]

**Proof of Theorem 1.4.** It follows from Theorem A that \( (1) \iff (2) \). From Theorem 1.3, we conclude that \( (1) \iff (3) \). Lemma 3.1 gives \( (4) \iff (5) \) and \( (4) \iff (2) \). Thus it remains to show that \( (3) \implies (1) \) and \( (3) \implies (5) \).

We first prove that \( (3) \implies (5) \). Let \( h \) be a quasisymmetric homeomorphism on the unit circle \( S^1 \). Then there exists a unique pair of conformal mappings \( f \in S_Q \) on \( \Delta \) and \( g \) on \( \Delta^* \), such that \( f(0) = f'(0) = 1 = 0 \), \( g(\infty) = \infty \) and \( h = f^{-1} \circ g \) on \( S^1 \) (see [11, Lemma 1.1 in Chapter III]). Suppose \( f \) can be extended to a quasiconformal mapping of the whole plane \( \mathbb{C} \), which is also denoted by \( f \), such that its complex dilatation \( \mu \) satisfies \( \frac{|\mu(z)|^p}{(|z|^2 - 1)^{p-2}} \) \( dx \, dy \in CM_s(\Delta^*) \). It is noted that \( \tilde{H} = g^{-1} \circ f \) is a quasiconformal extension of \( h^{-1} \) to \( \Delta^* \) and has the same complex dilatation \( \mu \) as \( f \).
Then $H = j \circ \hat{H} \circ j$, where $j(z) = 1/\overline{z}$, is a quasiconformal extension of $h^{-1}$ to $\Delta$ with complex dilatation $\nu(z)$ satisfying $|\nu(z)| = |\mu(1/\overline{z})|$. A computation shows that \[
abla^{p} \int \frac{\nu(z)^{p}}{(1-|z|^{2})^{2-s}} \, dx \, dy \in CM_{s,0}(\Delta). \] By Lemma 2.2 and Lemma 3.1, we conclude that (5) holds.

We now show that $(1) \implies (3)$. Suppose that (1) holds. Noting that $F_{0}(p, s)$ is a subspace of the little Bloch space $B_{1}^{1}$, we have
\[
\lim_{|z| \to 1} (1 - |z|^{2}) \frac{|f''(z)|}{|f'(z)|} = 0.
\]
Becker and Pommerenke (see [4]) constructed a quasiconformal extension of the conformal mapping $f$ to the whole plane $\mathbb{C}$ by the following formula
\[
f(z) = f(1/\overline{z}) + f'(1/\overline{z})(z - 1/\overline{z}), \quad z \in \Delta^{*}.
\]
By some computations we have
\[
|\mu(z)| = |1/z|^{2}(1 - |1/z|^{2})|f''(1/\overline{z})|/|f'(1/\overline{z})|.
\]
For $z, b \in \Delta^{*}$, we set $w = 1/\overline{z}$ and $b = 1/\overline{a}$. A change of variable gives
\[
\begin{align*}
\int_{\Delta} |\mu(z)|^{p} \frac{(|b|^{2} - 1)^{s}}{|1 - b\overline{z}|^{2s}} \, dx \, dy \\
\leq \int_{\Delta} |f''(w)|/|f'(w)|^{p} (1 - |w|^{2})^{p-2+s} \frac{(1 - |a|^{2})^{s}}{|1 - \overline{a}w|^{2s}} \, du \, dv.
\end{align*}
\]
Noting that $\log f' \in F_{0}(p, s)$ and $|b| \to 1$ if and only if $|a| \to 1$, we conclude that
\[
\int_{\Delta} \frac{|\mu(z)|^{p}}{(1 - |z|^{2})^{2-s}} \, dx \, dy \in CM_{s,0}(\Delta^{*}).
\]
The proof follows.

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References


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