ON THE HAUSDORFF MEASURE OF THE JULIA SET
AND THE ESCAPING SET OF ENTIRE FUNCTIONS
WITH REGULARLY GROWING MAXIMUM MODULUS

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Abstract. We prove that the Hausdorff measure of the escaping set and the Julia set of an
entire function \( f \) is infinite with respect to certain gauge functions, provided that \( f \) is outside of the
Eremenko–Lyubich class, and that the maximum modulus \( M(r, f) \) of \( f \) satisfies a certain regularity
condition.

1. Introduction

Let \( f \) be a transcendental entire function, and denote by

\[ f^n := f \circ \cdots \circ f \]

the \( n \)-th iterate of \( f \), for \( n \in \mathbb{N} \). The Fatou set \( F(f) \) is the set of points in \( \mathbb{C} \) such that \( \{ f^n \} \) forms a normal family in the sense of Montel (or, equivalently, is equicontinuous). The complement \( J(f) \) of \( F(f) \) is called the Julia set of \( f \). Both sets are completely invariant. For an introduction to the basic properties of these sets, we refer to the survey [4] and the books [3, 18, 26].

A gauge function is a monotonically increasing function \( h : [0, \varepsilon) \to [0, +\infty) \) which is continuous from the right and satisfies \( h(0) = 0 \).

Definition 1.1. Let \( A \subset \mathbb{R}^n \) be a set, \( \delta > 0 \) a constant, and let \( h \) be a gauge function. Then we call

\[ H^h(A) := \liminf_{\delta \to 0} \left\{ \sum_{j=1}^{\infty} h(\text{diam}(A_j)) : A \subset \bigcup_{j=1}^{\infty} A_j \text{ and } \text{diam}(A_j) < \delta \right\} \]

the Hausdorff measure with respect to \( h \), where

\[ \text{diam}(A_j) = \sup_{x, y \in A_j} |x - y| \]

is the diameter of \( A_j \).

The Hausdorff measure is an outer measure for measurable sets. In particular, when \( h^s(r) = r^s \) (\( s > 0 \)), then \( H^{h^s}(A) \) is the \( s \)-dimension Hausdorff measure of \( A \). If \( H^{h^s}(A) < \infty \) and \( t > s \), then \( H^{h^t}(A) = 0 \); if \( H^{h^s}(A) > 0 \) and \( t < s \), then \( H^{h^t}(A) = \infty \). Moreover, there exists a constant \( s \) such that \( H^{h^t}(A) = 0 \) for all \( t > s \)

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and $H^{h'}(A) = \infty$ for all $t < s$. The above $s$ is called Hausdorff dimension of $A$, and we denote $s = \dim(A)$.

In 1987, McMullen [17] proved that $\dim(J(E_\lambda)) = 2$ for $\lambda \neq 0$, where $E_\lambda = \lambda \exp(z)$. He also remarked that $H^h(J(E_\lambda)) = \infty$ when $h(t) = t^2 \log^n(1/t)$ and $n \in \mathbb{N}$. In his proofs, he first showed that these results hold for the escaping set $I(f) := \{ z ; f^n(z) \to \infty \text{ as } n \to \infty \}$, and then $I(f) \subset J(f)$ for the functions $E_\lambda$.

There is a close relation between the Julia set and the escaping set $I(f)$, which is studied for a general transcendental entire function $f$ by Eremenko. In [10], he proved that

$$J(f) = \partial I(f).$$

Let $\text{sing}(f^{-1})$ denote the set of singular values of $f$, which consists of critical and finite asymptotic values. The Eremenko–Lyubich class

$$\mathcal{B} := \{ f \text{ is transcendental entire function: } \text{sing}(f^{-1}) \text{ is bounded} \}$$

plays an important role in complex dynamics. In [11], Eremenko and Lyubich introduced a logarithmic change of variable, which has become a standard tool for studying entire functions in class $\mathcal{B}$. Using this method, they showed that $I(f) \subset J(f)$ for $f \in \mathcal{B}$. It is easy to check that $E_\lambda \in \mathbb{B}$, thus $I(E_\lambda) \subset J(E_\lambda)$.

There are many results on the Hausdorff dimension of entire functions, see [2, 5, 6, 21, 22, 24, 25]. In [2] and [22], Barański and Schubert independently proved that $\dim(J(f)) = 2$ if $f \in \mathcal{B}$ has finite order of growth. For more details, we refer to surveys [14, 23].

Given $\lambda_0 \in (0, 1/e)$, the function $E_{\lambda_0}$ has two fixed points $\alpha_{\lambda_0}$ and $\beta_{\lambda_0}$, where $\alpha_{\lambda_0}$ is attracting and $\beta_{\lambda_0} > e$ is repelling. Recall that a classical result of Koenigs says that there exists a function $\Phi_{\lambda_0}$ holomorphic in a neighborhood $D(\lambda_0)$ of $\beta_{\lambda_0}$ which satisfies $\Phi_{\lambda_0}(\beta_{\lambda_0}) = 0$, $\Phi'_{\lambda_0}(\beta_{\lambda_0}) = 1$ and

$$\Phi_{\lambda_0}(E_{\lambda_0}(z)) = \beta_{\lambda_0}\Phi_{\lambda_0}(z), \quad z, E_{\lambda_0}(z) \in U. \quad (1.1)$$

It is easy to see that $\Phi_{\lambda_0}(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Recently, Peter [19, 20] studied the Hausdorff measure on Julia set of exponential functions and entire functions in class $\mathcal{B}$ by introducing such a $\Phi$ and proving the next result.

