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L^1-ELLIPTIC REGULARITY AND $H = W$ ON THE WHOLE L^p-SCALE ON ARBITRARY MANIFOLDS

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Abstract. We define abstract Sobolev type spaces on L^p-scales, $p \in [1, \infty)$, on Hermitian vector bundles over possibly noncompact manifolds, which are induced by smooth measures and families $\mathfrak{P}$ of linear partial differential operators, and we prove the density of the corresponding smooth Sobolev sections in these spaces under a generalised ellipticity condition on the underlying family. In particular, this implies a covariant version of Meyers–Serrin's theorem on the whole L^p-scale, for arbitrary Riemannian manifolds. Furthermore, we prove a new local elliptic regularity result in L^1 on the Besov scale, which shows that the above generalised ellipticity condition is satisfied on the whole L^p-scale, if some differential operator from $\mathfrak{P}$ that has a sufficiently high (but not necessarily the highest) order is elliptic.

1. Introduction

Let us recall that a classical result of Meyers and Serrin [14] states that for any open subset U of the Euclidean $\mathbf{R}^m$ and any $k \in \mathbf{N}_{\geq 0}$, $p \in [1, \infty)$, one has $W^{k,p}(U) = H^{k,p}(U)$, where $W^{k,p}(U)$ is given as the complex Banach space of all $f \in L^1_{\mathrm{loc}}(U)$ such that

$$(1) \qquad \|f\|_{k,p} := \left(\int_U |f(x)|^p \, \mathrm{d}x + \sum_{1 \leq |\alpha| \leq k} \int_U |\partial^\alpha f(x)|^p \, \mathrm{d}x \right)^{1/p} < \infty,$$

and where $H^{k,p}(U)$ is defined as the closure of $W^{k,p}(U) \cap C^\infty(U)$ with respect to the norm $\|\cdot\|_{k,p}$.

A natural question is whether this theorem can be generalised to more complicated situations: let us consider, for example, a smooth possibly noncompact smooth manifold X, equipped with a smooth measure μ. Let

$$\mathfrak{P} = \{P_1, \ldots, P_s\}$$

be a finite family of differential operators on X and $p \in [1, \infty)$. Given f in $L^p_\mu(X)$, we can define, for each $j \in \{1, \ldots, s\}$, $P_j f$ in a weak sense and consider the class $W^{\mathfrak{P},p}_\mu(X)$ of elements f in $L^p_\mu(X)$ such that $P_j f \in L^p_\mu(X)$ for each $j \in \{1, \ldots, s\}$. Then $W^{\mathfrak{P},p}_\mu(X)$ admits a natural norm, making of it a Banach space (cf. Definition 2.8), and the aim of this paper is to prescribe conditions on $\mathfrak{P}$ such that the subspace $W^{\mathfrak{P},p}_\mu(X) \cap C^\infty(X)$ is dense in $W^{\mathfrak{P},p}_\mu(X)$.

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On the other hand, the class of scalar differential operators on X is greatly enlarged (even from the point of view of applications) if we consider differential operators on sections of vector bundles. So we can consider Hermitian vector bundles $E, F_1, \dots, F_s$ on the manifold X and the Banach space $\Gamma_{L^p_\mu}(X,E)$ of (equivalence classes of) Borel sections f in $E \to X$ such that $\int_X |f(x)|^p_{E_x} \mu(\mathrm{d}x) < \infty$ (usual modification in case $p = \infty$). Of course, $|\cdot|_{E_x}$ stands for the norm induced by the Hermitian structure in the fiber E_x. Suppose that, for each $j \in \{1, \dots, s\}$, a linear differential operator P_j, mapping sections of the vector bundle E into sections of the vector bundle F_j, is given. Then we can consider

$$\Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) := \{f \in \Gamma_{L^p_\mu}(X,E) \colon P_j f \in \Gamma_{L^p_\mu}(X,F_j) \quad \forall j \in \{1, \dots, s\}\},$$

which, again, admits a natural structure of Banach space. In order to make the paper simpler to read, we shall specify in the next section what we mean with $P_j f$. Ultimately, the question we address here is:

Under which assumptions on $\mathfrak{P}$ is the space of smooth Sobolev sections

$$\Gamma_{C^\infty}(X,E) \cap \Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) \text{ dense in } \Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) \text{ w.r.t. } \|\cdot\|_{\mathfrak{P},p,\mu}? \tag{2}$$

To this end, the highest differential order $k := \max\{k_1, \dots, k_s\}$ of the system $\mathfrak{P}$, plays an essential role: Namely, it turns out that even on an entirely local level (cf. Lemma 2.10), the machinery of Friedrichs mollifiers precisely applies

$$\text{if } \Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) \subset \Gamma_{W^{k-1,p}_{\mathrm{loc}}}(X,E). \tag{3}$$

With this observation, our basic abstract result *Theorem 2.9 precisely states that the local regularity (3) implies (2), and that furthermore any compactly supported element of $\Gamma_{W^{\mathfrak{P},p}_\mu}(X,E)$ can be even approximated by a sequence from $\Gamma_{C^\infty_c}(X,E)$.*

This result turns out to be optimal in the following sense (cf. Example 2.11): *There are differential operators P such that for any $q > 1$ one has*

$$W^{P,q} \subset W^{\mathrm{ord}(P)-2,q}_{\mathrm{loc}}, \quad W^{P,q} \not\subset W^{\mathrm{ord}(P)-1,q}_{\mathrm{loc}},$$
$$C^\infty \cap W^{P,q} \text{ is not dense in } W^{P,q}.$$

Thus it remains to examine the regularity assumption (3) in applications, where of course we can assume $k \geq 2$.

To this end, it is clear from classical local elliptic estimates that for $p > 1$, (3) is satisfied whenever there is some elliptic P_j with $k_j \geq k - 1$. However, the L^1-case $p = 1$ is much more subtle, since the usual local elliptic regularity is well-known to fail here (cf. Remark 3.2). However, *in Theorem 3.1 we prove a new modified local elliptic regularity result on the scale of Besov spaces, which implies that, in the L^1-situation, one loses exactly one differential order of regularity when compared with the usual local elliptic L^p, $p > 1$, estimates.* This in turn shows that for $p = 1$, (3) is satisfied whenever there is some elliptic P_j with $k_j = k$. These observations are collected in Corollary 3.4. The proof of Theorem 3.1 relies on *a new existence and uniqueness result, (cf. Proposition A.3 in Section A) for certain systems of linear elliptic PDE's on the Besov scale,* which is certainly also of an independent interest. Indeed, we point out that the Besov scale turns out to be the natural framework for settling the regularity theory in the case $p = 1$ and this leads to heavier technical difficulties than in the case $p > 1$. On the other hand, the application to the $H = W$ result follows from a much simpler consequence of our general result (see Corollary 3.4 b) below) that does not even require the knowledge of Besov spaces and can be stated in terms of Sobolev spaces.

Finally, we would like to point out that the regularity (3) does not require the ellipticity of any P_j at all. Indeed, *in Corollary 3.6 we prove that if (M, g) is a possibly noncompact Riemannian manifold and $E \to M$ a Hermitian vector bundle with a (not necessarily Hermitian) covariant derivative ∇, then for any $s \in \mathbf{N}$ and $p \in (1, \infty)$, the Sobolev space*

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) := \Gamma_{W^{\{\nabla^1_g,\dots,\nabla^s_g\},p}_{\mathrm{vol}_g}}(M, E).$$

satisfies

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) \subset \Gamma_{W^{s,p}_{\mathrm{loc}}}(M, E),$$

which means that we do not even have to use the full strength of Theorem 2.9 here. To the best of our knowledge, the resulting density of

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) \cap \Gamma_{C^\infty}(M, E) \text{ in } \Gamma_{W^{s,p}_{\nabla,g}}(M, E)$$

is entirely new in this generality (cf. [16] for the scalar case).

2. Preliminaries and main results

Throughout, let X be *a smooth m-manifold (without boundary, and with a countable basis)* which is allowed to be noncompact. For subsets $Y_1, Y_2 \subset X$ we write

$$Y_1 \Subset Y_2, \text{ if and only if } Y_1 \text{ is open, } \overline{Y_1} \subset Y_2, \text{ and } \overline{Y_1} \text{ is compact.}$$

We abbreviate that for any $k \in \mathbf{N}_{\geq 0}$, we denote with $\mathbf{N}^m_k$ the set of multi-indices $\alpha \in (\mathbf{N}_{\geq 0})^m$ with $|\alpha| := \sum_{j=1}^m \alpha_j \leq k$. Note that $(0, \dots, 0) \in \mathbf{N}^m_k$ by definition, for any k.

In order to be able to deal with Banach structures that are not necessarily induced by Riemannian structures [3], *we fix a smooth measure μ on X*, that is, μ is a Borel measure on X such that for any chart $(\Phi = (x^1, \cdots, x^m), U)$ for X there is a (necessarily unique) $0 < \mu_\Phi \in C^\infty(U)$ with the property that for all Borel sets $N \subset U$ one has

$$\mu(N) = \int_N \mu_\Phi \,\mathrm{d}x := \int_{\Phi(N)} \mu_\Phi \circ \Phi^{-1}(x^1, \cdots, x^m) \,\mathrm{d}x^1 \cdots \mathrm{d}x^m, \tag{4}$$

where $\mathrm{d}x = \mathrm{d}x^1 \cdots \mathrm{d}x^m$ stands for Lebesgue integration.

We always understand our linear spaces to be complex-valued, and an index "c" in spaces of sections or functions stands for "compact support", where in the context of equivalence classes (with respect to some/all μ as above) of Borel measurable sections, compact support of course means "compact essential support".

Let $\pi \colon E \to X$ be a smooth complex vector bundle over X with rank ℓ, i.e., π is a smooth surjective map such that:

(i) each fiber $E_x := \pi^{-1}(\{x\})$ is an ℓ-dimensional complex vector space;
(ii) for each $x_0 \in X$ there are an open neighbourhood $U \subset X$ and a smooth diffeomorphism $\Psi \colon U \times \mathbf{C}^\ell \to \pi^{-1}(U)$, which is referred to as a *smooth trivialization of* $E \to X$, such that $\pi \circ \Psi$ is the projection onto the first slot and $\Psi_{|\{x\}} \colon \{x\} \times \mathbf{C}^\ell \to E_x$ is an isomorphism of complex vector spaces for all $x \in U$.

As we have already done, whenever there is no danger of confusion, we shall omit the map π in the notation and simply denote the vector bundle by $E \to X$. A *section in* $E \to X$ *over a subset* $U \subset X$ is nothing but a map $f \colon U \to E$ such that $f(x) \in E_x$ for all x, and the complex linear space of smooth sections in $E \to X$ over an open subset $U \subset X$ is denoted by $\Gamma_{C^\infty}(U, E)$, where remark that $U \mapsto \Gamma_{C^\infty}(U, E)$ defines a sheaf. The complex linear space of equivalence classes of Borel sections in $E \to X$ over a Borel set $U \subset X$ is simply written as $\Gamma(U, E)$.

Notice that the assumption (ii) above on the existence of local trivializations can be conveniently rephrased in an equivalent way in terms of *frames* as follows

(ii)' for each $x_0 \in X$ there is an open neighbourhood $U \subset X$ which admits a smooth frame $e_1, \ldots, e_\ell \in \Gamma_{C^\infty}(U, E)$, that is, e_j are smooth sections of $E \to X$ such that $e_1(x), \ldots, e_\ell(x)$ is a basis for E_x, for every $x \in U$ (we further recall here that if the vector bundle is *Hermitian*, that is, if it comes equipped with a smooth family of Hermitian inner product[a] on its fibers (its *Hermitian structure*), then a frame as above is called *orthogonal* or *orthonormal* if the basis above has this property for all $x \in U$).

We also recall that given another smooth complex vector bundle $F \to X$, a *morphism* $S\colon E \to F$ is understood to be a smooth map which preserves the fibers in a complex linear way, and smooth vector bundles over X become a category this way. Any smooth functor on the category of complex linear spaces canonically induces a functor on the category of smooth vector bundles over X, so that for example we get the dual bundle $E^* \to X$, tensor bundles, and so on.

The complex linear space $\Gamma_{W^{k,p}_{\mathrm{loc}}}(X, E)$ of *local L^p-Sobolev sections in $E \to X$ with differential order k* is defined to be the space of $f \in \Gamma(X, E)$ such that for all charts (Φ, U) for X such that there is a smooth frame $e_1, \ldots, e_{\ell_0} \in \Gamma_{C^\infty}(U, E)$ for $E \to X$ on U, one has

$$(f^1, \ldots, f^{\ell_0}) \in W^{k,p}_{\mathrm{loc}}(U, \mathbf{C}^{\ell_0}) := W^{k,p}_{\mathrm{loc}}(\Phi(U), \mathbf{C}^{\ell_0}), \quad \text{if } f = \sum_{j=1}^{\ell_0} f^j e_j \text{ in } U.$$

In particular, we have the space of locally p-integrable sections

$$\Gamma_{L^p_{\mathrm{loc}}}(X, E) := \Gamma_{W^{0,p}_{\mathrm{loc}}}(X, E).$$

We continue by listing some conventions and some notation concerning linear differential operators on manifolds. We start by adding the following two classical definitions on linear differential operators for the convenience of the reader, who can find these and the corresponding basics in [17, 20, 4, 13]. We also refer the reader to [11] (and the references therein) for the jet bundle aspects of (possibly nonlinear) partial differential operators. Assume that smooth complex vector bundles $E \to X$, $F \to X$, with $\mathrm{rank}(E) = \ell_0$ and $\mathrm{rank}(F) = \ell_1$ are given.

