FRANKE–JAWERTH EMBEDDINGS FOR BESOV AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS

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Abstract. The classical Jawerth and Franke embeddings

$$F_{p_0,q}^{s_0}(\mathbf{R}^n) \hookrightarrow B_{p_1,p_0}^{s_1}(\mathbf{R}^n) \quad \text{and} \quad B_{p_0,p_1}^{s_0}(\mathbf{R}^n) \hookrightarrow F_{p_1,q}^{s_1}(\mathbf{R}^n)$$

are versions of Sobolev embedding between the scales of Besov and Triebel–Lizorkin function spaces for $s_0 > s_1$ and $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$. We prove Jawerth and Franke embeddings for the scales of Besov and Triebel–Lizorkin spaces with all exponents variable

$$F_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow B_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n) \quad \text{and} \quad B_{p_0(\cdot),p_1(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow F_{p_1(\cdot),q(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n),$$

respectively, if $\inf_{x\in\mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ and

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

We work exclusively with the associated sequence spaces $b_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ and $f_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$, which is justified by well known decomposition techniques. We give also a different proof of the Franke embedding in the constant exponent case which avoids duality arguments and interpolation. Our results hold also for 2-microlocal function spaces $B_{p(\cdot),q(\cdot)}^{\boldsymbol{w}}(\mathbf{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{\boldsymbol{w}}(\mathbf{R}^n)$ which unify the smoothness scales of spaces of variable smoothness and generalized smoothness spaces.

1. Introduction

Spaces of variable integrability, also known as variable exponent function spaces $L_{p(\cdot)}(\mathbf{R}^n)$, can be traced back to Orlicz [36] 1931, but the modern development started with the papers [30] of Kováčik and Rákosník as well as [13] of Edmunds and Rákosník and [8] of Diening. The spaces $L_{p(\cdot)}(\mathbf{R}^n)$ have interesting applications in fluid dynamics, namely in the theory of electrorheological fluids [38], where $p(\cdot)$ is a function of the electric field. Further, these variable function spaces were used in image processing, PDEs and variational calculus, see the introduction of [11]. For an overview we refer to [10].

Sobolev and Besov spaces with variable smoothness but fixed integrability have been introduced in the late 60's and early 70's in the works of Unterberger [41], Višik and Eskin [43], Unterberger and Bokobza [42] and in the work of Beauzamy [4]. Leopold studied in [31] Besov spaces where the smoothness is determined by a symbol $a(x, \xi)$ of a certain class of hypoelliptic pseudodifferential operators. In the special case $a(x, \xi) = (1 + |\xi|^2)^{\sigma(x)/2}$ these spaces coincide with spaces of variable smoothness $B_{p,p}^{\sigma(x)}(\mathbf{R}^n)$.

A more general approach to spaces of variable smoothness are the so-called 2-microlocal function spaces $B_{p,q}^{\boldsymbol{w}}(\mathbf{R}^n)$ and $F_{p,q}^{\boldsymbol{w}}(\mathbf{R}^n)$. The smoothness in these scales gets measured by a weight sequence $\boldsymbol{w} = (w_j)_{j \in \mathbf{N}_0}$. Besov spaces with such weight sequences appeared first in the works of Peetre [37] and Bony [7]. Establishing a wavelet characterization for 2-microlocal Hölder–Zygmund spaces in [22] it turned out that 2-microlocal spaces are well adapted in connection to regularity properties of functions [23, 34, 33]. Spaces of variable smoothness are a special case of 2-microlocal function spaces and in [32] and [6] characterizations by differences have been given for certain classes of them.

Even in the case of constant exponents the integrability exponents $p, q \in (0, \infty]$ and the smoothness parameter s inherit a quite interesting interplay regarding embeddings and special cases, see [39] and [40].

If one considers at first Triebel–Lizorkin spaces $F_{p(\cdot),q}^{s}(\mathbf{R}^n)$ of [46], where only the integrability parameter $p(\cdot)$ is chosen to be variable, then it is convenient to choose the other exponents variable as well. This can already be seen by the Sobolev embedding from [45]

$$F_{p_0(\cdot),q}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow F_{p_1(\cdot),q}^{s_1(\cdot)}(\mathbf{R}^n)$$

under the usual condition, but now pointwise,

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Now also the smoothness parameter $s(\cdot)$ should be chosen variable. That also the third index $q(\cdot)$ should be variable can be seen by the following trace theorem. It was obtained by Diening, Hästö and Roudenko in [11] where variable smoothness and integrability were for the first time combined in one approach. They defined Triebel–Lizorkin spaces $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ and considered the trace theorem on $\mathbf{R}^{n-1}$. Here the usual result holds in a variable analogue

$$\operatorname{Tr} F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) = F_{p(\cdot),p(\cdot)}^{s(\cdot)-\frac{1}{p(\cdot)}}(\mathbf{R}^{n-1}), \text{ with } s(\cdot) - \frac{1}{p(\cdot)} > (n-1)\max\left(\frac{1}{p(\cdot)} - 1, 0\right),$$

[11, Theorem 3.13] and we see the necessity of taking s and q variable if p is not constant.

For the Besov spaces it is non-trivial to have also the parameter q as a variable one. Almeida and Hästö were able to introduce in [3] Besov spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ with all three indices variable and proved the Sobolev and other usual embeddings in this scale. These spaces need to be defined by using another modular which already uses the variable structure on $q(\cdot)$, see Section 2.

Interestingly, these variable Besov spaces fit very well to the constant exponent case theory which can be seen by the embedding

$$B_{p(\cdot),\min(p(\cdot),q(\cdot))}^{s(\cdot)}(\mathbf{R}^n) \hookrightarrow F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) \hookrightarrow B_{p(\cdot),\max(p(\cdot),q(\cdot))}^{s(\cdot)}(\mathbf{R}^n).$$

On the other hand, in [29] it has been shown that the triangle inequality in $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ is in general not true for exponents with $\min(p(\cdot),q(\cdot)) \geq 1$. This is in sharp contrast to the case of Triebel–Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and to the constant exponent spaces $B^s_{p,q}(\mathbf{R}^n)$ and $F^s_{p,q}(\mathbf{R}^n)$, which are always normed spaces if $\min(p(\cdot),q(\cdot)) \geq 1$, or $\min(p,q) \geq 1$, respectively.

For the full variable spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ different characterizations of the spaces as decompositions via atoms, molecules, local means and ball means of differences (see [11, 12, 28]) have been shown. Further, there also exist results on the extension operator from halfspaces in [35].

Furthermore, also for the more general scale of 2-microlocal function spaces $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ (see Section 5 for details) all the above mentioned characterizations have been obtained and there exist results on traces [17], pointwise multipliers [18] and Fourier multipliers [2].

Regarding Franke–Jawerth embeddings, they go back to Jawerth in [24] and Franke in [14]. Using interpolation techniques and duality, these authors proved the following.

Theorem 1.1. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 \leq \infty$ *and* $0 < q \leq \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

(i) *Then*

(1.1) $$F^{s_0}_{p_0,q}(\mathbf{R}^n) \hookrightarrow B^{s_1}_{p_1,p_0}(\mathbf{R}^n).$$

(ii) *If* $p_1 < \infty$, *then*

(1.2) $$B^{s_0}_{p_0,p_1}(\mathbf{R}^n) \hookrightarrow F^{s_1}_{p_1,q}(\mathbf{R}^n).$$

The surprising effect in (1.1) and (1.2) is that no conditions on q are necessary. Furthermore, the summability index of the Besov space in (1.1) must be equal (or larger than) p_0, i.e. the integrability index of the Triebel–Lizorkin space in (1.1). Similarly, the summability index of the Besov space in (1.2) has to be equal (or smaller than) p_1, i.e. again the integrability index of the Triebel–Lizorkin space in (1.2). This shows that Jawerth and Franke embeddings exploit the fine properties of Besov and Triebel–Lizorkin spaces and exhibit an interesting interplay between these two scales of function spaces.

Later Vybíral in [44] gave a new proof of Theorem 1.1. The author transferred the problem to the corresponding sequence spaces and, instead of interpolation, he used the technique of non-increasing rearrangements as well as duality. The developed technique via sequence spaces was also used in [21] to obtain the Franke–Jawerth embeddings in the Morrey space versions of Besov and Triebel–Lizorkin spaces and in [19] to get these embeddings for spaces with dominating mixed smoothness.

Our aim is to extend these results to the scale of Besov and Triebel–Lizorkin spaces with variable smoothness and integrability $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$, obtaining in this way a fine connection between those scales of function spaces. One can observe then that the somehow artificial definition of $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ seems to be well chosen.

The paper is organized as follows. We introduce in Section 2 the necessary notation and definitions which are needed afterwards. Furthermore, we also state some known theorems for the spaces with variable exponents. In Section 3 we present another proof for the Franke embedding in the constant exponent case. The novelty

of the technique here is that we totally avoid the use of interpolation and duality arguments. With the help of this result in the constant exponent case we state and prove the Jawerth and Franke embeddings in the scales of Besov $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ and Triebel–Lizorkin spaces $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ with variable exponents in Section 4. In Section 5 we transfer our results to 2-microlocal function spaces with variable exponents. Finally, in the last section we pose some open problems.

2. Notation and definitions

We shall adopt the following general notation: $\mathbf{N}$ denotes the set of all natural numbers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{Z}$ denotes the set of integers, $\mathbf{R}^n$ for $n \in \mathbf{N}$ denotes the n-dimensional real Euclidean space with $|x|$, for $x \in \mathbf{R}^n$, denoting the Euclidean norm of x.

