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PRIME ENDS IN THE HEISENBERG GROUP $\mathbf{H}_1$ AND THE BOUNDARY BEHAVIOR OF QUASICONFORMAL MAPPINGS

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Abstract. We investigate prime ends in the Heisenberg group $\mathbf{H}_1$, extending Näkki's construction for collared domains in Euclidean spaces. The corresponding class of domains is defined via uniform domains and the Loewner property. Using prime ends, we show the counterpart of Carathéodory's extension theorem for quasiconformal mappings, the Koebe theorem on arcwise limits, the Lindelöf theorem for principal points, and the Tsuji theorem.

1. Introduction

The cornerstone for the theory of prime ends, is a work by Carathéodory [21], who first defined prime ends for simply-connected domains in the plane. The main motivation for his studies came from the problem of continuous and homeomorphic extensions of conformal mappings. A result due to Carathéodory (and Osgood–Taylor [61]), ensures that a conformal map between Jordan domains in the plane extends to a homeomorphism between the closures. However, there are simple examples, for instance a slit-disk, for which an extension theorem fails. Nevertheless, by introducing the so-called prime end boundary, Carathéodory was able to show that *a conformal homeomorphism between bounded simply-connected planar domains U and V extends to a homeomorphism between $\overline{U}$ and the prime ends compactification of V*. The subsequent development of the prime ends theory has led to generalizations of prime ends for more general domains in the plane, and in higher dimensional Euclidean spaces, for instance see Kaufman [43], Mazurkiewicz [51], Freudenthal [29] and more recently Epstein [27] and Karmazin [42], see also [2] for a theory of prime ends in metric spaces. Applications of prime ends encompass: the theory of continua, see Carmona–Pommerenke [23, 24], the boundary behavior of solutions to elliptic PDEs, see Ancona [4] and the studies of the Dirichlet problem for p-harmonic functions in metric spaces, see Björn–Björn–Shanmugalingam [12].

In this work we follow the original motivation for studying prime ends and investigate extension problems and the related boundary behavior for quasiconformal mappings in the setting of Heisenberg group $\mathbf{H}_1$. Similar results of this type were obtained by Väisälä [68, Chapter 17], [70] and Näkki [57] in the Euclidean setting. The

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latter one introduced prime ends based on the notion of the n-modulus of curve families in $\mathbf{R}^n$. One of our goals is to generalize Näkki's results to the sub-Riemannian setting. If one seeks to explore these ideas in other geometric settings, then the Heisenberg group $\mathbf{H}_1$ together with the sub-Riemannian geometry is a natural candidate. The reason being that $\mathbf{H}_1$ has a large enough family of quasiconformal mappings to make it an interesting pursuit, see the discussion at the end of Section 2.3. It is perhaps surprising that such a generalization is not straightforward and requires a new approach. First we recall some basic definitions for the Heisenberg groups including rectifiable curves, contact and quasiconformal mappings, which we define also in terms of the modulus of curve families (the rudimentary properties of modulus in $\mathbf{H}_1$ are recalled and proved in the Appendix).

In Section 3.1 we define prime ends following the approach in [57]. Upon introducing a topology on the prime end boundary ∂_P, Definition 3.5, we prove the following extension result, allowing us to extend a quasiconformal mapping to a homeomorphism between the prime end boundaries.

Theorem 1.1. *Let Ω and Ω' be domains in $\mathbf{H}_1$ and let $f\colon \Omega \to \Omega'$ be a quasiconformal mapping of Ω onto Ω'. Then, the extension mapping $F\colon \Omega \cup \partial_P \Omega \to \Omega' \cup \partial_P \Omega'$, where*

$$F(p) = \begin{cases} f(p) & \text{if } p \in \Omega, \\ [f(E_k)] & \text{if } p = [E_k] \in \partial_P \Omega, \end{cases}$$

is a homeomorphism.

Some of the most important definitions required in our work are given in Sections 3.2 and 3.3. There we recall the Loewner spaces, uniform domains in $\mathbf{H}_1$, and observe in Lemma 3.3 that in uniform domains, our modulus-based definition of prime ends, has an equivalent form in terms of the Heisenberg distance. This result is the key-part of our Definition 3.8 of the so-called collared domains. The original definition introduced by Väisälä and Näkki cannot be applied directly in our setting due to a rigidity property of the conformal mappings in $\mathbf{H}_1$ that is not present in the Euclidean setting (see Section 2), and the lack of domains satisfying the Loewner condition (problems which do not arise in the Euclidean setting). Furthermore, in Section 3.3 we relate collaredness with another important class of domains finitely connected at the boundary, and prove the following result.

Denote by $\partial_{\mathrm{SP}}\Omega$, the part of the prime end boundary $\partial_{\mathrm{P}}\Omega$ consisting of singleton prime ends only.

Theorem 1.2. *Let $\Omega \subset \mathbf{H}_1$ be a collared domain and let $f\colon \Omega \to \Omega'$ be a quasiconformal map of Ω onto a domain $\Omega' \subset \mathbf{H}_1$. Then there exists a homeomorphic extension $\tilde{f}\colon \overline{\Omega} \to \Omega' \cup \partial_{\mathrm{SP}}\Omega'$ defined as follows:*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ [f(E_k^x)] & \text{if } x \in \partial\Omega, \end{cases}$$

where $[E_k^x]$ is the canonical prime end associated with $x \in \partial\Omega$.

For the definition of canonical prime ends, see the discussion following Observation 3.3. Moreover, Theorem 1.2 naturally corresponds to Theorem 4.1 in Näkki [57] and Section 3.1 in Väisälä [70].

The goal of Section 4 is to present yet another perspective on prime ends, and show that in the domains finitely (in particular, locally) connected at the boundary, one can construct singleton prime ends associated with every boundary point. We also relate our prime ends to those studied in [2] in metric spaces, see Section 4.1. There, we recall a notion of mod-uniform domains used in Section 5, see Definition 4.1.

The important results of this paper are presented in Section 5, where we study the boundary behavior of quasiconformal mappings. We first recall notions of accessibility and observe that one can assign a singleton prime end to each accessible boundary point, see Observation 5.2. In total we provide three methods to obtain canonical prime ends in $\mathbf{H}_1$: by employing collaredness (Observation 3.3), via the finite connectedness at the boundary (Lemma 4.1) and in Observation 5.2. We show the Koebe theorem providing conditions which imply that a quasiconformal mapping has arcwise limits along all end-cuts in domains finitely connected at the boundary.

Theorem 1.3. (The Koebe theorem in $\mathbf{H}_1$) *Let $f\colon \Omega \to \Omega_0$ be a quasiconformal map between a domain $\Omega \subset \mathbf{H}_1$ finitely connected at the boundary and a mod-uniform domain $\Omega_0 \subset \mathbf{H}_1$. Then f has arcwise limits along all end-cuts of Ω.*

This result corresponds to the classical observation for conformal mappings and generalizes similar result in $\mathbf{R}^n$ due to Näkki [57, Theorem 7.2]. We then prove a version of the Lindelöf theorem relating the principal points of prime ends to cluster sets of mappings along end-cuts (see the appropriate definitions in Section 5).

Theorem 1.4. (The Lindelöf theorem in $\mathbf{H}_1$) *Let f be a bounded quasiconformal mapping of a ball $B \subset \mathbf{H}_1$ onto a domain $\Omega_0 \subset \mathbf{H}_1$ with the property that*

$$\lim_{r\to 0} \operatorname{diam}_{\mathbf{H}_1}\big(f(\partial B(x_0, r) \cap B)\big) = 0 \quad \textit{for all } x_0 \in \partial B.$$

Then for all $x_0 \in \partial B$ it holds that for every angular end-cut γ of B from x_0 we have

$$C_\gamma(f, x_0) = \Pi(f(E_k^{x_0})).$$

The proof of this result requires developing some new observations and illustrates differences between the Euclidean and the Heisenberg settings. The corresponding results in $\mathbf{R}^n$ are due to Gehring [31, Theorem 6], Näkki [57, Theorem 7.4] and Vuorinen [74, Section 3]. Finally, we show the following variant of the Tsuji theorem on the Sobolev capacities of sets of arcwise limits.

Theorem 1.5. (The Tsuji theorem in $\mathbf{H}_1$) *Let f be a quasiconformal mapping of a ball $B = B_R \subset \mathbf{H}_1$ of radius R such that $f(B)$ is a mod-uniform domain (cf Definition 4.1) and let $F\colon \overline{B} \to f(B) \cup A_f$ be an arcwise extension of f. If A_f is compact and the Sobolev 4-capacity $C_4(A_f) = 0$, then $C_4'(F^{-1}(A_f)) = 0$.*

By C_4' we denote the Sobolev 4-capacity considered with respect to ball $\overline{B}$ (see Section 5 for details of definitions). The main reason to employ the capacity C_4' in the proof of Theorem 1.5, is that we face once again the lack of some techniques available in $\mathbf{R}^n$, namely the modulus symmetry property.

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2. Preliminaries

In this section we recall basic definitions and properties of the Heisenberg group $\mathbf{H}_1$, including a brief discussion on curves and their lengths, the Heisenberg and the sub-Riemannian metrics. Moreover, we recall notions of the horizontal Sobolev spaces and quasiregular and quasiconformal mappings in $\mathbf{H}_1$. Further discussion, including the definition and properties of the modulus of curve families, is presented in the Appendix.

2.1. The Heisenberg group $\mathbf{H}_1$. The Heisenberg group $\mathbf{H}_1$ is often presented using coordinates (z,t) where $z = x+iy \in \mathbf{C}$, $t \in \mathbf{R}$ with multiplication defined by

$$\begin{aligned}(z_1,t_1)(z_2,t_2) &= (z_1+z_2, t_1+t_2+2\,\mathrm{Im}\,(z_1\bar{z}_2)) \\ &= (x_1+x_2, y_1+y_2, t_1+t_2+2(x_2y_1-x_1y_2)).\end{aligned}$$

It follows that $(z,t)^{-1} = (-z,-t)$, and a natural basis for the left invariant vector fields is given by the following vector fields

$$\tilde{X} = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad \tilde{Y} = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t} \quad \text{and} \quad \tilde{T} = \frac{\partial}{\partial t},$$

where $[\tilde{X},\tilde{Y}] = -4\tilde{T}$. The horizontal bundle is given pointwise by $\mathcal{H}_p = \mathrm{span}\{\tilde{X}(p), \tilde{Y}(p)\}$ and a curve $\gamma\colon I \to \mathbf{H}_1$ is horizontal if for almost all $t_0 \in I$, $\gamma'(t_0)$ exists and belongs to $\mathcal{H}_{\gamma(t_0)}$.

The pseudonorm given by

$$||(z,t)|| = (|z|^4 + t^2)^{1/4}$$

gives rise to a left invariant distance defined by $d_{\mathbf{H}_1}(p,q) = \|p^{-1}q\|$ which is called the Heisenberg distance. More explicitly we have

$$\begin{aligned}d_{\mathbf{H}_1}((z_1,t_1),(z_2,t_2)) &= ||(-z_1,-t_1)(z_2,t_2)|| \\ &= ||(z_2-z_1, t_2-t_1-2\mathrm{Im}(z_1\bar{z}_2))|| \\ &= (|z_2-z_1|^4 + (t_2-t_1-2\mathrm{Im}(z_1\bar{z}_2))^2)^{1/4},\end{aligned}$$

and we use the notation $\tau_{(z_0,t_0)}$ to denote the isometric map $(z,t) \to (z_0,t_0)(z,t)$, called the left-translation (in (z_0,t_0)).

A dilation by $r \in \mathbf{R}$ is defined by $\delta_r(z,t) = (rz, r^2t)$, indeed $d_{\mathbf{H}_1}(\delta_r(p),\delta_r(q)) = |r|d_{\mathbf{H}_1}(p,q)$. The left invariant Haar measure λ is simply the 3-dimensional Lebesgue measure on $\mathbf{H}_1$ and $\delta_r^* d\lambda = r^4 d\lambda$. It follows that the Hausdorff dimension of the metric measure space $(\mathbf{H}_1, d_{\mathbf{H}_1}, \lambda)$ is $Q=4$ and the space is Q-Ahlfors regular, i.e., there exists a positive constant c such that for all balls B with radius r we have

$$\frac{1}{c}r^Q \le \mathcal{H}^Q(B) \le cr^Q, \tag{1}$$

where $\mathcal{H}^Q$ denotes the Q-dimensional Hausdorff measure induced by $d_{\mathbf{H}_1}$.

Definition 2.1. A Q-regular metric measure space will be a triple (X,d,μ) satisfying (1).

We note that Q-regularity implies that the metric space (X,d) has Hausdorff dimension Q and μ is a constant multiple of the Q-dimensional Hausdorff measure induced by d, see [67].

Examples are when X is a Carnot group with sub-Riemannian distance d_s and Haar measure. Indeed, the Haar measure is a multiple of Lebesgue measure which is a multiple of the Q-dimensional Hausdorff measure induced by d_s. We can replace d_s with any equivalent metric. On $\mathbf{H}_1$ in particular the measure $\mathcal{H}^4$ is a constant multiple of 3-dimensional Lebesgue measure, an inequality similar to (1) is valid with $\mathcal{H}^4$ replaced by λ.

2.2. Rectifiable curves. A curve γ in $\mathbf{H}_1$ is a continuous map $\gamma\colon I \to \mathbf{H}_1$ where I is an open or closed interval. If $I = [a,b]$ then the Heisenberg length of γ is given by

$$l(\gamma) = \sup \sum_{i=1}^{n} d_{\mathbf{H}_1}(\gamma(t_i), \gamma(t_{i+1})), \tag{2}$$

where the supremum is over all finite sequences $a = t_1 \leq t_2 \leq \cdots \leq t_n \leq t_{n+1} = b$. If I is not closed then

$$l(\gamma) = \sup l(\gamma|_J)$$

where the supremum is over all closed subintervals $J \subset I$. If $l(\gamma) < \infty$ we say that γ is rectifiable.

A curve $\gamma\colon I \to \mathbf{H}_1$ is locally rectifiable if each subcurve $\gamma|_{[\alpha,\beta]}$ is rectifiable for all closed intervals $[\alpha, \beta] \subseteq I$. For example the horizontal curve $\gamma : (-1, 1) \to \mathbf{H}_1$ defined by $\gamma(s) = (x(s) + iy(s), t(s))$ where

$$x(s) + iy(s) = s \sin(1/s) + is \cos(1/s) \quad \text{and} \quad t(s) = -2s \tag{3}$$

is not locally rectifiable since any subcurve $\gamma(s)|_{[\alpha,\beta]}$ such that $0 \in [\alpha, \beta]$ is not rectifiable, see (4) below. Conversely, the curve $\gamma|_{(0,1)}$ is locally rectifiable but not rectifiable.

$$l(\gamma|_{[\alpha,\beta]}) = \int_\alpha^\beta \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2} = \int_\alpha^\beta \sqrt{1 + \frac{1}{s^2}}\, ds. \tag{4}$$

When $\alpha = 0$ the integral diverges, since

$$\int_0^\beta \sqrt{1 + \frac{1}{s^2}}\, ds = \lim_{t\to\infty} \int_{1/\beta}^t \frac{1}{u}\sqrt{1 + \frac{1}{u^2}}\, du > \lim_{t\to\infty} \int_{1/\beta}^t \frac{1}{u}\, du = \infty.$$

The following theorem is proved in exactly the same way as Theorem 3.2 in [68] with the Euclidean metric replaced by the Heisenberg distance and so we omit the proof.

Theorem 2.1. *If $\gamma\colon (a, b) \to \mathbf{H}_1$ is rectifiable then it has a unique extension $\gamma^*\colon [a, b] \to \mathbf{H}_1$ such that $l(\gamma^*) = l(\gamma)$.*

For each rectifiable curve γ of a closed interval there is a unique arc length parametrization of γ arising from the arc length function $S_\gamma\colon [a, b] \to [0, l(\gamma)]$ given by $S_\gamma(t) = l(\gamma|_{[a,t]})$. In particular there is a unique 1-Lipschitz map $\bar{\gamma}\colon [0, l(\gamma)] \to \mathbf{H}_1$ called the arc length parametrization such that $\gamma(t) = \bar{\gamma} \circ S_\gamma(t)$. The arc length parametrization facilitates the definition of the line integral of a nonnegative Borel function $\varrho\colon \mathbf{H}_1 \to [0, \infty]$ as follows:

$$\int_\gamma \varrho\, dl := \int_0^{l(\gamma)} \varrho \circ \bar{\gamma}(s)\, ds. \tag{5}$$

If I is open, then we set

$$\int_\gamma \varrho\, dl := \sup_{\gamma'} \int_{\gamma'} \varrho\, dl,$$

where the supremum is over all closed subcurves γ' of γ.

By Pansu [63], Lipschitz mappings between open subsets of Carnot groups are Pansu differentiable almost everywhere. For locally rectifiable curves this means that $\lim_{s\to 0} \delta_{1/s} \circ \tau^{-1}_{\bar{\gamma}(s_0)} \circ \bar{\gamma}(s_0 + s)$ exists for almost all s_0 which implies that $\bar{\gamma}'(s_0)$ exists and is horizontal, moreover the same is true for $\gamma'(s_0)$. If $\gamma\colon [a, b] \to \mathbf{H}$ is a horizontal curve, then the sub-Riemannian length of γ is given by the integral

$$l_S(\gamma) = \int_a^b \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2}\, ds$$

and $l(\gamma) = l_S(\gamma)$ (see Koranýi [45]). Moreover, the change of variable $s = S_\gamma(t)$ in (5) shows that

$$\int_\gamma \varrho\, dl = \int_a^b \varrho(\gamma(s))|\gamma'(s)|\, ds$$

where $|\gamma'(s)| = \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2}$.