Definition 2.1. A morphism of complex linear sheaves

$$P\colon \Gamma_{C^\infty}(X, E) \longrightarrow \Gamma_{C^\infty}(X, F)$$

is called a *smooth complex linear partial differential operator of order at most k*, if for any chart $((x^1, \ldots, x^m), U)$ for X which admits frames $e_1, \ldots, e_{\ell_0} \in \Gamma_{C^\infty}(U, E)$, $f_1, \ldots, f_{\ell_1} \in \Gamma_{C^\infty}(U, F)$, and any $\alpha \in \mathbf{N}^m_k$, there are (necessarily uniquely determined) smooth functions

$$P_\alpha\colon U \longrightarrow \mathrm{Mat}(\mathbf{C}; \ell_0 \times \ell_1)$$

such that for all $(\phi^1, \ldots, \phi^{\ell_0}) \in C^\infty(U, \mathbf{C}^{\ell_0})$ one has

$$P \sum_{i=1}^{\ell_0} \phi^i e_i = \sum_{j=1}^{\ell_1} \sum_{i=1}^{\ell_0} \sum_{\alpha \in \mathbf{N}^m_k} P_{\alpha ij} \frac{\partial^{|\alpha|} \phi^i}{\partial x^\alpha} f_j \quad \text{in } U.$$

[a]Where w.l.o.g. we assume our Hermitian inner products to be antilinear in the *first* slot.

The complex linear space of smooth at most k-th order complex linear partial differential operators is denoted by $\mathscr{D}^{(k)}_{C^\infty}(X; E, F)$.

Definition 2.2. Let $P \in \mathscr{D}^{(k)}_{C^\infty}(X; E, F)$.

a) The (linear principal) symbol of P is the unique morphism of smooth complex vector bundles over X,

$$\sigma_P \colon (\mathrm{T}^*X)^{\odot k} \longrightarrow \mathrm{Hom}(E, F),$$

where $\odot$ stands for the symmetric tensor product, such that for all $x\colon U \to \mathbf{R}^m$, $e_1, \dots, e_{\ell_0}$, $f_1, \dots, f_{\ell_1}$, α as in Definition 2.1 one has

$$\sigma_P\left(\mathrm{d}x^{\odot\alpha}\right) e_i = \sum_{j=1}^{\ell_1} P_{\alpha ij} f_j \quad \text{in } U.$$

b) P is called *elliptic*, if for all $x \in X$, $v \in \mathrm{T}^*_x X \setminus \{0\}$, the complex linear map

$$\sigma_{P,x}(v) := \sigma_{P,x}(v^{\otimes k})\colon E_x \longrightarrow F_x \ \text{ is invertible.}$$

It is clear that the composition of an at most k-th order linear differential operator with a l-th order one yields a $l + k$-th order differential operator, and that the symbols respect this composition in the obvious way. Furthermore, any morphism of smooth vector bundles $f\colon E \to F$ induces the operator $P_f \in \mathscr{D}^{(0)}_{C^\infty}(X; E, F)$ given by $P_f\phi(x) := f(x)\phi(x)$, where of course $f(x) := f|_{E_x}\colon E_x \to F_x$, and the assignment $f \mapsto P_f$ is an isomorphism of complex linear spaces. There will be no danger to simply write f instead of P_f.

We continue with global descriptions of formal adjoints. To this end, in the sequel we will denote the canonical pairing of a linear space with its dual by $(\cdot,\cdot)$. One has:

Proposition and definition 2.3. *There is a unique differential operator $P^\mu \in \mathscr{D}^{(k)}_{C^\infty}(X; F^*, E^*)$ which satisfies*

$$\int_X (P^\mu\psi, \phi)\,\mathrm{d}\mu = \int_X (\psi, P\phi)\,\mathrm{d}\mu \tag{5}$$

for all $\psi \in \Gamma_{C^\infty}(X, F^*)$, $\phi \in \Gamma_{C^\infty}(X, E)$ with either ϕ or ψ compactly supported. The operator P^μ is called the formal adjoint w.r.t. μ. *An explicit local formula for P^μ can be found in the proof (cf. formula (6) below).*

Proof. It is clear that there can be at most one operator satisfying (5). In order to prove the existence, using a standard partition of unity argument and the fact that differential operators are local, it is sufficient to define P^μ locally. Now, in the situation of Definition 2.2 a) let e^*_i and f^*_j be the dual smooth frames over U for $E \to X$, and $F \to X$, respectively. Then for all $(\psi^1, \dots, \psi^{\ell_1}) \in C^\infty(U, \mathbf{C}^{\ell_1})$ we define

$$P^\mu \sum_{j=1}^{\ell_1} \psi^j f^*_j := \frac{1}{\mu} \sum_{i=1}^{\ell_0} \sum_{j=1}^{\ell_1} \sum_{\alpha \in \mathbf{N}^m_k} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}\left(P_{\alpha ij}\mu\psi^j\right)}{\partial x^\alpha} e^*_i \quad \text{in } U. \tag{6}$$

Let $\psi := \sum_j \psi^j f^*_j$ and $\phi = \sum_i \phi^i e_i$ be smooth sections in $F^* \to X$ and $E \to X$ over U, respectively, one of which having a compact support. Integrating by parts we can calculate

$$\int_U (P^\mu\psi, \phi)\,\mathrm{d}\mu = \sum_{i=1}^{\ell_0} \sum_{j=1}^{\ell_1} \sum_{\alpha \in \mathbf{N}^m_k} \int_U \frac{1}{\mu} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}(P_{\alpha ij}\mu\psi^j)}{\partial x^\alpha}\, \phi^i \mu\,\mathrm{d}x$$

$$= \sum_{i=1}^{\ell_0} \sum_{j=1}^{\ell_1} \sum_{\alpha \in \mathbf{N}_k^m} \int_U \psi_j P_{\alpha ij} \frac{\partial^{|\alpha|} \phi^i}{\partial x^\alpha} \, \mu \, \mathrm{d}x = \int_U (\psi, P\phi) \, \mathrm{d}\mu$$

which proves (5). □

We continue with:

Proposition and definition 2.4. *Given $f \in \Gamma_{L^1_{\mathrm{loc}}}(X, E)$ and $g \in \Gamma_{L^1_{\mathrm{loc}}}(X, F)$, we write $Pf = g$, if and only if for all smooth measures ν on X it holds that*

$$\int_X (P^\nu \psi, f) \, \mathrm{d}\nu = \int_X (\psi, g) \, \mathrm{d}\nu \quad \text{for all } \psi \in \Gamma_{C^\infty_c}(X, F^*). \tag{7}$$

The latter property is equivalent to (7) being true for some *smooth measure ν.*

Proof. Assume that there is a smooth measure ν with (7), and let ν' be an arbitrary smooth measure. In order to see that one also has (7) with respect to ν', let $0 < \frac{\mathrm{d}\nu'}{\mathrm{d}\nu} \in C^\infty(X)$ be the Radon–Nikodym derivative of ν' with respect to ν. We have, for all $h_1 \in \Gamma_{C^\infty_c}(X, E)$ and all $h_2 \in \Gamma_{C^\infty_c}(X, F^*)$:

$$\begin{aligned} \int_X (h_2, Ph_1) \, \mathrm{d}\nu' &= \int_X \frac{\mathrm{d}\nu'}{\mathrm{d}\nu} (h_2, Ph_1) \, \mathrm{d}\nu = \int_X \left(P^\nu \left(\frac{\mathrm{d}\nu'}{\mathrm{d}\nu} h_2 \right), h_1 \right) \mathrm{d}\nu \\ &= \int_X \frac{\mathrm{d}\nu}{\mathrm{d}\nu'} \left(P^\nu \left(\frac{\mathrm{d}\nu'}{\mathrm{d}\nu} h_2 \right), h_1 \right) \mathrm{d}\nu', \end{aligned}$$

so that $P^{\nu'} h = \frac{\mathrm{d}\nu}{\mathrm{d}\nu'} P^\nu \big(\frac{\mathrm{d}\nu'}{\mathrm{d}\nu} h \big)$ for all $h \in \Gamma_{C^\infty_c}(X, F^*)$. Thus if we have (7) with respect to ν, it follows that

$$\int_X \left(P^{\nu'} \psi, f \right) \mathrm{d}\nu' = \int_X \left(P^\nu \left(\frac{\mathrm{d}\nu'}{\mathrm{d}\nu} \psi \right), f \right) \mathrm{d}\nu = \int_X (\psi, g) \frac{\mathrm{d}\nu'}{\mathrm{d}\nu} \, \mathrm{d}\nu = \int_X (\psi, g) \, \mathrm{d}\nu',$$

as claimed. □

Accordingly, in the sequel, the assumption $Pf \in \Gamma_{L^p_{\mathrm{loc}}}(X, F)$, $p \in [1, \infty]$, is equivalent to the existence of some (necessarily unique) $g \in \Gamma_{L^p_{\mathrm{loc}}}(X, F)$ such that $Pf = g$ in the sense of Proposition 2.4.

In typical applications, $E \to X$ and $F \to X$ come equipped with smooth Hermitian structures $h_E(\cdot, \cdot)$ and $h_F(\cdot, \cdot)$, respectively. Then, analogously to Proposition 2.3, one has:

Proposition and definition 2.5. *There is a uniquely determined operator $P^{\mu, h_E, h_F} \in \mathscr{D}^{(k)}_{C^\infty}(X; F, E)$ which satisfies*

$$\int_X h_E(P^{\mu, h_E, h_F} \psi, \phi) \, \mathrm{d}\mu = \int_X h_F(\psi, P\phi) \, \mathrm{d}\mu$$

for all $\psi \in \Gamma_{C^\infty}(X, F)$, $\phi \in \Gamma_{C^\infty}(X, E)$ with either ϕ or ψ compactly supported. The operator P^{μ, h_E, h_F} is called the formal adjoint of P w.r.t. (μ, h_E, h_F). *An explicit local formula for P^{μ, h_E, h_F} can be found in the proof.*

Proof. Again, it is sufficient to prove the local existence. To this end, in the situation of Definition 2.2 a), we assume that e_i and f_j are orthonormal w.r.t. h_E and h_F, respectively. Then analogously as done in the proof of Proposition 2.3 one finds that

$$P^{\mu,h_E,h_F} \sum_{j=1}^{\ell_1} \psi^j f_j := \frac{1}{\mu} \sum_{i=1}^{\ell_0} \sum_{j=1}^{\ell_1} \sum_{\alpha \in \mathbf{N}_k^m} (-1)^{|\alpha|} \frac{\partial^{|\alpha|} \left(\overline{P_{\alpha j i}} \mu \psi^j \right)}{\partial x^\alpha} e_i \quad \text{in } U \tag{8}$$

does the job. □

We add:

Lemma 2.6. *Given $f \in \Gamma_{L^1_{\mathrm{loc}}}(X,E)$, $g \in \Gamma_{L^1_{\mathrm{loc}}}(X,F)$ one has $Pf = g$, if and only if for all triples (ν, h_E, h_F) as above (that is, ν is a smooth measure and h_E, h_F are smooth Hermitian structures) it holds that*

$$\int_X h_E(P^{\mu,h_E,h_F}\psi, f)\, \mathrm{d}\nu = \int_X h_F(\psi, g)\, \mathrm{d}\nu \quad \textit{for all } \psi \in \Gamma_{C^\infty_{\mathrm{c}}}(X,F), \tag{9}$$

and this property is furthermore equivalent to (9) being true for some *such triple (ν, h_E, h_F).*

Proof. In view of Proposition 2.4 it is sufficient to prove that if there exists a triple (ν, h_E, h_F) with (9) and if h'_E and h'_F are new smooth Hermitian structures on $E \to X$ and $F \to X$, respectively, then one also has (9) with respect to (ν, h'_E, h'_F). To this end, define the isomorphisms of smooth complex vector bundles over X given by

$$\begin{aligned} S_E &\colon E \longrightarrow E, \quad h'_E(S_E\phi_1, \phi_2) := h_E(\phi_1, \phi_2), \\ S_F &\colon F \longrightarrow F, \quad h'_F(S_F\psi_1, \psi_2) := h_F(\psi_1, \psi_2). \end{aligned}$$

Note that $h_E(S_E^{-1}\phi_1, \phi_2) = h'_E(\phi_1, \phi_2)$, and likewise for h_F. Now as in the proof of Lemma 2.4 one finds

$$P^{\nu,h'_E,h'_F} = S_E^{-1} P^{\nu,h'_E,h'_F} S_F,$$

and using this formula one easily proves the claim. □

Remark 2.7. 1. P^{μ,h_E,h_F} can be constructed from P^μ by means of the commutative diagram

$$\begin{array}{ccc} \Gamma_{C^\infty}(X,F^*) & \xrightarrow{\;P^\mu\;} & \Gamma_{C^\infty}(X,E^*) \\ \uparrow \tilde{h}_F & & \downarrow \tilde{h}_E^{-1} \\ \Gamma_{C^\infty}(X,F) & \xrightarrow[P^{\mu,h_E,h_F}]{} & \Gamma_{C^\infty}(X,E) \end{array}$$

where $\tilde{h}_E$ and $\tilde{h}_F$ stand for the complex linear isomorphisms which are induced by h_E and h_F, respectively (that is $\tilde{h}_E(\phi) := h_E(\cdot, \phi)$ and likewise for $\tilde{h}_F$).