For $q \in (0, \infty]$, ℓ_q stands for the linear space of all complex sequences $a = (a_j)_{j \in \mathbf{N}_0}$ endowed with the quasi-norm

$$\|a \mid \ell_q\| = \Big(\sum_{j=0}^{\infty} |a_j|^q \Big)^{1/q},$$

with the usual modification if $q = \infty$. By c, C, etc. we denote positive constants independent of appropriate quantities. For two non-negative expressions (i.e., functions or functionals) $\mathcal{A}$, $\mathcal{B}$, the symbol $\mathcal{A} \lesssim \mathcal{B}$ (or $\mathcal{A} \gtrsim \mathcal{B}$) means that $\mathcal{A} \leq c\,\mathcal{B}$ (or $c\,\mathcal{A} \geq \mathcal{B}$), for some $c > 0$. If $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{A} \gtrsim \mathcal{B}$, we write $\mathcal{A} \sim \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Before introducing the function spaces under consideration we still need to recall some notation. By $\mathcal{S}(\mathbf{R}^n)$ we denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and by $\mathcal{S}'(\mathbf{R}^n)$ its dual space of all tempered distributions on $\mathbf{R}^n$. For $f \in \mathcal{S}'(\mathbf{R}^n)$ we denote by $\widehat{f}$ the Fourier transform of f and by $f^{\vee}$ the inverse Fourier transform of f.

Let $\varphi_0 \in \mathcal{S}(\mathbf{R}^n)$ be such that

(2.3) $\varphi_0(x) = 1$ if $|x| \leq 1$ and   supp $\varphi_0 \subset \{x \in \mathbf{R}^n \colon |x| \leq 2\}$.

Now define $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) := \varphi(2^{-j}x)$ for all $j \in \mathbf{N}$. Then the sequence $(\varphi_j)_{j \in \mathbf{N}_0}$ forms a smooth dyadic partition of unity.

By $\mathcal{P}(\mathbf{R}^n)$ we denote the class of exponents, which are measurable functions $p \colon \mathbf{R}^n \to (c, \infty]$ for some $c > 0$. Let $p \in \mathcal{P}(\mathbf{R}^n)$. Then, $p^+ := \text{ess-sup}_{x \in \mathbf{R}^n} p(x)$, $p^- := \text{ess-inf}_{x \in \mathbf{R}^n} p(x)$ and $L_{p(\cdot)}(\mathbf{R}^n)$ is the variable exponent Lebesgue space, which consists of all measurable functions f such that for some $\lambda > 0$ the modular $\varrho_{p(\cdot)}(f/\lambda)$ is finite, where

$$\varrho_{p(\cdot)}(f) := \int_{\mathbf{R}_0^n} |f(x)|^{p(x)} \, dx + \text{ess-sup}_{x \in \mathbf{R}_\infty^n} |f(x)|.$$

Here $\mathbf{R}_\infty^n$ denotes the subset of $\mathbf{R}^n$ where $p(x) = \infty$ and $\mathbf{R}_0^n = \mathbf{R}^n \setminus \mathbf{R}_\infty^n$. The Luxemburg norm of a function $f \in L_{p(\cdot)}(\mathbf{R}^n)$ is given by

$$\|f \mid L_{p(\cdot)}(\mathbf{R}^n)\| := \inf \left\{ \lambda > 0 \colon \varrho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

In order to define the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)})$, we need to define another modular. For $p, q \in \mathcal{P}(\mathbf{R}^n)$ and a sequence $(f_\nu)_{\nu \in \mathbf{N}_0}$ of complex-valued Lebesgue measurable

functions on $\mathbf{R}^n$, we define

$$(2.4) \qquad \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \inf \left\{ \lambda_\nu > 0 \colon \varrho_{p(\cdot)}\left(\frac{f_\nu}{\lambda_\nu^{1/q(\cdot)}} \right) \leq 1 \right\}.$$

If $q^+ < \infty$, then we can replace (2.4) by the simpler expression

$$(2.5) \qquad \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}(f_\nu) = \sum_{\nu=0}^{\infty} \left\| |f_\nu|^{q(\cdot)} \mid L_{\frac{p(\cdot)}{q(\cdot)}}(\mathbf{R}^n) \right\|.$$

The (quasi-)norm in the $\ell_{q(\cdot)}(L_{p(\cdot)})$ spaces is defined as usual by

$$(2.6) \qquad \| f_\nu \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n)) \| = \inf \left\{ \mu > 0 \colon \varrho_{\ell_{q(\cdot)}(L_{p(\cdot)})}\left(\frac{f_\nu}{\mu} \right) \leq 1 \right\}.$$

For the sake of completeness, we state also the definition of the space $L_{p(\cdot)}(\ell_{q(\cdot)})$. At first, one just takes the norm $\ell_{q(\cdot)}$ of $(f_\nu(x))_{\nu \in \mathbf{N}_0}$ for every $x \in \mathbf{R}^n$ and then the $L_{p(\cdot)}$-norm with respect to $x \in \mathbf{R}^n$, i.e.

$$(2.7) \qquad \| f_\nu \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbf{R}^n)) \| = \left\| \left(\sum_{\nu=0}^{\infty} |f_\nu(x)|^{q(x)} \right)^{1/q(x)} \mid L_{p(\cdot)}(\mathbf{R}^n) \right\|.$$

The following regularity classes for the exponents are sufficient to make the definition of the spaces independent on the chosen decomposition of unity.

Definition 2.1. Let $g \in C(\mathbf{R}^n)$. We say that g is locally log-Hölder continuous, abbreviated $g \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$, if there exists $c_{\log}(g) > 0$ such that

$$(2.8) \qquad |g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x-y|)} \quad \text{for all } x, y \in \mathbf{R}^n.$$

We say that g is globally log-Hölder continuous, abbreviated $g \in C^{\log}(\mathbf{R}^n)$, if g is locally log-Hölder continuous and there exists $g_\infty \in \mathbf{R}$ such that

$$(2.9) \qquad |g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)} \quad \text{for all } x \in \mathbf{R}^n.$$

We use the notation $p \in \mathcal{P}^{\log}(\mathbf{R}^n)$ if $p \in \mathcal{P}(\mathbf{R}^n)$ and $1/p \in C^{\log}(\mathbf{R}^n)$. It was proved in [9] that the maximal operator $\mathcal{M}$ is bounded in $L_{p(\cdot)}(\mathbf{R}^n)$ provided that $p \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $1 < p^- \leq p^+ \leq \infty$.

We recall the definition of the spaces $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ and $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$, as given in [11] and [3].

Definition 2.2. Let $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $s \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$.

(i) If $p^+, q^+ < \infty$, then the space $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\| f \mid F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) \| := \left\| \left(2^{js(\cdot)}(\varphi_j \widehat{f})^\vee \right)_{j \in \mathbf{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbf{R}^n)) \right\|$$

is finite.

(ii) The space $B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\| f \mid B_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) \| := \left\| \left(2^{js(\cdot)}(\varphi_j \widehat{f})^\vee \right)_{j \in \mathbf{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n)) \right\|$$

is finite.

Remark 2.3. The independence of the resolution of unity in the definition of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ can be justified by characterizations of the spaces, for example by local means (see [26] for Triebel–Lizorkin and [28] for Besov spaces), if $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$.

Remark 2.4. These spaces include very well known spaces. In particular, if $p(\cdot) = p$, $q(\cdot) = q$ and $s(\cdot) = s$ are constants, we get back to the classical Besov and Triebel–Lizorkin spaces $B^s_{p,q}(\mathbf{R}^n)$ and $F^s_{p,q}(\mathbf{R}^n)$.

The spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ are isomorphic to sequence spaces. The underlying theorems are characterizations of the spaces above by atoms, wavelets and the φ-transform. The assertions can be found in [11], [27] and [12].

In this work we opt to make use of the atomic characterization and to this end we shall introduce some notation. Let $\mathbf{Z}^n$ stand for the lattice of all points in $\mathbf{R}^n$ with integer-valued components, $Q_{j,m}$ denotes a cube in $\mathbf{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m = (2^{-j}m_1, \ldots, 2^{-j}m_n)$ and with side length 2^{-j}, where $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$ and $j \in \mathbf{N}_0$. If Q is a cube in $\mathbf{R}^n$ and $r > 0$ then rQ is the cube in $\mathbf{R}^n$ concentric with Q and with side length r times the side length of Q. By χ_E we denote the characteristic function of the measurable set E. However, when E is the cube $Q_{j,m}$, the characteristic function of $Q_{j,m}$ is simply denoted by $\chi_{j,m}$.

For $p, q \in \mathcal{P}(\mathbf{R}^n)$, we put

$$\sigma_p := n \left(\frac{1}{\min(1, p^-)} - 1 \right) \quad \text{and} \quad \sigma_{p,q} := n \left(\frac{1}{\min(1, p^-, q^-)} - 1 \right).$$

We present now the definition of the atoms, which are the building blocks for the atomic decomposition, as well as the definition of the sequence spaces.

Definition 2.5. Let $K, L \in \mathbf{N}_0$ and let $d > 1$. A K-times continuously differentiable complex-valued function $a \in C^K(\mathbf{R}^n)$ is called $[K, L]$-atom centered at $Q_{j,m}$, for all $j \in \mathbf{N}_0$ and $m \in \mathbf{Z}^n$, if

$$(2.10) \qquad\qquad \operatorname{supp} a \subset d\, Q_{j,m},$$

$$(2.11) \qquad\qquad |D^\beta a(x)| \leq 2^{|\beta|j} \quad \text{for} \quad |\beta| \leq K$$

and

$$(2.12) \qquad\qquad \int_{\mathbf{R}^n} x^\beta a(x)\, dx = 0 \quad \text{for} \quad 0 \leq |\beta| < L \quad \text{and} \quad j \geq 1.$$

Remark 2.6. If an atom a is centered at $Q_{j,m}$, i.e., if it fulfills (2.10), then we denote it by $a_{j,m}$. If $L = 0$ or $j = 0$, then no moment conditions (2.12) are required.