The sub-Riemannian distance $d_s(p_1, p_2)$ is defined as the infimum of sub-Riemannian lengths of all horizontal curves joining points p_1 and p_2. The Heisenberg metric and the sub-Riemannian metric are equivalent, to be precise

$$\frac{1}{\sqrt{\pi}}\, d_s(p_1, p_2) \le d_{\mathbf{H}_1}(p_1, p_2) \le d_s(p_1, p_2),$$

see Bellaïche [9].

2.3. Horizontal Sobolev space on $\mathbf{H}_1$, contact, quasiregular and quasiconformal mappings on $\mathbf{H}_1$. Below we recall some basic definitions in the theory of the Sobolev spaces in $\mathbf{H}_1$ and contact mappings, and apply them to define the main classes of mappings we study in the paper, namely quasiregular and quasiconformal mappings.

Definition 2.2. Let $U \subset \mathbf{H}_1$ be an open subset of $\mathbf{H}_1$. For $1 \le p < \infty$, we say that a function $u\colon U \to \mathbf{R}$ belongs to the *horizontal Sobolev space* $HW^{1,p}(U)$ if $u \in L^p(U)$ and the horizontal derivatives $\tilde{X}u$ and $\tilde{Y}u$ exist in the distributional sense and are represented by elements of $L^p(U)$. The space $HW^{1,p}(U)$ is a Banach space with respect to the norm

$$\|u\|_{HW^{1,p}(U)} = \|u\|_{L^p(U)} + \|(\tilde{X}u, \tilde{Y}u)\|_{L^p(U)}.$$

In the similar way we define the local spaces $HW^{1,p}_{\mathrm{loc}}(U)$. We define space $HW^{1,p}_0(U)$ as a closure of $C_0^\infty(U)$ in $HW^{1,p}(U)$.

The horizontal gradient $\nabla_0 u$ of $u \in HW^{1,p}_{\mathrm{loc}}(U)$ is given by the equation

$$\nabla_0 u = (\tilde{X}u)\tilde{X} + (\tilde{Y}u)\tilde{Y}.$$

A contact form on $\mathbf{H}_1$ is given by $\omega = dt + 2(x\, dy - y\, dx)$, in particular $\omega \wedge d\omega$ is a volume form and $\mathcal{H}_p = \ker \omega_p$.

Definition 2.3. Let $\Omega, \Omega' \subset \mathbf{H}_1$ be domains in $\mathbf{H}_1$. We say that a diffeomorphism $f\colon \Omega \to \Omega'$ is a *contact transformation* if it preserves the contact structure, i.e.

$$f^*\omega = \Lambda\omega,$$

where $\Lambda\colon \Omega \to \mathbf{R}$ and $\Lambda \ne 0$ in Ω.

Note that the definition implies that f_* preserves the horizontal bundle, moreover we can weaken the regularity assumption to $HW^{1,Q}_{\mathrm{loc}}(\Omega, \Omega')$ and simply require statements to hold λ-a.e. The contact maps that will be of relevance in our work here will be those which are quasiconformal.

Quasiconformality has a number of quantitatively equivalent definitions, although the distortion factor need not be the same for each definition. We begin with the metric definition. Let $f\colon \Omega \to \Omega'$ be a homeomorphism where Ω and Ω' are domains in $\mathbf{H}_1$, and let the distortion function of f be given by

$$H_f(p_1, r) = \frac{\sup\{d(f(p_1), f(p_2))\colon d(p_1, p_2) \le r\}}{\inf\{d(f(p_1), f(p_2))\colon d(p_1, p_2) \ge r\}}.$$

Definition 2.4. Let Ω, Ω' be domains in $\mathbf{H}_1$. A homeomorphism $f\colon \Omega \to \Omega'$ is *quasiconformal*, if

$$\tilde{H}_f(q) := \limsup_{r\to 0} H_f(q,r)$$

is bounded on Ω.

It was shown by Mostow that a quasiconformal map on a domain in $\mathbf{H}^1$ is *absolutely continuous on lines* (ACL), see the discussion in Korányi–Reimann [47]. This property is defined analogously as the ACL property for mappings on open subsets of $\mathbf{R}^n$, but in terms of the fibrations given by the left invariant horizontal vector fields $\tilde{X}$ and $\tilde{Y}$ instead of lines parallel to the coordinate axes.

Quasiconformal maps are Pansu differentiable almost everywhere, see [63], which implies that they are $HW^{1,Q}_{\mathrm{loc}}(\Omega,\Omega')$ regular contact maps. We recall that the Pansu differential $Df(p_1)$ is the automorphism of $\mathbf{H}_1$ defined as

$$Df(p_1)\,p_2 := \lim_{t\to 0} \delta_t^{-1}\circ \tau^{-1}_{f(p_1)} \circ f \circ \tau_{p_1} \circ \delta_t(p_2),$$

where $p_1, p_2 \in \mathbf{H}_1$ and $\delta_t(p_2)$ stands for the dilation by t at p_2, while τ_{p_1} denotes the left-translation by p_1.

It follows that quasiconformality can be expressed analytically by the requirement that the map is ACL, Pansu differentiable a.e. and there exists $1 \le K_a < \infty$ such that

$$\|Hf_*\|^4_\infty \le K_a \det f_*$$

where

$$\|Hf_*\|_\infty = \max\{|f_*(V)|\colon V \in \mathcal{H}_q,\ |V| = \sqrt{dx(V)^2 + dy(V)^2} = 1\}$$

and the subscript a indicates analytic. We note that K_a-analytically quasiconformal is equivalent to metrically quasiconformal with $\operatorname{ess\,sup}_\Omega \tilde{H}_f = \sqrt{K_a}$, however in practice we will not need this distinction and so we will simply use the term K-quasiconformal.

As pointed out above, quasiconformal mappings are absolutely continuous on almost all locally rectifiable curves in the sense that the family Γ_f consisting of rectifiable curves whose image under a quasiconformal map f is not rectifiable satisfies $\mathrm{Mod}_4(\Gamma_f) = 0$ (see Appendix for definitions and some properties of the modulus of curve families). See Theorem 9.8 in Heinonen–Koskela–Shanmugalingma–Tyson [39] for a proof in the setting of spaces with locally bounded geometry. Stated in terms specific to the Heisenberg setting we have:

Theorem 2.2. (Theorem 18 in Balogh–Fässler–Platis [7]) *If $f\colon \Omega \to \Omega'$ is a quasiconformal map between two domains in $\mathbf{H}_1$ and Γ is a curve family in Ω, then*

$$\frac{1}{K_a}\mathrm{Mod}_4(\Gamma) \le \mathrm{Mod}_4(f\Gamma) \le K_a \mathrm{Mod}_4(\Gamma).$$

We remark that inequalities in Theorem 2.2 are also a sufficient condition for quasiconformality which is usually referred to as the geometric definition of quasiconformality.

In Capogna–Cowling [15], the authors prove that 1-quasiconformal maps (1-qc for short) are C^∞ and, consequently from Koranýi–Reimann [46], the following Liouville theorem holds: an orientation preserving 1-qc map of a domain $\Omega \subseteq \mathbf{H}_1$ is given by the action of an element in $SU(1,2)$. In particular, a 1-qc map is always a composition of the following five basic types of 1-qc maps:

(1) Left translation (isometry),

(2) Dilation (1-qc),
(3) Rotation: $R_\theta(z,t) = (e^{i\theta}z, t)$ (isometry),
(4) Inversion in the unit sphere: $J(z,t) = \frac{-1}{|z|^4+t^2}(z(|z|^2+it), t) = \left(\frac{z}{it-|z|^2}, \frac{-t}{|z|^4+t^2}\right)$ (1-qc),
(5) Reflection: $S(z,t) = (\overline{z}, -t)$ (isometry).

Note that the maps 1 to 4 generate $SU(1,2)$.

We remark that via the Cayley transformation, the inversion arises from the fact that the one point compactification of $\mathbf{H}_1$ can be realised as the unit sphere in $\mathbf{C}^2$, see [46]. The inversion facilitates the definition of stereographic projection of any sphere in $\mathbf{H}_1$ to the complex plane. Using translations and dilations, the given sphere is mapped to the sphere with center $(0,-3/2)$ and radius $1/\sqrt{2}$ and then inverted in the unit sphere centered at $(0,-1)$, i.e., apply

$$\tau_{(0,-1)} \circ J \circ \tau_{(0,1)}. \tag{6}$$

We note that, unlike the case of $\mathbf{R}^n$, we do not have the freedom to normalize, since left translations do not preserve the complex plane. More explicitly, in the Euclidean case we can post compose the stereographic projection with a translation to produce a conformal map with the property that a given point on the sphere is mapped to a chosen point in the complex plane, whereas in the Heisenberg group this not possible, since left translations do not preserve the complex plane.

Since the 1-qc maps are given by the action of a finite dimensional Lie group, we say that $\mathbf{H}_1$ is 1-qc rigid. In such cases, a Carathéodory extension theorem for 1-qc mappings is somewhat trivial. Similarly, if we are going to consider a nontrivial Carathéodory extension theorem for quasiconformal maps, we at least need to avoid Carnot groups that are contact rigid, i.e., the contact maps are given by the actions of a finite dimensional Lie group, see [62]. Following Euclidean space, the most nonrigid of all Carnot groups is $\mathbf{H}_1$. Indeed, the pseudo group of local contact mappings is large and so a reasonably interesting theory can be expected. In fact, in [46] the authors produce an infinite dimensional family of quasiconformal maps as flows of vector fields as well as developing a Beltrami type equation. However, there is no existence theorem for this equation. On the other hand, in Balogh [5], it is shown that quasiconformal maps exist on $\mathbf{H}_1$ that are not bi-Lipschitz. Furthermore, quasiconformal mappings which are not global arise from the winding maps discussed by Balogh–Fässler–Peltonen [6].

3. Prime ends in the Heisenberg group $\mathbf{H}_1$

In this section we give basic definitions of the prime ends theory in the sub-Riemannian setting. First, following the modulus approach of Näkki, we define prime ends and a topology on the prime end boundary. Using prime ends, we show the first extension result for quasiconformal mappings, see Theorem 1.1. The remaining part of this section is devoted to study the so-called *collared domains*. Näkki [57] and Väisälä [68] defined collared domains in order to study extension properties and the prime end boundary. It turns out that the structure of the Heisenberg group does not allow us to follow their approach. Namely, the Loewner property of collaring domains, crucial for the properties of prime ends, need not hold for natural counterparts of collaring domains in $\mathbf{H}_1$. Therefore, we need a new definition, in particular we impose an additional uniformity assumption on the collaring neighborhood. See details in Section 3.3 and Section 3.2 for Loewner and uniform domains in the Heisenberg setting. Using collared domains, we obtain another extension result for quasiconformal mappings, namely Theorem 1.2.

3.1. Prime ends according to Näkki. Näkki in [57] introduced a theory of prime ends for domains in $\mathbf{R}^n$ based on the notion of n-modulus. We follow his idea and develop the appropriate theory in the Heisenberg setting based on the notion of Q-modulus where $Q = 4$ is the Hausdorff dimension of $\mathbf{H}_1$.

Definition 3.1. (cf. Section 3.1 in [57]) A connected subset E of a domain $\Omega \subset \mathbf{H}_1$ is called a *cross-set* if:

(1) E is relatively closed in Ω,
(2) $\overline{E} \cap \partial\Omega \neq \emptyset$,
(3) $\Omega \setminus E$ consists of two components whose boundaries intersect $\partial\Omega$.

Definition 3.2. A collection $\{E_k\}_{k=1}^{\infty}$ of cross-sets is called a *chain* if E_k separates E_{k-1} and E_{k+1} within Ω for all k. We denote the component of $\Omega \setminus E_k$ containing E_{k+1} by $D(E_k)$ and define an *impression* of a chain $\{E_k\}_{k=1}^{\infty}$ as follows

$$I[E_k] := \bigcap_{k=1}^{\infty} \overline{D(E_k)}.$$

The definition immediately implies that the impression of a chain is either a continuum or a point. (We remark that by a continuum we always mean a nondegenerate continuum, i.e. a connected compact set containing at least two points.) The set of all chains is in some sense too large so the following additional conditions are imposed to cut it down.

Recall that by $\mathrm{Mod}_4(E, F, \Omega)$ we denote a 4-modulus of a curve family of rectifiable curves γ with one endpoint in E and another in F and $\gamma \setminus (E \cup F) \subset \Omega$ (cf. the discussion before Lemma A.1 and also Lemma 5.1).

Definition 3.3. A chain is a *prime chain* if:

(a) $\mathrm{Mod}_4(E_{k+1}, E_k, \Omega) < \infty$,
(b) For any continuum $F \subset \Omega$ we have that

$$\lim_{k\to\infty} \mathrm{Mod}_4(E_k, F, \Omega) = 0.$$

In view of Theorem 2.2, conditions (a) and (b) are quasiconformally invariant, and under certain restrictions on Ω, imply stronger separation of the cross sets as well as control over their diameter. In particular, we will discuss domains Ω so that (a) implies $\mathrm{dist}_{\mathbf{H}_1}(E_k, E_{k+1}) > 0$ and (b) implies $\mathrm{diam}_{\mathbf{H}_1}(E_k) \to 0$, cf. Lemma 3.3.

It is crucial for our further work to know that the impression of a (prime) chain is a subset of the topological boundary $\partial\Omega$. This follows from Lemma A.10 in [2] together with Part (b) of Definition 3.3. Lemma A.10 is formulated for the so-called acceptable sets (cf. Definition 4.2 below) but for the sake of convenience we will state it without appealing to the definition of prime ends as in [2] and specialize to the setting of $\mathbf{H}_1$. This is due to the fact that $\mathbf{H}_1$ satisfies the main assumptions of [2], that is $\mathbf{H}_1$ is a complete metric measure space with a doubling measure, supporting a $(1, 4)$-Poincaré inequality.

Lemma 3.1. (Lemma A.10 in [2]) *Let $\{D_k\}_{k=1}^{\infty}$ be a sequence of open bounded connected sets $D_k \subsetneq \Omega$ such that $\overline{D_k} \cap \partial\Omega \neq \emptyset$ satisfying $\overline{D}_{k+1} \cap \Omega \subset D_k$ for each k. If $\lim_{k\to\infty} \mathrm{Mod}_4(D_k, B, \Omega) = 0$ for some ball $B \subset \Omega \setminus D_1$, then $I = \bigcap_{k=1}^{\infty} \overline{D}_k \subset \partial\Omega$.*

Let now $\{E_k\}_{k=1}^{\infty}$ be a (prime) chain in a domain $\Omega \subset \mathbf{H}_1$. Notice that sets $D(E_k)$, as in Definition 3.2, for all k satisfy assumptions of Lemma 3.1. Moreover, $\Gamma(E_k, F, \Omega) < \Gamma(D(E_k), F, \Omega)$ for all k sufficiently large, and thus, Part 5 of Lemma A.1 implies that property (b) of Definition 3.3 holds for $\{D(E_k)\}_{k=1}^{\infty}$ and any continuum $F \subset \Omega$ as well. If B is a ball with $B \subset \Omega \setminus D(E_1)$, then $F = \overline{B}$ is a continuum in Ω. Furthermore, $\mathrm{Mod}_4(D(E_k), B, \Omega) \leq \mathrm{Mod}_4(E_k, B, \Omega)$. Thus, Lemma 3.1 implies that $I := \bigcap_{k=1}^{\infty} \overline{D(E_k)} \subset \partial\Omega$.

It turns out that one can define an equivalence relation on the set of prime chains in a given domain. This gives rise to one of the main notions of our work, the so-called prime ends.

Definition 3.4. We say that a chain $\{F_k\}_{k=1}^{\infty}$ *divides* the chain $\{E_k\}_{k=1}^{\infty}$ if each domain $D(E_k)$ contains all but a finite number of the cross-sets F_l. If also the opposite holds, that is each domain $D(F_l)$ contains all but a finite number of the cross-sets E_k, then we say that chains $\{E_k\}_{k=1}^{\infty}$ and $\{F_k\}_{k=1}^{\infty}$ are *equivalent*. The equivalence classes of prime chains are called *prime ends* of Ω and the set of all prime ends is denoted $\partial_{\mathrm{P}}\Omega$ and called the *prime end boundary*. We use the notation $[E_k]$ to denote the prime end defined by the prime chain $\{E_k\}_{k=1}^{\infty}$.

If $[E_k] \in \partial_{\mathrm{P}}\Omega$, then the impression of any representative of $[E_k]$ is the same, and so the impression $I[E_k]$ of $[E_k]$ is well defined. By Theorem 2.2, a quasiconformal map $f\colon \Omega \to \Omega'$, naturally extends to the prime ends by setting $f([E_k]) := [f(E_k)]$, see also Theorem 1.1.

We introduce a topology on the prime end boundary of a domain in $\mathbf{H}_1$. A similar construction in $\mathbf{R}^n$ is presented in [57], see also [2, Section 8] for a discussion in metric spaces. We then apply this topology in studying the extension of a quasiconformal map to a map between the prime ends closures of the underlying domains.

Definition 3.5. A topology on $\Omega \cup \partial_{\mathrm{P}}\Omega$ is given by extending the relative topology of Ω by defining neighborhoods of prime ends as follows: A neighborhood of a prime end $[E_k] \in \partial_{\mathrm{P}}\Omega$ has the form $U \cup U_P$ where

(a) $U \subset \Omega$ is open,
(b) $\partial U \cap \partial\Omega \neq \emptyset$,
(c) $U \cup (\partial U \cap \partial\Omega)$ is relatively open,
(d) $D(E_k) \subset U$ for k sufficiently large,
(e) $U_P = \{[F_l] \in \partial_P\Omega\colon D(F_l) \subset U$ for all sufficiently large$\}$.