2. The assignment $P \mapsto P^\mu$ is a complex linear map, whereas $P \mapsto P^{\mu,h_E,h_F}$ is a complex antilinear map.

3. Somewhat more generally, using the density bundle $|X| \to X$ one finds that (cf. Proposition 1.2.12 in [20], or [4]) for any $P \in \mathscr{D}^{(k)}_{C^\infty}(X;E,F)$ there is a unique transpose

$$P^t \in \mathscr{D}^{(k)}_{C^\infty}(X; F^* \otimes |X|, E^* \otimes |X|),$$

which satisfies

$$\int_X \big(P^t\psi, \phi\big) = \int_X (\psi, P\phi) \quad \text{for all } \psi \in \Gamma_{C^\infty}(X, F^* \otimes |X|),$$

and all $\phi \in \Gamma_{C^\infty}(X,E)$, with either ϕ or ψ compactly supported.

The operator P^μ (and thus also P^{μ,h_E,h_F}) can be constructed from P^t. As we will not make any particular use of density bundles in the sequel, our approach has the advantage of being more explicit and self-contained.

From now on, given a smooth *Hermitian* vector bundle $E \to X$ and $p \in [1,\infty]$, abusing the notation as usual, $(\cdot,\cdot)_x$ denotes the Hermitian structure on the fibers E_x, with $|\cdot|_x = \sqrt{(\cdot,\cdot)_x}$ the corresponding norm, and we get a complex Banach space

$$\Gamma_{L^p_\mu}(X,E) := \{f \mid f \in \Gamma(X,E),\ \|f\|_{p,\mu} < \infty\},$$

where

$$\|f\|_{p,\mu} := \begin{cases} \Big(\int_X |f(x)|_x^p \mu(\mathrm{d}x)\Big)^{1/p}, & \text{if } p < \infty, \\ \inf\{C \mid C \geq 0,\ |f| \leq C \ \mu\text{-a.e.}\}, & \text{if } p = \infty. \end{cases}$$

Of course, $\Gamma_{L^2_\mu}(X,E)$ becomes a complex Hilbert space with its canonical inner product.

The following definition is in the centre of this paper:

Definition 2.8. Let $p \in [1,\infty)$, $s \in \mathbf{N}$, $k_1 \dots, k_s \in \mathbf{N}_{\geq 0}$, and for each $i \in \{1,\dots,s\}$ let $E \to X$, $F_i \to X$ be smooth Hermitian vector bundles and let $\mathfrak{P} := \{P_1,\dots,P_s\}$ with $P_i \in \mathscr{D}^{(k_i)}_{C^\infty}(X;E,F_i)$. Then the complex Banach space

$$\Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) := \big\{f \mid f \in \Gamma_{L^p_\mu}(X,E),\ P_i f \in \Gamma_{L^p_\mu}(X,F_i) \text{ for all } i \in \{1,\dots,s\}\big\}$$

$$\subset \Gamma_{L^p_\mu}(X,E), \quad \text{with norm } \|f\|_{\mathfrak{P},p,\mu} := \left(\|f\|^p_{p,\mu} + \sum_{i=1}^s \|P_i f\|^p_{p,\mu} \right)^{1/p},$$

is called the $\mathfrak{P}$*-Sobolev space of* L^p_μ*-sections* in $E \to X$.

Note that in the above situation, $\Gamma_{W^{\mathfrak{P},2}_\mu}(X,E)$ is a complex Hilbert space with the obvious inner product, and we have the complex linear space

$$\Gamma_{W^{\mathfrak{P},p}_{\mathrm{loc}}}(X,E) := \{f \mid f \in \Gamma_{L^p_{\mathrm{loc}}}(X,E), P_i f \in \Gamma_{L^p_{\mathrm{loc}}}(X,F_i) \text{ for all } i \in \{1,\dots,s\}\}$$

of *locally p-integrable sections in* $E \to X$ *with differential structure* $\mathfrak{P}$, which of course does not depend on any Hermitian structures.

Our first main result is the following abstract Meyers–Serrin type theorem:

Theorem 2.9. *Let* $p \in [1,\infty)$, $s \in \mathbf{N}$, $k_1 \dots, k_s \in \mathbf{N}_{\geq 0}$, *and let* $E \to X$, $F_i \to X$, *for each* $i \in \{1,\dots,s\}$, *be smooth Hermitian vector bundles, and let* $\mathfrak{P} := \{P_1,\dots,P_s\}$ *with* $P_i \in \mathscr{D}^{(k_i)}_{C^\infty}(X;E,F_i)$ *be such that in case* $k := \max\{k_1,\dots,k_s\} \geq 2$ *one has* $\Gamma_{W^{\mathfrak{P},p}_\mu}(X,E) \subset \Gamma_{W^{k-1,p}_{\mathrm{loc}}}(X,E)$. *Then for any* $f \in \Gamma_{W^{\mathfrak{P},p}_\mu}(X,E)$ *there is a sequence*

$$(f_n) \subset \Gamma_{C^\infty}(X,E) \cap \Gamma_{W^{\mathfrak{P},p}_\mu}(X,E),$$

which can be chosen in $\Gamma_{C^\infty_c}(X,E)$ *if* f *is compactly supported, such that* $\|f_n - f\|_{\mathfrak{P},p,\mu} \to 0$ *as* $n \to \infty$.

The following vector-valued and higher order result on Friedrichs mollifiers is the main tool for the proof of Theorem 2.9, and should in fact be of an independent interest. As many generalisations of Friedrichs result, it lies on a local level; in a different vein, we quote the Meyers–Serrin type results proved in [5, 6] for generalised Sobolev spaces defined by first-order differential operators with Lipschitz continuous coefficients.

Proposition 2.10. *Let $0 \leq h \in C_c^\infty(\mathbf{R}^m)$ be such that $h(x) = 0$ for all x with $|x| \geq 1$, $\int_{\mathbf{R}^m} h(x)\,\mathrm{d}x = 1$. For any $\epsilon > 0$ define $0 \leq h_\epsilon \in C_c^\infty(\mathbf{R}^m)$ by $h_\epsilon(x) := \epsilon^{-m} h(\epsilon^{-1}x)$. Furthermore, let $U \subset \mathbf{R}^m$ be open, let $k \in \mathbf{N}_{\geq 0}$, $\ell_0, \ell_1 \in \mathbf{N}$, $p \in [1, \infty)$, and let $P \in \mathscr{D}^{(k)}_{C^\infty}(U; \mathbf{C}^{\ell_0}, \mathbf{C}^{\ell_1})$,*

$$P = \sum_{\alpha \in \mathbf{N}_k^m} P_\alpha \partial^\alpha, \quad \text{with } P_\alpha \colon U \longrightarrow \mathrm{Mat}(\mathbf{C}; \ell_0 \times \ell_1) \text{ in } C^\infty.$$

a) *Assume that $f \in L^p(U, \mathbf{C}^{\ell_0})$, $Pf \in L^p(U, \mathbf{C}^{\ell_1})$ with compact support, and that $f \in W^{k-1,p}_{\mathrm{loc}}(U, \mathbf{C}^{\ell_0})$. Then one has $Pf_\epsilon \to Pf$ as $\epsilon \to 0+$ in $L^p_{\mathrm{loc}}(U, \mathbf{C}^{\ell_1})$, where for sufficiently small $\epsilon > 0$ we have set $f_\epsilon := f * h_\epsilon \in C^\infty(U, \mathbf{C}^{\ell_0})$.*
b) *If $f \in C^k(U, \mathbf{C}^{\ell_0})$ with compact support, then $Pf_\epsilon \to Pf$ as $\epsilon \to 0+$, uniformly over U.*

Proof. a) The case $k = 1$ is the classical Friedrichs' theorem, but it is known to hold in an even more general situation, see [7, Eq. (3.8)] and also [15, Lemma 6.1]: If $a \in C^1(U)$ is bounded with its derivatives and $u \in L^p(U)$, with $1 \leq p < \infty$, denoting by $\partial_j u$ the derivative in the sense of distributions and defining $a\partial_j u := \partial_j(au) - \partial_j a \cdot u$, we have

$$\lim_{\epsilon \to 0} \|(a\partial_j u)_\epsilon - a\partial_j u_\epsilon\|_{L^p(U)} = 0.$$

The same holds also in case $p = \infty$ if u is uniformly continuous and bounded. Observe that the above statement is not the case $k = 1$ of the proposition, as it does not require that $a\partial_j u \in L^p(U)$.

Now let $k \geq 2$: the proof is an easy consequence of the above statement: it is well known that $(Pf)_\epsilon \to Pf$ in $L^p(U, \mathbf{C}^{\ell_1})$. So it suffices to prove that

$$\|(Pf)_\epsilon - P(f_\epsilon)\|_{L^p(U, \mathbf{C}^{\ell_1})} \to 0.$$

To this aim, let us show that

$$\|(P_\alpha \partial^\alpha f)_\epsilon - P_\alpha \partial^\alpha (f_\epsilon)\|_{L^p(U, \mathbf{C}^{\ell_1})} \to 0$$

for every $\alpha \in \mathbf{N}_k^m$. In fact, for some $j \in \{1, \ldots, m\}$ and some $\alpha' \in \mathbf{N}_{k-1}^m$,

$$(P_\alpha \partial^\alpha f)_\epsilon - P_\alpha \partial^\alpha (f_\epsilon) = (P_\alpha \partial_j (\partial^{\alpha'} f))_\epsilon - P_\alpha \partial_j (\partial^{\alpha'} f)_\epsilon$$

and $\partial^{\alpha'} f \in L^p(U, \mathbf{C}^{\ell_0})$.

b) This is an elementary property of mollifiers. □

Proof of Theorem 2.9. Let

$$\ell_0 := \mathrm{rank}(E), \quad \ell_j := \mathrm{rank}(F_j), \quad \text{for any } j \in \{1, \ldots, s\}.$$

We take a relatively compact, locally finite atlas $\bigcup_{n \in \mathbf{N}} U_n = X$ such that each U_n admits smooth frames for

$$E \longrightarrow X,\ F_1 \longrightarrow X, \ldots, F_s \longrightarrow X.$$

Let (φ_n) be a partition of unity which is subordinate to (U_n), that is,

$$0 \leq \varphi_n \in C_c^\infty(U_n), \quad \sum_n \varphi_n(x) = 1 \quad \text{for all } x \in X,$$

where the latter is a locally finite sum. Now let $f \in \Gamma_{W^{\mathfrak{P},p}_{\mu}}(X,E)$, and $f_n := \varphi_n f$. Let us first show that $f_n \in \Gamma_{W^{\mathfrak{P},p}_{\mu,c}}(U_n,E)$. Indeed, let $j \in \{1,\dots,s\}$ and let $e_1,\dots,e_{\ell_0} \in \Gamma_{C^\infty}(U_n,E)$ denote a frame for $E \to X$ on U_n. Then, as elements in the space of distributions $\Gamma_{D'}(U_n,E)$ defined to be all maps $T\colon \Gamma_{C^\infty_c}(U_n,E) \to \mathbf{C}$ such that the induced map

$$C^\infty_c(U_n,\mathbf{C}^{\ell_0}) \ni (\psi_1,\dots,\psi_{\ell_0}) \longmapsto T\sum_j \psi_j e_j \in \mathbf{C}$$

is[b] in $D'(U_n,\mathbf{C}^{\ell_0})$, one has

$$P_j f_n = \varphi_n P_j f + [P_j,\varphi_n]f, \quad \text{but} \quad [P_j,\varphi_n] \in \mathscr{D}^{k_j-1}_{C^\infty}(U_n;E,F_j),$$

and as we have $f \in \Gamma_{W^{k-1,p}_{\mathrm{loc}}}(X,E)$, it follows that

$$(\partial^\alpha f_1,\dots,\partial^\alpha f_{\ell_0}) \in L^p_{\mathrm{loc}}(U_n,\mathbf{C}^{\ell_0}) \ \text{ for all } \alpha \in \mathbf{N}^m_{k-1},$$

where $f = \sum_j f_j e_j$ on U_n. Thus we get

$$[P_j,\varphi_n]f \in \Gamma_{L^p_\mu}(U_n,F_j)$$

as the coefficients of $[P_j,\varphi_n]$ have a compact support in U_n and $0 < \mu_{U_n} \in C^\infty(U_n)$. As of course $\varphi_n P_j f \in \Gamma_{L^p_\mu}(X,F_j)$, the proof of $f_n \in \Gamma_{W^{\mathfrak{P},p}_{\mu,c}}(U_n,E)$ is complete. But now, given $\epsilon > 0$, we may appeal to Proposition 2.10 a) to pick an $f_{n,\epsilon} \in \Gamma_{C^\infty_c}(X,E)$ with support in U_n such that

$$\|f_n - f_{n,\epsilon}\|_{\mathfrak{P},p,\mu} < \epsilon/2^{n+1}.$$

Finally, $f_\epsilon(x) := \sum_n f_{n,\epsilon}(x)$, $x \in X$, is a locally finite sum and thus defines an element in $\Gamma_{C^\infty}(X,E)$ which satisfies

$$\|f_\epsilon - f\|_{\mathfrak{P},p,\mu} \le \sum_{n=1}^\infty \|f_{n,\epsilon} - f_n\|_{\mathfrak{P},p,\mu} < \epsilon,$$

which proves the first assertion. If f is compactly supported, then picking a *finite* covering of the support of f with $U_n' s$ as above, the above proof also shows the second assertion. □