Definition 2.7. Let $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$.

(i) If $p^+ < \infty$, then the sequence space $f^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ consists of those complex-valued sequences $\lambda = (\lambda_{j,m})_{j \in \mathbf{N}_0, m \in \mathbf{Z}^n}$ such that

$$\|\lambda \mid f^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| := \left\| \left(\sum_{m \in \mathbf{Z}^n} |\lambda_{j,m}|\, 2^{js(\cdot)} \chi_{j,m} \right)_{j \in \mathbf{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbf{R}^n)) \right\|$$

is finite.

(ii) The sequence space $b^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ consists of those complex-valued sequences $\lambda = (\lambda_{j,m})_{j\in\mathbf{N}_0,m\in\mathbf{Z}^n}$ such that

$$\|\lambda \mid b^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| := \left\| \left(\sum_{m\in\mathbf{Z}^n} |\lambda_{j,m}| \, 2^{js(\cdot)} \chi_{j,m} \right)_{j\in\mathbf{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n)) \right\|$$

is finite.

At this point we can state the atomic decomposition theorems for the spaces under consideration. The result for Triebel–Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ was firstly proved in [11], but assuming $s(\cdot) \geq 0$ and $s(\cdot)$ with a limit at infinity. This result was later extended in [27] for functions $s : \mathbf{R}^n \to \mathbf{R}$, and reads as follows.

Theorem 2.8. *Let $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$ with $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. Furthermore, let $d > 1$, $K, L \in \mathbf{N}_0$ with $K > s^+$ and $L > \sigma_{p,q} - s^-$ be fixed. Then every $f \in \mathcal{S}'(\mathbf{R}^n)$ belongs to $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ if, and only if, it can be represented as*

$$(2.13) \qquad f = \sum_{j=0}^{\infty} \sum_{m\in\mathbf{Z}^n} \lambda_{j,m} \, a_{j,m}, \quad \text{convergence being in } \mathcal{S}'(\mathbf{R}^n),$$

for $(a_{j,m})_{j\in\mathbf{N}_0,m\in\mathbf{Z}^n}$ $[K,L]$-atoms according to Definition 2.5 and $\lambda \in f^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$. Moreover,

$$\|f \mid F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| \sim \inf \|\lambda \mid f^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\|,$$

where the infimum is taken over all possible representations (2.13) of f.

As for the Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$, an atomic representation theorem was derived in [12] for $q^+ < \infty$. This condition was suppressed in [1], where the authors proved the corresponding result for the (more general) scale of 2-microlocal spaces with variable exponents $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$, but with a stronger condition on L. Due to our needs, we opt by using the result from [12], which we state below.

Theorem 2.9. *Let $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ with $q^+ < \infty$ and $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$. Furthermore, let $d > 1$, $K, L \in \mathbf{N}_0$ with $K > s^+$ and $L > \sigma_p - 1 - s^-$ be fixed. Then every $f \in \mathcal{S}'(\mathbf{R}^n)$ belongs to $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ if, and only if, it can be represented as*

$$(2.14) \qquad f = \sum_{j=0}^{\infty} \sum_{m\in\mathbf{Z}^n} \lambda_{j,m} \, a_{j,m}, \quad \text{convergence being in } \mathcal{S}'(\mathbf{R}^n),$$

for $(a_{j,m})_{j\in\mathbf{N}_0,m\in\mathbf{Z}^n}$ $[K,L]$-atoms according to Definition 2.5 and $\lambda \in b^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$. Moreover,

$$\|f \mid B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| \sim \inf \|\lambda \mid b^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\|,$$

where the infimum is taken over all possible representations (2.14) of f.

To prove our main results, we will also make use of the Sobolev embedding for $b^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$, proved in [3]. The counterpart for $f^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ was proved by Vybíral in [45].

Theorem 2.10. *Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ and $s_0, s_1 \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$. Let $s_0(x) \geq s_1(x)$ and $p_0(x) \leq p_1(x)$ for all $x \in \mathbf{R}^n$ with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Then we have

$$b_{p0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow b_{p1(\cdot),q(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n).$$

3. Franke embedding - constant exponents case

The Franke embedding (1.2) was shown in [14] using duality and interpolation. An alternative proof avoiding interpolation was given in [44]. The main tools used were the technique of non-increasing rearrangements and duality. Here, we give a new proof of the Franke embedding for the scale of spaces with constant exponents, which still relies on non-increasing rearrangements, but we avoid using duality.

We start by introducing the concept of non-increasing rearrangement and some of its important properties. We refer to [5] to an extensive treatment of this subject.

Definition 3.1. Let μ be the Lebesgue measure in $\mathbf{R}^n$. If h is a measurable function on $\mathbf{R}^n$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\left\{\lambda > 0 \colon \mu\{x \in \mathbf{R}^n \colon |h(x)| > \lambda\} > t\right\}, \quad t \in (0, \infty).$$

The following lemma can be found in [5, Chapter 2, Prop. 1.8].

Lemma 3.2. *If $0 < p \leq \infty$, then*

$$\|h \mid L_p(\mathbf{R}^n)\| = \|h^* \mid L_p(0, \infty)\|$$

for every measurable function h.

The next lemma is a simple corollary of results included in [5]. We give a proof for reader's convenience.

Lemma 3.3. *Let $(h_j)_{j \in \mathbf{N}_0}$ be a sequence of non-negative measurable functions on $\mathbf{R}^n$. If $1 \leq p \leq \infty$, then*

$$\left\|\sum_{j=0}^{\infty} h_j \mid L_p(\mathbf{R}^n)\right\| \leq \left\|\sum_{j=0}^{\infty} h_j^* \mid L_p(0, \infty)\right\|.$$

Proof. By [5, Chapter 2, Theorem 3.4],

$$\int_0^t (h_0 + h_1)^*(u)du \leq \int_0^t h_0^*(u)du + \int_0^t h_1^*(u)du$$

holds for every $0 < t < \infty$. By induction, this can be further generalized to every finite number of functions and, by monotone convergence theorem, also to infinite number of terms, i.e.

$$(3.15) \quad \int_0^t \left(\sum_{j=0}^{\infty} h_j\right)^*(u)du \leq \sum_{j=0}^{\infty} \int_0^t h_j^*(u)du = \int_0^t \left(\sum_{j=0}^{\infty} h_j^*(u)\right)du, \qquad 0 < t < \infty.$$

Lemma 3.2 then gives

$$\left\|\sum_{j=0}^{\infty} h_j \mid L_p(\mathbf{R}^n)\right\| = \left\|\left(\sum_{j=0}^{\infty} h_j\right)^* \mid L_p(0, \infty)\right\|$$

and the result follows by (3.15) and the Hardy–Littlewood–Pólya Principle, cf. [5, Chapter 2, Corollary 4.7]. $\square$

The main result of this section is an alternative proof of the sequence space version of (1.2). In the contrary to [44], it avoids the use of duality.

Theorem 3.4. *Let* $-\infty < s_1 < s_0 < \infty$, $0 < p_0 < p_1 < \infty$ *and* $0 < q \leq \infty$ *with*

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then

$$b_{p_0,p_1}^{s_0}(\mathbf{R}^n) \hookrightarrow f_{p_1,q}^{s_1}(\mathbf{R}^n).$$

Proof. By lifting properties it is enough to show the Franke embedding with $s_1 = 0$, i.e.

$$(3.16) \qquad b_{p_0,p_1}^{n\left(\frac{1}{p_0} - \frac{1}{p_1}\right)}(\mathbf{R}^n) \hookrightarrow f_{p_1,q}^0(\mathbf{R}^n).$$

Plugging in Definition 2.7, we get

$$(3.17) \qquad \|\gamma|f_{p_1,q}^0(\mathbf{R}^n)\| = \left\|\left(\sum_{j=0}^{\infty}\sum_{m\in\mathbf{Z}^n}|\gamma_{j,m}|^q\chi_{j,m}(\cdot)\right)^{\frac{1}{q}}|L_{p_1}(\mathbf{R}^n)\right\|$$

and

$$(3.18) \qquad \|\gamma|b_{p_0,p_1}^{n\left(\frac{1}{p_0} - \frac{1}{p_1}\right)}(\mathbf{R}^n)\| = \left(\sum_{j=0}^{\infty}2^{-jn}\left(\sum_{m\in\mathbf{Z}^n}|\gamma_{j,m}|^{p_0}\right)^{\frac{p_1}{p_0}}\right)^{\frac{1}{p_1}}.$$

To prove (3.16), we will use $0 < p_0 < p_1 < \infty$ and also $0 < q \leq \min(1, p_0)$, since it follows by elementary embeddings that we can take q arbitrarily small.

We write

$$\|\gamma|f_{p_1,q}^0(\mathbf{R}^n)\| = \left\|\left(\sum_{j=0}^{\infty}\sum_{m\in\mathbf{Z}^n}|\gamma_{j,m}|^q\chi_{j,m}(\cdot)\right)^{\frac{1}{q}}|L_{p_1}(\mathbf{R}^n)\right\|$$

$$= \left\|\sum_{j=0}^{\infty}\sum_{m\in\mathbf{Z}^n}|\gamma_{j,m}|^q\chi_{j,m}(\cdot)|L_{\frac{p_1}{q}}(\mathbf{R}^n)\right\|^{1/q}$$

$$\leq \left\|\sum_{j=0}^{\infty}\left(\sum_{m\in\mathbf{Z}^n}|\gamma_{j,m}|^q\chi_{j,m}\right)^*(\cdot)|L_{\frac{p_1}{q}}(0,\infty)\right\|^{1/q}$$

$$= \left\|\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}(\gamma_{j,l}^*)^q\tilde{\chi}_{j,l}(\cdot)|L_{\frac{p_1}{q}}(0,\infty)\right\|^{1/q}.$$

Here, we have used Lemma 3.3 due to $\frac{p_1}{q} \geq 1$ and $\tilde{\chi}_{j,l}$ is the characteristic function of $[2^{-jn}l, 2^{-jn}(l+1))$. Furthermore, $(\gamma_{j,l}^*)_{l=0}^{\infty}$ is a non-increasing rearrangement of $(|\gamma_{j,m}|)_{m\in\mathbf{Z}^n}$, i.e. a non-increasing sequence of non-negative real numbers, such that the sets

$$\{m \in \mathbf{Z}^n \colon |\gamma_{j,m}| > t\} \quad \text{and} \quad \{l \in \mathbf{N}_0 \colon \gamma_{j,l}^* > t\}$$

have the same cardinality for all $t > 0$. Formally, it can be defined for every $j \in \mathbf{N}_0$ by

$$\gamma_{j,l}^* = \sup\{\lambda > 0 \colon \#\{m \in \mathbf{Z}^n \colon |\gamma_{j,m}| > \lambda\} > l\}, \qquad l \in \mathbf{N}_0,$$

where $\#A$ denotes the number of elements of the set A.