**Theorem A.** Define $\lambda_0 \in (0, 1/e), \beta_{\lambda_0}, \Phi_{\lambda_0}$ as above, let $K_{\lambda_0} = \log 2/ \log \beta_{\lambda_0}$ and $h(t) = t^2 g(t)$ be a gauge function. If

$$\liminf_{t \to 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K_{\lambda_0},$$

then $H^h(J(E_\lambda)) = \infty$ for all $\lambda \in \mathbb{C} \setminus \{0\}$.

**Theorem B.** Let $\lambda_0 \in (0, 1/e)$. There exists $K > 0$ with the following property: If $f \in \mathcal{B}$ and $\rho(f) = \rho > 1/2$, then $H^h(J(f)) = \infty$, where $h(t) = t^2 (\Phi_{\lambda_0}(1/t))^\kappa$ and $\kappa > (\log(\rho) + K) / \log \beta_{\lambda_0}$.

**Remark 1.2.** Peter [20] has obtained a necessary condition for a gauge function $h'$ such that $H^{h'}(J(e^{\pi z})) = 0$.

Bergweiler and Karpińska [5] considered entire functions $f \not\in \mathcal{B}$ for which there exist constant $A, B, C, r_0 > 1$ such that

$$A \log M(r, f) \leq \log M(Cr, f) \leq B \log M(r, f) \quad \text{for all } r > r_0,$$

and proved the following result.
Theorem C. If $f$ is an entire function satisfying (1.2), then $\dim(I(f) \cap J(f)) = 2$.

2. Main results

The first result is in the spirit of Theorem A, but for functions $f$ satisfying (1.2).

**Theorem 2.1.** Let $\lambda_0 \in (0, 1/e)$ and $\beta_{\lambda_0}, \Phi_{\lambda_0}$ be as above. Let $\Delta > 0$ be a constant and $\kappa > \frac{\log(1/\Delta)}{\log \beta_{\lambda_0}}$. If $h(t) = t^2g(t)$ is a gauge function satisfying
\[
\liminf_{t \to 0} \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > \kappa,
\]
then for every entire function $f$ satisfying (1.2), we have $H^h(I(f) \cap J(f)) = \infty$.

**Corollary 2.2.** Let $f$ be an entire function satisfying (1.2), and let $h(t) = t^2\log^m \frac{1}{t}$ for $m \in \mathbb{N}$. Then $H^h(I(f) \cap J(f)) = \infty$.

As is mentioned in [5], the hypothesis (1.2) is satisfied if there exist constants $c_1, c_2, \rho > 0$ such that
\[c_1r^\rho \leq \log M(r, f) \leq c_2r^\rho,
\]
for large $r$, and thus in particular if there exist $c, \rho > 0$ such that
\[\log M(r, f) \sim cr^\rho,
\]
as $r \to \infty$. Hence there are many entire functions satisfying the condition (1.2). We may prove this assertion by relying on the following theorem of Clunie [9]: Let $\phi(r)$ be increasing and convex in $\log r$ with $\phi(r) \neq O(\log r)$ $(r \to \infty)$. (This condition is imposed to exclude certain trivial cases.) Then there is an entire function $f(z)$ such that
\[\log M(r, f) \sim \phi(r) \quad \text{and} \quad T(r, f) \sim \phi(r),
\]
as $r \to \infty$.

Theorem 2.3 below is of crucial importance. We define the set $T(f, \alpha, \beta, \delta, \lambda)$ consisting of points $z$ such that
\[
\alpha \log M(|z|, f) \leq \left| \frac{zf''(z)}{f'(z)} \right| \leq \beta \log M(|z|, f), \tag{2.1}
\]
\[|f(z)| \geq \exp(|z|^{\delta}), \tag{2.2}
\]
and
\[
\left| \frac{\zeta f''(\zeta)}{f'(\zeta)} \right| \leq \beta \log M(|\zeta|, f), \quad \text{for} \; |\zeta - z| \leq \lambda \frac{|z|}{\log M(|z|, f)}. \tag{2.3}
\]
In this definition, the conditions (2.1) and (2.3) are the same as those appearing in [5]. Condition (2.2) concerning the escaping rate of point $z$ is different from the one in [5]. Indeed, we consider a subset with much faster escaping rate.

For $R > 0$, let $A(R) = \{z \in \mathbb{C} : R < |z| < 2R\}$. For measurable sets $X, Y \subset \mathbb{C}$ the density of $X$ in $Y$ is defined by
\[\text{dens}(X, Y) = \frac{\text{area}(X \cap Y)}{\text{area}(Y)}.
\]

**Theorem 2.3.** Let $f$ be an entire function satisfying (1.2). Then there exist positive constants $\alpha, \beta, \delta$ and $\eta$ such that if $\lambda \geq 0$, we have $\text{dens}(T(f, \alpha, \beta, \delta, \lambda), A(R)) > \eta$ for all sufficiently large $R$. 

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3. Proof of Theorem 2.3

Throughout this article, denote by $T(r, f)$, $M(r, f)$ and $L(r, f)$ the Nevanlinna characteristic function, maximum modulus and minimum modulus of $f$, respectively. By $n(r, a)$ we denote the number of zeros of $f - a$ in the disc $\{z : |z| < r\}$. For an entire function $f$, the growth order $\rho(f)$ and lower order $\lambda(f)$ are respectively defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \lambda(f) = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$  

For more details, we refer the reader to the books [12, 13].