We close this section with two examples that illustrate the assumption

$$\Gamma_{W^{\mathfrak{P},p}_{\mu}}(X,E) \subset \Gamma_{W^{k-1,p}_{\mathrm{loc}}}(X,E) \tag{10}$$

in Theorem 2.9. The first example shows that our assumptions are optimal in a certain sense:

Example 2.11. Consider the third order differential operator

$$A := -x\partial^3 + (x-1)\partial^2 = (1-\partial)\circ x \circ \partial^2 \in \mathscr{D}^{(3)}_{C^\infty}(\mathbf{R})$$

on $\mathbf{R}$ (with its Lebesgue measure). *Then for any* $p \in (1,\infty)$ *one has*

$$W^{A,p}(\mathbf{R}) \subset W^{1,p}_{\mathrm{loc}}(\mathbf{R}), \quad W^{A,p}(\mathbf{R}) \not\subset W^{2,p}_{\mathrm{loc}}(\mathbf{R})$$

and $W^{A,p}(\mathbf{R}) \cap C^\infty(\mathbf{R})$ *is not dense in* $W^{A,p}(\mathbf{R})$: Indeed, we first observe that

$$W^{A,p}(\mathbf{R}) = \{u \mid u \in L^p(\mathbf{R}),\ x\partial^2 u \in W^{1,p}(\mathbf{R})\}.$$

To see this, if $f = Au$ and $v = x\partial^2 u$, $v \in S'(\mathbf{R})$, $(1-\partial)v = f$, so that $(1-i\xi)\hat v = \hat f$, so that $v = \mathcal{F}^{-1}[(1-i\xi)^{-1}\hat f] \in W^{1,p}(\mathbf{R})$. Here, $\mathcal{F}$ is the Fourier transformation and $\hat\Psi := \mathcal{F}\Psi$.

[b]Note that $\Gamma_{D'}(U_n,E)$ does not depend on a particular choice of a frame for $E \to X$ on U_n.

Next we show $W^{A,p}(\mathbf{R}) \subset W^{1,p}_{\mathrm{loc}}(\mathbf{R})$. In fact, let $u \in W^{A,p}(\mathbf{R})$ and set $x\partial^2 u = g \in W^{1,p}(\mathbf{R})$. We write g in the form $g = g(0) + \int_0^x \partial g(y)\,\mathrm{d}y$. Then

$$\partial^2 u(x) = \frac{g(0)}{x} + h(x), \quad x \in \mathbf{R} \setminus \{0\},$$

with $h(x) = \frac{1}{x}\int_0^x \partial g(y)\,\mathrm{d}y$. As $p > 1$, it is a well known consequence of Hardy's inequality that $h \in L^p(\mathbf{R})$. So

$$\partial^2 u = g(0)\, p.v.\left(\frac{1}{x}\right) + h + k,$$

with $k \in D'(\mathbf{R})$, $\mathrm{supp}(k) \subseteq \{0\}$. We deduce that

$$g(x) = g(0) + xh(x) + xk(x),$$

implying $xk(x) = 0$. From $k(x) = \sum_{j=0}^m a_j \delta^{(j)}$ it follows that $xk(x) = -\sum_{j=1}^m j a_j \delta^{(j-1)} = 0$ if and only if $k(x) = a_0\delta$, whence

$$\partial^2 u = g(0) p.v.\left(\frac{1}{x}\right) + h + a_0\delta,$$

so that

$$\partial u(x) = g(0)\ln(|x|) + \int_0^x \partial g(y)\,\mathrm{d}y + a_0 H(x) + C \in L^p_{\mathrm{loc}}(\mathbf{R}),$$

where H is the Heaviside function, and we have proved that $W^{A,p}(\mathbf{R}) \subset W^{1,p}_{\mathrm{loc}}(\mathbf{R})$.

In order to see $W^{A,p}(\mathbf{R}) \not\subset W^{2,p}_{\mathrm{loc}}(\mathbf{R})$, consider the function $u(x) = \phi(x)\ln(|x|)$, with $\phi \in C^\infty_c(\mathbf{R})$, $\phi(x) = x$ in some neighbourhood of 0. Then $x\partial^2 u \in W^{1,p}(\mathbf{R})$, but $u \notin W^{2,p}_{\mathrm{loc}}(\mathbf{R})$, since one has

$$\partial^2 u(x) = p.v.\left(\frac{1}{x}\right)$$

in a neighborhood of 0. So Theorem 2.8 is not applicable.

To see that $W^{A,p}(\mathbf{R}) \cap C^\infty(\mathbf{R})$ is not dense in $W^{A,p}(\mathbf{R})$, let again $u(x) := \phi(x)\ln(|x|)$ with ϕ as above. Assume (by contradiction) that there exists $(u_n)_{n\in\mathbf{N}}$ with $u_n \in W^{A,p}(\mathbf{R}) \cap C^\infty(\mathbf{R})$, such that

$$\|u_n - u\|_{L^p(\mathbf{R})} + \|Au_n - Au\|_{L^p(\mathbf{R})} \to 0 \ \text{ as } n \to \infty.$$

We set $v = x\partial^2 u$, $v_n = x\partial^2 u_n$. Then

$$\|v_n - v\|_{L^p(\mathbf{R})} + \|\partial v_n - \partial v\|_{L^p(\mathbf{R})} \to 0 \ \text{ as } n \to \infty,$$

so that (considering the continuous representative of any $W^{1,p}(\mathbf{R})$ equivalence class) $v_n(0) \to v(0)$. However, one has $v_n(0) = 0$ for all $n \in \mathbf{N}$, while $v(0) = 1$, a contradiction.

The second example shows that the assumption (10) in Theorem 2.9 is in general not necessary for the conclusion, which can be seen by using differential operators with constant coefficients, a situation which, however, can be seen as very special in our geometric context, as it does not make any sense on manifolds.

Example 2.12. In fact, if $X = \mathbf{R}^m$ and $\mathfrak{P} = \{P_1, \dots, P_s\}$ is a family of differential operators with constant coefficients, it is clear that $C^\infty(\mathbf{R}^m; \mathbf{C}^{\ell_0}) \cap W^{\mathfrak{P},p}(\mathbf{R}^m, \mathbf{C}^{\ell_0})$ is dense in $W^{\mathfrak{P},p}(\mathbf{R}^m, \mathbf{C}^{\ell_0})$, because $\forall f \in W^{\mathfrak{P},p}(\mathbf{R}^m, \mathbf{C}^{\ell_0})$, (with the notation of Proposition 2.10) we have $P_j f_\epsilon = (P_j f)_\epsilon$. On the other hand, in general $W^{\mathfrak{P},p}(\mathbf{R}^m, \mathbf{C}^{\ell_0})$ is, in general, not contained in $W^{k-1,p}_{\mathrm{loc}}(\mathbf{R}^m; \mathbf{C}^{\ell_1})$. Consider, for example, the operator $P = \partial^2_{x_1}$ in $X = \mathbf{R}^2$. If $f(x_1, x_2) = g(x_1)h(x_2)$, with $g \in W^{2,p}(\mathbf{R})$ and

$h \in L^p(\mathbf{R})$, $f \in L^p(\mathbf{R}^2)$, $Pf(x_1,x_2) = g''(x_1)h(x_2) \in L^p(\mathbf{R}^2)$, but, in general, $\partial_2 f(x_1,x_2) \notin L^p_{\mathrm{loc}}(\mathbf{R}^2)$.

3. Applications of Theorem 2.9

3.1. The elliptic case. In this subsection we first state some regularity results for elliptic operators and then we apply them to the Meyers–Serrin approximation. We first record the following local elliptic regularity result, whose L^p_{loc}-case, $p \in (1,\infty)$, is classical (see for example Theorem 10.3.6 in [17]), while the L^1_{loc}-case seems to be entirely new, and can be considered as our second main result:

Theorem 3.1. *Let $U \subset \mathbf{R}^m$ be open, let $k \in \mathbf{N}_{\geq 0}$, $\ell \in \mathbf{N}$, and let $P \in \mathscr{D}^{(k)}_{C^\infty}(U;\mathbf{C}^\ell,\mathbf{C}^\ell)$,*

$$P = \sum_{\alpha \in \mathbf{N}^m_k} P_\alpha \partial^\alpha, \quad \text{with } P_\alpha \colon U \longrightarrow \mathrm{Mat}(\mathbf{C};\ell\times\ell) \text{ in } C^\infty$$

be elliptic. Then the following results hold true:

a) *If $p \in (1,\infty)$, then for any $f \in L^p_{\mathrm{loc}}(U,\mathbf{C}^\ell)$ with $Pf \in L^p_{\mathrm{loc}}(U,\mathbf{C}^\ell)$ one has $f \in W^{k,p}_{\mathrm{loc}}(U,\mathbf{C}^\ell)$.*

b) *For any $f \in L^1_{\mathrm{loc}}(U,\mathbf{C}^\ell)$ with $Pf \in L^1_{\mathrm{loc}}(U,\mathbf{C}^\ell)$ it holds that $f \in W^{k-1,1}_{\mathrm{loc}}(U,\mathbf{C}^\ell)$.*

Before we come to the proof, a few remarks are in order:

Remark 3.2. In fact, we are going to prove the following much stronger statement in part b): Under the assumptions of Theorem 3.1 b), for any $f \in L^1_{\mathrm{loc}}(U,\mathbf{C}^\ell)$ with $Pf \in L^1_{\mathrm{loc}}(U,\mathbf{C}^\ell)$, one has that for any $\psi \in C^\infty_{\mathrm{c}}(U)$, the distribution ψf is in the Besov space

$$B^k_{1,\infty}(\mathbf{R}^m,\mathbf{C}^\ell) \subset W^{k-1,1}(\mathbf{R}^m,\mathbf{C}^\ell).$$

This in turn is proved using a new existence and uniqueness result (cf. Proposition A.3 in Section A) for certain systems of linear elliptic PDE's on the Besov scale. We refer the reader to Section A for the definition and essential properties of the Besov spaces $B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell) \subset S'(\mathbf{R}^m,\mathbf{C}^\ell)$ (with $S'(\mathbf{R}^m)$ the Schwartz distributions), where $\beta \in \mathbf{R}$, $p,q \in [1,\infty]$. Note that in the situation of Theorem 3.1 b), the assumptions $f, Pf \in L^1_{\mathrm{loc}}(U,\mathbf{C}^\ell)$, do not imply $f \in W^{k,1}_{\mathrm{loc}}(U,\mathbf{C}^\ell)$: An explicit counterexample has been given in [18] for the Euclidean Laplace operator. In fact, it follows from results of [10] that for any strongly elliptic differential operator P in $\mathbf{R}^m$, $m \geq 2$, with constant coefficients and order $2k$, there is a f with $f, Pf \in L^1_{\mathrm{loc}}(\mathbf{R}^m)$, and $f \notin W^{2k,1}_{\mathrm{loc}}(\mathbf{R}^m)$. In this sense, the above k-th order Besov regularity can be considered to be optimal.