We discretize the last norm, i.e. we use that

$$\|\varphi|L_r(0,\infty)\| \sim \left(\sum_{k=-\infty}^{\infty}2^{-kn}\varphi(2^{-kn})^r\right)^{1/r}$$

holds for $1 \leq r < \infty$ and every non-negative non-increasing function φ. Further, we observe that $\tilde{\chi}_{j,l}(2^{-kn}) = 1$ only for $l = 2^{(j-k)n}$ if $j - k \geq 0$ and for $l = 0$ if $j - k < 0$.

This can be summarized as $l = \lfloor 2^{(j-k)n} \rfloor$, where $\lfloor x \rfloor$ is the integer part of a real number x.

This leads to

$$\|\gamma|f^0_{p_1,q}(\mathbf{R}^n)\| \lesssim \Big(\sum_{k=-\infty}^{\infty} 2^{-kn} \Big[\Big(\sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (\gamma^*_{j,l})^q \tilde{\chi}_{j,l}(\cdot) \Big) (2^{-kn}) \Big]^{\frac{p_1}{q}} \Big)^{\frac{1}{p_1}}$$

$$= \Big(\sum_{k=-\infty}^{\infty} 2^{-kn} \Big(\sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (\gamma^*_{j,l})^q \tilde{\chi}_{j,l}(2^{-kn}) \Big)^{\frac{p_1}{q}} \Big)^{\frac{1}{p_1}}$$

$$= \Big(\sum_{k=-\infty}^{\infty} 2^{-kn} \Big(\sum_{j=0}^{\infty} (\gamma^*_{j,\lfloor 2^{(j-k)n} \rfloor})^q \Big)^{\frac{p_1}{q}} \Big)^{\frac{1}{p_1}}$$

$$\lesssim \Big[\sum_{k=-\infty}^{0} 2^{-kn} \Big(\sum_{j=0}^{\infty} (\gamma^*_{j,2^{(j-k)n}})^q \Big)^{\frac{p_1}{q}} \Big]^{\frac{1}{p_1}} + \Big[\sum_{k=1}^{\infty} 2^{-kn} \Big(\sum_{j=0}^{k-1} (\gamma^*_{j,0})^q \Big)^{\frac{p_1}{q}} \Big]^{\frac{1}{p_1}}$$

$$+ \Big[\sum_{k=1}^{\infty} 2^{-kn} \Big(\sum_{j=k}^{\infty} (\gamma^*_{j,2^{(j-k)n}})^q \Big)^{\frac{p_1}{q}} \Big]^{\frac{1}{p_1}} = I + II + III.$$

We estimate all the three terms separately.

The first term can be estimated in the following way. Starting with Hölder's inequality with $p_0/q \geq 1$ for the sum over j and with $\beta > 0$, we get

$$I = \Big[\sum_{k=-\infty}^{0} 2^{-kn} \Big(\sum_{j=0}^{\infty} (\gamma^*_{j,2^{(j-k)n}})^q \Big)^{\frac{p_1}{q}} \Big]^{\frac{1}{p_1}}$$

$$\lesssim \Big[\sum_{k=-\infty}^{0} \Big(\sum_{j=0}^{\infty} 2^{-kn\frac{p_0}{p_1}} 2^{jn\beta\frac{p_0}{q}} (\gamma^*_{j,2^{(j-k)n}})^{p_0} \Big)^{\frac{p_1}{p_0}} \Big]^{\frac{p_0}{p_1}\frac{1}{p_0}}.$$

Now, after using the triangle inequality with $\frac{p_1}{p_0} > 1$ and the embedding $\ell_{\frac{p_0}{p_1}} \hookrightarrow \ell_1$ in the sum over k, we substitute $l = j - k$ and get

$$I \lesssim \Big[\sum_{j=0}^{\infty} \Big(\sum_{k=-\infty}^{0} 2^{-kn} 2^{jn\beta\frac{p_1}{q}} (\gamma^*_{j,2^{(j-k)n}})^{p_1} \Big)^{\frac{p_0}{p_1}} \Big]^{\frac{1}{p_0}}$$

$$\leq \Big[\sum_{j=0}^{\infty} \sum_{k=-\infty}^{0} 2^{-kn\frac{p_0}{p_1}} 2^{jn\beta\frac{p_0}{q}} (\gamma^*_{j,2^{(j-k)n}})^{p_0} \Big]^{\frac{1}{p_0}}$$

$$= \Big[\sum_{j=0}^{\infty} 2^{jn\beta\frac{p_0}{q}} \sum_{l=j}^{\infty} 2^{(l-j)n\frac{p_0}{p_1}} (\gamma^*_{j,2^{ln}})^{p_0} \Big]^{\frac{1}{p_0}}$$

$$= \Big[\sum_{j=0}^{\infty} 2^{jn(\beta\frac{p_0}{q} - \frac{p_0}{p_1})} \sum_{l=j}^{\infty} 2^{ln} 2^{ln(\frac{p_0}{p_1}-1)} (\gamma^*_{j,2^{ln}})^{p_0} \Big]^{\frac{1}{p_0}}.$$

Since $\frac{p_0}{p_1} - 1 < 0$, we have

$$I \lesssim \Big[\sum_{j=0}^{\infty} 2^{jn(\beta\frac{p_0}{q} - \frac{p_0}{p_1})} 2^{jn(\frac{p_0}{p_1}-1)} \sum_{l=j}^{\infty} 2^{ln} (\gamma^*_{j,2^{ln}})^{p_0} \Big]^{\frac{1}{p_0}}$$

$$= \Big[\sum_{j=0}^{\infty} 2^{-jn\frac{p_0}{p_1}} 2^{jn(\beta\frac{p_0}{q} + \frac{p_0}{p_1}-1)} \sum_{l=j}^{\infty} 2^{ln} (\gamma^*_{j,2^{ln}})^{p_0} \Big]^{\frac{1}{p_0}}.$$

We finish this estimate choosing β with $\beta\frac{p_0}{q}+\frac{p_0}{p_1}-1<0$, i.e., $0<\beta<q(\frac{1}{p_0}-\frac{1}{p_1})$ and get with (3.18)

$$I \lesssim \Big[\sum_{j=0}^{\infty}2^{-jn}\Big(\sum_{l=j}^{\infty}2^{ln}(\gamma_{j,2^{ln}}^*)^{p_0}\Big)^{\frac{p_1}{p_0}}\Big]^{\frac{1}{p_1}} \lesssim \|\gamma|b_{p_0,p_1}^{n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}(\mathbf{R}^n)\|.$$

To estimate the second term, we start by using Hölder's inequality with $p_1/q>1$ for the sum over j with $\beta>0$

$$II^{p_1} \lesssim \sum_{k=1}^{\infty}2^{-kn}2^{kn\beta\frac{p_1}{q}}\sum_{j=0}^{k-1}2^{-jn\beta\frac{p_1}{q}}(\gamma_{j,0}^*)^{p_1}$$

$$=\sum_{j=0}^{\infty}2^{-jn\beta\frac{p_1}{q}}(\gamma_{j,0}^*)^{p_1}\sum_{k=j+1}^{\infty}2^{kn(\beta\frac{p_1}{q}-1)}.$$

Now, choosing $\beta>0$ with $\beta\frac{p_1}{q}-1<0$, i.e., $0<\beta<\frac{q}{p_1}$, we get again by (3.18)

$$II^{p_1} \lesssim \sum_{j=0}^{\infty}2^{-jn\beta\frac{p_1}{q}}(\gamma_{j,0}^*)^{p_1}2^{jn\beta\frac{p_1}{q}}2^{-jn}$$

$$=\sum_{j=0}^{\infty}(\gamma_{j,0}^*)^{p_1}2^{-jn} \lesssim \|\gamma|b_{p_0,p_1}^{n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}(\mathbf{R}^n)\|^{p_1}.$$

Finally, we estimate the third term starting again by using a parameter $\delta>0$ and Hölder's inequality

$$III^{p_1}=\sum_{k=1}^{\infty}2^{-kn}\Big(\sum_{j=k}^{\infty}2^{jn\delta}2^{-jn\delta}(\gamma_{j,2^{(j-k)n}}^*)^q\Big)^{\frac{p_1}{q}}$$

$$\lesssim \sum_{k=1}^{\infty}2^{-kn}2^{-kn\delta\frac{p_1}{q}}\sum_{j=k}^{\infty}2^{jn\delta\frac{p_1}{q}}(\gamma_{j,2^{(j-k)n}}^*)^{p_1}$$

$$=\sum_{j=1}^{\infty}2^{jn\delta\frac{p_1}{q}}\sum_{k=1}^{j}2^{-kn(1+\delta\frac{p_1}{q})}(\gamma_{j,2^{(j-k)n}}^*)^{p_1}.$$