We comment that another definition of a topology can be given if one defines the convergence of points and prime ends to a prime end. The above definition is similar in construction to the one given in Proposition 8.5 in [2], however, there, the constructed topology fails to be Hausdorff, cf. Example 8.9 in [2]. However, notice that since cross-sets are assumed to be connected, one cannot consider boundaries of acceptable sets (cf. Definitinion 4.1 in [2]) as in that example to be cross-sets. The topology as in Definition 3.5 is Hausdorff at least for collared domains (according to the Definition 3.8) or for more general domains finitely connected at the boundary. This statement follows from Observation 3.3 and Lemma 4.1, respectively. In both cases every boundary point is the impression of a prime end. Therefore, since interior points of Ω are separated and points in the interior of Ω are separated from points in the prime end boundary $\partial_{\mathrm{P}}\Omega = \partial_{\mathrm{SP}}\Omega$, it remains to see that any pair of distinct points in $\partial_{\mathrm{SP}}\Omega$ are separated. To this end let $[E_j], [F_k] \in \partial_{\mathrm{SP}}\Omega$ be distinct prime ends, then it follows that there exists $n \in \mathbf{N}$ such that $D(E_j) \cap D(F_k) = \emptyset$ for all

$j, k \geq n$. If $U = D(E_n)$ and $V = D(F_n)$, then $(U \cup U_P) \cap (V \cup V_P) = \emptyset$, hence $[E_j]$ and $[F_k]$ are separated and the topology is indeed Hausdorff as claimed.

An important question to consider is: *when does $\Omega \cup \partial_P \Omega$ together with the topology described above become a compact space?* Obviously, it is necessary that Ω is relatively compact in the metric topology of $\mathbf{H}_1$ but delicate issues can arise with regards to $\partial_P \Omega$. In particular, if Ω is relatively compact and $\{U^\alpha\}$ is a covering of Ω by relatively open sets, then we can select a finite collection $\{U^\beta\}$ which covers Ω and write $\{U^\beta\} = \{U^{\beta_0}\} \cup \{U^{\beta_1}\}$ where each element of the collection $\{U^{\beta_1}\}$ satisfies $\partial U^{\beta_1} \cap \partial \Omega \neq \emptyset$. Then, one considers if $\{U^{\beta_0}\} \cup \{U^{\beta_1} \cup U_P^{\beta_1}\}$ is a cover of $\Omega \cup \partial_P \Omega$ which requires that every prime end of Ω belongs to $U_P^{\beta_1}$ for some β_1. If an underlying domain is finitely connected at the boundary and so every boundary point is the impression of a prime end, then this requirement is fulfilled (see the discussion below, in particular Observation 3.3 and Lemma 4.1). In the context of extension of quasiconformal mappings, the domains of interest, the so-called collared domains, will be seen to have the property that all the prime ends have singleton impressions (see Section 3.3).

Example 1. The following variant of the topologist's comb is an example of a domain which fails to be finitely conntected at the boundary and its prime end boundary is noncompact, see also Example 5.1 and the presentation in Sections 7–10 of [2], and Björn [10].

Let $L_k = (0, 1/2] \times \{1/2^k\} \times (0, 1)$ and let $\Omega = (0, 1)^3 \setminus \bigcup_{k=1}^\infty L_k$ be a topologist's comb in $\mathbf{H}_1$. Define

$$E_k = \Big(\{1/2 + 1/2^{k+1}\} \times (0, 1/2^k] \times (0, 1)\Big) \cup \Big((1/2, 1/2 + 1/2^{k+1}) \times \{1/2^k\} \times (0, 1)\Big).$$

Then $\{E_k\}_{k=1}^\infty$ is a chain in Ω with impression $I[E_k] = [0, 1/2] \times \{0\} \times [0, 1]$. However, $\{E_k\}_{k=1}^\infty$ is not a prime chain. It satisfies part (a) of Definition 3.3 but fails part (b). Indeed, since Ω has finite measure and $\operatorname{dist}_{\mathbf{H}_1}(E_{k+1}, E_k) > 0$ one has a positive lower bound for the length of curves in the family $\Gamma(E_{k+1}, E_k, \Omega)$. Then, the standard estimate provides an upper bound for $\operatorname{Mod}_4(E_{k+1}, E_k, \Omega)$, see e.g. [39, Lemma 5.3.1], showing that Definition 3.3(a) holds for all k.

Let D_k denote domains as in Definition 3.2 and B be a ball in $\Omega \setminus D_1$. Then,

$$\operatorname{Mod}_4(E_k, B, \Omega) \geq \operatorname{Mod}_4(D_k, B, \Omega) > 0$$

for all large enough k and the same holds for the limit, when $k \to \infty$. Indeed, the positivity of the modulus is a direct consequence of the fact that all D_k contain a copy of a line segement $F := (0, 1)$ whose Sobolev capacity $C_4(F) > 0$. Hence, by Lemma 2.24 in [11], also $C_4'(F) > 0$ and $C_4'(D_k) > 0$ (cf. (16) in Section 5). Thus, the modulus of curves in Ω passing through D_k is positive (Proposition 1.48 in [11]). By a variant of Theorem 3.1 in [56] we get that $\operatorname{Mod}_4(D_k, B, \Omega) > 0$ is equivalent to $\operatorname{Mod}_4(D_k, B', \Omega) > 0$ for all balls $B' \subset \Omega \setminus D_1$ (cf. also Lemma A.11 in [2]). If the latter failed, then also the modulus of curves in Ω passing through D_k would be zero.

Therefore, $[E_k]$ is a not a prime end in Ω. Similar reasoning shows that there is no prime end associated with a set $[0, 1/2] \times \{0\} \times [0, 1] \subset \partial \Omega$. However, any other point of $\partial \Omega$ is an impression of a singleton prime end.

Below we present the proof of our first extension result, Theorem 1.1 allowing us to extend a quasiconformal mapping between domains in $\mathbf{H}_1$ to a homeomorphism between the prime ends closures.

Proof of Theorem 1.1. The map F is well defined. Indeed, for $p \in \Omega$ this follows from f being a homeomorphism. For $p = [E_k] \in \partial_{\mathrm{P}}\Omega$, the discussion following Definition 3.4 gives us that the value of $F([E_k])$ is independent on the representative of $[E_k]$.

The extended map is a bijection. If $[F'_l] \in \partial_P\Omega'$ and $F_l = f^{-1}(F'_l)$ for all l, then $F([F_l]) = [F'_l]$ and that $\{F_l\}_{l=1}^{\infty}$ defines a (prime) chain and, thus, a prime end in $\partial_{\mathrm{P}}\Omega$, follows from f being a homeomorphism and Theorem 2.2. If $F([E_k]) = F([F_l])$, then by the definition of F, it holds that $[f(E_k)] = [f(F_l)]$, which again by Theorem 2.2, implies that $[E_k] = [F_l]$.

The map F is continuous. If $V \cup V_P$ is a neighborhood contained in $\Omega' \cup \partial_P\Omega'$ such that $f([E_k]) \in V_P$ for some $[E_k] \in \partial_{\mathrm{P}}\Omega$, then $D(f(E_k)) \subset V$ for k sufficiently large, and since $D(f(E_k)) = f(D(E_k))$, we have that $D(E_k) \subset f^{-1}(V)$. It follows that the preimage $F^{-1}(V \cup V_P)$ is contained in $f^{-1}(V) \cup f^{-1}(V)_P$. Moreover, if $[F_l] \in f^{-1}(V)_P$ then $D(F_l) \subset f^{-1}(V)$ for l sufficiently large and $f(D(F_l)) = D(f(F_l)) \subset V$. Hence $[f(F_l)] \in V_P$ and we conclude that

$$F^{-1}(V \cup V_P) = f^{-1}(V) \cup f^{-1}(V)_P.$$

It follows that the preimage $F^{-1}(V \cup V_P)$ is open, thus implying that F is continuous.

The extended map is open. Let $U \cup U_P$ be a neighborhood in $\Omega \cup \partial_P\Omega$ and let $[E_k] \in U_P$. It follows that $F([E_k]) \in f(U)_P$ since $f(D(E_k)) = D(f(E_k))$. Furthermore, if $[F_l] \in f(U)_P$ then $f^{-1}(D(F_l)) \subset U$ for l sufficiently large. Hence $F(U \cup U_P) = f(U) \cup f(U)_P$. □

We illustrate the above theorem with the following examples. At present, even the classical theory on boundary behavior of quasiconformal mappings in Euclidean spaces is not replete with explicit examples of QC-mappings between the two given domains, e.g. see Chapter 17 in Väisälä [68] and references therein for $\mathbf{R}^n$, see also Meyer [52] for some examples in $\mathbf{R}^3$. This circumstance is due to the lack of the measurable Riemann mapping theorem in $\mathbf{R}^n$ for $n \geq 3$ and in the setting of the Heisenberg group $\mathbf{H}_1$.

In the example below we show a class of mappings with no extensions to homeomorphisms between the topological closures but with extensions to the prime ends boundaries as in Theorem 1.1. Then, in the remark we discuss the subtleties related to the quasiconformal mappings on domains with inaccessible boundary points.

Example 2. (Winding map) A significant family of nontrivial quasiconformal extension problems arises from the winding maps studied by Balogh–Fässler–Peltonen in [6]. In particular, their work leads to examples where we have a quasiconformal map of a domain Ω such that $\partial\Omega$ contains points at which f does not extend to a homeomorphism. In such cases a homeomorphic extension to a prime end compactification is all that is available and thus in a sense essential.

We give a brief description of the aforementioned examples and refer the reader to [6] for the detailed analysis. Let us recall that the cylindrical coordinate projection $\Phi\colon [0,\infty) \times \mathbf{R} \times \mathbf{R} \to \mathbf{H}_1$ is defined by $\Phi(r,\theta,t) = (re^{i\theta}, t)$. The restriction $\Phi|_{[0,\infty)\times(-\pi,\pi]\times\mathbf{R}}$ is injective and is denoted Φ_0. It follows from the definition that

$$\Phi_0^{-1}(z,t) = (|z|, \arg(z), t),$$

where the argument is chosen so that $\arg(z) \in (-\pi, \pi]$.

The winding map $\tilde{f}_k\colon [0,\infty) \times \mathbf{R} \times \mathbf{R} \to [0,\infty) \times \mathbf{R} \times \mathbf{R}$, of degree $k \in \mathbf{Z}$, is given by

$$\tilde{f}_k(r,\theta,t) = (r, k\theta, kt).$$

By virtue of the fact that k is an integer, we can define a function $f_k\colon \mathbf{H}_1 \to \mathbf{H}_1$ by setting

$$f_k(z,t) = \Phi \circ \tilde{f}_k \circ \Phi_0^{-1}(z,t) = (|z|^{1-k} z^k, kt). \tag{7}$$

Note that the restriction of f_k to the domain

$$\Omega_k = \{(z,t)\colon z \neq 0,\ \arg(z) \in (-\pi/k, \pi/k)\}$$

is a diffeomorphism. However, f_k is not invertible at any point on the t-axis. Furthermore, as shown in [6], f_k is K-quaisconformal on Ω_k with $K = k^2$ according to the analytic definition of quasiconformality.

Let $r > 0, T > 0$ and let us define the following bounded subdomain of Ω_k:

$$\Omega_{k,r,T} = \{(z,t)\colon 0 < |z| < r,\ \arg(z) \in (-\pi/k, \pi/k),\ |t| < T\}.$$

Then, $f_k|_{\Omega_{k,r,T}}$ is quasiconformal and

$$\Omega'_{k,r,T} = f_k(\Omega_{k,r,T}) = \{(z,t)\colon 0 < |z| < r,\ \arg(z) \in (-\pi, \pi),\ |t| < T\}.$$

We denote $f_k(\Omega_{k,r,T})$ by $\Omega'_{k,r,T}$.

Note that $f_k|_{\Omega_{k,r,T}}$ is quasiconformal, and has a continuous extension to $\Omega_{k,r,T} \cup t$-axis which is injective. However on the faces given by $\arg(z) = \pm\pi/k$ the continuous extension fails to be injective and, thus, cannot be homeomorphic. Nevertheless, Theorem 1.1 applied with $f = f_k|_{\Omega_{k,r,T}}$ and $\Omega = \Omega_{k,r,T}$, $\Omega' = \Omega'_{k,r,T}$ gives the existence of a homeomorphism between $\Omega_{k,r,T} \cup \partial_P \Omega_{k,r,T}$ and $\Omega'_{k,r,T} \cup \partial_P \Omega'_{k,r,T}$. In particular, all prime ends are singletons and each point p in the set

$$\{(z,t) \in \partial\Omega'_{k,r,T}\colon\ \arg(z) = -\pi \text{ or } \arg(z) = \pi\} \cup t\text{-axis}$$

corresponds to two prime ends with p as their impression.

We remark that another example with a bounded domain is given by

$$f_k \circ \tau_{(1,0)}^{-1} \circ J\colon J \circ \tau_{(1,0)}(\Omega_k) \to \Omega'.$$

Remark 1. (Domains with inaccessible boundary points) In the planar setting one studies several examples of simply-connected domains with inaccessible boundary points and such domains are known to be conformally, and thus quasiconformally, equivalent to a unit disc via the Riemann mapping theorem. Consequently, the classical theory of prime ends due to Carathéodory applies (see e.g. Sections 3 and 5 in [2], Chapter 9 in Collingwood–Lohwater [25]). Hence it follows that in the planar case, one has at hand a number of explicit examples to illustrate the boundary behavior of quasiconformal maps. The situation changes in the setting of higher dimensional Euclidean spaces and for the Heisenberg groups, where a viable counterpart of the Riemann mapping theorem is still to be discovered. This leads to several problems when studying quasiconformal mappings between domains without inaccessible boundary points, e.g., a ball and domains with inaccessible boundary points.

A counterpart of a topologist's comb in $\mathbf{H}_1$ is possible however constructing a nonextendable quasiconformal map of the comb presents challenges which appear to be difficult to overcome. At the very least, the comb must be the domain of a contact map which does not extend homeomorphically to the topological boundary. A potential candidate for such a map is the prolongation (in the jet space sense) of the Riemann mapping of the two dimensional comb to the unit disc, see Chapter 4 in Olver [60] or [76] for details on prolongation in jet spaces. Whilst it is always the case that the prolongation is locally diffeomorphic and contact on its domain of definition, it is not necessarily injective which is already a nontrivial obstacle. To address the quasiconformality of the prolongation is equally if not more challenging.

3.2. The Loewner condition and uniform domains. Let (X, d, μ) be a rectifiably connected metric measure space of Hausdorff dimension Q, equipped with a locally finite Borel regular measure μ. Following Definition 8.1 in Chapter 8 of Heinonen [34], we define a *Loewner function* $\Psi_X\colon (0,\infty) \to [0,\infty)$ by the formula

$$\Psi_X(t) := \inf\{\mathrm{Mod}_Q\Gamma(E,F,X)\colon \Delta(E,F) \le t\}, \tag{8}$$

where $E, F \subset X$ are nondegenerate disjoint continua in X and

$$\Delta(E,F) = \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}E, \mathrm{diam}F\}}$$

denotes the relative distance between sets E and F. We note that by definition Ψ_X is decreasing.

Definition 3.6. A rectifiably connected metric measure space (X, d, μ) is said to be Q-Loewner if $\Psi_X(t) > 0$ for all $t > 0$.

In general, if a metric measure space (X, d, μ) satisfies some connectivity and volume growth conditions, then it is Q-Loewner if and only if it supports a weak $(1, Q)$-Poincaré inequality (see Chapter 9 in [34]). A Carnot group G equipped with the sub-Riemannian metric d_s and Lebesgue measure is such a metric measure space, and in fact a Q-Loewner space where Q is the Hausdorff dimension of (G, d_s) (see Proposition 11.17 in Hajłasz–Koskela [33]).

From the point of view of our studies, where we use the Heisenberg metric in preference to the sub-Riemannian, we have the following result:

Theorem 3.1. (Proposition 14.2.9 in Heinonen–Koskela–Shanmugalingam–Tyson [39]) *The metric measure space* $(\mathbf{H}_1, d_{\mathbf{H}_1}, \lambda)$ *is* 4*-Loewner.*

We remark that the previous theorem is also a consequence of Theorem 9.27 in [34].

Next, we recall the definition of uniform domains. Such domains play an important role in analysis and PDEs, see Heinonen [34], Martio–Sarvas [50], Näkki–Väisälä [59] and Väisälä [69] for more information on uniform domains. Examples of uniform domains encompass bounded Lipschitz domains, quasidisks including some domains with fractal boundaries such as the von Koch snowflake, see also Capogna–Garofalo [17], Capogna–Tang [19], Monti–Morbidelli [53] for further examples of uniform domains in the setting of Heisenberg groups. In particular, Theorem 1.3 in [53] states that every $C^{1,1}$ domain in a step 2 homogeneous group is an NTA domain and thus uniform.

Definition 3.7. A domain $\Omega \subset \mathbf{H}_1$ is called uniform, if there exist two positive constants α and β, such that each pair of points $x, y \in \Omega$ can be joined by a rectifiable curve γ, such that:

(a) $l(\gamma) \le \beta d_{\mathbf{H}_1}(x,y)$,

(b) $\alpha \min\{l(\gamma_{xz}), l(\gamma_{yz})\} \le \mathrm{dist}_{\mathbf{H}_1}(z, \partial\Omega)$ for all $z \in \gamma$, where γ_{xz} (γ_{yz}) denote subarcs of γ joining x and z (y and z).

An important example of uniform domains in $\mathbf{H}_1$ is provided by the following result.

