A NOTE ON A MULTILINEAR LOCAL $Tb$ THEOREM FOR CALDERÓN–ZYGMUND OPERATORS

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Abstract. In this short note, we extend a local $Tb$ theorem that was proved in [4] to a full multilinear local $Tb$ theorem.

1. Introduction

In [4], we proved (in collaboration with Grau de la Herran) a multilinear local $Tb$ theorem for square functions, and applied it to prove a local $Tb$ theorem for singular integrals. The local $Tb$ theorem for singular integrals had “local $Tb$ type” testing conditions for the operator $T$ on pseudo-accretive collections $b = \{b_Q\}_{1 \leq i \leq m}$, but tested the adjoints of $T$ on the constant function $1$; see [4] for more details on this.

There has been interest recently in multilinear local $Tb$ theorems for square function and singular integrals, for example in the works [6], [1], [5] and [7]. In particular, in [7], the authors are interested in multilinear local $Tb$ theorems that test all adjoints of a multilinear operator $T$ on pseudo-accretive systems, rather than only on the operator itself; two examples are in [7] and the authors cite these as a feature of their multilinear local $Tb$ result for a class of $n$-linear forms known as perfect Calderón–Zygmund operators. In this note, we show that our result from [4] can be easily extended to a local $Tb$ theorem for Calderón–Zygmund operators where $T$ and all of its adjoints are tested on pseudo-accretive systems.

The following local $Tb$ theorem for multilinear singular integral operators, which is an extension of Theorem 1.2 from [4], is the main result of the article.

**Theorem 1.1.** Let $T$ be a continuous $m$ linear operator from $\mathcal{S} \times \cdots \times \mathcal{S}$ into $\mathcal{S}'$ with a standard Calderón–Zygmund kernel $K$. Suppose that $T \in WBP$ and for each $j = 0, 1, \ldots, m$ there exist $2 \leq q < \infty$ and $1 < q_{i,j} < \infty$ with $\frac{1}{q_j} = \sum_{i=1}^{m} \frac{1}{q_{i,j}}$ and an $m$-compatible collection of pseudo-accretive systems $b_j = \{b_Q^{i,j}\}_{1 \leq i \leq m}$ indexed by dyadic cubes $Q$ such that

\[
\int_Q \left( \int_0^{\ell(Q)} |Q_t T^{**j}(P_t b_Q^{1,j}, \ldots, P_t b_Q^{m,j})(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \lesssim |Q|.
\]

Then $T$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^p$ for all $1 < p_i < \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Here we assume that this estimate holds for any approximation to

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the identity $P_t$ with smooth compactly supported kernels and any Littlewood–Paley–Stein projection operators $Q_t$ whose kernels also are smooth compactly supported function.

The precise meaning of (1) is the following: For any $\varphi, \psi \in C_0^\infty$ such that $\widehat{\varphi}(0) = 1$ and $\widehat{\psi}(0) = 0$, (1) holds for $P_tf = \varphi_t * f$, $Q_tf = \psi_t * f$, $\varphi_t(x) = \frac{1}{m} \varphi\left(\frac{x}{t}\right)$, and $\psi_t(x) = \frac{1}{m} \psi\left(\frac{x}{t}\right)$, where the constant is independent of the dyadic cube $Q$, but may depend on $\varphi$ and $\psi$.

We prove this theorem by applying the square function estimates that were also proved in [4], but with a few minor modifications to allow for the extension to Theorem 1.1. This note is intended to be an addendum to the article [4]. So the reader should refer to that article for definitions, discussion, and history related to these results.

2. A few definitions and results from [4]

Define the family of multilinear of operators $\{\Theta_t\}_{t > 0}$

\[\Theta_t(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^m} \theta_t(x, y_1, \ldots, y_m) \prod_{i=1}^m f_i(y_i) \, dy_i\]

where $\theta_t : \mathbb{R}^{(m+1)n} \to \mathbb{C}$ and the square function

\[S(f_1, \ldots, f_m)(x) = \left( \int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}\]

associated to $\{\Theta_t\}_{t > 0}$, where $f_i$ for $i = 1, \ldots, m$ are initially functions in $C_0^\infty(\mathbb{R}^n)$. Also assume that $\theta_t$ satisfies for all $x, y_1, \ldots, y_m, x', y_1', \ldots, y_m' \in \mathbb{R}^n$