We substitute $l=j-k$ and obtain

$$III^{p_1} \lesssim \sum_{j=1}^{\infty}2^{jn\delta\frac{p_1}{q}}\sum_{l=0}^{j-1}2^{(l-j)n(1+\delta\frac{p_1}{q})}(\gamma_{j,2^{ln}}^*)^{p_1}$$

$$=\sum_{j=1}^{\infty}2^{-jn}\Big[\sum_{l=0}^{j-1}2^{ln(\frac{q}{p_1}+\delta)\frac{p_1}{q}}(\gamma_{j,2^{ln}}^*)^{p_1}\Big]$$

$$\leq \sum_{j=1}^{\infty}2^{-jn}\Big[\sum_{l=0}^{j-1}2^{ln(\frac{q}{p_1}+\delta)\frac{p_0}{q}}(\gamma_{j,2^{ln}}^*)^{p_0}\Big]^{\frac{p_1}{p_0}},$$

where the last step comes from the elementary embedding $\ell_{p_0}\hookrightarrow\ell_{p_1}$. Choosing now $\delta>0$ such that $(\frac{q}{p_1}+\delta)\frac{p_0}{q}=1$, i.e. $\delta=\frac{q}{p_0}-\frac{q}{p_1}>0$, we get

$$III^{p_1} \lesssim \sum_{j=1}^{\infty}2^{-jn}\Big[\sum_{l=0}^{j-1}2^{ln}(\gamma_{j,2^{ln}}^*)^{p_0}\Big]^{\frac{p_1}{p_0}} \lesssim \|\gamma|b_{p_0,p_1}^{n\left(\frac{1}{p_0}-\frac{1}{p_1}\right)}(\mathbf{R}^n)\|^{p_1},$$

which concludes the proof. $\qquad\square$

4. Franke–Jawerth embeddings - variable exponent case

We now return to the scale of Besov and Triebel–Lizorkin spaces with variable exponents. First, we state the result in the form of embeddings of sequence spaces under the assumptions we really need in the proof. After that, we combine those with the conditions required in Theorems 2.9 and 2.8, and we present the results on function spaces in the form of a corollary.

Before we come to the main aim of this section, we define a certain lifting operator between sequence spaces and establish its basic properties.

Lemma 4.1. *Let* $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ *and let* $s, \sigma \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$. *Then the lifting operator* $I_{\sigma(\cdot)}$ *given by*

$$(I_{\sigma(\cdot)}\lambda)_{j,m} = 2^{j\sigma(2^{-j}m)}\lambda_{j,m}, \quad j \in \mathbf{N}_0, \quad m \in \mathbf{Z}^n,$$

is an isomorphism (i.e. bounded linear operator with bounded inverse) of $b_{p(\cdot),q(\cdot)}^{s(\cdot)}$ *onto* $b_{p(\cdot),q(\cdot)}^{s(\cdot)-\sigma(\cdot)}$ *and, if* $p^+ < \infty$, *also of* $f_{p(\cdot),q(\cdot)}^{s(\cdot)}$ *onto* $f_{p(\cdot),q(\cdot)}^{s(\cdot)-\sigma(\cdot)}$.

Proof. First of all we observe that for $s \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$ the norms of the sequence spaces from Definition 2.7 can equivalently be expressed by

$$(4.19) \qquad \|\lambda \mid b_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)\| \sim \left\| \left(\sum_{m \in \mathbf{Z}^n} |\lambda_{j,m}| \, 2^{js(2^{-j}m)} \chi_{j,m} \right)_{j \in \mathbf{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbf{R}^n)) \right\|$$

and

$$\|\lambda \mid f_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)\| \sim \left\| \left(\sum_{m \in \mathbf{Z}^n} |\lambda_{j,m}| \, 2^{js(2^{-j}m)} \chi_{j,m} \right)_{j \in \mathbf{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n)) \right\|.$$

This simply follows from Lemma 4.1.6 in [10] which gives for $s \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$

$$(4.20) \qquad\qquad 2^{js(x)} \sim 2^{js(2^{-j}m)} \quad \text{for all } x \in Q_{j,m}.$$

Using the definition of $I_{\sigma(\cdot)}$ and (4.19) twice leads to

$$\|I_{\sigma(\cdot)}\lambda \mid b_{p(\cdot),q(\cdot)}^{s(\cdot)-\sigma(\cdot)}(\mathbf{R}^n)\|$$
$$\sim \left\| \left(\sum_{m \in \mathbf{Z}^n} 2^{js(2^{-j}m)-j\sigma(2^{-j}m)} 2^{j\sigma(2^{-j}m)} |\lambda_{j,m}|\chi_{j,m} \right)_{j \in \mathbf{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n)) \right\|$$
$$\sim \|\lambda \mid b_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)\|.$$

The proof in the f-case follows in the same way. $\qquad\square$

We shall use the lifting operator in the connection with embeddings of sequence spaces.

Lemma 4.2. *Let* $p, \tilde{p}, q, \tilde{q} \in \mathcal{P}^{\log}(\mathbf{R}^n)$, $s, \tilde{s} \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$ *and* $a, \tilde{a} \in \{b, f\}$. *We assume that* $p^+ < \infty$ *if* a *equals* f *and* $\tilde{p}^+ < \infty$ *if* $\tilde{a}$ *equals* f. *Then* $a_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) \hookrightarrow \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot)}(\mathbf{R}^n)$ *if, and only if,* $a_{p(\cdot),q(\cdot)}^{s(\cdot)-\tilde{s}(\cdot)}(\mathbf{R}^n) \hookrightarrow \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{0}(\mathbf{R}^n)$.

Proof. The proof follows by standard arguments involving two applications of Lemma 4.1. Let, for example, $a_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n) \hookrightarrow \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot)}(\mathbf{R}^n)$. Then

$$\|\lambda \mid \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{0}(\mathbf{R}^n)\| = \|I_{\tilde{s}(\cdot)} \circ I_{-\tilde{s}(\cdot)}(\lambda) \mid \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{0}(\mathbf{R}^n)\| \lesssim \|I_{-\tilde{s}(\cdot)}(\lambda) \mid \tilde{a}_{\tilde{p}(\cdot),\tilde{q}(\cdot)}^{\tilde{s}(\cdot)}(\mathbf{R}^n)\|$$
$$\lesssim \|I_{-\tilde{s}(\cdot)}(\lambda) \mid a_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)\| \lesssim \|\lambda \mid a_{p(\cdot),q(\cdot)}^{s(\cdot)-\tilde{s}(\cdot)}(\mathbf{R}^n)\|.$$

The other direction follows in the same way. $\qquad\square$

4.1. Jawerth embedding.

Theorem 4.3. *Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ with $p_0^+ < \infty$ and $s_0, s_1 \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$. Let $\inf_{x \in \mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ with*

$$(4.21) \qquad s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Then

$$(4.22) \qquad f_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow b_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n).$$

Proof. Let us put

$$\varepsilon' = \inf_{x \in \mathbf{R}^n}(s_0(x) - s_1(x)) = \inf_{x \in \mathbf{R}^n}\left(\frac{n}{p_0(x)} - \frac{n}{p_1(x)}\right) > 0.$$

Then

$$\frac{p_1(x)}{p_0(x)} - 1 = p_1(x)\left(\frac{1}{p_0(x)} - \frac{1}{p_1(x)}\right) \geq p_1(x)\frac{\varepsilon'}{n} \geq \frac{p_1^- \varepsilon'}{n}.$$

Putting $\varepsilon = \frac{p_1^- \varepsilon'}{2n} > 0$ we get for every $x \in \mathbf{R}^n$

$$(4.23) \qquad p_0(x) < (1+\varepsilon)p_0(x) < p_1(x).$$

The proof of (4.22) will be the result of the following chain of embeddings

$$(4.24) \qquad \begin{aligned} f_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) &\hookrightarrow f_{p_0(\cdot),\infty}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow b_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}^{s_0(\cdot) - \frac{n}{p_0(\cdot)} + \frac{n}{(1+\varepsilon)p_0(\cdot)}}(\mathbf{R}^n) \\ &= b_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}^{s_1(\cdot) - \frac{n}{p_1(\cdot)} + \frac{n}{(1+\varepsilon)p_0(\cdot)}}(\mathbf{R}^n) \hookrightarrow b_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n). \end{aligned}$$

The first embedding in (4.24) is an elementary statement about the monotonicity of f-spaces in the summability index q and the last embedding follows from Theorem 2.10 and (4.23). The identity in (4.24) is a simple consequence of (4.21). Hence, it remains to prove the second embedding in (4.24), which is actually a special case of (4.22) with $q = \infty$ and $p_1(x) = (1+\varepsilon)p_0(x)$. Finally, by the lifting property from Lemma 4.2, we may consider only the case when the smoothness exponent of the target space is zero.

The proof of (4.22) will therefore follow from

$$(4.25) \qquad f_{p_0(\cdot),\infty}^{\frac{n}{p_0(\cdot)} \cdot \frac{\varepsilon}{1+\varepsilon}}(\mathbf{R}^n) \hookrightarrow b_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}^{0}(\mathbf{R}^n)$$

or, equivalently,

$$(4.26) \qquad \|\gamma \mid b_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}^{0}(\mathbf{R}^n)\| \leq C \,\|\gamma \mid f_{p_0(\cdot),\infty}^{\frac{n}{p_0(\cdot)} \cdot \frac{\varepsilon}{1+\varepsilon}}(\mathbf{R}^n)\|,$$

for some constant $C > 0$ and $\gamma = (\gamma_{j,m})_{j,m}$, for $j \in \mathbf{N}_0, m \in \mathbf{Z}^n$.