Lemma 3.2. (Corollary 1 in Capogna–Garofalo [17]) *Balls in the Heisenberg metric are uniform domains.*

We note that the class of uniform domains is independent of the choice between the Heisenberg metric $d_{\mathbf{H}_1}$ or the sub-Riemannian metric d_s. Indeed, this is clear once it is observed that the length of a curve in the sub-Riemannian metric is the same as the length in the Heisenberg metric, and that the two metrics are equivalent. Similarly, the class of quasiconformal mappings is the same regardless of which metric we use. We can, thus, state the following theorem for the Heisenberg metric even though in Capogna–Tang [19] it is proved only for the sub-Riemannian metric (cf. Theorem 2.15 in Martio–Sarvas [50] for the prototypical result in the Euclidean setting).

Theorem 3.2. (Theorem 3.1 in [19]) *Let $\Omega \subset \mathbf{H}_1$ be a uniform domain with constants α and β. If $f\colon \mathbf{H}_1 \to \mathbf{H}_1$ is a global K-quasiconformal map, then $f(\Omega)$ is a uniform domain with constants α' and β' depending on α, β, K and the homogeneous dimension $Q = 4$.*

The following theorem uses the concept of a space being locally Q-Loewner which is somewhat technical in its definition and of no concern anywhere else in the discussion, so we direct the reader to Bonk–Heinonen–Koskela [14] and Herron [41] for details rather than provide them here.

Theorem 3.3. (Theorem 6.47 in [14]) *An open connected subset Ω of a locally compact Q-regular Q-Loewner space is locally Q-Loewner. In particular, uniform subdomains of such spaces are Q-Loewner.*

The following consequences of Theorem 3.3 will be of vital importance from the point of view of the notation of collaredness and prime ends, cf. Definition 3.8.

Theorem 3.4. (Fact 2.12 in [41]) *Let Ω be a uniform subdomain of a locally compact Q-regular Q-Loewner space and let E and F be nondegenerate connected subsets of $\overline{\Omega}$ with $\overline{E} \cap \overline{F} \neq \emptyset$, then $\operatorname{Mod}_Q(E, F, \Omega) = \infty$.*

Lemma 3.3. *If $\{E_k\}$ is a prime chain in a uniform subdomain $\Omega \subset \mathbf{H}_1$, then conditions (a) and (b) in Definition 3.3 are equivalent to the following ones, respectively:*

(a) $\operatorname{dist}_{\mathbf{H}_1}(E_k, E_{k+1}) > 0$ *for* $k = 1, 2, \ldots,$
(b) $\lim_{k\to\infty} \operatorname{diam}_{\mathbf{H}_1}(E_k) = 0$.

Proof. Property (a) is an immediate consequence of Theorem 3.4. To see that (b) holds we first note that condition (b) in Definition 3.3, and $\Delta(E_k, F) \leq t_0$ for some $t_0 > 0$ and all k, together imply that $\Psi_X(t_0) = 0$. This contradicts the Loewner condition, and hence $\Delta(E_k, F) \to \infty$ which in turn implies (b).

For the opposite direction, notice that for any $k = 1, 2, \ldots$, condition $\operatorname{dist}_{\mathbf{H}_1}(E_k, E_{k+1}) > 0$ implies that one can inscribe an annulus intersected with the domain Ω and apply argument similar to the last estimate in (12) to conclude part (a) in Definition 3.3. As for the part (b), notice that $\operatorname{diam}_{\mathbf{H}_1}(E_k) \to 0$, for $k \to \infty$ implies that for any $N \geq 1$ it holds that $E_k \subset B(I[E_k], 1/N)$ for large enough $k = k(N)$. From this observation the standard curve family argument gives part (b) in Definition 3.3. □

3.3. Collared domains. The notion of collared domains in $\mathbf{R}^n$ was introduced by Väisälä, see Definition 17.5 in [68] in the context of the boundary behavior of quasiconformal mappings, see also Näkki [54, 55, 56]. Moreover, in [57], Näkki employed collaredness in the studies of prime ends based on the conformal modulus in Euclidean domains. In this section we introduce collared domains in the setting of the Heisenberg groups equipped with the Heisenberg metric (the Korányi metric). This notion will be subsequently used to develop prime ends theory in $\mathbf{H}_1$.

In some sense a collared domain, according to Väisälä's definition, is a manifold with boundary, where the coordinate maps of charts containing boundary components are quasiconformal with target in the closed upper half space. In the spirit of Väisälä one could consider the following definition which we provide for the sake of discussion, but which will not be our working definition: a domain $\Omega \subset \mathbf{H}_1$ is said to be *locally quasiconformally collared* at $x \in \partial\Omega$ if there exists a neighborhood $U \subset \mathbf{H}_1$ of x such that $U \cap \Omega$ is uniform and there exists a homeomorphism h of $U \cap \overline{\Omega}$ onto a half ball

$$B((x_0,0),r)_+ := B((x_0,0),r) \cap \{(z,t) \in \mathbf{H}_1 \colon t \geq 0\}$$

such that $h(x) = x_0 \in \mathbf{C}$ and $h|_{U\cap\Omega}$ is quasiconformal. We call (U,h) a *collaring* coordinate and note that $U \cap \partial\Omega$ is mapped onto the open disc $B((x_0,0),r)_+ \cap \mathbf{C}$.

Note that this definition differs from Definition 17.5 in [68] in the following ways: firstly we do not require $x_0 = 0$ due to the restrictions inherent in the stereographic projection, in particular we cannot follow up with a normalization to $x_0 = 0$ with left translation, since left multiplication does not stabilize $\mathbf{C}$, and secondly and perhaps most importantly, we impose the assumption of some local uniformity which is required so that Lemma 3.3 is applicable. However, there is one major drawback in this definition, namely, we need to know that half balls in $\mathbf{H}_1$ are uniform, which is difficult to verify since half balls are not mapped to half balls under left translations, while the definition of a ball is a left translation of the ball at the origin.

Moreover, there is another problem with mimicking [68]. Namely, in Theorem 17.10, and in [57, Lemma 2.3], it is proved that in the Euclidean case, collaring coordinates satisfy a local Loewner property without assuming local uniformity. However, their proof relies on estimates involving the modulus of curve families contained in spheres [68, Section 10], which are not available to us in the Heisenberg setting, due to the fact that spheres contain very few horizontal curves.

Therefore, we will work with the following alternate definition for collaredness.

Definition 3.8. A domain $\Omega \subset \mathbf{H}_1$ is said to be *locally quasiconformally collared* at $x \in \partial\Omega$ if there exists a uniform subset $U \subseteq \Omega$ and a quasiconformal map $h\colon U \to B(0,1)$ such that $h(U) = B(0,1)$ and:

(a) $x \in \partial U$,
(b) h extends to a map $H\colon U \cup (\partial U \cap \partial\Omega) \to \overline{B(0,1)}$, homeomorphic onto its image and such that $H(\partial U \cap \partial\Omega) \subseteq \partial B(0,1)$ is connected, closed, and contains $H(x)$ in the interior of $\partial B(0,1)$ (in the topology of $\partial B(0,1)$).

We call (U,h) a *collaring* coordinate of x.

Definition 3.9. A domain Ω is said to be *collared* if every boundary point of Ω is locally quasiconformally collared.

From now on, the term *collared domains* will refer to domains collared in the sense of Definitions 3.8 and 3.9.

Example 3. A ball B in $\mathbf{H}_1$ is collared. Indeed, since B is a uniform domain, by Lemma 3.2, we are allowed to take $U = B$ and $h = \mathrm{Id}$ in Definition 3.8.

Observation 3.1. *Let $\Omega \subset \mathbf{H}_1$ be a bounded uniform domain, such that Ω is the image of the unit ball $B(0,1)$ under a quasiconformal mapping f, with a continuous extension to a map $F\colon \overline{B(0,1)} \to \overline{\Omega}$. Then, Ω is collared.*

Proof. In Definition 3.8 let $U := \Omega$ and $h := f$. The assumptions of Observation 3.1 imply that F is, in fact, a homeomorphic extension of f. Indeed, since balls are locally connected at the boundary and by Observation 4.1 a uniform domain is mod-uniform, then by the Koebe theorem, see Theorem 1.3, mapping f has arcwise limits along all end-cuts in $B(0, 1)$. Therefore, by Theorem 4 and Example 5 in [1] (specialized to the setting of $\mathbf{H}_1$), and by the comment following [1, Theorem 4], f has a homeomorphic extension $F\colon \overline{B(0,1)} \to \overline{\Omega}$.

Then (a) in Definition 3.8 holds trivially, while $F(\partial U \cap \partial\Omega) = F(\partial U) = \partial B(0, 1)$ and thus Definition 3.8(b) holds true as well (for $H := F$). □

Remark 2. The following variant of the above definition of collaredness at $x \in \partial\Omega$ can be used as well. As in Definition 3.8, we consider a neighborhood $U \subset \Omega$, not necessarily uniform, such that $x \in \partial U$, but we require (b) to hold with respect to a global quasiconformal map $h\colon \mathbf{H}_1 \to \mathbf{H}_1$ such that $h(U) = B(0, 1)$. Although this definiton is more restrictive, it takes advantage of the fact that we know from Lemma 3.2 together with Proposition 4.2 and Theorem 4.4 in [19], that balls are uniform and their images under global quasiconformal mappings are uniform.

In either the definition which we have settled on, or this alternative case, since $\mathbf{H}_1$ admits a large family of locally quasiconformal mappings, there is a large family of collared domains. Moreover, in [5], Balogh constructs global quasiconformal maps on $\mathbf{H}_1$ which are not Lipschitz and distort Hausdorff dimension. It is therefore possible that a collared domain can have a complicated boundary, i.e., not rectifiable.

In what follows we will appeal also to the following connectedness properties of the boundaries.

Definition 3.10. We say that $\Omega \subset \mathbf{H}_1$ is *finitely connected at a point* $x \in \partial\Omega$ if for every $r > 0$ there exists a bounded open set V in $\mathbf{H}_1$ containing x such that $x \in V \subset B(x, r)$ and $V \cap \Omega$ has only finitely many components. If Ω is finitely connected at every boundary point, then we say it is finitely connected at the boundary.

In particular, if $V \cap \Omega$ has exactly one component, then we say that Ω is *locally connected at* $x \in \partial\Omega$.

The following result extends discussion in Sections 6.3 and 6.4 in Näkki [56], and Theorem 17.10 in Väisälä [68], to the setting of $\mathbf{H}_1$ and relates the notions of collaredness and boundary connectivity of the domain.

Observation 3.2. *Collared domains in* $\mathbf{H}_1$ *are locally connected at the boundary.*

We remark that using Näkki's definition of collaredness, the above lemma says that a collared domain is finitely connected at the boundary, see Theorem 6.4 and Corollary 6.6 in [56].

Proof. Let $\Omega \subset \mathbf{H}_1$ be a collared domain and consider any $x \in \partial\Omega$. Let U be as in Definition 3.8 with $x \in \partial U$. Let $B(x, r) \subset \mathbf{H}_1$ be as in Definition 3.10. Recall that $h(U) = B(0, 1)$ is locally connected at the boundary, since $B(0, 1)$ is uniform by Corollary 1 in Capogna–Garofalo [17] and uniform domains are locally connected at the boundary, see Proposition 11.2 in [2]. Therefore, we can choose an open connected set $V \subset \mathbf{H}_1$ with $h(x) \in \partial(V \cap B(0, 1))$. Since h is a homeomorphism, we obtain that $h^{-1}(V \cap B(0, 1))$ is a connected subset of $U \cap B(x, r)$ and $x \in \partial(U \cap h^{-1}(V \cap B(0, 1)))$. Thus, Ω is locally connected at x. □

The following observation will play a fundamental role in our studies.

Observation 3.3. *Let $\Omega \subset \mathbf{H}_1$ be collared. Then for every boundary point $x \in \partial\Omega$, there exists a singleton prime end $[E_k]$ such that $I[E_k] = \{x\}$.*

Proof. Let $x \in \partial\Omega$ and (U, h) be its collaring coordinate as in Definition 3.8. Let $x_0 = H(x) \in \partial B(0,1)$ and define sets E_k by

$$E_k := H^{-1}(\partial B(x_0, 1/k) \cap B(0,1)) \tag{9}$$

for $k = 1, 2, \ldots$. It is an immediate observation that $F_k := \partial B(x_0, 1/k) \cap B(0,1)$ are cross-sets for all k, cf. Definition 3.1. Moreover, since by Lemma 3.2 we have that $B(0,1)$ is a uniform domain, then the definition of the sets F_k, and Lemma 3.3, imply that $\{F_k\}_{k=1}^{\infty}$ is a prime chain in $B(0,1)$ as in Definition 3.3. By construction $I[F_k] = \{x_0\}$. The definition of E_k in (9) together with the uniformity of U, imply that every E_k satisfies conditions (a) and (b) of Lemma 3.3, and hence, by Theorem 3.4, we conclude that $\{E_k\}_{k=1}^{\infty}$ is a prime chain in Ω. Clearly $I[E_k] \subset \partial\Omega$ and $x \in I[E_k]$. By Lemma 3.3 it holds that $\operatorname{diam}_{\mathbf{H}_1} E_k \to 0$ for $k \to \infty$ and, hence $I[E_k]$ is a singleton. □

In what follows, we will call such a prime chain a *canonical prime chain* associated with $x \in \partial\Omega$, and similarly the associated prime end will also be called a *canonical prime end* associated with x.

Recall, that by $\partial_{\mathrm{SP}}\Omega$ we denote the part of the prime end boundary $\partial_{\mathrm{P}}\Omega$ consisting of singleton prime ends only.

Theorem 3.5. *If Ω is a collared domain, then the impression map $I\colon \partial_{\mathrm{P}}\Omega \to \partial\Omega$ is a surjection. Moreover, $I|_{\partial_{\mathrm{SP}}\Omega}$ is a bijection.*

Proof. By Observation 3.3, the impression map is onto so we only need to show that $I|_{\partial_{\mathrm{SP}}\Omega}$ is injective. If the impression map is not injective then we can find distinct prime ends $[E_i]$ and $[F_j]$ such that $I[E_i] = I[F_j] = \{x\}$ for some $x \in \partial\Omega$ or equivalently

$$\bigcap_i \overline{D(E_i)} = \bigcap_j \overline{D(F_j)} = \{x\}.$$

Since $[E_i] \neq [F_j]$, we can assume that for each j there exists $n_j \in \mathbf{N}$ such that $n_j \geq j$ and $E_i \not\subseteq D(F_j)$ for all $i \geq n_j$ (note that it in the last assertion it may be necessary to pass to a subsequence of the $\{E_i\}_{i=1}^{\infty}$ denoted again, for simplicity, by $\{E_i\}_{i=1}^{\infty}$). If in this case we have $E_i \cap D(F_j) = \emptyset$, then by choosing i larger if necessary, we may assume by (a) in Lemma 3.3 that $\overline{E}_i \cap \overline{D(F_j)} = \emptyset$. Then it follows that $\overline{D(E_i)} \cap \overline{D(F_j)} = \emptyset$ which contradicts $I[E_i] = I[F_j] = \{x\}$. Therefore, we must assume that $E_i \cap D(F_j) \neq \emptyset$ for all $i \geq n_j$ which implies $E_i \cap \partial D(F_j) \neq \emptyset$ for all $i \geq n_j$. For each $i \geq n_j$, choose $x_i \in E_i \cap \partial D(F_j)$, then it follows that x is a limit point of $\Omega \setminus D(F_j)$. Indeed, since $x_i \to x$, we have that $x \in \overline{F_j} \cap \partial\Omega$ which contradicts item (a) in Definition 3.3 of prime chain, namely $\mathrm{Mod}_4(F_{j+1}, F_j, \Omega) < \infty$. In particular, if (U, h) is a collaring coordinate at x, then for j sufficiently large, we have $\overline{F}_j \subset U \cap \overline{\Omega}$ and, since $\operatorname{dist}_{\mathbf{H}_1}(F_{j+1}, F_j) = 0$, Theorem 3.4 implies that $\mathrm{Mod}_4(F_{j+1}, F_j, U \cap \Omega) = \infty$. By the monotonicity of the 4-modulus it follows that $\mathrm{Mod}_4(F_{j+1}, F_j, \Omega) = \infty$, which contradicts the assumption that $\{F_j\}_{j=1}^{\infty}$ is a prime chain. □

We are now in a position to prove Theorem 1.2, where we study the extension of a quasiconformal map, to a homeomorphic transformation between the topological and the prime ends closures of a domain and the target domain, respectively. Results of this kind have a long history going back to Carathéodory's idea of prime ends.

Väisälä [70, Section 3] studied the special case of a ball in $\mathbf{R}^n$, whereas Näkki [57, Theorem 4.1] studied the setting of collared domains in $\mathbf{R}^n$.

Proof of Theorem 1.2. As a consequence of Theorem 1.1 and Theorem 3.5 above, we need only to check that the extension of the identity map $I_\Omega\colon \Omega \to \Omega$, to a map $\tilde{I}_\Omega\colon \Omega \cup \partial\Omega \to \Omega \cup \partial_{\mathrm{SP}}\Omega$, where

$$\tilde{I}_\Omega(x) = \begin{cases} x & \text{if } x \in \Omega, \\ [E_k^x] & \text{if } x \in \partial\Omega, \end{cases}$$

is continuous and open. To this end we need only examine the behavior at the boundary.

Let $U \subset \Omega$ have the property that $\partial U \cap \partial\Omega \neq \emptyset$ and the property that $U \cup (\partial U \cap \partial\Omega)$ is relatively open, then for U_P as in Definition 3.5, we have

$$\tilde{I}_\Omega\left(U \cup (\partial U \cap \partial\Omega)\right) = U \cup U_P,$$

since by collaredness, every $[F_l] \in U_P$ satisfies $[F_l] = [E_k^x]$ for some $x \in \partial U \cap \partial\Omega$. Hence $\tilde{I}_\Omega$ is open.