Proof of Theorem 3.1 b). In this proof, we denote with $(\cdot,\cdot)$ the standard inner product in each $\mathbf{C}^n$, and with $|\cdot|$ the corresponding norm and operator norm, and $\mathrm{B}_r(x)$ stands for the corresponding open ball of radius r around x. Let us consider the formally self-adjoint elliptic partial differential operator

$$T := P^\dagger P = \sum_{\alpha \in \mathbf{N}^m_{2k}} T_\alpha \partial^\alpha \in \mathscr{D}^{(2k)}_{C^\infty}(\mathbf{R}^m;\mathbf{C}^\ell,\mathbf{C}^\ell).$$

Here, $P^\dagger \in \mathscr{D}^{(k)}_{C^\infty}(U;\mathbf{C}^\ell,\mathbf{C}^\ell)$ denotes the usual formal adjoint of P, which is well-defined by

$$\int_U (P^\dagger \varphi_1, \varphi_2)\,\mathrm{d}x = \int_U (\varphi_1, P\varphi_2)\,\mathrm{d}x,$$

for all $\varphi_1, \varphi_2 \in C^\infty(U, \mathbf{C}^\ell)$ one of which having a compact support In other words, $P^\dagger$ is nothing but the operator P^{μ,h_E,h_F} with respect to the Lebesgue measure and the canonical Hermitian structures on the trivial bundles. By a standard partition of unity argument, it suffices to prove that if $\psi \in C^\infty_c(U)$ with

$$\mathrm{supp}(\psi) \subset \mathrm{B}_{t_0}(x_0) \subset U \tag{11}$$

for some $x_0 \in U,\ t_0 > 0$ we have $\psi f \in B^k_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$. The proof consists of two steps: We first construct a differential operator Q^ψ which satisfies the assumptions of Proposition A.3, and which coincides with T near $\mathrm{supp}(\psi)$, and then we apply Proposition A.3 together with a maximality argument to Q^ψ to deduce the assertion.

We can assume that there are $t_0 > 0$, $x_0 \in U$ such that $\mathrm{B}_{t_0}(x_0) \Subset U$. We also take some $\phi \in C^\infty_c(U)$ with $\phi = 1$ on $\mathrm{B}_{t_0}(x_0)$, and for any $0 < t < t_0$ we set

$$C_t := \overline{\max_{y \in \mathrm{B}_t(x_0), \alpha \in \mathbf{N}^m_{2k}}} |T_{\alpha ij}(y) - T_{\alpha ij}(x_0)|,$$

and we pick a $\chi_t \in C^\infty_c(\mathbf{R}^2, \mathbf{R}^2)$ with $\chi_t(z) = z$ for all z with $|z| \leq C_t$, and $|\chi_t(z)| \leq 2C_t$ for all z. We define a differential operator

$$Q^{(t)} = \sum_{\alpha \in \mathbf{N}^m_{2k}} Q^{(t)}_\alpha \partial^\alpha \in \mathscr{D}^{(2k)}_{C^\infty}(\mathbf{R}^m; \mathbf{C}^\ell, \mathbf{C}^\ell),$$

$$Q^{(t)}_{\alpha ij}(x) := T_{\alpha ij}(x_0) + \chi_t\big(\phi(x)(T_{\alpha ij}(x) - T_{\alpha ij}(x_0))\big) =: T_{\alpha ij}(x_0) + A^{(t)}_{\alpha ij}(x)$$

(with the usual extension of $\phi(T_{\alpha ij} - T_{\alpha ij}(x_0))$ to zero away from U being understood, so in particular we have $Q^{(t)}_{\alpha ij}(x) = T_{\alpha ij}(x_0)$, if $x \in \mathbf{R}^m \setminus U$). Let $\zeta \in \mathbf{R}^m \setminus \{0\}$, $\eta \in \mathbf{C}^\ell$ be arbitrary. Then using $\sigma_{T,x_0} = \sigma^\dagger_{P,x_0}\sigma_{P,x_0}$, and that

$$\mathbf{R}^m \setminus \{0\} \ni \zeta' \longmapsto \sigma_{P,x_0}(\mathrm{i}\zeta') = \sum_{\alpha \in \mathbf{N}^m_k, |\alpha| = k} P_\alpha(x_0)(\mathrm{i}\zeta')^\alpha \in \mathrm{GL}(\mathbf{C}; \ell \times \ell)$$

is well-defined and positively homogeneous of degree k, one finds

$$\Re(\sigma_{T,x_0}(\mathrm{i}\zeta), \eta, \eta) = (\sigma_{T,x_0}(\mathrm{i}\zeta), \eta, \eta) \geq D_1 |\zeta|^{2k} |\eta|^2,$$

where

$$D_1 := \min_{\zeta' \in \mathbf{R}^m, \eta' \in \mathbf{C}^\ell, |\zeta'| = 1 = |\eta'|} |\sigma_{P,x_0}(\mathrm{i}\zeta')\eta'|^2 > 0.$$

Furthermore, for $x \in U$ one easily gets

$$\Re(\sigma_{A^{(t)},x}(\mathrm{i}\zeta), \eta, \eta) \geq -D(k,m) \max_{\alpha \in \mathbf{N}^m_{2k}} |A^{(t)}_\alpha(x)||\zeta|^{2k}|\eta|^2,$$

for some $D(k,m) > 0$. From now one we fix some small t such that

$$\sup_{x \in U} \max_{\alpha \in \mathbf{N}^m_{2k}} |A^{(t)}_\alpha(x)| \leq D_1/(2D(k,m)).$$

Then we get the estimate

$$\Re(\sigma_{Q^{(t)}(\mathrm{i}\zeta),x}, \eta, \eta) \geq \frac{D_1}{2} |\zeta|^{2k} |\eta|^2 \text{ for all } x \in \mathbf{R}^m\ ,$$

thus

$$\left| \left(r^{2k} + \sigma_{Q^{(t)},x}(\mathrm{i}\xi)\right)^{-1} \right| \leq \frac{D_1}{2}(r + |\xi|)^{-2k},$$

which is valid for all

$$(x, \xi, r) \in \mathbf{R}^m \times (\mathbf{R}^m \times [0,\infty)) \setminus \{(0,0)\}).$$

In other words, $Q^\psi := Q^{(t)}$ satisfies the assumptions of Proposition A.3 with $\theta_0 = \pi$, and by construction one has

$$Q^\psi_\alpha = T_\alpha \text{ for all } \alpha \in \mathbf{N}^m_{2k}, \text{ in a open neighbourhood of } \operatorname{supp}(\psi). \tag{12}$$

Since $L^1(\mathbf{R}^m, \mathbf{C}^\ell) \hookrightarrow B^0_{1,\infty}(\mathbf{R}^n, \mathbf{C}^\ell)$, the assumption $f \in L^1_{\mathrm{loc}}(U, \mathbf{C}^\ell)$ implies

$$\beta_0 := \sup\big\{\beta \mid \beta \in \mathbf{R},\ \tilde\psi f \in B^\beta_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell) \text{ for all } \tilde\psi \in C^\infty_{\mathrm{c}}(U)\big\} \geq 0.$$

We also know that $Pf \in L^1_{\mathrm{loc}}(U, \mathbf{C}^\ell)$. Then $P(\psi f) = \psi Pf + P_1 f$, where the commutator $P_1 := [P, \psi] \in \mathscr{D}^{(k-1)}_{C^\infty}(U; \mathbf{C}^\ell, \mathbf{C}^\ell)$ has coefficients with compact support in U, and using (12) we get

$$Q^\psi(\psi f) = T(\psi f) = P^\dagger P(\psi f) = P^\dagger(\psi Pf) + P^\dagger P_1 f,$$

all equalities understood in the sense of distributions with compact support in U. We fix $R \geq 0$ so large that the conclusions of Proposition A.3 hold for $Q = Q^\psi$, $\theta_0 := \pi$, $r = R$,

$$\beta \in \big\{-2k, \min\big\{\beta_0 + \tfrac{1}{2} - 2k, -k\big\}\big\}.$$

So ψf coincides with the unique solution w in $B^0_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$ of

$$R^{2k} w + Q^\psi w = R^{2k}\psi f + P^\dagger(\psi Pf) + P^\dagger P_1 f. \tag{13}$$

On the other hand, as $\tilde\psi f \in B^{\beta_0 - \frac{1}{2}}_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$ for all $\tilde\psi \in C^\infty_{\mathrm{c}}(U)$ (by the very definition of β_0), we get

$$R^{2k}\psi f + P^\dagger(\psi Pf) + P^\dagger P_1 f \in B^{\min\{-k, \beta_0 + \frac{1}{2} - 2k\}}_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell).$$

So (13) has a unique solution $\tilde w$ in $B^{\min\{\beta_0 + \frac{1}{2}, k\}}_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$, evidently coinciding with ψf, by the uniqueness of the solutions of (13) in the class $B^0_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$. We deduce that $\psi f \in B^{\min\{\beta_0 + \frac{1}{2}, k\}}_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$, so that, ψ being arbitrary, $\min\big\{\beta_0 + \frac{1}{2}, k\big\} \leq \beta_0$, implying $k \leq \beta_0$ and $\min\{\beta_0 + \frac{1}{2}, k\} = k$. We have thus shown that $\psi f \in B^k_{1,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$. □

Keeping Lemma 2.6 in mind, we immediately get the following characterisation of local Sobolev spaces:

Corollary 3.3. *Let $E \to X$ be a smooth complex vector bundle, and let $k \in \mathbf{N}_{\geq 0}$.*

a) *If $p \in (1, \infty)$, then for any elliptic operator $Q \in \mathscr{D}^{(k)}_{C^\infty}(X; E, E)$ one has*

$$\Gamma_{W^{k,p}_{\mathrm{loc}}}(X, E) = \Gamma_{W^{Q,p}_{\mathrm{loc}}}(X, E).$$

b) *For any elliptic $Q \in \mathscr{D}^{(k+1)}_{C^\infty}(X; E, E)$ one has*

$$\Gamma_{W^{Q,1}_{\mathrm{loc}}}(X, E) \subset \Gamma_{W^{k,1}_{\mathrm{loc}}}(X, E).$$

Theorem 3.1 in combination with Theorem 2.9 immediately imply:

Corollary 3.4. *Let $s \in \mathbf{N}$, $k_1 \ldots, k_s \in \mathbf{N}_{\geq 0}$, let $E \to X$, $F_i \to X$, $i \in \{1, \ldots, s\}$, be smooth Hermitian vector bundles, and $\mathfrak{P} := \{P_1, \ldots, P_s\}$ with $P_i \in \mathscr{D}^{(k_i)}_{C^\infty}(X; E, F_i)$, and let $k := \max\{k_1, \ldots, k_s\}$.*

a) *Let $p \in (1,\infty)$. If one either has $k < 2$, or the existence of some $j \in \{1,\dots,s\}$ with P_j elliptic and $k_j \geq k-1$, then the assumptions from Theorem 2.9 are satisfied by $\mathfrak{P}$. In particular for any $f \in \Gamma_{W^{\mathfrak{P},p}_{\mu}}(X,E)$ there is a sequence*

$$(f_n) \subset \Gamma_{C^\infty}(X,E) \cap \Gamma_{W^{\mathfrak{P},p}_{\mu}}(X,E),$$

which can be chosen in $\Gamma_{C^\infty_c}(X,E)$ if f is compactly supported, such that $\|f_n - f\|_{\mathfrak{P},p,\mu} \to 0$ as $n \to \infty$.

b) *If one either has $k < 2$, or the existence of some $j \in \{1,\dots,s\}$ with P_j elliptic and $k_j = k$, then the assumptions from Theorem 2.9 are satisfied by $\mathfrak{P}$. In particular for any $f \in \Gamma_{W^{\mathfrak{P},1}_{\mu}}(X,E)$ there is a sequence*

$$(f_n) \subset \Gamma_{C^\infty}(X,E) \cap \Gamma_{W^{\mathfrak{P},1}_{\mu}}(X,E),$$

which can be chosen in $\Gamma_{C^\infty_c}(X,E)$ if f is compactly supported, such that $\|f_n - f\|_{\mathfrak{P},1,\mu} \to 0$ as $n \to \infty$.

3.2. A covariant Meyers–Serrin Theorem on arbitrary Riemannian manifolds. The aim of this section is to apply Theorem 2.9 in the context of covariant Sobolev spaces on Riemannian manifolds. These spaces have been considered in this full generality, for example in [19], and in the scalar case, in [2, 12]. The point we want to make here is that Theorem 2.9 can be applied in many situations, even if none of the underlying P_j's is elliptic.