3.1. Some lemmas.

**Lemma 3.1.** [5, Theorem 1.2] Let $f$ be an entire function satisfying (1.2). Then there exist $\alpha_0, \beta_0$ and $\eta_0$ such that $\text{dens}(T(f, \alpha_0, \beta_0, 0, \lambda, A(R))) > \eta_0$ for all large enough $R$.

**Lemma 3.2.** [12, Theorem 3.4] (Borel Theorem) The order of the Weierstrass canonical product $f(z)$ is equal to the order of $n(r, 0)$, i.e.,

$$\rho(f) = \limsup_{r \to \infty} \frac{\log n(r, 0)}{\log r}.$$

**Lemma 3.3.** [12, Theorem 3.2] Let $\{\alpha_k\}, k = 1, 2, \ldots$, be a sequence of complex numbers satisfying $0 < |\alpha_1| \leq |\alpha_2| \leq \ldots$, such that

$$\sum_{k=1}^{\infty} \frac{1}{|\alpha_k|^p} < \infty,$$

where $p$ is a positive integer. Suppose that an entire function $g(z)$ has a power series representation of the form

$$g(z) = 1 + c_p z^p + c_{p+1} z^{p+1} + \cdots$$

Then the product

$$f(z) = \prod_{k=1}^{\infty} g\left(\frac{z}{\alpha_k}\right)$$

converges absolutely and uniformly on each bounded disc to an entire function.

**Lemma 3.4.** [16, Theorem 1] Given $n$ points $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the complex plane (repetitions being allowed), and an arbitrary number $H > 0$, there exists a set of at most $n$ circles, whose radii $h_k$ satisfy the inequality

$$\sum_{k=1}^{n} h_k^2 \leq 4H^2$$

with the property that, if $z$ is outside these circles, then

$$\sum_{k=1}^{n} \frac{1}{|z - \alpha_k|} \leq \frac{2n}{H}.$$

**Lemma 3.5.** [15, p. 19] (Boutrox–Cartan lemma) Given any constant $H > 0$ and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, there is a series of circles in the complex plane, with the sum of the radii equal to $2H$, such that for each point $z$ lying outside these circles,

$$|z - \alpha_1||z - \alpha_2| \cdots |z - \alpha_n| > (H/e)^n.$$
For an entire function $f$ which satisfies the regularly growth condition (1.2), the order is bounded from above and the lower order is bounded from below.

**Lemma 3.6.** [5, p. 533] Let $f$ be an entire function satisfying (1.2). Then

$$0 < \frac{\log A}{\log C} \leq \lambda(f) \leq \rho(f) \leq \frac{\log B}{\log C} < \infty,$$

and there exists a constant $K > 0$ such that

$$n(2r, a) \leq Kn(r, a).$$

**3.2. The proof.** Lemma 3.1 implies that the theorem holds in the special case $\delta = \lambda = 0$.

Replacing $f$ with $f - a$, if necessary, we may assume that $0$ is not a Valiron deficiency of $f$. By the discussions in [5], we can find a constant $C > 16$ such that $n(Cr, 0) > 2n(r, 0)$. Given $\theta \in (0, 1)$, we may thus choose a subsequence of $\{z_j\}$ of zeros of $f$ such that there are $\{\theta n(C^k r_0, 0)\}$ zeros in the annuli $A_k = \{z: C^{k+1} r_0 > |z| \geq C^k r_0\}$, denoted by $\{z_{j(k)}\}$, for $r_0 > 1$ and all $k \in \mathbb{N}$. Then

$$\sum_j \frac{1}{|z_j|} = \sum_{k=1}^\infty \sum_{z_{j(k)} \in A_k} \frac{1}{|z_{j(k)}|} \leq \sum_{k=1}^\infty \frac{\theta n(C^k r_0, 0)}{C^k r_0} = \sum_{k=1}^\infty \frac{(1 + \theta)^k n(C^k r_0, 0)}{C^k r_0} \leq \frac{\theta n(r_0, 0)}{r_0} \sum_{k=1}^\infty \left(1 + \frac{1}{8}\right)^k < \infty.$$ 

Consequently, $h(z) = \prod_{k=1}^\infty (1 - \frac{z}{r})$ is an entire function by Lemma 3.3. Recall that $r > r_0$. Hence there exists an integer $k$ such that $C^k r_0 \leq r < C^{k+1} r_0$. It follows from Lemma 3.2 that

$$\rho(h) = \limsup_{r \to \infty} \frac{\log n(r, 0)}{\log r} = \limsup_{k \to \infty} \frac{\log n(C^{k+1} r_0, 0)}{\log(C^{k+1} r_0)}$$

$$= \limsup_{k \to \infty} \frac{\log(1 + \theta)^{k+1} n(r_0, 0)}{\log(C^{k+1} r_0)} = \frac{\log(1 + \theta)}{\log C},$$

while

$$\rho(h) = \limsup_{r \to \infty} \frac{\log n(r, 0)}{\log r} = \limsup_{k \to \infty} \frac{\log n(C^k r_0, 0)}{\log(C^k r_0)}$$

$$= \limsup_{k \to \infty} \frac{\log(1 + \theta)^k n(r_0, 0)}{\log(C^{k+1} r_0)} = \frac{\log(1 + \theta)}{\log C}.$$