Now let $U \cup U_P \subset \Omega \cup \partial_P\Omega$ where $U \cup (\partial U \cap \partial\Omega)$ is relatively open. By collaredness, every $[F_l] \in U_P$ satisfies $[F_l] = [E_k^x]$ for some $x \in \partial U \cap \partial\Omega$. Hence,

$$U \cup U_P = \tilde{I}_\Omega(U \cup (\partial U \cup \partial\Omega))$$

and so $\tilde{I}_\Omega$ is continuous. □

4. Further properties of prime ends. Relations to the theory of prime ends on metric spaces

In this section we develop and discuss further properties of prime ends as defined in Section 3.1 in the setting of the Heisenberg group $\mathbf{H}_1$. Moreover, for domains in $\mathbf{H}_1$, we present relations between Näkki's prime ends and the theory of Mod_Q-ends as well as Mod_Q-prime ends as developed in [2]. We restrict our discussion to $\mathbf{H}_1$ mainly for the sake of uniformity of the presentation. However, most of the results in this section can be stated for the higher order Heisenberg groups and even for more general Carnot–Carathéodory groups under natural modifications of the statements below.

The following definition is due to Näkki, see [55, 56]. Näkki uses the term uniform domains, whereas we call them mod-uniform domains in order to distinguish from uniform domains as in Martio–Sarvas [50], see comments below.

Definition 4.1. We say that a domain $\Omega \subset \mathbf{H}_1$ is *mod-uniform* if for every $t > 0$ there is $\epsilon > 0$ such that if $\min\{\mathrm{diam}(E), \mathrm{diam}(F)\} \geq t$, then $\mathrm{Mod}_4(\Gamma(E, F, \Omega)) \geq \epsilon$ for any nondegenerate connected sets $E, F \subset \Omega$.

As observed by Näkki, mod-uniform domains in $\overline{\mathbf{R}^n}$ are finitely connected at the boundary, see Theorem 6.4 in [56]. Moreover, a domain $\Omega \subset \overline{\mathbf{R}^n}$ which is finitely connected at the boundary, is mod-uniform if and only if Ω can be mapped quasiconformally onto a collared domain, see [56, Section 6.5]. From the point of view of our discussion, it is important that Theorem 6.4 in [56] easily extends to the $\mathbf{H}_1$ setting, and we omit the proof of this observation.

We further remark that one should not confuse Definition 4.1 with the uniform domains studied by [50], Näkki–Väisälä [59] and Väisälä [69], see Definition 3.7 and Section 3.2 for the importance of uniform domains in our studies. For instance, the latter uniform domains are necessarily locally connected at the boundary, see e.g. Proposition 11.2 in [2]. In fact the following holds.

Observation 4.1. *A uniform bounded domain $\Omega \subset \mathbf{H}_1$ is mod-uniform.*

Proof. By Theorem 3.3 we get that Ω is 4-Loewner. Let then $E, F \subset \Omega$ be nondegenerate disjoint continua and suppose that $\min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \geq t$ for some $t > 0$. Then

$$\Delta(E,F) \leq \frac{1}{t} \operatorname{dist}_{\mathbf{H}_1}(E,F) \leq \frac{1}{t} \operatorname{diam}_{\mathbf{H}_1} \Omega$$

and hence by Definition 3.6, it holds that

$$\operatorname{Mod}_4(E,F,\Omega) \geq \Psi_\Omega(\Delta(E,F)) > 0.$$

Moreover, since Ω is bounded and Ψ_Ω is a nonincreasing function, we in fact obtain that there exists a uniform lower bound

$$\Psi_\Omega(\Delta(E,F)) \geq \Psi_\Omega\left(\frac{1}{t} \operatorname{diam}_{\mathbf{H}_1} \Omega\right) := \epsilon > 0$$

and the proof is completed. $\square$

In the discussion following Definition 3.9, we noticed that every boundary point of a collared domain is an impression of the singleton prime end, the so-called canonical prime end. In fact, the following stronger result holds, cf. Observation 3.2.

Lemma 4.1. *If $\Omega \subset \mathbf{H}_1$ is a domain finitely connected at the boundary, then every $x \in \partial\Omega$ is the impression of a prime end.*

The proof of this observation is based on the following topological result.

Lemma 4.2. (Lemma 10.5 in [2]) *Assume that Ω is finitely connected at $x_0 \in \partial\Omega$. Let $A_k \subsetneq \Omega$ be such that:*

(1) $A_{k+1} \subset A_k$,
(2) $x_0 \in \overline{A_k}$,
(3) $\operatorname{dist}_{\mathbf{H}_1}(x_0, \Omega \cap \partial A_k) > 0$ *for each* $k = 1, 2, \ldots$.

Furthermore, let $0 < r_k < \operatorname{dist}_{\mathbf{H}_1}(x_0, \Omega \cap \partial A_k)$ be a sequence decreasing to zero. Then for each $k = 1, 2, \ldots$ there is a component $G_{j_k}(r_k)$ of $B(x_0, r_k) \cap \Omega$ intersecting A_l for each $l = 1, 2, \ldots$, and such that $x_0 \in \overline{G_{j_k}(r_k)}$ and $G_{j_k}(r_k) \subset A_k$.

Proof of Lemma 4.1. Following the notation of Lemma 4.2, we let $x_0 \in \partial\Omega$ and set $A_k = \Omega \setminus \{x\}$ for some $x \in \Omega$ and all $k = 1, \ldots$. First, we construct a nested sequence of connected sets

$$F_k^{x_0} = G_{j_k}(r_k) \subset B(x_0, r_k) \cap \Omega$$

with $\operatorname{diam}_{\mathbf{H}_1}(F_k^{x_0}) \to 0$ as $k \to \infty$. The idea of such construction is based on the proof of Lemma 10.6 in [2] and, therefore, we present only a sketch of the reasoning.

Let us consider the rooted tree with vertices $G_j(r_k)$, $j = 1, 2, \ldots, N(r_k)$, $k = 1, 2, \ldots$, where two vertices are connected by an edge provided that they are $G_j(r_k)$ and $G_i(r_{k+1})$ for some i, j and k with $G_i(r_{k+1}) \subset G_j(r_k)$. Denote by $\mathcal{P}$ the collection of all descending paths starting from the root and define a metric function measuring the distance between branches of the tree. Namely, let $t(p,q) = 2^{-n}$, where n is the level where paths p and q branch (or end), i.e. n is the largest integer such that p and q have a common vertex $G_j(r_n)$. For each $k = 1, 2, \ldots$, we consider the subcollection $\mathcal{P}_k$ consisting of all paths $p \in \mathcal{P}$ for which there exists a component $G_j(r_k) \subset A_k$ such that p passes through the vertex $G_j(r_k)$. By Lemma 4.2 all $\mathcal{P}_k$

are nonempty, $\mathcal{P}_{k+1} \subset \mathcal{P}_k$ for $k = 1, \dots$ and each $\mathcal{P}_k$ is complete in t. Since $\mathcal{P}$ is totally bounded in t, we get that all $\mathcal{P}_k$ are compact. As a consequence, we obtain an infinite path $q \in \bigcap_{k=1}^{\infty} \mathcal{P}_k$. The vertices through which it passes define the sequence of sets $\{F_k^{x_0}\}_{k=1}^{\infty}$ such that $F_k^{x_0} = G_{j_k}(r_k)$, $k = 1, 2, \dots$. Moreover,

$$\operatorname{diam}_{\mathbf{H}_1} F_k^{x_0} \le \operatorname{diam}_{\mathbf{H}_1}(B(x_0, r_k) \cap \Omega) \le 2r_k \to 0 \quad \text{as } k \to \infty.$$

Next, we use sets $F_k^{x_0}$ to define a prime chain $[E_k]$ with impression $I[E_k] = \{x_0\}$. Define

$$E_k := \left(\overline{F_{2k-1}^{x_0}} \cap \Omega\right) \setminus \left(\overline{F_{2k}^{x_0}} \cap \Omega\right), \quad k = 1, \dots. \tag{10}$$

Then, E_k are connected, relatively closed in Ω for all k, also $\overline{E_k} \cap \partial\Omega \neq \emptyset$ and $\Omega \setminus E_k$ has exactly two components. By construction we get that E_k separates E_{k-1} and E_{k+1}. Furthermore, since $\operatorname{dist}_{\mathbf{H}_1}(E_k, E_{k+1}) > 0$ it holds that $\operatorname{Mod}_4(E_k, E_{k+1}, \Omega) < \infty$.

Finally, let $K \subset \Omega$ be a continuum. Note that

$$E_k \subset F_{2k-1}^{x_0} \subset B(x_0, r_k) \cap \Omega \quad \text{and} \quad \lim_{k \to \infty} \operatorname{dist}_{\mathbf{H}_1}(B(x_0, r_k) \cap \Omega, \{x_0\}) = 0.$$

Therefore, $\lim_{k\to\infty} \operatorname{Mod}_4(E_k, K, \Omega) = 0$, as the family of curves passing through the fixed point has zero p-modulus for $1 \le p \le Q = 4$, cf. (12) below, for the similar argument. Thus, $[E_k]$ is a prime chain and defines a prime end. □

We illustrate the above construction of prime ends with the following observation similar to Theorem 1.2. However, here we do not require the domain to be collared.

Lemma 4.3. *Let $\Omega_0 \subset \mathbf{H}_1$ be a domain locally connected at the boundary and let f be a quasiconformal mapping from Ω_0 onto a domain $\Omega \subset \mathbf{H}_1$. Then, there exists an extension map $F\colon \Omega_0 \cup \partial\Omega_0 \to \Omega \cup \partial_{\mathrm{SP}}\Omega$, such that $F|_{\Omega_0} = f$.*

Let us remark that in the Euclidean setting, Theorem 4.2 in Näkki [54] shows that for Ω_0 being a ball and collared Ω (in the sense of Definition 17.5 in [68]), map F in the lemma is a homeomorphism. The same assertion holds also when $\Omega \subset \mathbf{R}^n$ if finitely connected at the boundary, see Section 3.1 in Väisälä [70]. For more general type domains in $\mathbf{H}_1$ (and more general metric measure spaces) we refer to Theorem 5 in [1].

Proof. Since Ω is 1-connected at the boundary, it is in particular finitely connected at the boundary and, hence, Lemmas 4.2 and 4.1 can be applied with sets

$$F_k^{x_0} := B(x_0, 1/k) \cap \Omega_0 \quad \text{for } k = 1, \dots$$

and any $x_0 \in \partial\Omega_0$. As in the proof of Lemma 4.1 we construct a prime end $[E_k^x]$ in Ω_0 following (10). Define $F\colon \partial\Omega_0 \to \partial_{\mathrm{SP}}\Omega$ as follows:

$$F(x) = \begin{cases} f(x) & \text{for } x \in \Omega_0, \\ [f(E_k^x)] & \text{for } x \in \partial\Omega_0. \end{cases}$$

The proof of the observation will be completed once we show that $\{f(E_k^x)\}_{k=1}^{\infty}$ defines a prime chain (end) in $\Omega = f(\Omega_0)$ with singleton impression $I[f(E_k^x)] := \{y\} \subset \partial\Omega$. Indeed, since f is a homeomorphism, it holds that $f(E_k^x)$ are cross-sets in Ω for all k. In particular, since for all k cross-sets E_k^x divide Ω_0 into exactly two domains, then so do $f(E_k^x)$ for all k. Similarly, by topology we have that if E_{k+1} separates E_k and E_{k+2}, then the same holds for their images under homeomorphism f. Next, if $\operatorname{dist}_{\mathbf{H}_1}(E_k^x, E_{k+1}^x) > 0$, then by the injectivity of f we have that $\operatorname{dist}_{\mathbf{H}_1}(f(E_k^x), f(E_{k+1}^x)) > 0$ for all k. Since f is quasiconformal we infer from $\operatorname{Mod}_Q(E_k^x, E_{k+1}^x, \Omega_0) < \infty$ that $\operatorname{Mod}_Q(f(E_k^x), f(E_{k+1}^x), \Omega) < \infty$.

Finally, since $[E_k^x]$ is a prime end in Ω_0, we have that for any continuum $C \subset \Omega_0$

$$0 \le \lim_{k\to\infty} \mathrm{Mod}_Q(f(E_k^x), f(C), \Omega) \le K \lim_{k\to\infty} \mathrm{Mod}_Q(E_k^x, C, \Omega_0) = 0$$

by quasiconformality of f. Note that every continuum $C' \subset \Omega$ is an image under f of some continuum in $C \subset \Omega_0$, as we can set $C := f^{-1}(C')$. This argument, together with Lemma 3.1 imply that $I[fE_n] \subset \partial\Omega$. The proof of Lemma 4.3 is therefore completed. □

4.1. Näkki's prime ends and prime ends on metric spaces. In this section we compare a variant of Näkki's theory of prime ends introduced in previous sections, to a theory of prime ends developed for a general metric measure spaces in [2]. First, we recall some building blocks of this metric measure space theory.

Let $\Omega \subset X$ be a domain in a complete metric measure space with a doubling measure, supporting the $(1,p)$-Poincaré inequality for $1 < p < \infty$. For the significance of these assumptions we refer to [2]. Here we only note that these conditions hold for the Heisenberg groups $\mathbf{H}_n$ and more general Carnot–Carathéodory groups, see e.g. Section 11 in Hajłasz–Koskela [33].

Definition 4.2. We say that a bounded connected set $E \subsetneq \Omega$ is an *acceptable* set if $\overline{E} \cap \partial\Omega$ is nonempty.

Since an acceptable set E is bounded and connected, it holds that $\overline{E}$ is compact and connected. Moreover, E is infinite, otherwise we would have $\overline{E} = E \subset \Omega$. Therefore, $\overline{E}$ is a continuum.

Definition 4.3. A sequence $\{E_k\}_{k=1}^{\infty}$ of acceptable sets is a *chain* if

(1) $E_{k+1} \subset E_k$ for all $k = 1, 2, \ldots$,
(2) $\mathrm{dist}_{\mathbf{H}_1}(\Omega \cap \partial E_{k+1}, \Omega \cap \partial E_k) > 0$ for all $k = 1, 2, \ldots$,
(3) the *impression* $\bigcap_{k=1}^{\infty} \overline{E}_k \subset \partial\Omega$.

We further comment that a variant of this definition can be considered as well with the Heisenberg distance in condition (2) substituted with the Mazurkiewicz distance, see Definition 2.3 in Estep–Shanmugalingam [28].

Definition 4.4. Similarly to the setting of Näkki's prime chains, we define the *division* of chains and say that two chains are *equivalent* if they divide each other. A collection of mutually equivalent chains is called an *end* and denoted $[E_k]$, where $\{E_k\}_{k=1}^{\infty}$ is any chain in the equivalence class. An end $[E_k]$ is called a *prime end* if any other end dividing it must be equivalent to it, i.e., if $[E_k]$ is not divisible by any other end.

For further definitions and properties of prime ends as in Definition 4.4, we refer to Sections 3–5 and 7 of [2]. Among the topics studied in [2] are also notions of Mod_p-ends and Mod_p-prime ends for $1 \le p < \infty$, see Section 6 in [2]. However, here we confine our discussion to the setting of $p = Q$ only with $Q = 4$, the Ahlfors dimension of $\mathbf{H}_1$.

Definition 4.5. A chain $\{E_k\}_{k=1}^{\infty}$ is called a Mod_4-*chain* if

$$\lim_{k\to\infty} \mathrm{Mod}_4(E_k, K, \Omega) = 0 \tag{11}$$

for every compact set $K \subset \Omega$.

In fact Lemma A.11 in [2] allows us to require (11) to hold only for some compact set K_0 with the Sobolev capacity $C_4(K_0) > 0$, see also Lemma 3.1 above.

Definition 4.6. An end $[E_k]$ is a Mod_4-*end* if it contains a Mod_4-chain representing it. A Mod_4-end $[E_k]$ is a Mod_4-*prime end* if the only Mod_4-end dividing it is $[E_k]$ itself.

Remark 3. Similar to the prime chains studied in Section 3.1, it holds that the impression is either a point or a continuum, since $\{\overline{E_k}\}_{k=1}^{\infty}$ is a decreasing sequence of continua. Furthermore, Properties 1 and 2 of Definition 4.3, imply that $E_{k+1} \subset \mathrm{int} E_k$. In particular, $\mathrm{int} E_k \neq \emptyset$.

Theorem 4.1. *Let* $\Omega \subset \mathbf{H}_1$ *be a collared domain. Then, a prime chain defines a* Mod_4-*chain as in Definition 4.5, while a prime end defines a* Mod_4-*end according to Definition 4.6. Moreover, a singleton prime end defines a* Mod_4-*prime end according to Definition 4.6.*

Proof. Let $\{E_k\}_{k=1}^{\infty}$ be a prime chain in Ω. Recall, that by $D(E_k) := \Omega \setminus E_k$ we denote the component of Ω containing E_{k+1} for $k = 1, \ldots$. Then $D(E_k)$ are acceptable sets as in Definition 4.2, cf. Lemma 3.1. Moreover, $D(E_{k+1}) \subset D(E_k)$ for all k.

Claim: The definition of a prime chain together with Lemma 3.3 show that $\mathrm{dist}_{\mathbf{H}_1}(E_k, E_{k+1}) > 0$ since $\mathrm{Mod}_4(E_k, E_{k+1}, \Omega) < \infty$, and it follows that $\mathrm{dist}_{\mathbf{H}_1}(\Omega \cap \partial D(E_k), \Omega \cap \partial D(E_{k+1})) > 0$.

Indeed, by Observation 3.2 collared domains are 1-connected at the boundary, which in turn gives us that every boundary point of $\partial\Omega$ is the impression of a singleton prime end, cf. Lemma 4.1. Let us apply this observation to point $x = I[E_k]$. Since Ω is collared, there exists a uniform set $U \subset \Omega$ associated with x, such that $x \in \partial U \cap \partial\Omega$. Then, Lemma 3.3 can be applied to U concluding the proof of the claim for k large enough.