Let us put

$$h(x) = \sup_{j,m} 2^{\frac{jn}{p_0(x)} \cdot \frac{\varepsilon}{1+\varepsilon}} |\gamma_{j,m}| \chi_{j,m}(x), \quad x \in \mathbf{R}^n.$$

Then for every $x \in Q_{j,m}$ we have $2^{\frac{jn}{p_0(x)} \cdot \frac{\varepsilon}{1+\varepsilon}} |\gamma_{j,m}| \leq h(x)$ and

$$|\gamma_{j,m}| \leq \inf_{y \in Q_{j,m}} 2^{\frac{-jn}{p_0(y)} \cdot \frac{\varepsilon}{1+\varepsilon}} h(y).$$

Using this notation,

$$\|\gamma| f_{p_0(x),\infty}^{\frac{n}{p_0(x)} \cdot \frac{\varepsilon}{1+\varepsilon}}(\mathbf{R}^n)\| = \|h \mid L_{p_0(\cdot)}(\mathbf{R}^n)\|$$

and (4.26) reads as

$$\|\gamma \mid b^0_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}(\mathbf{R}^n)\| \le C \, \|h \mid L_{p_0(\cdot)}(\mathbf{R}^n)\|.$$

We assume that $\|h \mid L_{p_0(\cdot)}(\mathbf{R}^n)\| \le 1$, which is by the unit ball property (cf. [10, Lemma 2.1.4] equivalent to

$$(4.27) \qquad \int_{\mathbf{R}^n} h(x)^{p_0(x)} \, dx \le 1,$$

and need to prove that

$$\|\gamma \mid b^0_{(1+\varepsilon)p_0(\cdot),p_0(\cdot)}(\mathbf{R}^n)\| \le C.$$

This we will show by finding $C > 0$ with

$$\varrho_{\ell_{p_0(\cdot)}(L_{(1+\varepsilon)p_0(\cdot)})} \left(\sum_{m \in \mathbf{Z}^n} \frac{|\gamma_{j,m}|}{C} \chi_{j,m} \right) \le 1,$$

i.e.

$$\sum_{j=0}^{\infty} \left\| \frac{1}{C^{p_0(\cdot)}} \left(\sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}| \chi_{j,m}(\cdot) \right)^{p_0(\cdot)} \mid L_{(1+\varepsilon)}(\mathbf{R}^n) \right\| \le 1.$$

Due to $0 < p_0^- \le p_0^+ < \infty$, this in turn is equivalent to

$$\sum_{j=0}^{\infty} \left\| \left(\sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}| \chi_{j,m}(\cdot) \right)^{p_0(\cdot)} \mid L_{(1+\varepsilon)}(\mathbf{R}^n) \right\| \le C$$

with possibly a different value of $C > 0$. To show this, we write

$$\sum_{j=0}^{\infty} \left\| \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{p_0(\cdot)} \chi_{j,m} \mid L_{1+\varepsilon}(\mathbf{R}^n) \right\|$$

$$\le \sum_{j=0}^{\infty} \left\| \sum_{m \in \mathbf{Z}^n} \left(\inf_{y \in Q_{j,m}} 2^{\frac{-jn}{p_0(y)} \cdot \frac{\varepsilon}{1+\varepsilon}} h(y) \right)^{p_0(\cdot)} \chi_{j,m} \mid L_{1+\varepsilon}(\mathbf{R}^n) \right\|$$

$$= \sum_{j=0}^{\infty} \left\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} \left(\inf_{y \in Q_{j,m}} 2^{\frac{-jn}{p_0(y)} \cdot \frac{\varepsilon}{1+\varepsilon}} h(y) \right)^{(1+\varepsilon)p_0(x)} dx \right\}^{\frac{1}{1+\varepsilon}}$$

$$= \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \left\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} \left(\inf_{y \in Q_{j,m}} 2^{-jn\varepsilon \left(\frac{p_0(x)}{p_0(y)} - 1 \right)} h(y)^{(1+\varepsilon)p_0(x)} \right) dx \right\}^{\frac{1}{1+\varepsilon}}.$$

We use the regularity of p_0 to obtain for $x, y \in Q_{j,m}$

$$2^{-jn\left(\frac{p_0(x)}{p_0(y)} - 1 \right)} = \left[2^{j\left(\frac{1}{p_0(x)} - \frac{1}{p_0(y)} \right)} \right]^{np_0(x)} \le 2^{np_0(x)c_{\log}(1/p_0)} \le c'.$$

So, it is enough to prove

$$(4.28) \qquad \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \left\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} h_{j,m}^{(1+\varepsilon)p_0(x)} dx \right\}^{\frac{1}{1+\varepsilon}} \le C,$$

where we denoted

$$h_{j,m} = \inf_{y \in Q_{j,m}} h(y).$$

We split the left-hand side of (4.28) into

$$\sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} h_{j,m}^{(1+\varepsilon)p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{\{m:h_{j,m} \leq 1\}} \int_{Q_{j,m}} h_{j,m}^{(1+\varepsilon)p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$+ \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{\{m:h_{j,m} > 1\}} \int_{Q_{j,m}} h_{j,m}^{(1+\varepsilon)p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}} = I + II.$$

The first term can be estimated by (4.27)

$$I \leq \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{\{m:h_{j,m} \leq 1\}} \int_{Q_{j,m}} h_{j,m}^{p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \int_{\mathbf{R}^n} h(x)^{p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}} \leq c.$$

To estimate II, we make first a couple of observations. Let $h_{j,m} \geq 1$. Then

$$1 \geq \int_{\mathbf{R}^n} h(x)^{p_0(x)} dx \geq \int_{Q_{j,m}} h(x)^{p_0(x)} dx \geq \int_{Q_{j,m}} h_{j,m}^{p_0(x)} dx \geq |Q_{j,m}| h_{j,m}^{p_0^-} = 2^{-jn} h_{j,m}^{p_0^-}$$

and we get

$$1 \leq h_{j,m} \leq 2^{jn/p_0^-}.$$

Hence there is an $0 \leq \alpha \leq 1$ such that $h_{j,m} = 2^{\alpha jn/p_0^-}$. Since $p_0^+ < \infty$, we have that $1/p_0 \in C_{\text{loc}}^{\log}(\mathbf{R}^n)$ implies $p_0 \in C_{\text{loc}}^{\log}(\mathbf{R}^n)$ and we can use its regularity. Hence, there is a constant $c > 1$ such that for any $x, y \in Q_{j,m}$ it holds $c^{-1} \leq 2^{j(p_0(x)-p_0(y))} \leq c$ and therefore

$$(4.29) \qquad c^{-\alpha n/p_0^-} \leq 2^{\frac{\alpha jn}{p_0^-}(p_0(x)-p_0(y))} = h_{j,m}^{(p_0(x)-p_0(y))} \leq c^{\alpha n/p_0^-} \leq c^{n/p_0^-}.$$

If we also denote $p_{j,m} = \inf_{y \in Q_{j,m}} p_0(y)$, we obtain

$$h_{j,m}^{p_0(x)} = h_{j,m}^{p_0(x)-p_{j,m}} \cdot h_{j,m}^{p_{j,m}} \leq C h_{j,m}^{p_{j,m}} \leq C \inf_{y \in Q_{j,m}} \big(h(y)^{p_0(y)} \big).$$

The last fact we shall use is that

$$(4.30) \qquad \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} \big(\inf_{y \in Q_{j,m}} \varphi(y) \big)^{1+\varepsilon} dx \Big\}^{\frac{1}{1+\varepsilon}} \leq c \|\varphi \mid L_1(\mathbf{R}^n)\|$$

for each non-negative $\varphi \in L_1(\mathbf{R}^n)$. The proof follows easily using the technique of non-increasing rearrangement:

$$\sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} \big(\inf_{y \in Q_{j,m}} \varphi(y) \big)^{1+\varepsilon} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{l=1}^{\infty} 2^{-jn} \varphi^*(l2^{-jn})^{1+\varepsilon} \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{k=0}^{\infty} 2^{(k-j)n} \varphi^*(2^{(k-j)n})^{1+\varepsilon} \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \sum_{k=0}^{\infty} 2^{(k-j)\frac{n}{1+\varepsilon}} \varphi^*(2^{(k-j)n})$$

$$= \sum_{l=-\infty}^{\infty} \varphi^*(2^{-ln}) 2^{-\frac{ln}{1+\varepsilon}} \sum_{j=l}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \lesssim \sum_{l=-\infty}^{\infty} \varphi^*(2^{-ln}) 2^{-ln} \sim \|\varphi \mid L_1(\mathbf{R}^n)\|.$$

We apply (4.30) with $\varphi(y) = h(y)^{p_0(y)}$ to estimate II

$$II = \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{\{m:h_{j,m}>1\}} \int_{Q_{j,m}} h_{j,m}^{(1+\varepsilon)p_0(x)} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{\{m:h_{j,m}>1\}} \int_{Q_{j,m}} (\inf_{y\in Q_{j,m}} h(y)^{p_0(y)})^{1+\varepsilon} dx \Big\}^{\frac{1}{1+\varepsilon}}$$

$$\lesssim \sum_{j=0}^{\infty} 2^{-jn\frac{\varepsilon}{1+\varepsilon}} \Big\{ \sum_{m\in\mathbf{Z}^n} \int_{Q_{j,m}} (\inf_{y\in Q_{j,m}} h(y)^{p_0(y)})^{1+\varepsilon} dx \Big\}^{\frac{1}{1+\varepsilon}} \lesssim \int_{\mathbf{R}^n} h(y)^{p_0(y)} dy \leq C$$

and finish the proof. $\qquad \Box$

Using the correspondence of sequence and function spaces from Theorems 2.8 and 2.9 we obtain the Jawerth embedding for the variable function spaces.