Then

$$\rho(h) = \frac{\log(1 + \theta)}{\log C} < \frac{\log 2}{\log 16} < 1. \tag{3.1}$$

If $\theta$ is sufficiently small, then Lemma 3.6 implies $\rho(h) < \frac{\log A}{\log C} \leq \lambda(f)$. Hence, the definitions of $\rho(h)$ and $\lambda(h)$ yield

$$T(r, h) = o(T(r, f)),$$

as $r \to \infty$ without an exceptional set.
Let
\[ g(z) = \frac{f(z)}{h(z)}. \]

Using the formula (6.6) in [12, p. 29] and the Nevanlinna first fundamental theorem, we have \( T(r, g) \leq T(r, f) + T(r, h) = O(T(r, f)). \) Moreover, from the standard inequalities \( T(r, f) \leq \log^+ M(r, f) \leq 3T(2r, f), \) we can deduce that
\[ \log M(r, g) = O(\log M(2r, f)), \quad as \ r \to \infty. \]

So \( g \) satisfies the assumption of Lemma 3.1 with the constants \( A \) and \( B \) being chosen suitably. That is,
\[ \text{dens}(T(g, \alpha^*, \beta^*, 0, 0), A(R)) \geq \eta \]
for sufficiently large \( R \) and for \( 0 < \alpha^* < \alpha_0, \beta^* > \beta_0. \)

Now \( \frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} - \frac{zh'(z)}{h(z)}, \) so that
\begin{equation}
(3.2) \quad \left| \frac{zh'(z)}{h(z)} \right| = \left| \frac{zf'(z)}{f(z)} \right| - \left| \frac{zh'(z)}{h(z)} \right|. \end{equation}

Since \( h(0) = 1, \) it follows from the formula (1.3') in [12, p. 88] that
\[ \left| \frac{h'(z)}{h(z)} \right| \leq \frac{4sT(s, h)}{(s - |z|)^2} + \sum_{|z| < s} \frac{2}{|z - z_j|} \]
for \( s > |z|. \) Considering \( s = 2|z|, \) we get, for every \( z \in A(R) \) and \( \varepsilon > 0, \)
\[ \frac{4sT(s, h)}{(s - |z|)^2} \leq \frac{8T(2|z|, h)}{|z|} = O\left(r^{\rho(h)-1+\varepsilon}\right), \]
where \( r = |z|. \) Applying Lemma 3.4 to \( \sum_{|z| < s} \frac{2}{|z - z_j|} \) with \( H = \frac{\sqrt{3r}R}{2} \), where \( \varepsilon \) is a small constant, we deduce that
\[ \sum_{|z| < s} \frac{2}{|z - z_j|} \leq \frac{2n(2r, 0)}{H} = O\left(\frac{T(2r, h)}{r}\right) = O\left(r^{\rho(h)-1+\varepsilon}\right), \]
for all \( z \) outside set \( E_1 \) which is a union of finite discs and satisfies \( \text{area}(E_1) = \varepsilon \text{ area } A(R). \) So,
\begin{equation}
(3.3) \quad \left| \frac{zh'(z)}{h(z)} \right| = O\left(|z|^{\rho(h)+\varepsilon}\right) = o(T(r, f)) = o(\log M(r, f)), \end{equation}
for all \( z \in T(g, \alpha^*, \beta^*, 0, 0) \setminus E_1. \) Thus, there are constants \( \alpha^{**}, \beta^{**} \) such that
\[ \alpha^{**} \log M(|z|, f) \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \beta^{**} \log M(|z|, f) \]
for all \( z \in T(g, \alpha^*, \beta^*, 0, 0) \setminus E_1. \) Therefore,
\begin{equation}
(3.4) \quad T(g, \alpha^*, \beta^*, 0, 0) \setminus E_1 \subset T(f, \alpha^{**}, \beta^{**}, 0, 0). \end{equation}

For any \( R \) and \( z \in A(R), \) we have
\begin{equation}
(3.5) \quad h(z) = \prod_{|z| \leq \frac{R}{4}} \left(1 - \frac{z}{z_j}\right) \cdot \prod_{\frac{R}{4} < |z| \leq 4R} \left(1 - \frac{z}{z_j}\right) \cdot \prod_{|z| > 4R} \left(1 - \frac{z}{z_j}\right)
= h_1(z)h_2(z)h_3(z). \end{equation}
We now estimate $\log |h_i(z)|$, $i = 1, 2, 3$, starting with

\begin{equation}
\log |h_1(z)| \geq \sum_{|z_j| \leq 4} \log \left( \frac{|z_j|}{z_j} - 1 \right) \geq \log(3)n \left( \frac{R}{4}, 0 \right).
\end{equation}

Using the Boutrox–Cartan lemma to $h_2(z)$, we get

$$|h_2(z)| = \prod_{\frac{R}{4} < |z_j| \leq 4R} \left( 1 - \frac{z_j}{z} \right) \geq \frac{1}{(4R)^m} \left( \frac{H}{e} \right)^m = \left( \frac{H}{4cR} \right)^m$$

except for a set $E_2$ which is a union of disks with sum of radii less than $2H$, where $m = n(4R, 0) - n \left( \frac{R}{4}, 0 \right)$. For a constant $c < 8e$, put $2H = cR$. Thus

$$|h_2(z)| \geq \left( \frac{H}{4cR} \right)^m \geq \left( \frac{c}{8e} \right)^m.$$