Again by Lemma 3.3, and the discussion following Definition 3.3 (see Lemma 3.1), it holds that the impression $\bigcap_{k=1}^{\infty} \overline{D(E_k)} \subset \partial\Omega$. Hence, $\{D(E_k)\}_{k=1}^{\infty}$ defines a chain as in Definition 4.3. The Mod_4-condition for all continua assumed in Part (b) of Näkki's Definition 3.3, implies that $\{D(E_k)\}_{k=1}^{\infty}$ is in fact a Mod_4-chain. Finally, since $[E_k]$ is a class of equivalent prime chains, we obtain that $[D(E_k)]$ is a Mod_4-end.

Let $[E_k]$ be additionally a singleton prime end. Then, Proposition 7.1 in [2] says that a singleton end is a prime end (in the sense of Definition 4.4). As a consequence, $[D(E_k)]$ is a Mod_4-prime end (as in Definition 4.6). □

5. Boundary behavior of quasiconformal mappings in $\mathbf{H}_1$

The main purpose of this section is to employ the theory of prime ends in the studies of the boundary behavior of quasiconformal mappings in the Heisenberg group $\mathbf{H}_1$. Our results extend the corresponding ones proved in Näkki [57, Section 7]. We provide counterparts of the following three results from the theory of conformal and quasiconformal mappings in $\mathbf{R}^n$:

- the Koebe theorem on existence of arcwise limits along end-cuts (Theorem 1.3),
- the Lindelöf theorem on relation between asymptotic values of a map and sets of principal points for prime ends (Theorem 1.4),
- the Tsuji theorem on the Sobolev capacities of sets of arcwise limits (Theorem 1.5).

These results require some definitions and auxiliary results, which we now present. Recall that if $\{E_k\}_{k=1}^{\infty}$ is a chain of cross-sets in Ω, then by $D(E_k)$ we denote the component of $\Omega \setminus E_k$ containing E_{k+1} (cf. Definition 3.2).

Definition 5.1. A point $x \in \partial\Omega$ is an *accessible* boundary point if there exists a curve $\gamma\colon [0,1] \to \mathbf{H}_1$ such that $\gamma(1) = x$ and $\gamma([0,1)) \subset \Omega$. We call γ an *end-cut of* Ω *from* x. Moreover, if $[E_k]$ is a prime end and there is a curve γ as above such that for every k there is $t_k \in (0,1)$ with $\gamma([t_k,1)) \subset D(E_k)$, then $x \in \partial\Omega$ is *accessible through* $[E_k]$.

Remark 4. Note that $x \in \partial\Omega$ can be accessible through $[E_k]$ only if x belongs to the impression of $[E_k]$.

The following result relates connectivity of the boundary of a domain to accessibility of points.

Observation 5.1. *Let $\Omega \subset \mathbf{H}_1$ be a domain finitely connected at the boundary. Then every $x \in \partial\Omega$ is accessible and accessible through some prime end $[E_n]$.*

Proof. Lemma 4.1 allows us to assign with every $x \in \partial\Omega$ a prime end, denoted $[E_n]$, with $I[E_n] = \{x\}$. Moreover, x is accessible through $[E_n]$ (cf. Definition 5.1). To see this choose $x_n \in D(E_n)$ for $n = 1, 2, \ldots$. Since both x_n and x_{n+1} belong to the pathconnected set $D(E_n)$, there exists a curve γ_n connecting x_n to x_{n+1}. Let γ denote the concatenation of all curves γ_n, with $\gamma([0,1)) \subset \Omega$ and $\gamma(1) = x$. From the proof of Lemma 4.1 we infer that $\lim_{n\to\infty} \operatorname{diam}_{\mathbf{H}_1}(E_n) = 0$ and so γ is continuous at 1. Hence, x is accessible and accessible through $[E_k]$. Moreover, γ is an end-cut of Ω from x. $\square$

Using Definition 5.1 we may provide another method to associate with every accessible boundary point a prime end. The following result will play a particular role in the studies of cluster sets of quasiconformal mappings (cf. Lemma 7.7 in [2]). One can consider Observation 5.2 as a complimentary result to Lemma 4.1.

Observation 5.2. *Let $\Omega \subset \mathbf{H}_1$ and $x \in \partial\Omega$ be an accessible point. Let further r_n for $n = 1, 2, \ldots$ be a strictly decreasing sequence converging to zero as $n \to \infty$. Then there exist a sequence t_n for $n = 1, 2, \ldots$ with $0 < t_n < 1$ and a prime end $[E_n]$ such that:*

(1) $I[E_n] = \{x\}$,
(2) $\gamma([t_n, 1)) \subset D(E_n)$,
(3) $D(E_n)$ *is a component of* $\Omega \cap B(x, r_n)$ *for all* $n = 1, 2, \ldots$.

In particular, x is accessible through $[E_n]$. Moreover, $[E_n]$ is s singleton prime end.

Proof. Let γ be an end-cut of Ω from x as in Definition 5.1. It is easy to notice that continuity of γ implies existence of a sequence $t_n \in (0,1)$ for $n = 1, 2, \ldots$, with a property that
$$\gamma([t_n, 1)) \subset \Omega \cap B(x, r_n).$$
For $n = 1, 2, \ldots$ we define D_n as the component of $\Omega \cap B(x, r_n)$ containing $\gamma(t_n)$ and set
$$E_n := (\overline{D_n} \setminus D_n) \cap \Omega.$$
We show that $\{E_n\}_{n=1}^{\infty}$ is a prime chain and, thus, gives rise to a prime end as in Definition 3.4.

By the definition, sets E_n for all n are relatively closed in Ω and
$$\overline{E_n} \cap \partial\Omega = \overline{(\partial D_n \cap \Omega)} \cap \partial\Omega \neq \emptyset.$$

Moreover, the choice of sets D_n implies that every $\Omega \setminus E_n$ consists of exactly two components whose boundaries intersect $\partial\Omega$. Hence, every E_n is a cross-set as in Definition 3.1.

By construction $D_{n+1} \subset D_n \subset D_{n-1}$ and, since the radii r_n are strictly decreasing, we obtain that E_n separates E_{n-1} and E_{n+1} for all $n = 2, \ldots$. Hence, $\{E_n\}_{n=1}^{\infty}$ fulfills conditions of a chain, cf. Definition 3.2.

Since $E_n = \Omega \cap \partial D_n \subset \partial B(x, r_n)$, it follows that for all $n = 1, 2, \ldots,$

$$\operatorname{dist}_{\mathbf{H}_1}(E_n, E_{n+1}) \geq r_n - r_{n+1} > 0.$$

As a consequence $\mathrm{Mod}_4(E_{n+1}, E_n, \Omega) < \infty$ for all n. Finally, let $F \subset \Omega$ be any continuum. Then for any n we have that

$$\mathrm{Mod}_4(E_n, F, \Omega) \leq \mathrm{Mod}_4(\partial B(x, r_n), F, \Omega) \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $\lim_{n\to\infty} \mathrm{Mod}_4(E_n, F, \Omega) = 0$ and, thus, conditions (a) and (b) of Definition 3.3 are satisfied for $\{E_n\}_{n=1}^{\infty}$ and $[E_n]$ defines a prime end in Ω.

Since $E_n = (\overline{D_n} \setminus D_n) \cap \Omega$ for $n = 1, 2, \ldots$, then $\overline{E_n} \subset \overline{\Omega \cap B(x, r_n)}$ for all n. Hence,

$$\operatorname{diam}_{\mathbf{H}_1} \overline{E_n} \leq \operatorname{diam}_{\mathbf{H}_1} \overline{\Omega \cap B(x, r_n)} \to 0 \qquad \text{for } n \to \infty$$

by assumptions. Thus, $I[E_n] \subset \partial\Omega$ and $I[E_n]$ is a singleton prime end. In fact $I[E_n] = \{x\}$, as $x \in \overline{D_n}$ for all n completing the proof of Observation 5.2. $\square$

Recall the following notion of cluster sets.

Definition 5.2. Let $\Omega \subset \mathbf{H}_1$ be a domain, $f\colon \Omega \to \mathbf{H}_1$ be a mapping and $x \in \partial\Omega$. We define *the cluster set of f at x* as follows:

$$C(f, x) := \bigcap_U \overline{f(U \cap \Omega)},$$

where the intersection ranges over all neighborhoods of x in $\mathbf{H}_1$.

Cluster sets can be further generalized to capture the behavior of a mapping along a curve in a more subtle way.

Definition 5.3. Let $\Omega \subset \mathbf{H}_1$ be a domain, $f\colon \Omega \to \mathbf{H}_1$ be a mapping and $x \in \partial\Omega$. We say that a sequence of points $\{x_n\}_{n=1}^{\infty}$ in Ω *converges along an end-cut γ at x* if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $0 < t_n < 1$ and $\lim_{n\to\infty} t_n = 1$ such that $x_n = \gamma(t_n)$ and

$$\lim_{n\to\infty} d_{\mathbf{H}_1}(x_n, x) = 0.$$

We say that a point $x' \in \mathbf{H}_1$ belongs to *the cluster set of f at x along an end-cut γ from x*, denoted by $C_\gamma(f, x)$, if there exists a sequence of points $\{x_n\}_{n=1}^{\infty}$ converging along an end-cut γ at x, such that

$$\lim_{n\to\infty} d_{\mathbf{H}_1}(f(x_n), x') = 0.$$

If $C_\gamma(f, x) = \{y\}$, then y is called an *arcwise limit (asymptotic value)* of f at x.

In other words, y is an asymptotic value of f at $x \in \Omega$, if there exists a curve $\gamma\colon [0, 1) \to \Omega$ such that $\gamma(t) \to x$ and $f(\gamma(t)) \to y$ for $t \to 1$.

The Koebe theorem. In 1915 Koebe [44] proved that a conformal mapping between a simply-connected planar domain Ω onto the unit disc has arcwise limits along all end-cuts of Ω. Theorem 1.3 extends Koebe's theorem and Theorem 7.2 in Näkki [57] to the setting of quasiconformal mappings in $\mathbf{H}_1$. Moreover, we study more general end-cuts than in [57] and more general assumptions on the underlying domains.

Proof of Theorem 1.3. We follow the idea of the proof of [57, Theorem 7.2]. Since Ω is finitely connected at every boundary point $x \in \partial\Omega$, then Observation 5.1 implies that all $x \in \partial\Omega$ are accessible. Let γ be an end-cut in Ω from $x \in \partial\Omega$. Let $K \subset \Omega$ be a continuum and let U_k be neighborhoods of x such that

$$\bigcap_{k=1}^{\infty} U_k = \{x\} \quad \text{and} \quad \gamma_k := U_k \cap \Omega \cap \gamma$$

are connected sets for all $k \geq k_0$ and some k_0. Since $\operatorname{diam}_{\mathbf{H}_1} U_k \to 0$ for $k \to \infty$ and $\gamma_k \subset U_k$, it follows that

$$\lim_{k\to\infty} \operatorname{Mod}_4(K, \gamma_k, \Omega) \leq \lim_{k\to\infty} \operatorname{Mod}_4(K, U_k, \Omega) = 0.$$

In order to see the latter equality, let us assume that $R > 0$ is sufficiently small and such that $B(x,R) \subset \mathbf{H}_1 \setminus K$. By the decay property of diameters for the sets U_k as $k \to \infty$ we have that, passing to a subsequence if necessary, $U_k \subset B(x, 1/k) \cap \Omega$ for all $k \geq k_0$. Since $\mathbf{H}_1$ is a path-connected metric measure space with doubling measure, Theorem 3.1 in [3] (see also Korányi–Reimann [47, Proposition 10]) together with arguments involving minorized families of curves (cf. Lemma A.1(5)) and the fact that the modulus of curve families is an outer measure, imply that for k large enough so that $B(x, 1/k) \subset B(x,R)$, it holds that

$$\begin{aligned}
\operatorname{Mod}_4(U_k, K, \Omega) &\leq \operatorname{Mod}_4(B(x,1/k) \cap \Omega, K, \Omega) \\
&\leq \operatorname{Mod}_4(\overline{B(x,1/k)}, \mathbf{H}_1 \setminus B(x,R), \mathbf{H}_1) \\
&\leq \operatorname{Mod}_4(\overline{B(x,1/k)}, \mathbf{H}_1 \setminus B(x,R), B(x,R)) \\
&\leq C(R) \left(\log \frac{R}{1/k}\right)^{-3} \to 0, \quad \text{for } k \to \infty.
\end{aligned} \tag{12}$$

To obtain the second estimate we use that Mod_4 is an outer measure and $\Gamma(B(x,1/k) \cap \Omega, K, \Omega) \subset \Gamma(\overline{B(x,1/k)}, \mathbf{H}_1 \setminus B(x,R), \mathbf{H}_1)$. Moreover, the fourth inequality relies on the fact that the family of curves $\Gamma(\overline{B(x,1/k)}, \mathbf{H}_1 \setminus B(x,R), \mathbf{H}_1)$ is minorized by $\Gamma(\overline{B(x,1/k)}, \mathbf{H}_1 \setminus B(x,R), B(x,R))$, see Section A.2.

The quasiconformality of f implies that

$$\lim_{k\to\infty} \operatorname{Mod}_4(f(K), f(U_k), f(\Omega)) = 0.$$

Since Ω_0 is a mod-uniform domain, Definition 4.1 gives us that $\lim_{k\to\infty} \operatorname{diam}(f(\gamma_k)) = 0$ and thus the cluster set $C_\gamma(f,x)$ is a singleton meaning that f has an arcwise limit along γ. $\square$

The Lindelöf theorem. A bounded analytic function of the unit disc having a limit y_0 along an end-cut at a boundary point x_0 has angular limit y_0 according to the classical theorem of Lindelöf. By an angular limit we mean that the limit is y_0 along any "angular" end-cut at x_0, that is an end-cut contained in some fixed cone in the unit disc with apex at x_0.

In [31, Theorem 6], Gehring proved a Lindelöf type theorem for quasiconformal mappings on balls in $\mathbf{R}^3$ which Näkki generalized to $\mathbf{R}^n$ in [57, Theorem 7.4]. In this context the theorem is stated in terms of angular end-cuts and principal points.

Definition 5.4. Let $[E_k]$ be a prime end in a domain $\Omega \subset \mathbf{H}_1$ and $x \in I[E_k]$. We say that x is a *principal point* relative to the prime end $[E_k]$, if every neighborhood of x contains a cross-set of a chain in $[E_k]$, i.e., x is a limit of a convergent chain in $[E_k]$. The set of principal points of a prime end $[E_k]$ is denoted by $\Pi(E_k)$. A point in $I[E_k]$ which is not principal is called a *subsidiary point*.

For the main ideas and definitions of principal (and subsidiary points, see below), we refer to Collingwood–Lohwater [25, Chapter 9.7] and Näkki [57, Section 7]. The importance of such notions in the classification of prime ends in $\mathbf{R}^n$ is described e.g. in [57, Section 8]. See also Carmona–Pommerenke [23, 24] for results regarding the theory of continua and principal points.

The proof of our Lindelöf type theorem proceeds in the same way as the Euclidean proof in [57], where the setting is transformed to the upper half space by the extended stereographic projection sending x_0 to the origin and then employing the fact that a cone in $\mathbf{R}^n$ with apex at 0 is invariant under Euclidean dilation. The geometry of $\mathbf{H}_1$ imposes some obstacles in following this approach. Although we can transform a ball to the upper half space using left translation and the mapping defined at (6), we do not have the luxury of choosing the destination of x_0. Secondly, the notion of a dilation invariant cone with apex at x_0 is complicated by the fact that the invariance is with respect to dilations centered at x_0, that is maps of the form

$$g_r = \tau_{x_0} \circ \delta_r \circ \tau_{x_0}^{-1}.$$

To begin the construction of our cones we first consider how dilations behave when they are centered at a boundary point $x_0 \in \partial B$, in particular we address the complications arising from the fact that the boundary of B is not preserved by such maps. Using left translation and the mapping defined at (6), we can consider the normalized situation where B is the upper half space and the image of x_0 is a point $w_0 = u_0 + iv_0 \in \mathbf{C}$. It follows that if g_r is a dilation centered at $(w_0, 0)$, then

$$\begin{aligned} &g_r(x+iy,t) \\ &= \big(u_0 + r(x-u_0) + i(v_0 + r(y-v_0)), r\big(rt + 2(1-r)(xv_0 - u_0y)\big)\big). \end{aligned} \tag{13}$$

Hence if $t > 0$, then $g_r(x+iy,t)$ is a point in the upper half space for all $r \in (0,1]$ provided that $xv_0 - u_0y > 0$. Let $\mathcal{K}_{x_0}$ denote the subset of B which corresponds to

$$\mathcal{K}_{(w_0,0)} = \{(x+iy,t) \colon xv_0 - u_0y > 0,\ t > 0\}$$

under the stereographic projection. Then

$$g_r(\overline{\mathcal{K}}_{x_0}) \subset \overline{\mathcal{K}}_{x_0}$$

when g_r is a dilation centered at x_0 and $r \in [0,1]$. Indeed, let $(x+iy,t) \in \mathcal{K}_{(w_0,0)}$ and g_r be a dilation centered at $(w_0,0)$ with $r \in [0,1]$. It follows that if $(x_r + iy_r, t_r) = g_r(x+iy,t)$, then $x_rv_0 - u_0y_r = r(xv_0 - u_0y)$.