Corollary 4.4. *Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ with $p_0^+, q^+ < \infty$ and $s_0, s_1 \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$. Let $\inf_{x\in\mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Then

$$F_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow B_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n).$$

Proof. Let $d > 1$ and consider $K, L \in \mathbf{N}_0$ such that

$$K > s_0^+ \quad \text{and} \quad L > \max\left(\sigma_{p_0,q} - s_0^-, \sigma_{p_1} - 1 - s_1^-\right).$$

By Theorem 2.8, each $f \in F_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n)$ can be written as

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbf{Z}^n} \lambda_{j,m} a_{j,m} \quad (\text{convergence in } \mathcal{S}'(\mathbf{R}^n))$$

with

(4.31) $$\|\lambda(f) \mid f_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n)\| \leq c \|f \mid F_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n)\|,$$

where $c > 0$ does not depend on f and $(a_{j,m})_{j\in\mathbf{N}_0, m\in\mathbf{Z}^n}$ are $[K, L]$-atoms. From Theorem 4.3, we know that

(4.32) $$\|\lambda(f) \mid b_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n)\| \leq c' \|\lambda(f) \mid f_{p_0(\cdot),q(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n)\|,$$

with $c' > 0$ independent of $\lambda(f)$. Due to our choice of K and L, we can use now the atomic characterization for Besov spaces stated in Theorem 2.9 and conclude not only that f also belongs to $B_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n)$, but also that the following inequality holds

(4.33) $$\|f \mid B_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n)\| \leq c'' \|\lambda(f) \mid b_{p_1(\cdot),p_0(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n)\|,$$

with $c'' > 0$ independent of f. Then, the combination of (4.31)–(4.33) gives us the desired estimate. $\qquad\square$

4.2. Franke embedding. In this section we prove the Franke embedding for function spaces with variable exponents. We have to avoid duality arguments in the variable exponent setting and therefore reduce the proof to the constant exponent case and apply Theorem 3.4, which we have shown in the previous section.

Theorem 4.5. *Let* $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ *with* $p_1^+ < \infty$ *and* $s_0, s_1 \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$. *Let* $\inf_{x \in \mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ *with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Then

$$b_{p_0(\cdot),p_1(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow f_{p_1(\cdot),q(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n).$$

Proof. By the lifting property from Lemma 4.2 we may suppose again $s_1 = 0$, and the elementary embeddings between f-spaces allow one to set

$$q(x) = \frac{1}{r}p_1(x), \quad \text{with } r > 1 \text{ chosen big enough.}$$

So it suffices to prove the embedding

$$(4.34) \qquad b_{p_0(\cdot),p_1(\cdot)}^{n\left(\frac{1}{p_0(\cdot)} - \frac{1}{p_1(\cdot)}\right)}(\mathbf{R}^n) \hookrightarrow f_{p_1(\cdot),\frac{1}{r}p_1(\cdot)}^{0}(\mathbf{R}^n).$$

Similarly to (4.23), $p_0(x) < p_1(x)$ are again well separated and we may find $\varepsilon > 0$ with

$$p_1(x) > (1 - \varepsilon)p_1(x) > p_0(x).$$

By the Sobolev embedding from Theorem 2.10 we obtain

$$b_{p_0(\cdot),p_1(\cdot)}^{n\left(\frac{1}{p_0(\cdot)} - \frac{1}{p_1(\cdot)}\right)}(\mathbf{R}^n) \hookrightarrow b_{(1-\varepsilon)p_1(\cdot),p_1(\cdot)}^{\frac{n}{p_1(\cdot)}\frac{\varepsilon}{1-\varepsilon}}(\mathbf{R}^n).$$

Hence, instead of (4.34) we show

$$(4.35) \qquad b_{(1-\varepsilon)p_1(\cdot),p_1(\cdot)}^{\frac{n}{p_1(\cdot)}\frac{\varepsilon}{1-\varepsilon}}(\mathbf{R}^n) \hookrightarrow f_{p_1(\cdot),\frac{1}{r}p_1(\cdot)}^{0}(\mathbf{R}^n).$$

We assume that $\|\gamma \mid b_{(1-\varepsilon)p_1(\cdot),p_1(\cdot)}^{\frac{n}{p_1(\cdot)}\frac{\varepsilon}{1-\varepsilon}}(\mathbf{R}^n)\| \le 1$, which is equivalent to

$$\sum_{j=0}^{\infty}\Big\| \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{p_1(\cdot)} 2^{jn\frac{\varepsilon}{1-\varepsilon}} \chi_{j,m} \mid L_{1-\varepsilon}(\mathbf{R}^n)\Big\| \le 1$$

or even

$$(4.36) \qquad \sum_{j=0}^{\infty} 2^{jn\frac{\varepsilon}{1-\varepsilon}} \left(\int_{\mathbf{R}^n} \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{(1-\varepsilon)p_1(x)} \chi_{j,m}(x)\, dx \right)^{\frac{1}{1-\varepsilon}} \le 1.$$

This implies that, for every $j \in \mathbf{N}_0$, we have the inequality

$$(4.37) \qquad \sum_{m \in \mathbf{Z}^n} \int_{Q_{j,m}} |\gamma_{j,m}|^{(1-\varepsilon)p_1(x)}\, dx \le 2^{-jn\varepsilon}.$$

Our aim is to prove that $\|\gamma \mid f_{p_1(\cdot),\frac{1}{r}p_1(\cdot)}^{0}(\mathbf{R}^n)\| \le C$. By Definition 2.7 and (2.7), it is sufficient to find $C > 0$, such that

$$\left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{p_1(\cdot)/r} \chi_{j,m}(\cdot) \right)^{r/p_1(\cdot)} \mid L_{p_1(\cdot)}(\mathbf{R}^n) \right\| \le C,$$

or, equivalently,

$$\int_{\mathbf{R}^n} \frac{1}{C^{p_1(x)}} \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{p_1(x)/r} \chi_{j,m}(x) \right)^r dx \leq 1.$$

Finally, due to the restriction $0 < p_1^- \leq p_1^+ < \infty$, it is enough to show

$$(4.38) \qquad \left\| \sum_{j=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{\frac{p_1(\cdot)}{r}} \chi_{j,m} \mid L_r(\mathbf{R}^n) \right\| \leq C$$

with possibly a different value of $C > 0$.

We have

$$\left\| \sum_{j=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\gamma_{j,m}|^{\frac{p_1(\cdot)}{r}} \chi_{j,m} \mid L_r(\mathbf{R}^n) \right\|$$

$$\leq \left\| \sum_{j=0}^{\infty} \sum_{\{m:|\gamma_{j,m}|\leq 1\}} |\gamma_{j,m}|^{\frac{p_1(\cdot)}{r}} \chi_{j,m} \mid L_r(\mathbf{R}^n) \right\| + \left\| \sum_{j=0}^{\infty} \sum_{\{m:|\gamma_{j,m}|>1\}} |\gamma_{j,m}|^{\frac{p_1(\cdot)}{r}} \chi_{j,m} \mid L_r(\mathbf{R}^n) \right\|$$

$$= I + II.$$

The estimate of I follows by (4.37)

$$I \leq \sum_{j=0}^{\infty} \left(\sum_{\{m:|\gamma_{j,m}|\leq 1\}} \int_{Q_{j,m}} |\gamma_{j,m}|^{p_1(x)} dx \right)^{\frac{1}{r}} \leq \sum_{j=0}^{\infty} \left(\sum_{\{m:|\gamma_{j,m}|\leq 1\}} \int_{Q_{j,m}} |\gamma_{j,m}|^{(1-\varepsilon)p_1(x)} dx \right)^{\frac{1}{r}}$$

$$\leq \sum_{j=0}^{\infty} 2^{-jn\varepsilon\frac{1}{r}} \leq c.$$

To estimate II, we observe that (4.37) implies for every $j \in \mathbf{N}_0$ and $m \in \mathbf{Z}^n$ with $|\gamma_{j,m}| > 1$

$$2^{-jn\varepsilon} \geq \int_{Q_{j,m}} |\gamma_{j,m}|^{(1-\varepsilon)p_1(x)} dx \geq 2^{-jn} |\gamma_{j,m}|^{(1-\varepsilon)p_1^-}$$

and therefore

$$1 \leq |\gamma_{j,m}| \leq 2^{jn\frac{1}{p_1^-}}.$$

Similarly to (4.29), for every $x \in Q_{j,m}$ we have $|\gamma_{j,m}|^{p_1(x)} \sim |\gamma_{j,m}|^{p_{j,m}}$, where $p_{j,m}$ is the value of p_1 in the middle of $Q_{j,m}$. Denoting $\alpha_{j,m} = |\gamma_{j,m}|^{p_{j,m}}$, we get that (4.36) implies

$$(4.39) \qquad \sum_{j=0}^{\infty} 2^{jn\frac{\varepsilon}{1-\varepsilon}} \left(2^{-jn} \sum_{\{m:|\gamma_{j,m}|>1\}} |\gamma_{j,m}|^{(1-\varepsilon)p_{j,m}} \right)^{\frac{1}{1-\varepsilon}} \leq c,$$

or, equivalently,

$$(4.40) \qquad \sum_{j=0}^{\infty} 2^{-jn} \left(\sum_{\{m:|\alpha_{j,m}|>1\}} |\alpha_{j,m}|^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} \leq c.$$

To estimate the second term II it is therefore sufficient to show that

$$(4.41) \qquad \left\| \sum_{j=0}^{\infty} \sum_{\{m:|\gamma_{j,m}|>1\}} |\gamma_{j,m}|^{p_{j,m}/r} \chi_{j,m} | L_r(\mathbf{R}^n) \right\| \leq C,$$

which in turn is equivalent to

$$(4.42) \qquad \left\| \sum_{j=0}^{\infty} \sum_{\{m: |\alpha_{j,m}| > 1\}} |\alpha_{j,m}|^{1/r} \chi_{j,m} | L_r(\mathbf{R}^n) \right\| \leq C.$$