Let us start by recalling some facts on covariant derivatives: A smooth covariant derivative ∇ on a smooth vector bundle $E \to X$ is a complex linear map

$$\nabla \colon \Gamma_{C^\infty}(X,E) \longrightarrow \Gamma_{C^\infty}(X, E \otimes \mathrm{T}^*X)$$

which satisfies the Leibniz rule

$$\nabla(f\psi) = f\nabla\psi + \psi \otimes \mathrm{d}f \quad \text{for all } f \in C^\infty(X),\ \psi \in \Gamma_{C^\infty}(X,F). \tag{14}$$

The Leibniz rule implies that any two smooth covariant derivatives ∇ and ∇' on $E \to X$ differ by a smooth 1-form which takes values in the endomorphisms of $E \to X$:

$$\nabla - \nabla' \in \Omega^1_{C^\infty}(X, \mathrm{End}(E)) = \Gamma_{C^\infty}(X, \mathrm{T}^*X \otimes \mathrm{End}(E)).$$

In particular, since the usual exterior derivative d is a covariant derivative on (vector-valued) functions, one has the following local description of covariant derivatives: If $\ell := \mathrm{rank}(E)$, and if $e_1,\dots,e_\ell \in \Gamma_{C^\infty}(U,E)$ is a smooth frame for $E \to X$, then there is a unique matrix

$$A \in \mathrm{Mat}\big(\Gamma_{C^\infty}(U, \mathrm{T}^*X); \ell \times \ell\big)$$

such that $\nabla = \mathrm{d} + A$ in U with respect to (e_j), in the sense that for all $(\psi^1,\dots,\psi^\ell) \in C^\infty(U, \mathbf{C}^\ell)$ one has

$$\nabla \sum_j \psi^j e_j = \sum_j (\mathrm{d}\psi^j) \otimes e_j + \sum_j \sum_i \psi^j A_{ij} \otimes e_i.$$

In particular, it becomes obvious that

$$\nabla \in \mathscr{D}^{(1)}_{C^\infty}(X; E, \mathrm{T}^*X \otimes E)\,.$$

The following result will make Theorem 2.9 accessible to covariant Riemannian Sobolev spaces:

Lemma 3.5. *Let $E \to X$ be a smooth complex vector bundle with a smooth covariant derivative ∇ defined on it. Then for any $p \in [1,\infty)$ one has $\Gamma_{W^{\nabla,p}_{\mathrm{loc}}}(X,E) = \Gamma_{W^{1,p}_{\mathrm{loc}}}(X,E)$.*

Proof. Let $\ell := \mathrm{rank}(E)$, and pick Hermitian structures on E and T^*X. Given $f \in \Gamma_{W^{\nabla,p}_{\mathrm{loc}}}(X,E)$, we have to prove $f \in \Gamma_{W^{1,p}_{\mathrm{loc}}}(X,E)$. To this end, it is sufficient to prove that if $V \Subset W \Subset X$ are such that there is a chart $(x^1,\dots,x^m)$ on W for X in which $E \to X$ admits an orthonormal frame $e_1,\dots,e_\ell \in \Gamma_{C^\infty}(W,E)$, then with the components $f^j := (f,e_j)$ of f one has

$$\sum_{k,j} \int_V |\partial_k f^j(x)|^p \,\mathrm{d}x < \infty. \tag{15}$$

To this end, we pick a matrix of 1-forms

$$A \in \mathrm{Mat}\big(\Gamma_{C^\infty}(W,\mathrm{T}^*X);\ell\times\ell\big)$$

such that with respect to the frame (e_j) one has $\nabla = \mathrm{d} + A$ in the above sense. It follows that in W one has

$$\sum_j \mathrm{d}f^j \otimes e_j = \mathrm{d}f = \nabla f - Af,$$

so using $|A_{ij}| \le C$ in V and that (e_j) is orthonormal we arrive at

$$\sum_j \int_V |\,\mathrm{d}f^j(x)|_x^p \,\mathrm{d}x \le \tilde{C} \int_V |\nabla f(x)|_x^p \,\mathrm{d}x < \infty. \tag{16}$$

But it is elementary and in fact well-known that the integrability (16) implies (15) (see for example Excercise 4.11 b) in [8]). □

If $E_j \to X$ is a smooth vector bundle and

$$\nabla_j \in \mathscr{D}^{(1)}_{C^\infty}(X;E_j,\mathrm{T}^*X\otimes E_j)$$

a smooth covariant derivative on $E_j \to X$ for $j = 1,2$, then one defines (cf. Section 3.3.1 in [17]) the *tensor covariant derivative of* ∇_1 *and* ∇_2 as the uniquely determined smooth covariant derivative

$$\nabla_1 \tilde{\otimes} \nabla_2 \in \mathscr{D}^{(1)}_{C^\infty}(X;E_1\otimes E_2,\mathrm{T}^*X\otimes E_1\otimes E_2)$$

on $E_1 \otimes E_2 \to X$ which satisfies

$$\nabla_1 \tilde{\otimes} \nabla_2 (f_1 \otimes f_2) = \nabla_1(f_1)\otimes f_2 + f_1 \otimes \nabla_2(f_2) \tag{17}$$

for all $f_1 \in \Gamma_{C^\infty}(X,E_1)$, $f_2 \in \Gamma_{C^\infty}(X,E_2)$ (the canonical complex linear isomorphism

$$\Gamma_{C^\infty}(X,\mathrm{T}^*X\otimes E_1\otimes E_2) \longrightarrow \Gamma_{C^\infty}(X,\mathrm{T}^*X\otimes E_2\otimes E_1)$$

being understood).

Now let (M,g) be a possibly noncompact smooth Riemannian manifold without boundary and let $\mu(\,\mathrm{d}x) = \mu_g(\,\mathrm{d}x)$ be the Riemannian volume measure, which is the uniquely determined smooth measure on M such that for each chart $((x^1,\dots,x^m),U)$, and each Borel set $N \subset U$ one has

$$\mu_g(N) = \int_N \sqrt{\det(g_{ij}(x))}\,\mathrm{d}x^1\cdots\mathrm{d}x^m, \quad \text{where } g_{ij} := g(\partial_i,\partial_j).$$

Recall that the Levi–Civita connection ∇_g on $\mathrm{T}M$ (complexified!) is the uniquely determined smooth covariant derivative

$$\nabla_g \in \mathscr{D}^{(1)}(M; \mathrm{T}M, \mathrm{T}M \otimes \mathrm{T}^*M)$$

which is *torsion free* in the sense of

$$\nabla_{g,A}B - \nabla_{g,B}A = [A, B] \quad \text{(commutator)}$$

for all smooth vector fields A, B on M, and *Hermitian*, in the sense of

$$Cg(A, B) = g(\nabla_{g,C}A, B) + g(A\nabla_{g,C}B),$$

where C is another arbitrary smooth vector field on M, and $g(A, B)$ is regarded as a smooth function on M. The dual bundle T^*M canonically inherits a Hermitian structure from g, and the covariant derivative ∇^*_g from ∇_g, so that ∇^*_g is nothing but ∇_g under the isomorphism of smooth complex vector bundles $\mathrm{T}M \to \mathrm{T}^*M$ which is induced by g.

Next, we give ourselves a smooth Hermitian vector bundle $E \to M$ and let ∇ be a smooth covariant derivative defined on the latter bundle. For any $j \in \mathbf{N}$, the operator

$$\nabla^{(j)}_g \in \mathscr{D}^{(1)}_{C^\infty}\big(M; (\mathrm{T}^*M)^{\otimes j-1} \otimes E, (\mathrm{T}^*M)^{\otimes j} \otimes E\big)$$

is defined recursively by $\nabla^{(1)}_g := \nabla$, $\nabla^{(j+1)}_g := \nabla^{(j)}_g \tilde{\otimes} \nabla^*_g$, and we can further set

$$\nabla^j_g := \nabla^{(j)}_g \cdots \nabla^{(1)}_g \in \mathscr{D}^{(j)}_{C^\infty}\big(M; E, (\mathrm{T}^*M)^{\otimes j} \otimes E\big).$$

Note that if $\dim(M) > 1$, then each ∇^j_g is nonelliptic.

With these preparations, we can state the following covariant Meyers–Serrin theorem for Riemannian manifolds (which in the case of scalar functions, that is, if $E = M \times \mathbf{C}$ with $\nabla = \mathrm{d}$) has also been observed in [16, Lemma 3.1]):

Corollary 3.6. *Let $p \in [1, \infty)$, $s \in \mathbf{N}$, and define a global Sobolev space by*

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) := \Gamma_{W^{\{\nabla^1_g,\dots,\nabla^s_g\},p}_{\mu_g}}(M, E).$$

Then one has

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) \subset \Gamma_{W^{s,p}_{\mathrm{loc}}}(M, E),$$

in particular, for any $f \in \Gamma_{W^{s,p}_{\nabla,g}}(M, E)$ there is a sequence

$$(f_n) \subset \Gamma_{C^\infty}(M, E) \cap \Gamma_{W^{s,p}_{\nabla,g}}(M, E),$$

which can be chosen in $\Gamma_{C^\infty_c}(M, E)$ if f is compactly supported, such

$$\|f_n - f\|_{\nabla,g,p} := \|f_n - f\|_{\{\nabla^1_g,\dots,\nabla^s_g\},p,\mu_g} \to 0 \text{ as } n \to \infty.$$

Proof. Applying Lemma 3.5 inductively shows

$$\Gamma_{W^{s,p}_{\nabla,g}}(M, E) \subset \Gamma_{W^{s,p}_{\mathrm{loc}}}(M, E),$$

so that the other statements are implied by Theorem 2.9. □

4. A substitute result for the $p = \infty$ case

As C^∞ is not dense in L^∞, it is clear that Theorem 2.9 cannot be true for $p = \infty$. In this case, one can nevertheless smoothly approximate generalised C^k-type spaces given by families $\mathfrak{P}$, without any further assumptions on $\mathfrak{P}$, an elementary fact which we record for the sake of completeness:

Proposition 4.1. *Let $s \in \mathbf{N}$, $k_1 \ldots, k_s \in \mathbf{N}_{\geq 0}$, and let $E \to X$, $F_i \to X$, for each $i \in \{1, \ldots, s\}$, be smooth Hermitian vector bundles, and let $\mathfrak{P} := \{P_1, \ldots, P_s\}$ with $P_i \in \mathscr{D}^{(k_i)}_{C^\infty}(X; E, F_i)$. Then with $k := \max\{k_1, \ldots, k_s\}$, define the Banach space $\Gamma_{\mathfrak{P},\infty}(X, E)$ by*

$$\Gamma_{\mathfrak{P},\infty}(X, E) := \{f \mid f \in \Gamma_{C\cap L^\infty}(X, E),\ P_i f \in \Gamma_{C\cap L^\infty}(X, F_i) \text{ for all } i \in \{1, \ldots, s\}\}$$

with norm $\|f\|_{\mathfrak{P},\infty} := \|f\|_\infty + \sum_{i=1}^{s} \|P_i f\|_\infty$.

Assume that $\Gamma_{\mathfrak{P},\infty}(X, E) \subset \Gamma_{C^{k-1}}(X, E)$. Then $\Gamma_{C^\infty}(X, E) \cap \Gamma_{\mathfrak{P},\infty}(X, E)$ is dense in $\Gamma_{\mathfrak{P},\infty}(X, E)$.

Using Proposition 2.10 b), this result follows from the same localisation argument as in the proof of Theorem 2.9.

Appendix A. An existence and uniqueness result for systems of linear elliptic PDE's on the Besov scale

Throughout this section, let $\ell \in \mathbf{N}$ be arbitrary. We again use the notation $(\cdot, \cdot)$, $|\cdot|$, and $\mathrm{B}_r(x)$ for the standard Euclidean data in each $\mathbf{C}^n$. We start by recalling the definition of Besov spaces with a positive differential order:

Definition A.1. For any $\alpha \in (0, 1], p \in [1, \infty], q \in [1, \infty)$, one defines $B^\alpha_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$ to be the space of $u \in L^p(\mathbf{R}^m, \mathbf{C}^\ell)$ such that

$$\int_{\mathbf{R}^m} \|u(\cdot + x) - 2u + u(\cdot - x)\|^q_{L^p(\mathbf{R}^m,\mathbf{C}^\ell)} |x|^{-m-\alpha q}\, \mathrm{d}x < \infty,$$

and $B^\alpha_{p,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$ to be the space of $u \in L^p(\mathbf{R}^m, \mathbf{C}^\ell)$ such that

$$\sup_{x\in\mathbf{R}^m\setminus\{0\}} |x|^{-\alpha} \|u(\cdot + x) - 2u + u(\cdot - x)\|_{L^p(\mathbf{R}^m,\mathbf{C}^\ell)} < \infty.$$

For $\alpha \in (1, \infty)$, $p \in [1, \infty]$, $q \in [1, \infty]$, one defines $B^\alpha_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$ to be the space[c] of $u \in W^{[\alpha],p}(\mathbf{R}^m, \mathbf{C}^\ell)$ such that for all $\beta \in (\mathbf{N}_{\geq 0})^m$ with $|\beta| = [\alpha]$ one has $\partial^\beta u \in B^{\alpha-[\alpha],p}(\mathbf{R}^m, \mathbf{C}^\ell)$. These are Banach spaces with respect to their canonical norms.