We define a cone $\mathcal{C}_{(w_0,0)} \subset \mathcal{K}_{(w_0,0)}$ with apex at w_0 as follows: let

$$\begin{aligned} \Sigma_{w_0} = \{&(a+ib,c) \in S(0,1) \cap \tau_{(w_0,0)^{-1}}(\mathcal{K}_{(w_0,0)}) : \\ &\tau_{(w_0,0)} \circ \delta_s(a+ib,c) \in \mathcal{K}_{(w_0,0)} \text{ for all } s \in (0,\infty)\} \end{aligned}$$

and set

$$\mathcal{C}_{(w_0,0)} = \{\tau_{(w_0,0)} \circ \delta_s(a+ib,c) \colon (a+ib,c) \in \Sigma_{w_0},\ s \in (0,\infty)\}.$$

We note that curves of the form $s \to \delta_{e^s}(z,t)$, $s \in \mathbf{R}$, are integral curves of the vector field $V(z,t) = (z, 2t)$ and so our cone is an open simply connected subset of $\mathbf{H}_1$. More precisely, if $(a+ib,c)$ satisfies $av_0 - bu_0 > 0$ and $c = \sqrt{1-(a^2+b^2)^2}$, then $(a+ib,c) \in \Sigma_{w_0}$, and so $\mathcal{C}_{(w_0,0)} \neq \emptyset$. In geometric terms, Σ_{w_0} is the upper quarter hemisphere of $S(0,1)$ lying on the same side of vertical plane containing the Euclidean ray $xv_0 - u_0 y = 0$ as $\mathcal{K}_{(w_0,0)}$.

By definition, if g_r is a dilation centered at $(w_0,0)$ then $g_r(\mathcal{C}_{(w_0,0)}) = \mathcal{C}_{(w_0,0)}$. Moreover, we define

$$\mathcal{C}_{x_0} := \tau_{x_0}^{-1}(\mathcal{C}_{(w_0,0)}),$$

and then $\overline{\mathcal{C}}_{x_0} \subset \overline{\mathcal{K}}_{x_0}$ becomes invariant under dilations centered at x_0.

We note that unless $c=0$, the curves $\tau_{(w_0,0)} \circ \delta_s(a+ib,c)$ are not horizontal. However, the tangent to such a curve at $(w_0,0)$ is $(a,b,2(av_0-bu_0))$, which is horizontal. Our discussion leads to the following notion used in Theorem 1.4.

Definition 5.5. Let γ be an end-cut of a ball $B \subset \mathbf{H}_1$ from a point $x_0 \in \partial B$. We say that γ is *angular* if there is a cone $\mathcal{C}_{x_0}$ such that $\gamma|_{[1-\epsilon,1)} \subset \mathcal{C}_{x_0}$ for all sufficiently small $\epsilon \in (0,1)$.

Remark 5. The $\mathbf{H}_1$-rays (see (19)) which join the origin to points $\partial B(0,1) \setminus (0,\pm 1)$ are angular. If $(z_0,t_0) \in \partial B(0,1)$, then it can be shown, that if $\gamma(s) = (x(s),y(s),t(s))$ is the image of the $\mathbf{H}_1$-ray $\phi(1-s,(z_0,t_0))$ under the stereographic projection of $B(0,1)$ onto the upper half space, then

$$x(s)y(0) - x(0)y(s) = s\frac{(1-t_0)+sA(s)}{(1-t_0)^2+sB(s)} > 0,$$

where A and B are continuous at $s=0$ and s is sufficiently small. The assumptions imply that $t_0 \neq 1$ and so we have strictly greater than 0 in the inequality above.

Theorem 1.4 is an analog of the Lindelöf theorem and corresponds to Theorem 6 in Gehring [31] and Theorem 7.4 in Näkki [57]. See also Vuorinen [74] for related studies in the context of angular limits for quasiregular mappings in $\mathbf{R}^n$ and Näkki [58] for further relations between angular end-cuts and various types of cluster sets.

In the proof of Theorem 1.4 we will need the following auxiliary result. Recall that if x is any boundary point of a collared domain, then we can associate with x a so-called canonical prime end, cf. the discussion following (9).

Observation 5.3. *Let $\Omega \subset \mathbf{H}_1$ be a collared domain and f be a quasiconformal embedding of Ω into $\mathbf{H}_1$. For any $x \in \partial\Omega$ and a canonical prime end $[E_k^x]$ with impression x, it follows that $[f(E_k^x)]$ is a prime end in $f(\Omega)$.*

Proof. It is an immediate consequence of the topological properties of the homeomorphism f, that $E_k = f(E_k^x)$ is a cross-set for $k=1,2,\ldots$ as in Definition 3.1. In order to show that E_k is a prime chain, and thus a prime end in $f(\Omega)$, we need to verify conditions (a) and (b) of Definition 3.3. By quasiconformality of f it holds that

$$\mathrm{Mod}_4(f(E_{k+1}), f(E_k), f(\Omega)) \leq K\mathrm{Mod}_4(E_{k+1}, E_k, \Omega) < \infty.$$

Similarly, if $F \subset \Omega$ is any continuum we have that

$$\lim_{k\to\infty} \mathrm{Mod}_4(f(E_k), f(F), f(\Omega)) \leq K \lim_{k\to\infty} \mathrm{Mod}_4(E_k, F, \Omega) = 0.$$

Hence, $[E_k]$ satisfies Definition 3.3. □

Note that by applying Observation 5.3 to a ball B we obtain that, if f is quasiconformal, then a chain $\{f(E_k^x)\}_{k=1}^{\infty}$ defines a prime end.

Recall that by $C_\gamma(f,x)$ we denote the cluster set of a map f along an end-cut γ from x (cf. Definition 5.3) and $\Pi(E_k)$ stands for a set of principal points of a prime end $[E_k]$ (cf. Definition 5.4).

Although we already stated the Lindelöf theorem in Section 1, see Theorem 1.4, for the sake of clarity and readers' convenience we recall its formulation:

Let f be a bounded quasiconformal mapping of a ball $B \subset \mathbf{H}_1$ onto a domain $\Omega_0 \subset \mathbf{H}_1$ with the property that

$$\lim_{r\to 0} \operatorname{diam}_{\mathbf{H}_1}\big(f(\partial B(x_0,r)\cap B)\big) = 0 \quad \textit{for all } x_0 \in \partial B.$$

Then for all $x_0 \in \partial B$ it holds that for every angular end-cut γ of B from x_0 we have

$$C_\gamma(f,x_0) = \Pi(f(E_k^{x_0})).$$

Remark 6. The assumption that $\operatorname{diam}_{\mathbf{H}_1}(f(\partial B(x_0,r)\cap B)) \to 0$ as $r \to 0$ is not needed in the Euclidean case, since it can be shown that $\operatorname{diam}(f(\partial B(x_0,r)\cap B)) \to 0$ in the spherical metric, see [57, Theorem 7.4] and Theorem 6 and Lemma 9 in [31]. In particular, see equation (33) on page 20 in [31], where Lemma 9 is applied. The proof of the aforementioned lemma relies on [32, Lemma 1] and does not carry over trivially to the setting of the Heisenberg group, since it relies on features of Euclidean geometry which are not obviously surmountable. However such a result in the Heisenberg setting seems plausible, for instance [8, Lemma 3.9] appears to be applicable, but as yet we do not have a precise analogue.

Proof of Theorem 1.4. Let us begin by noting that there is no loss of generality if we assume $B = B(0,1)$ and $f(0) = 0$.

Let $y \in \Pi(f(E_k^{x_0}))$ and let γ be an end cut in B from x_0, not necessarily angular. Since B is collared, there exists an end-cut from every point $x_0 \in \partial B$. We want to show that there exists a sequence $t_k \in (0,1)$ such that $t_k \to 1$, and $f(\gamma(t_k)) \to y$, i.e., $y \in C_\gamma(f,x_0)$. Since γ intersects $E_k^{x_0}$ for each k, it follows that $f\circ\gamma$ intersects $f(E_k^{x_0})$ at some point y_k for each k. By the definition of a principal point, for each $j \in \mathbf{N}$, the set $B(y,1/j)\cap\Omega_0$ contains $f(E_k^{x_0})$ for all k sufficiently large and so $y_k \to y$ in $d_{\mathbf{H}_1}$. By definition, each point $x_k = f^{-1}(y_k)$ lies in the intersection of γ and $E_k^{x_0}$ and so $x_k = \gamma(t_k)$ for some $t_k \in (0,1)$. Moreover, since $I[E_k^{x_0}] = x_0$, it follows that $x_k \to x_0$ and we conclude that $y \in C_\gamma(f,x_0)$.

Now we show that $C_\gamma(f,x_0) \subset \Pi(f(E_k^{x_0}))$ for every angular end cut from x_0. Let $y \in C_\gamma(f,x_0)$, i.e., there exists a sequence $t_n \to 1$ such that

$$s_n := d_{\mathbf{H}_1}(\gamma(t_n),x_0) \to 0, \quad \text{as } n\to\infty.$$

Let $E_{s_n}^{x_0} = S(x_0,s_n)\cap B$ for $n = 1,2,\ldots$ be a chain. As in the proof of Observation 5.2 we conclude that $[E_{s_n}^{x_0}]$ is a prime end in B and thus $[f(E_{s_n}^{x_0})]$ is a prime end in Ω_0, since f is quasiconformal. We want to show that y is a principal point of $[f(E_{s_n}^{x_0})]$, i.e., every neighborhood of y contains $f(E_{s_n}^{x_0})$ for some n. More precisely, we will show that for every $\epsilon > 0$, we have $f(E_{s_n}^{x_0}) \subset B(y,\epsilon)\cap f(B)$ for n sufficiently large.

Define a sequence of mappings on $\mathcal{K}_{x_0}$ by

$$f_n = f\circ g_{s_n} \quad \text{for } n = 1,2,\ldots,$$

where

$$g_{s_n} = \tau_{x_0}\circ\delta_{s_n}\circ\tau_{x_0}^{-1}.$$

Since f is bounded and $f(0) = 0$, it follows that all f_n avoid the values 0 and ∞, if we consider f as a mapping $f\colon B(0,1) \to \mathbf{H}_1\cup\{\infty\}$. By pg. 321 in [46] we have that $f_n|_{\mathcal{K}_{x_0}}$ correspond conformally to a sequence of K_f-quasiconformal mappings $\hat{f}_n\colon \hat{\mathcal{K}}_{x_0} \to S^3$, where K_f is the distortion of f, $S^3 = \partial\hat{B}(0,1) \subset \mathbf{C}^2$ and $\hat{\mathcal{K}}_{x_0} \subset S^3$ is the image of $\mathcal{K}_{x_0}$. The ball $\hat{B}(0,1)$ and the conformality are considered with respect to the spherical metric

$$d_S(u,w)^2 = 2|1-(u,w)| = ||u-w|^2 - 2i\mathrm{Im}\,(u,w)|,$$

where $(u,w) = u_1\overline{w_1}+u_2\overline{w_2}$. Moreover, every $\hat{f}_n$ avoids the values in S^3 corresponding with 0 and ∞ by a fixed positive distance for all n. By Theorem F in [47], the sequence $(\hat{f}_n)$ is normal. Hence, there exists a subsequence (f_{n_j}) which converges uniformly on compact subsets of $\mathcal{K}_{x_0}$ to a K_f-quasiconformal mapping h or a constant.

So as not to burden the notation with more subscripts, we assume that the sequence (s_n) is chosen so that the mappings f_n converge uniformly on compact subsets of $\mathcal{K}_{x_0}$.

Since γ is angular, there exists a cone $\mathcal{C}_{x_0} \subset \mathcal{K}_{x_0}$ such that $\gamma|_{[1-\epsilon,1)} \cap \mathcal{C}_{x_0} \neq \emptyset$ for all sufficiently small $\epsilon > 0$. Set $A_r := \overline{B(x_0,r)} \setminus B(x_0,r/2)$, then

$$\gamma(t_n) \subset A_{s_n} \cap \mathcal{C}_{x_0} = g_{s_n}(A_1 \cap \mathcal{C}_{x_0})$$

which implies that

$$f \circ \gamma(t_n) \subset f_n(A_1 \cap \mathcal{C}_{x_0}). \tag{14}$$

By the above discussion f_n converges uniformly on $A_1 \cap \overline{\mathcal{C}}_{x_0}$ to h or a constant. Since $f_n(A_1 \cap \mathcal{C}_{x_0}) \subset f(\overline{D(E^{x_0}_{s_n})})$ and $E^{x_0}_{s_n} = \partial B(x_0,s_n) \cap B$, it follows by assumption that

$$\operatorname{diam}_{\mathbf{H}_1}(f_n(A_1 \cap \mathcal{C}_{x_0})) \to 0$$

and so (f_n) converges to constant value which by (14) must be y. Moreover, the sets $f(E^{x_0}_{s_n})$ satisfy the requirements that qualify y as a principal point relative to the prime end $[f(E^{x_0}_{s_n})]$. Since $[E^{x_0}_k] = [E^{x_0}_{s_n}]$, it follows that $[f(E^{x_0}_k)] = [f(E^{x_0}_{s_n})]$, and hence we have $C_\gamma(f,x_0) \subset \Pi(f(E^{x_0}_k))$. □

The Tsuji theorem. Our next goal is to show the quasiconformal counterpart of the Tsuji theorem in $\mathbf{H}_1$. A theorem due to F. and M. Riesz states that if a planar bounded analytic function in the unit disk B^2 has the same radial limit in a set of positive Lebesgue measure in ∂B^2, then the function is constant, see e.g. Theorem 2.5 in Collingwood–Lohwater [25, Chapter 2]. The celebrated example due to Carleson [22] shows that the weaker version of that result, with radial limits existing in a boundary set of a positive logarithmic capacity, is false. However, Tsuji proved that the set of boundary points with the same radial limit α is of zero logarithmic capacity, provided that α is an ordinary point of the analytic function, see Theorem 5 in Tsuji [65] for details and Villamor [71] for further studies of Tsuji's result. In [65, Theorem 6] Tsuji also proved the following result: consider a conformal map between B^2 and a planar simply-connected domain Ω with the set $\mathcal{A}$ of accessible points in $\partial\Omega$ of zero capacity. Then the set of points in ∂B^2 corresponding to $\mathcal{A}$ has zero capacity as well. This result was extended to the setting of quasiconformal mappings in $\overline{\mathbf{R}}^n$ by Näkki [57, Theorem 7.12]. The following theorem generalizes Näkki's result in $\mathbf{H}_1$.

In the statement of Theorem 1.5 we use the notion of arcwise limit, cf. Definition 5.3. Furthermore, the Tsuji theorem in $\mathbf{H}_1$ relies on two notions which we now define: an arcwise extension of a quasiconformal mapping and the Sobolev capacity.

Let $\Omega \subset \mathbf{H}_1$ be a collared domain and f be a quasiconformal homeomorphism of Ω. Denote by $A_f \subset \partial f(\Omega)$ a set of arcwise limits of f at $\partial\Omega$. Similarly, denote by A a set of points in $\partial\Omega$ where f has an arcwise limit.

An *arcwise extension* of f, is the map $F\colon \Omega \cup A \to f(\Omega) \cup A_f$ defined as follows:

$$F(x) = \begin{cases} f(x), & x \in \Omega, \\ C_\gamma(f,x) \subset \partial f(\Omega), & x \in A. \end{cases}$$

The map F is well-defined, i.e. for every $x \in A$ it holds that $C_\gamma(f,x) = \{y\}$ for some $y \in A_f$ along any end-cut γ at x. Indeed, by Observation 3.2 and Observation 5.1, we have that every $x \in \partial\Omega$ is accessible, accessible through some prime end $[E_k^x]$ and there exists an end-cut of Ω from x. Furthermore, Theorem 1.2 shows that $C(f,x) = I[f(E_k^x)]$. This observation, combined with the immediate fact that $C_\gamma(f,x) \subset C(f,x)$ for any end-cut γ at x, gives us that $C_\gamma(f,x) \subset I[f(E_k^x)]$ for every x and its every end-cut. By Lemma 4.1 we know that $I[E_k^x] = \{x\}$. The argument will be completed once we show that $I[f(E_k^x)]$ is a singleton. By the definition of $C_\gamma(f,x)$, we have that for each $x \in A$ there is a sequence of the form $\{f(x_n)\}_{n=1}^\infty$ from which we can construct a curve γ' by connecting consecutive points in the sequence. By the constructions in Observations 5.1 and 5.2, we obtain a prime end denoted $[E_{\gamma'}]$, whose impression satisfies

$$C_{\gamma'}(f,x) \subset I[E_{\gamma'}] \subset I[f(E_k^x)]. \tag{15}$$

If a different end-cut γ_1 at x satisfies $C_{\gamma_1}(f,x) \neq C_{\gamma'}(f,x)$, then we obtain a different prime end, denoted $[E_{\gamma_1}]$, satisfying inclusions similar to (15). Therefore, both $[E_{\gamma'}]$ and $[E_{\gamma_1}]$ divide $[f(E_k^x)]$, cf. Definition 3.4. This observation contradicts the fact that $[f(E_k^x)]$ is a prime end. Indeed, if otherwise, then we would obtain that $[f^{-1}(E_{\gamma_1})]$ and $[f^{-1}(E_{\gamma'})]$ are prime ends in Ω which divide the singleton prime end $[E_k^x]$, and thus are equivalent to it. As a consequence, $[E_{\gamma'}]$ and $[E_{\gamma_1}]$ would also be equivalent, contradicting their definitions. Hence, $C_\gamma(f,x)$ is the same singleton set for all end-cuts γ at x, and so is $I[f(E_k^x)]$.

Thus, $C_\gamma(f,x)$ is a single point (asymptotic value) independent of the choice of end-cut γ and F is well-defined for every $x \in \partial A$.

Finally, we recall the notions of the Sobolev capacity and the condenser capacity specialized to the case of the Heisenberg group $\mathbf{H}_1$.

