LIGHT SIDE OF COMPACTNESS IN
LEBESGUE SPACES: SUDAKOV THEOREM

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Abstract. In this note we show that, in the case of bounded sets in metric spaces with
some additional structure, the boundedness of a family of Lebesgue $p$-summable functions follow
from a certain uniform limit norm condition. As a byproduct, the well known Riesz–Kolmogorov
compactness theorem can be formulated only with one condition.

1. Introduction

The classical theorem of Kolmogorov [11], sometimes also called Riesz–Kolmogorov theorem, characterizes the compactness of sets of functions in Lebesgue spaces. In the original formulation of Kolmogorov the theorem is the following:

Theorem 1.1. (Kolmogorov) Suppose $\mathcal{F}$ is a set of functions in $L^p([0,1])$ ($1 < p < \infty$). In order that this set be relatively compact, it is necessary and sufficient that both of the following conditions be satisfied:

(K1) the set $\mathcal{F}$ is bounded in $L^p$;
(K2) $\lim_{h \to 0} \| f_h - f \|_p = 0$ uniformly with respect to $f \in \mathcal{F}$,

where $f_h$ denotes the well-known Steklov function, viz.

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt.$$

After that, Tamarkin [18] extended the result to the case where the underlying space can be unbounded, with an additional condition related to the behaviour at infinity. Tulajkov [19] showed that Tamarkin’s result was true even when $p = 1$. Finally, Sudakov [16] showed that condition (K1) follows from condition (K2). All the previous results were proved in the framework of one dimensional Euclidean space.


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For a more detailed account regarding the history of the Riesz–Kolmogorov theorem, see [9].

In this small note we want to show that, whenever we are working in the general framework of metric measure spaces, condition (K1) is superfluous since it is a consequence of condition (K2).

2. Preliminaries

We shall denote the average of locally integrable function $f$ over the measurable set $A$ in the following manner

$$(f)_A = \frac{1}{\mu(A)} \int_A f \, d\mu.$$  

Let $(X, \rho, \mu)$ be a metric measure space equipped with a metric $\rho$ and a Borel regular measure $\mu$. We assume throughout the paper that the measure of every open nonempty set is positive and that the measure of every bounded set is finite. Additionally, we assume that the measure $\mu$ satisfies a doubling condition. This means that, there exists a constant $C_d > 0$ such that for every ball $B(x, r)$,

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)).$$

Now, let us recall the notion of continuity of a measure with respect to a metric (see [3, 1]).

**Definition 2.1.** Let $(X, \rho, \mu)$ be a metric measure space. The measure $\mu$ is said to be continuous with respect to the metric $\rho$ if for all $x \in X$ and $r > 0$ the following condition holds:

$$\lim_{y \to x} \mu(B(x, r) \Delta B(y, r)) = 0,$$

where $A \Delta B$ stands for the symmetric difference, i.e. $A \Delta B := A \setminus B \cup B \setminus A$.

For example, when $(X, \rho, \mu)$ is a geodesic space (cf. [10]) and the measure $\mu$ is doubling, then $\mu$ is continuous with respect to the metric $\rho$ (see [1]).

Now, we can recall the characterization of relatively compact sets in $L^p(X, \mu)$ from [6].

**Theorem 2.2.** Let $(X, \rho, \mu)$ be a metric measure space and $1 < p < \infty$. Suppose moreover, that there exists $\theta > 0$ such that $\mu(B(x, 1)) \geq \theta$. Let $x_0 \in X$, then the subset $\mathcal{F}$ of $L^p(X, \mu)$ is relatively compact in $L^p(X, \mu)$ if and only if the following conditions are satisfied:

1. $\mathcal{F}$ is bounded,
2. $\lim_{R \to \infty} \int_{X \setminus B(x_0, R)} |f(x)|^p \, d\mu(x) = 0$, uniformly for $f \in \mathcal{F}$,
3. $\lim_{r \to 0} \int_X |f(x) - (f)_{B(x, r)}|^p \, d\mu(x) = 0$, uniformly for $f \in \mathcal{F}$.