For negative differential orders, the definition is more subtle:

Proposition and definition A.2. *Let $t(\zeta) := |\zeta|$, $\zeta \in \mathbf{R}^m$, and for any $\gamma \in \mathbf{R}$ let*

$$J_\gamma := \mathcal{F}^{-1}(1 + t^2)^{-\gamma/2}$$

denote the Bessel potential of order γ. Let $\alpha \in (-\infty, 0]$, $p \in [1, \infty]$, $q \in [1, \infty)$, and pick some $\beta \in (0, \infty)$. Then one defines $B^\alpha_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$ to be the space of $u \in S'(\mathbf{R}^m, \mathbf{C}^\ell)$ such that $u = J_{\alpha-\beta} * f$ for some $f \in B^\beta_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$. This definition does not depend on the particular choice of β, and one defines

$$\|u\|_{B^\alpha_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} := \|J_{\alpha-1} * u\|_{B^1_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}\ ,$$

[c]Here, $[\alpha] := \max\{j | j \in \mathbf{N}, j < \alpha\}$.

which again produces a Banach space.

We are going to prove:

Proposition A.3. *Let $n \in \mathbf{N}_{\geq 0}$, $Q \in \mathscr{D}^{(n)}_{C^\infty}(\mathbf{R}^m; \mathbf{C}^\ell, \mathbf{C}^\ell)$,*

$$Q = \sum_{\alpha \in \mathbf{N}^m_n} Q_\alpha \partial^\alpha, \quad \text{with } Q_\alpha \colon \mathbf{R}^m \longrightarrow \mathrm{Mat}(\mathbf{C}; \ell \times \ell) \text{ in } W^{\infty,\infty},$$

that is, Q_α and all its derivatives are bounded. Suppose also that for some $\theta_0 \in (-\pi, \pi]$ and all

$$(x, \xi, r) \in \mathbf{R}^m \times (\mathbf{R}^m \times [0, \infty)) \setminus \{(0, 0)\}),$$

the complex $\ell \times \ell$ matrix $r^n \mathrm{e}^{\mathrm{i}\theta_0} - \sigma_{Q,x}(\mathrm{i}\xi)$ is invertible, and that there is $C > 0$ such that for all (x, ξ, r) as above one has

$$\left| \left(r^n \mathrm{e}^{\mathrm{i}\theta_0} - \sigma_{Q,x}(\mathrm{i}\xi) \right)^{-1} \right| \leq C(r + |\xi|)^{-n}. \tag{18}$$

We consider the system of linear PDE's given by

$$r^n \mathrm{e}^{\mathrm{i}\theta_0} u(x) - Qu(x) = g(x), \quad x \in \mathbf{R}^m, \ r \geq 0. \tag{19}$$

Then for any $\beta \in \mathbf{R}$, $p, q \in [1, \infty]$, there is a $R = R(\beta, p, q, Q) \geq 0$ with the following property: if $r \geq R$ and $g \in B^\beta_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$, then (19) has a unique solution $u \in B^{\beta+n}_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$.

Note that given some $Q \in \mathscr{D}^{(n)}_{C^\infty}(\mathbf{R}^m; \mathbf{C}^\ell, \mathbf{C}^\ell)$ which is strongly elliptic in the usual sense

$$\Re(\sigma_{Q,x}(\zeta)\eta, \eta) \geq \tilde{C}|\eta|^2 \ \text{ for all } x \in \mathbf{R}^m, \eta \in \mathbf{C}^\ell, \zeta \in \mathbf{C}^m \text{ with } |\zeta| = 1$$

with some $\tilde{C} > 0$ which is uniform in x, η, ζ, it is straightforward to see that the condition (18) is satisfied with $\theta_0 = \pi$, $C = \min\{1, \tilde{C}\}$ (see also the proof of Theorem 3.1 b)).

Before we come to the proof of Proposition A.3, we first collect some well known facts concerning Besov spaces. Unless otherwise stated, the reader may find these results in [9] and the references therein.

(i) For every $p \in [1, \infty]$ one has $B^0_{p,1}(\mathbf{R}^m) \hookrightarrow L^p(\mathbf{R}^m) \hookrightarrow B^0_{p,\infty}(\mathbf{R}^m)$.

(ii) Let $p, q \in [1, \infty]$, $\beta \in \mathbf{R}$. Then

$$B^{\beta+1}_{p,q}(\mathbf{R}^m) = \{f \mid f \in B^\beta_{p,q}(\mathbf{R}^m), \ \partial_j f \in B^\beta_{p,q}(\mathbf{R}^m) \text{ for all } j \in \{1, \dots, m\}\}.$$

So for all $k \in \mathbf{N}$ one has $B^k_{p,1}(\mathbf{R}^m) \hookrightarrow W^{k,p}(\mathbf{R}^m) \hookrightarrow B^k_{p,\infty}(\mathbf{R}^m)$.

(iii) As a consequence of (ii), we have the following particular case of Sobolev embedding theorem: if $\beta \in \mathbf{R}$, $1 \leq p, q \leq \infty$, $B^\beta_{p,q}(\mathbf{R}^m) \hookrightarrow B^{\beta - m/p}_{\infty,\infty}(\mathbf{R}^m)$.

(iv) Let us indicate with $(\cdot, \cdot)_{\theta,q}$ ($0 < \theta < 1$, $1 \leq q \leq \infty$) the real interpolation functor. Then, if $-\infty < \alpha_0 < \alpha_1 < \infty$, $1 \leq p, q_0, q_1 \leq \infty$, the real interpolation space $(B^{\alpha_0}_{p,q_0}(\mathbf{R}^m), B^{\alpha_1}_{p,q_1}(\mathbf{R}^m))_{\theta,q}$ coincides with $B^{(1-\theta)\alpha_0 + \theta\alpha_1}_{p,q}(\mathbf{R}^m)$, with equivalent norms.

(v) If $1 \leq p, q < \infty$ and $\beta \in \mathbf{R}$, the antidual space of $B^\beta_{p,q}(\mathbf{R}^m)$ can be identified with $B^{-\beta}_{p',q'}(\mathbf{R}^m)$ in the following sense: if $g \in B^{-\beta}_{p',q'}(\mathbf{R}^m)$, then the (antilinear) distribution $\langle \cdot, \overline{g} \rangle$ can be uniquely extended to a bounded antilinear functional in $B^\beta_{p,q}(\mathbf{R}^m)$ (we recall here also that, whenever $\max\{p, q\} < \infty$, then $C^\infty_{\mathrm{c}}(\mathbf{R}^m)$ is dense in each $B^\beta_{p,q}(\mathbf{R}^m)$). Moreover, all bounded antilinear functionals on $B^\beta_{p,q}(\mathbf{R}^m)$ can be obtained in this way.

(vi) Suppose that $a \in C^\infty(\mathbf{R}^m)$, and that for some $n \in \mathbf{R}$ and all $\xi \in \mathbf{R}^m$ one has

$$\max_{\alpha \in \mathbf{N}^m_{m+1}} |\partial^\alpha a(\xi)| \leq C(1+|\xi|)^{n-|\alpha|}.$$

Then for all

$$(\beta, p, q) \in \mathbf{R} \times [1,\infty] \times [1,\infty],$$

the Fourier multiplication operator $f \mapsto \mathcal{F}^{-1}(a\mathcal{F}f)$ maps $B^\beta_{p,q}(\mathbf{R}^m)$ into $B^{\beta-n}_{p,q}(\mathbf{R}^m)$, and the norm of the latter operator can be estimated by

$$C \sup_{\alpha \in \mathbf{N}^m_{m+1}, \xi \in \mathbf{R}^m} \left|(1+|\xi|)^{|\alpha|-n}\partial^\alpha a(\xi)\right|,$$

for some $C > 0$ independent of a (cf. [1]).

(vii) If $a \in W^{\infty,\infty}(\mathbf{R}^m)$ and $f \in B^\beta_{p,q}(\mathbf{R}^m)$, then one has $af \in B^\beta_{p,q}(\mathbf{R}^m)$. More precisely, there exist $C > 0$, $N \in \mathbf{N}$, independent of a and f, such that

$$\|af\|_{B^\beta_{p,q}(\mathbf{R}^m)} \leq C\left(\|a\|_{L^\infty(\mathbf{R}^m)}\|f\|_{B^\beta_{p,q}(\mathbf{R}^m)} + \|a\|_{W^{N,\infty}(\mathbf{R}^m)}\|f\|_{B^{\beta-1}_{p,q}(\mathbf{R}^m)}\right).$$

(viii) Let $0 \leq \chi_0 \in C^\infty_c(\mathbf{R}^m)$ be such that for some $\delta > 0$ one has

$$\mathrm{supp}(\chi_0) \subset [-\delta,\delta]^m,\ \chi_0 = 1 \text{ in } [-\delta/2,\delta/2]^m.$$

For any $j \in \mathbf{Z}^m$ set

$$\chi_j(x) := \chi_0(x - \delta j/2), \chi(x) := \sum_{j \in \mathbf{Z}^m} \chi_j(x), \psi_j(x) := \frac{\chi_j(x)}{\chi(x)}.$$

Then for all $\beta \in \mathbf{R}$, $p \in [1,\infty]$, there exist $C_1, C_2 > 0$ such that for all $f \in B^\beta_{p,p}(\mathbf{R}^m)$ it holds that

$$C_1\|f\|_{B^\beta_{p,p}(\mathbf{R}^m)} \leq \|(\|\psi_j f\|_{B^\beta_{p,p}(\mathbf{R}^m)})_{j \in \mathbf{Z}^m}\|_{\ell^p(\mathbf{Z}^m)} \leq C_2\|f\|_{B^\beta_{p,p}(\mathbf{R}^m)}.$$

With these preparations, we can now give the proof of Proposition A.3:

Proof of Proposition A.3. We prove the result in several steps.

Step 1 (constant coefficients): Let

$$Q = \sum_{\alpha \in \mathbf{N}^m_n} Q_\alpha \partial^\alpha, \quad \text{with } Q_\alpha \in \mathrm{Mat}(\mathbf{C}; \ell \times \ell),$$

and suppose that for some $\theta_0 \in (-\pi, \pi]$ and all

$$(\xi, r) \in (\mathbf{R}^m \times [0,\infty)) \setminus \{(0,0)\}),$$

the $\ell \times \ell$ matrix $r^n e^{i\theta_0} - i^n \sigma_Q(\xi)$ is invertible, and that there exists $C > 0$ such that for all (ξ, r) as above one has

$$|(r^n e^{i\theta_0} - \sigma_Q(i\xi))^{-1}| \leq C(r+|\xi|)^{-n}. \tag{20}$$

Then for any $\beta \in \mathbf{R}$, $1 \leq p, q \leq \infty$, there exists $R \geq 0$ such that, if $r \geq R$ and $g \in B^\beta_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$, the system (19) has a unique solution $u \in B^{\beta+n}_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$. Moreover, there exists a constant $C_0 > 0$, which only depends on β, p, q, the constant C in (20) and on $\max_{\alpha \in \mathbf{N}^m_n} |Q_\alpha|$, such that for all $r \geq R$ one has

$$r^n\|u\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq C_0\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}.$$

By interpolation, we obtain also, for every $\theta \in [0,1]$ and $r \geq R$,

$$\|u\|_{B^{\beta+\theta n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq C_0 r^{(\theta-1)n}\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}. \tag{21}$$

In order to prove the statement from Step 1, we start by assuming that Q coincides with its principal part $Q_n := \sum_{|\alpha|=n} Q_\alpha \partial^\alpha$. Then, employing the Fourier transform, it is easily seen that for any $r \geq 0$, $g \in S'(\mathbf{R}^m, \mathbf{C}^\ell)$, the only possible solution $u \in S'(\mathbf{R}^m, \mathbf{C}^\ell)$ of (19) is

$$u = \mathcal{F}^{-1}\left((r^n e^{i\theta_0} - \sigma_Q(i\xi))^{-1}\mathcal{F}g\right).$$

Observe that $(r^n e^{i\theta_0} - \sigma_Q(i\xi))^{-1}$ is positively homogeneous of degree $-n$ in the variables

$$(r,\xi) \in ([0,\infty) \times \mathbf{R}^m) \setminus \{(0,0)\}.$$

So for all $\alpha \in \mathbf{N}^m_n$, the matrix $\partial^\alpha_\xi (r^n e^{i\theta_0} - \sigma_Q(i\xi))^{-1}$ is positively homogeneous of degree $-n - |\alpha|$ in these variables, implying

$$\left|\partial^\alpha_\xi (r^n e^{i\theta_0} - \sigma_Q(i\xi))^{-1}\right| \leq C(\alpha)(r + |\xi|)^{-n-|\alpha|}.$$

It is easily seen that $C(\alpha)$ can be estimated in terms of the constant C in (20) and of $\max_{\alpha \in \mathbf{N}^m_n} |Q_\alpha|$. We deduce from (vi) that, for all $r \geq 0$, and all $g \in B^\beta_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$, the problem

$$r^n e^{i\theta_0} u(x) - Q_n u(x) = g(x), \quad x \in \mathbf{R}^m \tag{22}$$

has a unique solution u in $B^{\beta+n}_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$, and also that for all $r_0 > 0$ there is $C(r_0) > 0$ such that for all $r \geq r_0$ one has

$$\|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq C(r_0)\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}.$$

The latter inequality together with (22) also gives

$$\begin{aligned}
\|u\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} &\leq r^{-n}(\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|Q_n u\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}) \\
&\leq C_1(r_0) r^{-n}(\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{n+\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}) \\
&\leq C_2(r_0) r^{-n}\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)},
\end{aligned}$$

and now the estimate (21) follows directly by interpolation (see (iv)). Now we extend the previous facts from Q_n to Q, taking r sufficiently large. In fact, we write (19) in the form

$$r^n e^{i\theta_0} u(x) - Q_n u(x) = (Q - Q_n)u(x) + g(x).$$

Taking $h := r^n e^{i\theta_0} u - Q_n u$ as new unknown, we obtain

$$h - (Q - Q_n)(r^n e^{i\theta_0} - Q_n)^{-1} h = g. \tag{23}$$

We have

$$\begin{aligned}
&\|(Q - Q_n)(r^n e^{i\theta_0} - Q_n)^{-1} h\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \\
&\leq C_0\|(r^n e^{i\theta_0} - Q_n)^{-1} h\|_{B^{\beta+n-1}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq C_1 r^{-1}\|h\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}.
\end{aligned}$$

So, if $C_1 r^{-1} < 1$, then (23) has a unique solution $h \in B^\beta_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$ and, in case $C_1 r^{-1} \leq \frac{1}{2}$ such solution can be estimated in the form

$$\|h\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq 2\|g\|_{B^\beta_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}.$$

So the previous estimates and results can be extended from Q_n to Q.