\[|\theta_t(x, y_1, \ldots, y_m)| \lesssim t^{-mn} \prod_{i=1}^m (1 + t^{-1}|x - y_i|)^{N+\gamma},\]

\[|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x, y_1', \ldots, y_m')| \lesssim t^{-mn} (t^{-1}|y_i - y_i'|)^{-\gamma},\]

\[|\theta_t(x, y_1, \ldots, y_m) - \theta_t(x', y_1, \ldots, y_m)| \lesssim t^{-mn} (t^{-1}|x - x'|)^{-\gamma}\]

for some $N > n$ and $0 < \gamma \leq 1$. The following results were proved in [4].

**Theorem.** Let $\Theta_t$ and $S$ be defined as in (2) and (3) where $\theta_t$ satisfies (4)–(6). Suppose there exist $q, q > 1$ for $i = 1, \ldots, m$ with $\frac{1}{q} = \sum_{i=1}^m \frac{1}{q}$ and functions $b = \{b_Q^i\}_{1 \leq i \leq m}$ indexed by dyadic cubes $Q \subset \mathbb{R}^n$ such that for every dyadic cube $Q$

\[\int_{\mathbb{R}^n} |b_Q^i|^q \leq B_1|Q|,\]

\[\frac{1}{B_2} \lesssim \left| \frac{1}{|Q|} \int_Q \prod_{i=1}^m b_Q^i(x) \, dx \right|,\]

\[\frac{1}{|R|} \int_R \prod_{i=1}^m b_Q^i(x) \, dx \leq B_3 \prod_{i=1}^m \frac{1}{|R|} \int_R b_Q^i(x) \, dx\]

for all dyadic subcubes $R \subset Q$,

\[\int_Q \left( \int_0^\ell(Q) |\Theta_t(b_Q^1, \ldots, b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} \, dx \leq B_4|Q|,\]
Then $S$ satisfies
\begin{equation}
\|S(f_1, \ldots, f_m)\|_{L^p} \leq \prod_{i=1}^m \|f_i\|_{L^{p_i}}
\end{equation}
for all $1 < p_i < \infty$ and $2 \leq p < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

If $\{b_Q\}$ satisfies (7) and (8), we say that $b = \{b_Q^{ij}\}_{1 \leq i \leq m}$ is an $m$-compatible, or just compatible, collection of pseudo-accretive systems if they satisfy (7)–(9).