For the above $R$ there must exist an integer $k$ satisfying $r_k \leq \frac{R}{4} < r_{k+1}$, where $r_k = C^k r_0$. Recall that $C > 16$. Then $r_{k+2} = C r_{k+1} > 4R$ and

$$m = n(4R, 0) - n \left( \frac{R}{4}, 0 \right) \leq n(r_{k+2}, 0) - n(r_k, 0)$$

$$= (n(r_{k+2}, 0) - n(r_{k+1}, 0)) + (n(r_{k+1}, 0) - n(r_k, 0))$$

$$\leq \theta n(r_{k+1}, 0) + \theta n(r_k, 0) \leq \theta(\theta + 2)n(r_k, 0) \leq \theta(\theta + 2)n \left( \frac{R}{4}, 0 \right).$$

From the above argument, we can get

\begin{equation}
\log |h_2(z)| \geq - \log \left( \frac{8e}{c} \right)m \geq -\varepsilon' n \left( \frac{R}{4}, 0 \right),
\end{equation}

where $\varepsilon' = \log \left( \frac{8e}{c} \right) \theta(2 + \theta) > 0$. Using the inequality $\log(1 - x) \geq -x$, $x \leq \frac{1}{2}$, we obtain

$$\log |h_3(z)| \geq \sum_{|z_j| \geq 4R} \left( 1 - \frac{2R}{|z_j|} \right) \geq -2 \sum_{|z_j| \geq 4R} \frac{R}{|z_j|}.$$

Noting that there exists a constant $k_0$ such that $r_{k_0} \leq 4R < r_{k_0+1}$, we deduce

\begin{equation}
\sum_{|z_j| \geq 4R} \frac{R}{|z_j|} \leq \sum_{k=k_0}^{\infty} \sum_{z_j \in A_k} \frac{R}{|z_j|} \leq \sum_{k=k_0}^{\infty} (n(r_{k+1}, 0) - n(r_k, 0)) \frac{R}{r_k}
\end{equation}

$$\leq \sum_{k=k_0}^{\infty} \frac{R\theta(1 + \theta)^{k-k_0}}{C^{k-k_0} r_{k_0}} n(r_{k_0}, 0)$$

$$< \frac{C}{4} \sum_{k=k_0}^{\infty} \left( \frac{1 + \theta}{C} \right)^{k-k_0} \theta n(4R, 0) \leq 2CK^4 \theta n \left( \frac{R}{4}, 0 \right),$$

where $K$ is the constant as in Lemma 3.6. Therefore,

\begin{equation}
\log |h_3(z)| > -\varepsilon'' n \left( \frac{R}{4}, 0 \right),
\end{equation}

where $\varepsilon'' = \frac{4CK^4}{7}\theta$.

If $\theta$ is sufficiently small, then

$$\log 3 - \left( \log \left( \frac{8e}{c} \right)(2 + \theta) + \frac{4CK^4}{7} \right) \theta = \gamma \geq \frac{1}{2}.$$
It follows from (3.5), (3.6), (3.7) and (3.9) that \(\log |h(z)| \geq \gamma n(\frac{R}{4}, 0)\) for all \(z \in A(R)\) outside \(E_2\). Furthermore, (3.1) implies that

\[
(3.10) \quad \log |h(z)| \geq \gamma |z|^\delta \geq \frac{1}{2} |z|^\delta,
\]

where \(\delta < \frac{\log(1+\theta)}{\log C} = \rho(h)\).

From above argument, for any \(z \in T(g, \alpha^*, \beta^*, 0, 0)\backslash E_2\), we have

\[
|f(z)| = |g(z)h(z)| \geq \exp \left(\frac{1}{2} |z|^\delta\right).
\]

So \(T(g, \alpha^*, \beta^*, 0, 0)\backslash E \subset T(f, \alpha^*, \beta^*, \delta, 0)\), where \(E = E_1 \cup E_2\). This is the special case of Theorem 2.3. For the general \(\lambda\), we can use the same argument as Theorem 1.2 in [5]. Thus we finish the proof of Theorem 2.3.

**Remark 3.7.** There are many lower estimates known for the modulus of entire functions of order < 1, and, in particular, for functions of order < 1/2 in the book [8]. For our use, a more precise estimate is required for the function \(h\).

### 4. Proof of main theorems

#### 4.1. Preparation

First, we recall the Koebe distortion theorem.

**Lemma 4.1.** (Koebe Distortion Theorem) Let \(z_0 \in \mathbb{C}, r > 0\), and let \(f\) be a univalent function in \(D(z_0, r)\). If \(z \in D(z_0, r)\), then

\[
(4.1) \quad r^2 |f'(z_0)| \frac{|r - |z - z_0||}{(|r + |z - z_0||)^3} \leq |f'(z)| \leq r^2 |f'(z_0)| \frac{|r + |z - z_0||}{(|r - |z - z_0||)^3},
\]

and

\[
(4.2) \quad r^2 |f'(z_0)| \frac{|z - z_0|}{(|r + |z - z_0||)^2} \leq |f(z) - f(z_0)| \leq r^2 |f'(z_0)| \frac{|z - z_0|}{(|r - |z - z_0||)^2}.
\]

For our use, we also need the following consequence of Lemma 4.1.

**Lemma 4.2.** Let \(\Omega\) be a domain, and let \(Q \subset \Omega\) be compact. Then there exists a constant \(C' > 0\) such that if \(f\) is univalent in \(\Omega\) and \(z, \xi \in Q\), then \(|f'(\xi)| \leq C'|f'(z)|\).