3. Main result

The main result of this paper is the following.

**Theorem 3.1.** Assume that $(X, \rho, \mu)$ is a connected metric measure space with continuous measure satisfying the doubling condition. Suppose, moreover, that balls are relatively compact and there exists $\theta > 0$ such that $\mu(B(x, 1)) \geq \theta$. Let $1 < p < \infty$.
\( \infty \) and \( D \) be a bounded subset of \( X \) such that \( X \setminus D \neq \emptyset \). If the family \( \mathcal{F} \) in \( L^p(D, \mu) \) satisfies
\[
\lim_{r \to 0} \int_X |f(x) - (f)_{B(x,r)}|^p \, d\mu(x) = 0 \quad \text{uniformly for } f \in \mathcal{F},
\]
where we continue the function \( f \) by zero beyond \( D \), then \( \mathcal{F} \) is bounded.

In order to prove Theorem 3.1 we will need some auxiliary results. We start with the following lemma.

**Lemma 3.2.** Let \( h > 0 \) and denote by \( 1_D \) the characteristic function of the set \( D \). Then the operator \( U : L^p(X) \rightarrow L^p(X) \) given by
\[
Uf(x) = \frac{1}{f_{B(x,h)}} 1_D f \, d\mu
\]
is compact.

**Proof.** Let us take \( V = B(0,1) \subset L^p(X) \). We shall show that the set \( U(V) \) is relatively compact in \( L^p(X) \). For this purpose, we shall use the characterization of relatively compact sets in \( L^p(X) \) from Theorem 2.2. Since \( D \) is bounded, there exists a ball \( B(x_0, r) \) such that \( D \subset B(x_0, r) \). Thus, for \( f \in V \) we have \( \text{supp}(Uf) \subset B(x_0, r + h) =: W_h \). Hence, by the Jensen inequality, we get
\[
\|Uf\|_{L^p(X)}^p = \|Uf\|_{L^p(W_h)}^p = \int_{W_h} |Uf(x)|^p \, d\mu = \int_{W_h} \left| \frac{1}{f_{B(x,h)}} 1_D f \right|^p \, d\mu
\]
\[
\leq \int_{W_h} \int_{B(x,h)} |f|^p \, d\mu \, d\mu \leq \|f\|_{L^p(X)}^p \frac{\mu(W_h)}{\inf_{x \in X} \mu(B(x,h))}.
\]
Since \( \mu \) is doubling and \( \mu(B(x,1)) \geq \theta \), we have \( \inf_{x \in X} \mu(B(x,h)) > 0 \). Thus, we get that \( U(V) \) is bounded. Moreover, since \( \text{supp}(Uf) \subset W_h \), we get that \( \|Uf\|_{L^p(X \setminus W_h)} = 0 \). Finally, it remains to show that the family \( U(f) \) is uniformly \( L^p \)-equicontinuous. Let \( h > r > 0 \). We have
\[
\int_X \left| Uf(x) - \frac{1}{f_{B(x,r)}} Uf(y) \, d\mu(y) \right|^p \, d\mu(x)
\]
\[
= \int_{W_{2h}} \left| \int_{B(x,r)} \left( \frac{1}{f_{B(x,h)}} 1_D f \, d\mu \right) - \frac{1}{f_{B(y,h)}} 1_D f \, d\mu \right| \, d\mu(y) \left| \frac{1}{f_{B(x,h)}} 1_D f \, d\mu \right|^p \, d\mu(x).
\]
On the other hand, from the proof of Lemma 4.3 in [1] and by the Hölder inequality we have
\[
\left| \frac{1}{f_{B(x,h)}} 1_D f \, d\mu - \frac{1}{f_{B(y,h)}} 1_D f \, d\mu \right|
\]
\[
\leq \frac{1}{\mu(B(x,h))} \int_{B(y,h) \Delta B(y,h)} 1_D f \, d\mu + \frac{\mu(B(x,h) \Delta B(y,h))}{\mu(B(x,h)) \mu(B(y,h))} \int_{B(y,h)} 1_D f \, d\mu
\]
\[
\leq \|f\|_{L^p(X)} \left( \frac{\mu(B(x,h) \Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h) \Delta B(y,h))}{\mu(B(x,h)) \mu(B(y,h))^{1/p}} \right).
\]
Hence, we obtain
\[
\int_X \left| Uf(x) - \frac{1}{f_{B(x,r)}} Uf(y) \, d\mu(y) \right|^p \, d\mu(x) \leq \|f\|_{L^p(X)}^p \int_{W_{2h}} |I(x)|^p \, d\mu(x).
\]
where 
\[ I(x) = \int_{B(x,r)} \left( \frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y) \]