In Theorem 1.1, we only impose that $b_j = \{b_Q^{ij}\}_{1 \leq i \leq m}$ is an $m$-compatible pseudo-accretive system for each $j = 0, 1, \ldots, m$. In particular, $b_j$ must satisfy (7)–(9) for each $j = 0, 1, \ldots, m$, but there is no dependence between pseudo-accretive systems $b_j$ and $b_k$ for $j \neq k$.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $P_t$ be a smooth approximation to identity operators with smooth compactly supported kernels. Then it follows that $P_t^2 f \to f$ as $t \to 0^+$ and $P_t^2 f \to 0$ as $t \to \infty$ in $\mathcal{S}$ for $f \in \mathcal{S}_0$. Here $\mathcal{S}_0$ is the subspace of Schwartz functions satisfying $|\hat{f}(\xi)| \leq C_M |\xi|^M$ for all $M \in \mathbb{N}$. There exist Littlewood–Paley–Stein projection operators $Q_t^{(i)}$ for $i = 1, 2$ with smooth compactly supported kernels such that $t \frac{d}{dt} P_t^2 = Q_t^{(2)2} Q_t^{(1)} = Q_t$. Using these operators, we decompose $T$ for $f_i \in \mathcal{S}_0$, $i = 0, \ldots, m$
\begin{align*}
\langle T(f_1, \ldots, f_m), f_0 \rangle & = \int_0^\infty t \frac{d}{dt} \left\langle T(P_t^2 f_1, \ldots, P_t^2 f_m), P_t^2 f_0 \right\rangle \frac{dt}{t} \\
& = \sum_{i=0}^m \left\langle \int_0^\infty Q_t T^{(i)}(P_t^2 f_1, \ldots, P_t f_i, f_0, f_{i+1}, \ldots, P_t^2 f_m) \frac{dt}{t}, f_i \right\rangle \\
& = \sum_{i=0}^m \left\langle T_{i}(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots, f_m), f_i \right\rangle,
\end{align*}
(12) where we take the last line in (12) as the definition of $T_i$ for $i = 0, 1, \ldots, m$. It follows that $T_i$ is a multilinear singular integral operator with standard kernel
\begin{align*}
K_i(x, y_1, \ldots, y_m) & = \int_0^\infty \left\langle T^{(i)}(\varphi_1^{y_1}, \ldots, \varphi_1^{y_m}), \psi_t \right\rangle \frac{dt}{t},
\end{align*}
Also let $\Theta_i^t$ be the multilinear operator associated to
\begin{align*}
\theta_i^t(x, y_1, \ldots, y_m) & = Q_t^{(1)} T^{(i)}(\varphi_1^{y_1}, \ldots, \varphi_1^{y_m})(x),
\end{align*}
and let $S_i$ be the square function associated to $\Theta_i^t$. Note that $\theta_i^t(x, y_1, \ldots, y_m) \neq \left\langle T^{(i)}(\varphi_1^{y_1}, \ldots, \varphi_1^{y_m}), \psi_t \right\rangle$, and that $T_i$ is not actually the integral of $\Theta_i^t$ (one has $Q_t$ and the other has $Q_t^{(1)}$). Furthermore, by the hypotheses on $T^{(i)}$ and by the local $Tb$ theorem for square functions from [4], it follows that $S_i$ is bounded from $L^{2m} \times \cdots \times$
\(L^{2m}\) into \(L^2\). Therefore we have
\[
\langle T_i(f_1, \ldots, f_m), f_0 \rangle = \int_0^\infty \int_{\mathbb{R}^n} Q_i T^{*i}(P^i_tf_1, \ldots, P^i_tf_m)(x)f_0(x)dx\,dt \frac{dt}{t}
\]
\[
= \int_0^\infty \int_{\mathbb{R}^n} Q_i^{(1)} T^{*i}(P^i_tf_1, \ldots, P^i_tf_m)(x)Q_i^{(2)*}f_0(x)dx\,dt \frac{dt}{t}
\]
\[
\leq \|S_i(f_1, \ldots, f_m)\|_{L^2} \left\| \left( \int_0^\infty |Q_i^{(2)*}f_0|^2 \, dt \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{L^2}
\]
\[
\lesssim \|f_1\|_{L^{2m}} \cdots \|f_m\|_{L^{2m}} \|f_0\|_{L^2}.
\]
Hence \(T_i\) is bounded from \(L^{2m} \times \cdots \times L^{2m}\) into \(L^2\), and by the multilinear Calderón–Zygmund theory developed by Grafakos and Torres in [2, 3], it follows that \(T_i\) also bounded from \(L^{p_1} \times \cdots \times L^{p_m}\) into \(L^p\) for all \(1 < p_1, \ldots, p_m < \infty\) with \(\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}\) for each \(i = 0, 1, \ldots, m\). In particular, it follows that \(T_i\) is bounded from \(L^{m+1} \times \cdots \times L^{m+1}\) into \(L^{\frac{m+1}{m}}\) for each \(i = 0, 1, \ldots, m\). Then continuing from (12), we have
\[
| \langle T(f_1, \ldots, f_m), f_0 \rangle | \leq \sum_{i=0}^{m} | \langle T_i(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots, f_m), f_i \rangle |
\]
\[
\leq \sum_{i=0}^{m} \|T_i(f_1, \ldots, f_{i-1}, f_0, f_{i+1}, \ldots, f_m)\|_{L^{\frac{m+1}{m}}} \|f_i\|_{L^{m+1}}
\]
\[
\lesssim \prod_{j=0}^{m} \|f_j\|_{L^{m+1}}.
\]
Therefore \(T\) is bounded from \(L^{m+1} \times \cdots \times L^{m+1}\) into \(L^{\frac{m+1}{m}}\). Then it again follows from the multilinear Calderón–Zygmund theory in [2, 3] that \(T\) is bounded from \(L^{p_1} \times \cdots \times L^{p_m}\) into \(L^p\) for all \(1 < p_1, \ldots, p_m < \infty\) such that \(\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}\). □

References


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