Lemma 4.3 below plays an important role in proving that Hausdorff measure is \(\infty\). Before stating it, consider, for \(l \in \mathbb{N}\), a collection \(\mathcal{A}_l\) of compact, disjoint and connected subsets of \(\mathbb{C}\) with positive Lebesgue measure. Let \(A_l\) be the union of all elements of \(\mathcal{A}_l\). We say that \(\{\mathcal{A}_l\}\) is a series nesting intersection sets if it satisfies the following properties:

(a) Every element of \(\mathcal{A}_{l+1}\) is contained in a unique element of \(\mathcal{A}_l\).
(b) Every element of \(\mathcal{A}_l\) contains at least one element of \(\mathcal{A}_{l+1}\).
(c) For any \(F \in \mathcal{A}_l\), there exist two sequences of positive numbers \(\{\Delta_l\}\) and \(\{d_l\}\) \((d_l \to 0)\) such that

\[
dens(A_{l+1}, F) \geq \Delta_l; \quad \text{diam } F \leq d_l.
\]

The intersection \(A = \bigcap_{l=1}^\infty A_l\) is a non-empty and compact set.

**Lemma 4.3.** [20, Lemma 3.3] Let \(\{\mathcal{A}_l\}, A, \{d_l\}, \{\Delta_l\}\) be as above. Let \(\varepsilon > 0\) and \(\varphi: (0, \varepsilon) \to \mathbb{R}_{\geq 0}\) be a decreasing continuous function such that \(t^2 \varphi(t)\) is increasing.
Further, suppose that \( \lim_{t \to 0} t^2 \varphi(t) = 0 \) and

\[
\lim_{t \to \infty} \varphi(d_t) \prod_{j=1}^{l} \Delta_j = \infty.
\]

(4.3)

Define \( h : [0, \varepsilon) \to \mathbb{R} \) by setting

\[
h(t) = \begin{cases} 
  t^2 \varphi(t), & t > 0, \\
  0, & t = 0.
\end{cases}
\]

Then \( h(t) \) is a continuous gauge function, and \( H^h(A) = \infty \).

Let \( L \) be a constant such that \( \log M(2r, f) \leq L \log M(r, f) \) and let \( t(R) = \frac{M_r}{L \log M(r, f)} \). Bergweiler and Karpińska [5] applied the Ahlfors three islands theorem to domains

\[
D_v = \{ z \in \mathbb{C} : |\Re z| < 1, |3z - 8\pi v| < 3\pi \}, \quad v = 1, 2, 3,
\]

and showed that

**Lemma 4.4.** [5, Lemma 5.1] Let \( a \in T(f, \alpha, \beta, \delta, \lambda) \cap A(R) \) and \( v \in \{1, 2, 3\} \). If \( R \) is sufficiently large, then \( D(a, t(R)) \) contains a subdomain \( U \) such that \( \log f \) maps \( U \) bijectively onto one of the domains

\[
\Omega_v(a) = \log f(a) + D_v
\]

\[
= \{ z \in \mathbb{C} : |\Re z - \log f(a)| < 1, |3z - \log f(a) - 8\pi v| < 3\pi \}.
\]

Moreover, there exist \( \tau, q \) such that if \( V \) is the subset of \( U \) which is mapped onto

\[
Q_v(a) = \{ z \in \mathbb{C} : 0 \leq (\Re z - \log f(a)) < \log 2, |3z - \log f(a) - 8\pi v| \leq 2\pi \},
\]

then

\[
\text{area}(V) \geq \tau t(R)^2
\]

(4.4)

and

\[
\left| \frac{f'(z)}{f(z)} \right| \geq \frac{q}{t(R)} \quad \text{for } z \in V.
\]

(4.5)

The following lemma concerns with the number of discs \( D(a, t(R)) \) in the annulus

\[
A(R) = \{ z \in \mathbb{C} : R < |z| < 2R \}.
\]

**Lemma 4.5.** [5, Lemma 5.2] Let \( \eta \) be as in Theorem 2.3. For sufficiently large \( R \) there exists \( m(R) \in \mathbb{N} \) satisfying

\[
m(R) \geq \frac{\eta R^2}{2t^2(R)}
\]

such that there are \( m(R) \) points \( a_j \in T(f, \alpha, \beta, \delta, \lambda) \cap A(R) \), \( j = 1, 2, \cdots, m(R) \), satisfying \( D(a_j, t(R)) \subset A(R) \) for all \( j \) and \( D(a_j, t(R)) \) are pairwise disjoint.

**Lemma 4.6.** [27, Corollary 5] Let \( f \) be a transcendental meromorphic function with at most finitely many poles, and let \( d > 1 \) be a constant. If for all sufficiently large \( R > 0 \), we have

\[
\log M(2R, f) > d \log M(R, f),
\]

then \( J(f) \) has an unbounded component, and all components of \( F(f) \) are simply connected.
4.2. Proof of Theorem 2.1. The idea of construction of sets $A_l$ and part of the proof is from that of Theorem 2.1 in [5]. For completeness, we repeat it here.