The *Sobolev* 4*-capacity* of a set $E \subset \mathbf{H}_1$ is defined as follows:

$$C_4(E) := \inf \|u\|^4_{N^{1,4}(\mathbf{H}_1)}, \tag{16}$$

where the infimum is taken over all Newtonian functions $u \in N^{1,4}(\mathbf{H}_1)$ such that $u \geq 1$ on E (see e.g. [39] for definitions and properties of Newtonian spaces). If $\Omega \subset \mathbf{H}_1$ and $E \subset \Omega$, then in an analogous way we define $C'_4(E)$, the Sobolev 4-capacity of a set with respect to Ω (instead of $\mathbf{H}_1$), see the discussion in Chapter 2.5 in [11]. We further note that $C'_4(E) \leq C_4(E)$.

The following result relates the modulus of curve families to the condenser capacity, see Definition 1.4 and Remark 1.9 in Vuorinen [73], also Ziemer [77]; see Markina [49] for a discussion in Carnot groups.

Lemma 5.1. (cf. Lemma A.1 in [2] and Theorem in [49]) *For any choice of disjoint non-empty compact subsets E, F in the closure of a ball $B_R \subset \mathbf{H}_1$ we have that*

$$\mathrm{Mod}_4(E,F,B_R) = \mathrm{cap}_4(E,F,B_R), \tag{17}$$

where $\operatorname{cap}_4(E,F,B_R)$ *denotes the* 4*-capacity of the condenser* (E,F,B_R) *and is defined by*

$$\operatorname{cap}_4(E,F,B_R) := \inf_u \int_{B_R} g_u^4\, d\lambda, \tag{18}$$

where g_u *stands for a* 4*-weak upper gradient of* u *and the infimum is taken over all* $u\in N^{1,4}(B_R)$ *satisfying* $0\le u\le 1$ *in* B_R, $u=1$ *on* E *and* $u=0$ *on* F.

Let us comment that the equality between the 4-modulus and the 4-capacity for disjoint sets $E,F\subset B_R$ is a consequence of [2, Lemma A.1], while the case when E,F are allowed to be subsets of $\overline{B_R}$ is proven in [49]. In order to relate Definition 1.2 in [49] to the above definition of the condenser capacity, let us notice that by Proposition 11.6 and Theorem 11.7 in [33] the minimal 4-weak upper gradient of a Lipschitz function coincides almost everywhere with the norm of the horizontal gradient (cf. discussion in Chapter 14.2 in [39]). Moreover, Theorem 5.47 in [11] ensures the density of locally Lipschitz functions in $N^{1,4}(B_R)$.

Remark 7. The assumptions of Theorem 1.5 simplify if f is a global quasiconformal map $f\colon \mathbf{H}_1\to\mathbf{H}_1$. Then, since B is a uniform domain, by Lemma 3.2, also $f(B)$ is uniform, by Proposition 4.2 and Theorem 4.4 in [19]. Thus, $f(B)$ is mod-uniform by Observation 4.1.

Proof of Theorem 1.5. The main idea of the proof is similar to the one of Theorem 7.12 in [57]. However, we need to adjust several tools and auxiliary results to the Heisenberg setting. Moreover, we use techniques developed in recent years.

Let $E\subset f(B)$ be a closed set. Denote by $\Gamma(E,A_f,f(B))$ the family of (horizontal) curves γ in $f(B)$ with one endpoint in E, the other in A_f, and $\gamma\setminus(E\cup A_f)\subset f(B)$. By applying Proposition 1.48 in Björn–Björn [11] in the setting of $\mathbf{H}_1$, we have that $C_4(A_f)=0$ implies $\operatorname{Mod}_4(\Gamma_{A_f})=0$. Here, Γ_{A_f} denotes the family of all nonconstant (horizontal) curves in $\mathbf{H}_1$ passing through A_f. Since $\Gamma(E,A_f,f(B))\subset\Gamma_{A_f}$ it holds that

$$\operatorname{Mod}_4(E,A_f,f(B))\le \operatorname{Mod}_4(\Gamma_{A_f})=0.$$

We set $\Delta' := f^{-1}\Gamma(E,A_f,f(B))$ and use the quasiconformality of f to conclude that $\operatorname{Mod}_4(\Delta')=0$. Since B is collared and $f(B)$ is mod-uniform, then by the Koebe theorem, Theorem 1.3, we conclude that all curves in Δ' have the property that one of their ends belongs to $E'=F^{-1}(E)$ while the other one belongs to $A=F^{-1}(A_f)$. Let us define Δ'' by

$$\Delta'' := \Gamma(E',A,B)\setminus\Delta'$$

with a restriction that we consider open paths only. By the definition of A, we have that $F(A)=A_f$, and consequently all curves γ'' in $f(\Delta'')$ are such that f does not have an asymptotic value along them. It follows that all γ'' are nonrectifiable and Corollary A.1 implies that $\operatorname{Mod}_4(f(\Delta''))=0$. We again apply quasiconformality of f and obtain that $\operatorname{Mod}_4(\Delta'')=0$. Lemma A.2 and the subadditivity of the modulus result in the following observation:

$$\operatorname{Mod}_4(E',A,B)=\operatorname{Mod}_4(\Delta'\cup\Delta'')\le\operatorname{Mod}_4(\Delta')+\operatorname{Mod}_4(\Delta'')=0.$$

This implies that for the family Γ' consisting of nonconstant curves lying in $\overline{B}$ which have non-empty intersection with $A\subset\partial B$, the 4-modulus is zero, i.e.,

$$\operatorname{Mod}_4(\{\gamma\in\Gamma(\overline{B})\colon |\gamma|\cap A\neq\emptyset\})=0.$$

Indeed, suppose on the contrary that there exists a subfamily $\Gamma_1 \subset \Gamma'$ such that $0 < \mathrm{Mod}_4(\Gamma_1) < \infty$. Next, let us choose a sphere centered at the origin with radius r, contained in B, such that this sphere intersects curves in a subfamily of Γ_1 with positive 4-modulus, for large enough r (we denote this subfamily again by Γ_1). Upon taking the appropriate part of that sphere we obtain a continuum, denoted $E' \subset B$, intersecting all curves in Γ_1. Consider Γ_2, a family of curves starting at intersection points of curves in Γ_1 with E' and ending at A. Clearly, Γ_1 is minorized by Γ_2 and, therefore, it holds that

$$0 = \mathrm{Mod}_4(E', A, B) \geq \mathrm{Mod}_4(\Gamma_2) \geq \mathrm{Mod}_4(\Gamma_1) > 0$$

leading to a contradiction with the assumption on the positivity of $\mathrm{Mod}_4(\Gamma_1)$. Then, Proposition 1.48 in [11] applied to $\overline{B}$ and A gives us the assertion of the theorem. □

Appendix A. Appendix

In the section we provide some auxiliary results in the geometry of the Heisenberg group and the modulus of curve families in $\mathbf{H}_1$.

A.1. Polar coordinates. For each $(z,t) \in \mathbf{H}_1$, $z \neq 0$, the curve $\gamma_{(z,t)}(s) = \phi(s,(z,t))$, where

$$\phi(s,(z,t)) = \delta_s \left(\exp\left(-it\frac{\log(s)}{|z|^2} \right) z, t \right), \tag{19}$$

is a horizontal curve joining 0 to (z,t) which we refer to as an $\mathbf{H}_1$-ray. In particular, the parametrization (19) has the following properties (see Balogh–Tyson [8]):

(1) $\phi(s,(z,0)) = sz$,
(2) $\|\phi(s,(z,t))\| = \|\gamma_{(z,t)}(s)\| = s\|(z,t)\|$,
(3) If $\Phi_s(z,t) := \phi(s,(z,t))$, then $\det D\Phi_s(z,t) = s^4$ for $s > 0$ and $(z,t) \in \mathcal{Z} := \{(z,t) \in \mathbf{H}_1 \colon z \neq 0\}$.

By Theorem 3.7 in [8], there exists a unique Radon measure σ on $S \setminus \mathcal{Z}$ (for the unit sphere $S = S(0,1)$), such that for $u \in L^1(\mathbf{H}_1)$,

$$\int_{\mathbf{H}_1} u(z,t)\, d\lambda(z,t) = \int_{S\setminus\mathcal{Z}} \int_0^\infty u(\phi(s,v))\, s^3\, ds\, d\sigma(v). \tag{20}$$

Furthermore, by Proposition 2.18 in [8] we have $\lambda(\mathcal{Z}) = 0$,

$$S \setminus \mathcal{Z} = \{(\sqrt{\cos\alpha}\, e^{i\theta}, \sin\alpha) \colon \alpha \in (-\pi/2, \pi/2),\ \ \theta \in [0, 2\pi)\}, \tag{21}$$

and it follows that $d\sigma = d\alpha\, d\theta$ (see Example 3.11 in [8]).

A.2. Modulus of curve families in $\mathbf{H}_1$. The notion of the modulus of curve families is fundamental in the studies of geometry of metric spaces and mappings between domains in metric spaces. Since the modulus is a vital quantity throughout the text, we briefly discuss some of it's properties.

We now follow the standard way to define the modulus of curve families, see e.g. Chapter 6 in Väisälä [68]. Let Γ be a family of curves in a domain $\Omega \subset \mathbf{H}_1$. We say that a nonnegative Borel function $\varrho \colon \mathbf{H}_1 \to [0,\infty]$ is *admissible for* Γ if

$$\int_\gamma \varrho\, dl \geq 1,$$

for every locally rectifiable $\gamma \in \Gamma$. We denote the set of admissible functions by $F(\Gamma)$.

Let $1 \leq p < \infty$. Then the *p-modulus of curve family* Γ is defined as follows:

$$\mathrm{Mod}_p\Gamma := \inf_{\varrho \in F(\Gamma)} \int_{\mathbf{H}_1} \varrho^p \, d\lambda,$$

where λ is 3-dimensional Lebesgue measure on $\mathbf{H}_1 = \mathbf{R}^3$. If $F(\Gamma)$ is empty, then by convention we define $\mathrm{Mod}_p\Gamma = \infty$. If $\gamma \in \Gamma$ is a constant curve, then the condition $\int_\gamma \rho \, dl \geq 1$ is not satisfied and the set of admissible functions $F(\Gamma)$ is empty.

From the point of view of relating the p-modulus to other geometric data we often consider curve families joining subsets of a given domain. If $\Omega \subseteq \mathbf{H}^1$ is a domain such that E and F are subsets of Ω, then $\Gamma(E, F, \Omega)$ denotes the family of closed rectifiable curves γ in Ω which join E and F, i.e. one of the endpoints of γ belongs to E, the other to F and $\gamma \setminus (E \cup F) \subset \Omega$. If f is a homeomorphism of Ω, then we define $f\Gamma(E, F, \Omega) = \Gamma(f(E), f(F), f(\Omega))$.

The fundamental properties of the p-modulus that we require are summarized in the following lemma (see [37] section 2.3).

Lemma A.1. *The following properties hold for the p-modulus:*

(1) *The p-modulus of all curves that are not locally rectifiable is zero.*
(2) $\mathrm{Mod}_p \, \emptyset = 0$.
(3) *If* $\Gamma \subset \Gamma'$, *then* $\mathrm{Mod}_p\Gamma \leq \mathrm{Mod}_p\Gamma'$.
(4) *If* $\Gamma = \bigcup_{j=1}^\infty \Gamma_j$, *then* $\mathrm{Mod}_p\Gamma \leq \sum_{j=1}^\infty \mathrm{Mod}_p\Gamma_j$.
(5) *If every curve in* Γ' *contains a subcurve in* Γ, *then we say that* Γ' *is minorized by* Γ *and write* $\Gamma < \Gamma'$. *Then, it holds that* $\mathrm{Mod}_p\Gamma' \leq \mathrm{Mod}_p\Gamma$.

In the case $p = 4$ we can show that 4-modulus depends only on rectifiable curves. The following observations are analogues of Corollary 6.11 and Theorem 7.10 in [68] and are used in the proof of the Tsuji theorem 1.5 in Section 5. Since these results do not appear in the literature we provide their proof.

Given a curve family Γ, we denote by $F_r(\Gamma)$, the family of all nonnegative Borel functions $\varrho \colon \mathbf{H}_1 \to \mathbf{R}$ such that $\int_\gamma \varrho \, dl \geq 1$ for every rectifiable $\gamma \in \Gamma$. Note that $F(\Gamma) \subseteq F_r(\Gamma)$ with equality when Γ consists entirely of closed paths.

Theorem A.1. *If* Γ *is a curve family in* $\mathbf{H}_1$, *then*

$$\mathrm{Mod}_4\Gamma = \inf_{\varrho \in F_r(\Gamma)} \int_{\mathbf{H}_1} \varrho^4 \, d\lambda.$$

Proof. Since $F(\Gamma) \subseteq F_r(\Gamma)$ we have $\inf_{\varrho \in F_r(\Gamma)} \int_{\mathbf{H}_1} \varrho^4 \, d\lambda \leq \mathrm{Mod}_4\Gamma$. Let

$$\varrho_1(z,t) = \begin{cases} \frac{1}{\|(z,t)\| \log \|(z,t)\|} & \text{if } \|(z,t)\| \geq 2, \\ 1 & \text{if } \|(z,t)\| < 2, \end{cases}$$

then by (20) we have

$$\int_{\mathbf{H}_1} \varrho_1(z,t)^4 \, d\lambda(z,t) = 2\pi^2 \left(\frac{2^4}{4} + \frac{1}{3(\log 2)^3} \right).$$

Assume $\gamma \in \Gamma$ is locally rectifiable. If γ is bounded, then we have $\varrho_1(g) \geq a > 0$ for some a and so

$$\int_\gamma \varrho_1 \, dl = \infty.$$

If γ is unbounded, then there exists an increasing sequence $\{r_n\}_{n=0}^\infty \subset \mathbf{R}$ such that $\|\gamma(r_0)\| \geq 2$, $\|\gamma(r_n)\| \leq \|\gamma(r_{n+1})\|$ and $\|\gamma(r_n)\| \to \infty$ as $n \to \infty$. For each $n > 0$, let $\gamma_n = \gamma|_{[r_0, r_n]}$ and let $\bar{\gamma}_n$ denote the arc length parameterization of γ_n. Applying the triangle inequality to $d_{\mathbf{H}_1}$ we have $|\|pq\| - \|p\|| \le \|q\|$. Using this observation in (2) with $p = \bar\gamma_n(s_i)$ and $q = \bar\gamma_n(s_i)^{-1}\bar\gamma_n(s_{i+1})$ we see that $t \to \|\bar\gamma_n(t)\|$, $t \in [r_0, r_n]$, is rectifiable and by Theorem 5.7 in [68] it holds that

$$\int_{\bar\gamma_n} \varrho_1 \, dl \ge \int_{\|\bar\gamma_n\|} \varrho_0 \, |ds| \ge \log\log \|\gamma(r_n)\| - \log\log \|\gamma(r_0)\|,$$

where

$$\varrho_0(s) = \begin{cases} \frac{1}{s \log s} & \text{if } s \ge 2, \\ 1 & \text{if } s < 2. \end{cases}$$

It follows that we again have that $\int_\gamma \varrho_1 \, dl = \infty$.

Let $\varrho \in F_r(\Gamma)$ and set $\varrho_\epsilon = (\varrho^4 + \epsilon^4 \varrho_1^4)^{1/4}$, then $\varrho_\epsilon > \varrho$ and

$$\int_\gamma \varrho_\epsilon \, dl \ge \int_\gamma \varrho \, dl \ge 1$$

for every rectifiable $\gamma \in \Gamma$. If $\gamma \in \Gamma$ is not rectifiable, then

$$\int_\gamma \varrho_\epsilon \, dl \ge \epsilon \int_\gamma \varrho_1 \, dl = \infty.$$

It follows that $\varrho_\epsilon \in F(\Gamma)$ and

$$\mathrm{Mod}_4\Gamma \le \int_{\mathbf{H}_1} \varrho_\epsilon^4 \, d\lambda = \int_{\mathbf{H}_1} \varrho^4 \, d\lambda + \epsilon^4 \int_{\mathbf{H}_1} \varrho_1^4 \, d\lambda.$$

Since $\epsilon > 0$ and ϱ are arbitrary, we conclude that $\mathrm{Mod}_4\Gamma \le \inf_{\varrho \in F_r(\Gamma)} \int_{\mathbf{H}_1} \varrho^4 d\lambda$. □

Corollary A.1. *If* Γ_r *is the family of all rectifiable curves in* Γ*, then* $\mathrm{Mod}_4\Gamma = \mathrm{Mod}_4\Gamma_r$*. In particular, the family of all non-rectifiable curves in* $\mathbf{H}_1$ *has zero* 4*-modulus.*

Lemma A.2. *Let* $\Gamma_0 = \Gamma_0(E, F, \Omega)$ *denote the family of all curves* γ *in* Ω *with the property that the closure of the trace of* γ *has nonempty intersection with both* E *and* F*. If* $\Gamma = \Gamma(E, F, \Omega)$*, then*

$$\mathrm{Mod}_4(\Gamma_0) = \mathrm{Mod}_4(\Gamma).$$

Proof. Since Γ is minorized by Γ_0, we have $\mathrm{Mod}_4\Gamma \le \mathrm{Mod}_4\Gamma_0$. In order to prove the reverse inequality it suffices to prove that $F(\Gamma) \subset F_r(\Gamma_0)$.

Assume that $\varrho \in F(\Gamma)$ and that γ is a rectifiable path in Γ_0. If γ^* denotes the closed extension of γ given by Theorem 2.1, then the locus of γ^* meets both E and F. In particular, we may assume there exists $t_1 \le t_2$ such that $\gamma^*(t_1) \in E$ and $\gamma^*(t_2) \in F$. It follows that the curve $\beta = \gamma^*|_{[t_1,t_2]}$ belongs to Γ and

$$\int_\gamma \varrho \, dl = \int_{\gamma^*} \varrho \, dl \ge \int_\beta \varrho \, dl \ge 1.$$

We conclude that $\varrho \in F_r(\Gamma_0)$. □

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