By virtue of Lebesgue differentiation theorem (see e.g., [10]) we have
\[
\lim_{r \to 0} \int_{B(x,r)} \left( \frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y) = 0.
\]
Furthermore,
\[
\left| \int_{B(x,r)} \left( \frac{\mu(B(x,h)\Delta B(y,h))^{1-1/p}}{\mu(B(x,h))} + \frac{\mu(B(x,h)\Delta B(y,h))}{\mu(B(x,h))\mu(B(y,h))^{1/p}} \right) d\mu(y) \right|^p 
\leq \left( \frac{\mu(W_{4h})^{1-1/p}}{\inf_{x \in X} \mu(B(x,h))} + \frac{\mu(W_{4h})}{\inf_{x \in X} \mu(B(x,h))^{1+1/p}} \right)^p.
\]

Finally, the Lebesgue theorem finishes the proof. \qed

We will also need the following result.

\textbf{Lemma 3.3.} 1 is not an eigenvalue of \( U \).

\textbf{Proof.} Let us take \( f \in L^p(X) \) such that \( Uf = f \). We shall show that \( f = 0 \).

Since the measure \( \mu \) is continuous, from the proof of the previous lemma we have \( f \in C(X) \). Moreover, from the proof of the previous lemma we have that \( \text{supp} f \subset W_h = B(x_0, r + h) \). Next, let us take a ball \( B \) such that \( W_h \subset B \) and \( D \subset B \).

Suppose that \( M = \sup_{x \in B} f(x) > 0 \) and let
\[ C = \{ x \in X : f(x) = M \}. \]

Next, let us take \( x_0 \in \partial C \). Due to the fact that \( C \) is closed, we have that \( B(x_0, h) \cap (X \setminus C) \) is an open nonempty set. Thus \( \mu((X \setminus C) \cap B(x_0, h)) > 0 \). This contradicts our assumption that \( f(x_0) = Uf(x_0) \). \qed

We now prove the main result.

\textbf{Proof of Theorem 3.1.} For this purpose we use the Riesz–Schauder theory (see e.g., [21]). Since \( U \) is compact and 1 is not an eigenvalue of \( U \), we get that \( (U - I)^{-1} \) is bounded. On the other hand, we have \( \|Uf - f\|_{L^p(X)} \leq C \) for \( f \in \mathcal{F} \) and some positive constant \( C \). Thus,
\[ \|f\|_{L^p(X)} \leq C\|(U - I)^{-1}\|_{L^p(X) \to L^p(X)}, \]
and we obtain the desired result. \qed

As a corollary from Theorem 2.2 and Theorem 3.1 we obtain the following characterization of relatively compact sets.

\textbf{Theorem 3.4.} Assume that \((X, \rho, \mu)\) is a connected metric measure space with continuous measure satisfying the doubling condition. Suppose, moreover, that balls are relatively compact and there exists \( \theta > 0 \) such that \( \mu(B(x, 1)) \geq \theta \). Let \( 1 < p < \infty \) and \( D \) be a bounded subset of \( X \) such that \( X \setminus \overline{D} \neq \emptyset \). Then, the family \( \mathcal{F} \) in \( L^p(D, \mu) \) is relatively compact in \( L^p(D, \mu) \) if and only if
\[ \lim_{r \to 0} \int_X |f(x) - (f)_{B(x,r)}|^p d\mu(x) = 0 \text{ uniformly for } f \in \mathcal{F}, \]
where we continue the function \( f \) by zero beyond \( D \).
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References


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