Choose $R_0$ large enough, and let

$$A_0 = \{ A(R_0) \}.$$ 

By Lemma 4.4, there are domains $D(a(R_0), t(R_0)) \subset A(R_0), U(a(R_0))$ and $V(a(R_0))$ with $V(a(R_0)) \subset U(a(R_0)) \subset D(a(R_0), t(R_0))$ such that $\log f$ maps $U(a(R_0))$ onto the rectangles $\Omega_1(a(R_0))$ and $V(a(R_0))$ onto $Q_v(a(R_0))$. Since $f = \exp(\log f)$, we obtain that $f(U(a(R_0)))$ and $f(V(a(R_0)))$ are the annuli $\{ z : |f(a(R_0))|/e < |z| < e|f(a(R_0))| \}$ and $A(|f(a(R_0))|)$, respectively.

By Lemma 4.5, we note that there are at least $m(R_0) \geq t \frac{n R_0^2}{2R^2}$ many disjoint discs, say $\{ D(a_{j_1}(R_0), t(R_0)) \}_{j_1=1}^{m(R_0)}$, that are contained in $A(R_0)$. Consequently, there are $m(R_0)$ disjoint $V(a_{j_1}(R_0))$ having the above properties. Now, we can construct the sets

$$A_1 = \{ V(a_{j_1}(R_0)) : 1 \leq j_1 \leq m(R_0) \}.$$ 

For some $j$, put $R_{1, V_{j_1}} = R_{1, V(a_{j_1}(R_0))} = |f(a_{j_1}(R_0))|$. Since $R_0$ is large enough, it follows from (2.2) that $R_{1, V_{j_1}} > R_0$. Let $D(a(R_{1, V_{j_1}}), t(R_{1, V_{j_1}}))$ be a disc contained in $A(R_{1, V_{j_1}})$. Using Lemma 4.4 again, there are domains $V(a(R_{1, V_{j_1}}))$ and $U(a(R_{1, V_{j_1}}))$ such that $V(a(R_{1, V_{j_1}})) \subset U(a(R_{1, V_{j_1}})) \subset D(a(R_{1, V_{j_1}}), t(R_{1, V_{j_1}}))$. Therefore, $V(a(R_{1, V_{j_1}}))$ and $U(a(R_{1, V_{j_1}}))$ are mapped by $\log f$ bijectively onto $Q_v(a(R_{1, V_{j_1}}))$ and $\Omega_v(a(R_{1, V_{j_1}}))$ respectively. Then $f^2$ is a bijective mapping from a subset of $V_{j_1}(R_0)$ onto $Q_v(a(R_{1, V_{j_1}}))$. We define

$$A_2 = \bigcup_{V_{j_1} \in A_1} \{ \psi_{V_{j_1}}(Q_v(a_{j_2}(R_{1, V_{j_1}}))) : 1 \leq j_2 \leq m(R_{1, V_{j_1}}) \},$$

where $\psi_{V_{j_1}}$ is the inverse function of $\log f^2$ restricted on $V_{j_1}$ and $m(R_{1, V_{j_1}})$ is the number of domain $V(a(R_{1, V_{j_1}}))$ in $A(R_{1, V_{j_1}})$.

Inductively, $A_l$ consists of all sets $F$ which satisfy $f^l(F) = A(R_{l, F})$, where $R_{l, F} > R_0$. If $G$ is an element of $A_{l-1}$ which contains $F$, then for some $j \in \{ 1, 2, \ldots, m(R_{l-1, G}) \}$, we have

$$f^{l-1}(F) = V_j(R_{l-1, G}) \subset D(a_j(R_{l-1, G}), t(R_{l-1, G})) \subset A(R_{l-1, G}) = f^{l-1}(G).$$

Now we will construct $A_{l+1}$. By Lemma 4.4, there exists a domain $U(a_j(R_{l-1, G})) \subset D(a_j(R_{l-1, G}), t(R_{l-1, G}))$, which is mapped by $\log f$ bijectively onto $\Omega_v(a_j(R_{l-1, G}))$ and its subset $V_j(R_{l-1, F})$ is mapped by $\log f$ onto $Q_v(a_j(R_{l-1, G}))$. Thus $f^l$ is a bijective mapping from $F$ onto $Q_v(a_j(R_{l-1, G}))$. Denote the inverse function by $\psi$. We collect all domains $W_k \subset Q_v(a_j(R_{l-1, G}))$ which are mapped by the exponential function on $V(a_k(R_{l, F})) \subset A(R_{l, F})$ bijectively. Then

$$A_{l+1} = \bigcup_{F \in A_l} \{ \psi_F(W_k) \}.$$

Thus we have finished the construction of the sets $A_l$. To calculate the Hausdorff measure, both invariants $\Delta_k$ and $d_k$ mentioned above are needed.
Using (4.4) and Lemma 4.5, we deduce

\[
\text{area}\left(\bigcup_{k=1}^{m(R_{l,F})} W_{k,F}\right) = m(R_{l,F}) \int_{V_k(R_{l,F})} \frac{1}{|z|^2} \, dx \, dy \geq \frac{\eta \tau}{8}.
\]

Then (see [5, p. 549] for more details)

\[
(4.7) \quad \text{dens}(A_{l+1}, F) = \text{dens}\left(\bigcup_{k=1}^{m(R_{l,F})} \psi_F(W_k(R_{l,F}), \psi_F(Q_v(a_j(R_{l,G}))))\right)
\]

\[
\geq \frac{1}{(C')^2} \text{dens}\left(\bigcup_{k=1}^{m(R_{l,F})} W_k(R_{l,F}), Q_v(a_j(R_{l,G}))\right) \geq \frac{\eta \tau}{32(C')^2 \log 2} = \Delta,
\]

where \(C'\) is the constant as in Lemma 4.2.