To obtain (4.42) from (4.40), we employ the constant-index case, i.e. Theorem 3.4. Indeed, we put $\beta_{j,m} = |\alpha_{j,m}|^{1/r}$ if $|\alpha_{j,m}| > 1$ and zero otherwise and get

$$\left\| \sum_{j=0}^{\infty} \sum_{\{m: |\alpha_{j,m}| > 1\}} |\alpha_{j,m}|^{1/r} \chi_{j,m} | L_r(\mathbf{R}^n) \right\| = \left\| \sum_{j=0}^{\infty} \sum_{m \in \mathbf{Z}^n} |\beta_{j,m}| \chi_{j,m} | L_r(\mathbf{R}^n) \right\|$$

$$= \|\beta \mid f_{r,1}^0\| \lesssim \|\beta \mid b_{(1-\varepsilon)r,r}^{\frac{n\varepsilon}{r(1-\varepsilon)}}\| = \left\{ \sum_{j=0}^{\infty} 2^{\frac{jn\varepsilon}{(1-\varepsilon)r}r} \left(\sum_{m \in \mathbf{Z}^n} 2^{-jn} \beta_{j,m}^{(1-\varepsilon)r} \right)^{\frac{r}{(1-\varepsilon)r}} \right\}^{1/r}$$

$$= \left\{ \sum_{j=0}^{\infty} 2^{-jn} \left(\sum_{m: |\alpha_{j,m}| > 1} |\alpha_{j,m}|^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} \right\}^{1/r} \leq c^{1/r}. \qquad \square$$

Likewise the previous section, we can use the characterization of the spaces via atoms to transfer the result from the sequence spaces to the function spaces. We do not present the proof here because it follows similarly as the proof of Corollary 4.4.

Corollary 4.6. *Let $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ with $p_1^+, q^+ < \infty$ and $s_0, s_1 \in C_{\mathrm{loc}}^{\log}(\mathbf{R}^n)$. Let $\inf_{x \in \mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ with*

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}, \quad x \in \mathbf{R}^n.$$

Then

$$B_{p_0(\cdot), p_1(\cdot)}^{s_0(\cdot)}(\mathbf{R}^n) \hookrightarrow F_{p_1(\cdot), q(\cdot)}^{s_1(\cdot)}(\mathbf{R}^n).$$

5. Jawerth and Franke embedding in 2-microlocal spaces

The definition of Besov and Triebel–Lizorkin spaces of variable smoothness and integrability is a special case of the so-called 2-microlocal spaces of variable integrability. As all the proofs for spaces of variable smoothness do also serve for 2-microlocal spaces, we devote this section to present these results. We start by the definition of the spaces, which is based on the dyadic decomposition of unity as presented before combined with the concept of admissible weight sequences.

Definition 5.1. Let $\alpha \geq 0$ and $\alpha_1, \alpha_2 \in \mathbf{R}$ with $\alpha_1 \leq \alpha_2$. A sequence $\boldsymbol{w} = (w_j)_{j \in \mathbf{N}_0}$ of positive measurable functions in $\mathbf{R}^n$ belongs to the class $\mathcal{W}_{\alpha_1, \alpha_2}^{\alpha}(\mathbf{R}^n)$ if the following conditions are satisfied:

(i) There exists a constant $c > 0$ such that

$$0 < w_j(x) \leq c\, w_j(y)\, (1 + 2^j |x - y|)^{\alpha} \quad \text{for all } j \in \mathbf{N}_0 \text{ and all } x, y \in \mathbf{R}^n.$$

(ii) For all $j \in \mathbf{N}_0$ it holds

$$2^{\alpha_1} w_j(x) \leq w_{j+1}(x) \leq 2^{\alpha_2} w_j(x) \quad \text{for all } x \in \mathbf{R}^n.$$

Such a system $(w_j)_{j \in \mathbf{N}_0} \in \mathcal{W}_{\alpha_1, \alpha_2}^{\alpha}(\mathbf{R}^n)$ is called admissible weight sequence.

Properties of admissible weights may be found in [25, Remark 2.4]. Finally, here is the definition of the spaces under consideration.

Definition 5.2. Let $(\varphi_j)_{j \in \mathbf{N}_0}$ be a partition of unity as explained in Section 2, $\boldsymbol{w} = (w_j)_{j \in \mathbf{N}_0} \in \mathcal{W}_{\alpha_1, \alpha_2}^{\alpha}(\mathbf{R}^n)$ and $p, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$.

(i) The space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f \mid B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| := \|(w_j \, (\varphi_j \widehat{f})^{\vee})_{j \in \mathbf{N}_0} \mid \ell_{q(\cdot)}(L_{p(\cdot)}(\mathbf{R}^n))\|$$

is finite.

(ii) If $p^+, q^+ < \infty$, then the space $B^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f \mid F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbf{R}^n)\| := \|(w_j \, (\varphi_j \widehat{f})^{\vee})_{j \in \mathbf{N}_0} \mid L_{p(\cdot)}(\ell_{q(\cdot)}(\mathbf{R}^n))\|$$

is finite.

As before, the independence of the decomposition of unity for the 2-microlocal spaces follows from the local means characterization (see [26] for Triebel–Lizorkin and [28] for Besov spaces).

Remark 5.3. These 2-microlocal weight sequences are directly connected to variable smoothness functions $s\colon \mathbf{R}^n \to \mathbf{R}$ if we set

(5.43) $$w_j(x) = 2^{js(x)}, \quad x \in \mathbf{R}^n.$$

If $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$, then $\boldsymbol{w} = (w_j(x))_{j \in \mathbf{N}_0} = (2^{js(x)})_{j \in \mathbf{N}_0}$ belongs to $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbf{R}^n)$ with $\alpha_1 = s^-$ and $\alpha_2 = s^+$ and $\alpha = c_{\log}(s)$, where $c_{\log}(s)$ is the constant for $s(\cdot)$ from (2.8). That means that spaces of variable smoothness from Definition 2.2 are a special case of 2-microlocal function spaces from Definition 5.2. Both types of function spaces are very closely connected and the properties used in the proofs are either

(5.44) $$2^{j|s(x)-s(y)|} \le c \quad \text{or} \quad \frac{w_j(x)}{w_j(y)} \le c$$

for $|x - y| \le c\,2^{-j}$ and $j \in \mathbf{N}_0$. This property follows directly either from the definition of $s \in C^{\log}_{\mathrm{loc}}(\mathbf{R}^n)$ or from Definition 5.1.

Theorem 5.4. *Let $\boldsymbol{w}^0, \boldsymbol{w}^1 \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}(\mathbf{R}^n)$ and $p_0, p_1, q \in \mathcal{P}^{\log}(\mathbf{R}^n)$ with $q^+ < \infty$. Let $p_0(x) < p_1(x)$ with $\inf_{x \in \mathbf{R}^n}(p_1(x) - p_0(x)) > 0$ and*

$$1 < \frac{w^0_j(x)}{w^1_j(x)} = 2^{j\left(\frac{n}{p_0(x)} - \frac{n}{p_1(x)}\right)} \quad \text{for all } x \in \mathbf{R}^n \text{ and } j \in \mathbf{N}_0.$$

(i) *If $p_0^+ < \infty$, then*

$$F^{\boldsymbol{w}^0}_{p_0(\cdot),q(\cdot)} \hookrightarrow B^{\boldsymbol{w}^1}_{p_1(\cdot),p_0(\cdot)}.$$

(ii) *If $p_1^+ < \infty$, then*

$$B^{\boldsymbol{w}^0}_{p_0(\cdot),p_1(\cdot)} \hookrightarrow F^{\boldsymbol{w}^1}_{p_1(\cdot),q(\cdot)}.$$

Regarding the proof, one just needs to use the corresponding Sobolev embeddings for 2-microlocal spaces (see [17] and [1]) and follow exactly the same steps as before using always property (5.44) for the weight sequences.

6. Open problems

We close by listing several open problems, which are connected to the study of function spaces with variable exponents and their embeddings.

(1) Give an example that $\inf_{x \in \mathbf{R}^n}(s_0(x) - s_1(x)) > 0$ is really needed and can not be replaced by $s_0(x) > s_1(x)$ for all $x \in \mathbf{R}^n$. This seems to be feasible when working with function space on the whole $\mathbf{R}^n$, but it might get more tricky, when considering only functions with support in, say, the unit cube $[0,1]^n$.

(2) It is well known, that the Triebel–Lizorkin spaces $F_{p,q}^s(\mathbf{R}^n)$ with constant indices might depend on the chosen decomposition of unity if $p = \infty$. Therefore, the restriction $p^+ < \infty$ seems to be quite natural in Definition 2.2 (i). On the other hand, there is no such trouble for spaces $F_{p,q}^s(\mathbf{R}^n)$ with $p < \infty$ and $q = \infty$. It would be therefore highly interesting if the Triebel–Lizorkin spaces $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ can also be defined with $q^+ = \infty$ but with still variable q. To that end one needs to show that the spaces $F_{p(\cdot),q(\cdot)}^{s(\cdot)}(\mathbf{R}^n)$ are independent on the resolution of unity.

(3) Function spaces of Morrey type attracted recently a lot of attention in connection with the analysis of Navier–Stokes equations [16] and function spaces of Morrey type with variable exponents were introduced and studied already [15]. Is there a version of the Franke and Jawerth embedding for this scale of function spaces?

(4) The Franke and Jawerth embeddings were used in [20] to describe the fine properties of Besov and Triebel–Lizorkin spaces in term of the so-called envelopes. They determine the kind and size of singularities, which the functions from these spaces might posses. On the other hand, function spaces of variable exponents capture very well the local properties of functions and distributions. The interplay of the theory of envelopes and the function spaces with variable exponents would be therefore interesting.

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