Step 2 (a priori estimate for solutions in $B^{\beta+n}_{p,q}$ with small support): Let $\beta \in \mathbf{R}$, $1 \le p, q \le \infty$. Then there exist $r_0, \delta, C > 0$ with the following property: if $u \in B^{\beta+n}_{p,q}(\mathbf{R}^m, \mathbf{C}^\ell)$ satisfies

$$r^n \mathrm{e}^{\mathrm{i}\theta_0} u - Qu = g, \ \operatorname{supp}(u) \subset \prod_{j=1}^m [x_j^0 - \delta, x_j^0 + \delta] \ \text{ for some } x^0 \in \mathbf{R}^m,$$

then one has

$$r^n \|u\|_{B^{\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \le C\|g\|_{B^{\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}. \tag{24}$$

In order to prove this, we define the constant coefficient operator

$$Q(x_0, \partial) := \sum_{\alpha \in \mathbf{N}^m_n} Q_\alpha(x_0)\partial^\alpha$$

and observe that

$$r^n \mathrm{e}^{\mathrm{i}\theta_0} u(x) - Q(x^0, \partial)u(x) = (Q - Q(x^0, \partial))u(x) + g(x).$$

Let $\epsilon > 0$. For any $\phi \in C^\infty_{\mathrm{c}}(\mathbf{R}^m)$ which satisfies

$$\operatorname{supp}(\phi) \subset \prod_{j=1}^m [x_j^0 - 2\delta, x_j^0 + 2\delta],$$
$$\phi = 1 \ \text{ in } \ \prod_{j=1}^m [x_j^0 - \delta, x_j^0 + \delta], \quad \|\phi\|_{L^\infty(\mathbf{R}^m)} = 1,$$

we have

$$(Q - Q(x^0, \partial))u = \phi(Q - Q(x^0, \partial))u.$$

So, taking δ sufficiently small, from (iv) and (vii) we obtain

$$\|(Q - Q(x^0, \partial))u\|_{B^{\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \le \epsilon\|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + C(\epsilon)\|u\|_{B^{\beta+n-1}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)}.$$

Observe that δ can be chosen independent of x^0. So, from Step 1 with $\theta = (n-1)/n$ in (21), taking r sufficiently large (uniformly in x^0) we obtain

$$\begin{aligned} &r^n \|u\|_{B^{\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + r\|u\|_{B^{\beta+n-1}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \\ &\le C_0 \left(\epsilon\|u\|_{B^{\beta+n}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + C(\epsilon)\|u\|_{B^{\beta+n-1}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|g\|_{B^{\beta}_{p,q}(\mathbf{R}^m,\mathbf{C}^\ell)} \right). \end{aligned}$$

Taking ϵ so small that $C_0\epsilon \le \frac{1}{2}$ and r so large that $C_0 C(\epsilon) \le r$, we deduce (24).

Step 3 (a priori estimate for arbitrary solutions in $B^{\beta+n}_{p,p}$): For any $\beta \in \mathbf{R}$, $p \in [1, \infty)$, there exist $C_0, r_0 > 0$ such that if $r \ge r_0$ and $u \in B^{\beta+n}_{p,p}(\mathbf{R}^m; \mathbf{C}^\ell)$ is a solution to (19), then

$$r^n \|u\|_{B^{\beta}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \le C_0\|g\|_{B^{\beta}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}. \tag{25}$$

To see this, we take $\delta, r_0 > 0$ so that the conclusion in Step 2 holds. We consider a family of functions $(\psi_j)_{j \in \mathbf{Z}^m}$ as in (viii). Let $u \in B^{\beta+n}_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell)$ solve (19), with $r \ge r_0$. For each $j \in \mathbf{Z}^m$ we have

$$r^n \psi_j u - Q(\psi_j u) = \psi_j g + Q_j u,$$

with the commutator

$$Q_j := [Q, \psi_j] = \sum_{1 \le |\alpha| \le n} Q_\alpha \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} \partial^{\alpha-\gamma}\psi_j \partial^\gamma.$$

We set

$$\mathbf{Z}_j := \{i \mid i \in \mathbf{Z}^m,\ \operatorname{supp}(\psi_i) \cap \operatorname{supp}(\psi_j) \neq \emptyset\}.$$

Then $Q_j u = \sum_{i\in\mathbf{Z}_j} Q_j(\psi_i u)$, so that

$$\|Q_j u\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \leq C_1 \sum_{i\in\mathbf{Z}_j} \|\psi_i u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}.$$

with C_1 independent of j. So, from Step 2, we have, for each $j \in \mathbf{Z}^m$,

$$\begin{aligned} &r^n\|\psi_j u\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + r\|\psi_j u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|\psi_j u\|_{B^{\beta+n}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \\ &\leq C_2 \left(\|\psi_j g\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + \sum_{i\in\mathbf{Z}_j} \|\psi_i u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \right). \end{aligned} \tag{26}$$

We observe that $\mathbf{Z}_j$ has at most 7^m elements. So we have, in case $p < \infty$,

$$\Big(\sum_{i\in\mathbf{Z}_j} \|\psi_i u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \Big)^p \leq 7^{m(p-1)} \sum_{i\in\mathbf{Z}_j} \|\psi_i u\|^p_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}$$

and

$$\begin{aligned} \sum_{j\in\mathbf{Z}^m} \left(\sum_{i\in\mathbf{Z}_j} \|\psi_i u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \right)^p &\leq 7^{m(p-1)} \sum_{j\in\mathbf{Z}^m} \sum_{i\in\mathbf{Z}_j} \|\psi_i u\|^p_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \\ &= 7^{m(p-1)} \sum_{i\in\mathbf{Z}^m} \Big(\sum_{j\in\mathbf{Z}_i} 1 \Big) \|\psi_i u\|^p_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \\ &\leq 7^{mp} \left\| (\|\psi_i u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)})_{i\in\mathbf{Z}^m} \right\|^p_{\ell^p(\mathbf{Z}^m)}. \end{aligned}$$

So, from (26) and (viii), we deduce

$$\begin{aligned} &r^n\|u\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + r\|u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \\ &\leq C_3 \left(r^n \left\| \big(\|\psi_j u\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}\big)_{j\in\mathbf{Z}^m} \right\|_{\ell^p(\mathbf{Z}^m)} + r \left\| \big(\|\psi_j u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}\big)_{j\in\mathbf{Z}^m} \right\|_{\ell^p(\mathbf{Z}^m)} \right. \\ &\qquad \left. + \left\| \big(\|\psi_j u\|_{B^{\beta+n}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}\big)_{j\in\mathbf{Z}^m} \right\|_{\ell^p(\mathbf{Z}^m)} \right) \\ &\leq C_4 \left(\left\| \big(\|\psi_j g\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}\big)_{j\in\mathbf{Z}^m} \right\|_{\ell^p(\mathbf{Z}^m)} + \left\| \big(\|\psi_j u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)}\big)_{j\in\mathbf{Z}^m} \right\|_{\ell^p(\mathbf{Z}^m)} \right) \\ &\leq C_5 \left(\|g\|_{B^\beta_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} + \|u\|_{B^{\beta+n-1}_{p,p}(\mathbf{R}^m,\mathbf{C}^\ell)} \right). \end{aligned}$$

Taking $r \geq C_5$, we get the conclusion.

Step 4: For any $\beta \in \mathbf{R}$, $p \in [1, \infty)$, there exists $r_0 \geq 0$ such that if $r \geq r_0$, $g \in B^\beta_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell)$, then (19) has a unique solution $u \in B^{\beta+n}_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell)$.

The uniqueness follows from Step 3. We show the existence by a duality argument. We think of $r^n e^{i\theta_0} - Q$ as an operator from $B^{\beta+n}_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell)$ to $B^\beta_{p,p}(\mathbf{R}^m, \mathbf{C}^l)$. By Step 3, if r is sufficiently large, its range is a closed subspace of $B^\beta_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell)$. Assume that it does not coincide with the whole space. Then, applying a well known consequence of the theorem of Hahn–Banach and (v), there exists $h \in B^{-\beta}_{p',p'}(\mathbf{R}^m, \mathbf{C}^l)$, $h \neq 0$, such that

$$\big\langle (r^n e^{i\theta_0} - Q)u, \overline{h} \big\rangle = 0 \quad \text{for all } u \in B^{\beta+n}_{p,p}(\mathbf{R}^m, \mathbf{C}^\ell).$$

This implies that

$$(r^n e^{-i\theta_0} - Q^*)h = 0. \tag{27}$$

Now, it is easily seen that Q^* satisfies the assumptions of Proposition A.3 if we replace θ_0 with $-\theta_0$. We deduce from Step 3 that, if r is sufficiently large, (27) implies $h = 0$, a contradiction.

Step 5: For any $\beta \in \mathbf{R}$ there exists $r_0 \geq 0$ such that if $r \geq r_0$, $g \in B^{\beta}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$, then (19) has a unique solution $u \in B^{\beta+n}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$.

In the proof of Lemma 2.4 from [9] it is shown that for any $g \in B^{\beta}_{\infty,\infty}(\mathbf{R}^m)$, there is a sequence $(g_k)_{k\in\mathbf{N}}$ in $S(\mathbf{R}^m)$ converging to g in $S'(\mathbf{R}^m)$ and bounded in $B^{\beta}_{\infty,\infty}(\mathbf{R}^m)$. So we take a sequence $(g_k)_{k\in\mathbf{N}}$ in $S(\mathbf{R}^m, \mathbf{C}^\ell)$ converging to g in $S'(\mathbf{R}^m, \mathbf{C}^\ell)$ and bounded in $B^{\beta}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$. We fix γ larger than $\beta + \frac{m}{2}$ and think of g_k as an element of $B^{\gamma}_{2,2}(\mathbf{R}^m, \mathbf{C}^\ell)$. Then, by Step 4, if r is sufficiently large, the equation

$$r^n e^{i\theta_0} u_k - Q u_k = g_k$$

has a unique solution u_k in $B^{\gamma+n}_{2,2}(\mathbf{R}^m, \mathbf{C}^\ell)$. By (iii), $u_k \in B^{\beta+n}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$ and, by Step 3, if r is sufficiently large, the sequence $(u_k)_{k\in\mathbf{N}}$ is bounded in $B^{\beta+n}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$, because $(g_k)_{k\in\mathbf{N}}$ is bounded in $B^{\beta}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$. Then, by (v) and the theorem of Alaoglu, we may assume, possibly passing to a subsequence, that there exists $u \in B^{\beta+n}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell)$ such that

$$\lim_{k\to\infty} u_k = u \text{ in the weak topology } w(B^{\beta+n}_{\infty,\infty}(\mathbf{R}^m, \mathbf{C}^\ell), B^{-\beta-n}_{1,1}(\mathbf{R}^m, \mathbf{C}^\ell)).$$

Such convergence implies convergence in $S'(\mathbf{R}^m, \mathbf{C}^\ell)$. So

$$(r^n e^{i\theta_0} - Q)u_k \to (r^n e^{i\theta_0} - Q)u \text{ as } k \to \infty \text{ in } S'(\mathbf{R}^m, \mathbf{C}^\ell).$$

We deduce that $(r^n e^{i\theta_0} - Q)u = g$.

Step 6: Full statement. This is a simple consequence of Step 4, Step 5 and the interpolation property (iv). □

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