For calculating \(d_k\), it will be more convenient to choose any sequence of nested sets \(\{F_k\}_{k=0}^\infty\) which satisfies \(F_k \in A_k\) and \(F_{k+1} \subset F_k\) for every \(k\). Without loss of generality, let \(F_{l-1} = G\) and \(F_l = F\), where \(F, G\) are as above. In what follows, we use the abbreviated notation \(R_k = R_k, a_j = a_j(R_{l-1,G})\) and \(V_j(R_{l-1}) = V(a_j(R_{l-1,G})).\)

Recall the formula (4.6). Let \(\phi\) be the branch of the inverse of \(f^{l-1}\) which maps \(f^{l-1}(F)\) to \(F\). Then \(\phi\) is a univalent map in the domain \(D(\phi(a_j), t(R_{l-1}))\) and maps its subset \(V_j(R_{l-1})\) onto \(F\). Furthermore, \(\phi\) can extend to a univalent map in \(D(\phi(a_j), 2t(R_{l-1}))\) by Lemma 4.4. Koebe’s distortion theorem implies that if \(z \in D(\phi(a_j), t(R_{l-1}))\), then \(|\phi'(z)| \leq 12|\phi'(a_j)|\). So

\[
\text{diam}(F) \leq 12|\phi'(a_j)| \text{diam}(f^{l-1}(F)) \leq 24|\phi'(a_j)|t(R_{l-1}).
\]

It follows from (4.5) that

\[
|f'(f^k(z))| \geq \frac{|f^{k+1}(z)|}{t(R_k)} \geq \frac{R_{k+1}}{t(R_k)} = \tau_1 \frac{R_{k+1}}{R_k} \log M(R_k, f),
\]

where \(\tau_1 = \frac{R_k}{R_{l-1}}\). Since \(|\phi'(a_j)| = \frac{1}{|f^{l-1}(\phi(a_j))|}\) and \((f^{l-1})'(z) = \prod_{k=0}^{l-2} f'(f^k(z))\), we conclude that

\[
|\phi'(a_j)| \leq \frac{|R_0|}{R_{l-1}} \prod_{k=0}^{l-2} \frac{1}{\tau_1 \log M(R_k, f)},
\]

and thus

\[
\text{diam}(F) \leq 24|\phi'(a_j)| t(R_{l-1}) \leq 24 \frac{|R_0|}{R_{l-1}} \prod_{k=0}^{l-2} \frac{1}{\tau_1 \log M(R_k, f)} = \frac{\tau_2}{\prod_{k=0}^{l-1} \tau_1 \log M(R_k, f)},
\]

where \(\tau_2\) is a constant. From the condition (2.2), we have

\[
R_k \geq \exp(R_{k-1}^\delta) = E^k(R_0),
\]

where \(E^k(R_0)\) is the \(k\)-th iteration of \(\exp(z^\delta)\) in point \(R_0\).

For \(\lambda_0 \in (0, 1/e)\), we denote \(E_{\lambda_0}^k(z)\) by the \(k\)-th iteration of exponential \(\lambda_0 \exp(z)\). Fix \(r_0\), then for every \(\lambda_0 \in C\) and \(l \in N\), there exists \(r_1\) such that \(E^l(r_1) \geq E_{\lambda_0}^l(r_0)\).
Thus for \( l \geq 2 \) and sufficiently large \( R_0 \), we have

\[
\text{diam}(F_l) \leq \prod_{k=0}^{l-1} \frac{\tau_2}{\tau_1 \log M(R_k, f)} \leq \prod_{k=0}^{l-1} \frac{1}{\log R_k} \leq \frac{1}{\log E^{l-1}(R_0)} \leq \frac{1}{\log E^{l-1}(r_0)} = d_l,
\]

where \( r'_0 < r_0 \). Since we can take \( r'_0, r_0 \in U \), where \( U \) be as in (1.1). It follows from (1.1), (4.7) and (4.8) that

\[
\Phi \left( \frac{1}{d_l} \right) \prod_{j=1}^{l} \Delta_j = \Phi(E_{\lambda_0}^{l-2}(r'_0)) \delta \Delta_l = (\beta_{\lambda_0}^{l-2} \Phi(r'_0)) \delta \Delta_l = (\beta_{\lambda_0}^l \Delta) \delta \Phi(r'_0) \delta \Delta^2,
\]

which tends to \( \infty \) as \( l \) tends to \( \infty \) when \( \beta_{\lambda_0} \Delta > 1 \). Thus Lemma 4.3 implies that for \( \kappa > \frac{\log(1/\Delta)}{\log \beta_{\lambda_0}} \)

\[
H^h(A) = \infty, \quad \text{where } h(t) = t^2 \Phi \left( \frac{1}{t} \right)^\kappa.
\]

Moreover, from Lemma 4.6 we get \( A = \bigcap_{l=1}^{\infty} A_l \subset J(f) \). Hence \( A \subset I(f) \cap J(f) \) since \( A \subset I(f) \) by (2.2). Thus

\[
H^h(I(f) \cap J(f)) = \infty.
\]

This completes the proof of Theorem 2.1.

4.3. Proof of Corollary 2.2. Obviously, \( \log^m_\delta \left( \frac{1}{d_l} \right) \prod_{j=1}^{l} \Delta_j \) tends to infinity as \( l \to \infty \). Using Lemma 4.3 to it, we can complete the proof of Corollary 2.2.

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References

On the Hausdorff measure of the Julia set and the escaping set of entire functions


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