THE CLOSURE OF DIRICHLET SPACES
IN THE BLOCH SPACE

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Abstract. If $0 < p < \infty$ and $\alpha > -1$, the space of Dirichlet type $D^p_\alpha$ consists of those functions $f$ which are analytic in the unit disc $D$ and have the property that $f'$ belongs to the weighted Bergman space $A^p_\alpha$. Of special interest are the spaces $D^{p-1}_p (0 < p < \infty)$ and the analytic Besov spaces $B^p = D^{p-2}_p$ ($1 < p < \infty$). Let $B$ denote the Bloch space. It is known that the closure of $B^p$ ($1 < p < \infty$) in the Bloch norm is the little Bloch space $B^0$. A description of the closure in the Bloch norm of the spaces $H^p \cap B$ has been given recently. Such closures depend on $p$. In this paper we obtain a characterization of the closure in the Bloch norm of the spaces $D^p_\alpha \cap B$ ($1 \leq p < \infty$, $\alpha > -1$). In particular, we prove that for all $p \geq 1$ the closure of the space $D^{p-1}_p \cap B$ coincides with that of $H^2 \cap B$. Hence, contrary to what happens with Hardy spaces, these closures are independent of $p$. We apply these results to study the membership of Blaschke products in the closure in the Bloch norm of the spaces $D^p_\alpha \cap B$.

1. Introduction and main results

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ denote the open unit disc in the complex plane $\mathbb{C}$, $\partial D$ will be the unit circle. Also, $dA$ will denote the area measure on $D$, normalized so that the area of $D$ is 1. Thus $dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$. The space of all analytic functions in $D$ will be denoted by $\mathcal{H}ol(D)$. We also let $H^p$ ($0 < p \leq \infty$) be the classical Hardy spaces. We refer to [9] for the notation and results regarding Hardy spaces. The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\partial D$. The “little oh” version of $BMOA$ is the space $VMOA$. We refer to [15] for the theory of $BMOA$-functions.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space $A^p_\alpha$ consists of those functions $f \in \mathcal{H}ol(D)$ such that

$$
\| f \|_{A^p_\alpha} \overset{\text{def}}{=} \left( (\alpha + 1) \int_D (1 - |z|^2)^\alpha |f(z)|^p dA(z) \right)^{1/p} < \infty.
$$

The unweighted Bergman space $A^p_0$ is simply denoted by $A^p$. We refer to [10, 19, 31] for the notation and results about Bergman spaces. The space of Dirichlet type $D^p_\alpha$ ($0 < p < \infty$ and $\alpha > -1$) consists of those functions $f \in \mathcal{H}ol(D)$ such that $f' \in A^p_\alpha$. In other

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https://doi.org/10.5186/aasfm.2019.4402

2010 Mathematics Subject Classification: Primary 30H30; Secondary 46E15.

Key words: Bloch space, Dirichlet spaces, Besov spaces, weighted Bergman spaces, closure in the Bloch norm, Blaschke product.

This research is supported by grants from “El Ministerio de Economía y Competitividad, Spain” MTM2014-52865-P; and from “La Junta de Andalucía” (FQM210).
words, a function \( f \in \mathcal{H}ol(\mathbb{D}) \) belongs to \( \mathcal{D}_p^\alpha \) if and only if
\[
\|f\|_{\mathcal{D}_p^\alpha} \overset{\text{def}}{=} |f(0)| + \left( \alpha + 1 \right) \int_{\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|^p dA(z) \right)^{1/p} < \infty.
\]
If \( \alpha > p - 1 \) then it is well known that \( \mathcal{D}_p^\alpha = A_{\alpha-p} \) (see, e.g., [11, Theorem 6]).
For \( 1 < p < \infty \), the space \( \mathcal{D}_p^{p-2} \) is the analytic Besov space \( B^p \). The space \( B^1 \) requires a special definition: it is the space of all functions \( f \in \mathcal{H}ol(\mathbb{D}) \) such that \( f'' \in A^1 \).
Although the corresponding semi-norm is not conformally invariant, the space itself is. Another possible definition (with a conformally invariant semi-norm) is given in the fundamental article [3], where \( B^1 \) was denoted by \( \mathcal{M} \). The spaces \( B^p \), \( 1 \leq p < \infty \), form a nested scale of conformally invariant spaces which are contained in \( VMOA \) and show up naturally in different settings (see [3], [8] and [30]).

In particular, \( \mathcal{D}_0^2 = B^2 \) is the classical Dirichlet space \( \mathcal{D} \).

Finally, we recall that a function \( f \in \mathcal{H}ol(\mathbb{D}) \) is said to be a Bloch function if
\[
\|f\|_B \overset{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.
\]
The space of all Bloch functions will be denoted by \( B \). It is a non-separable Banach space with the norm \( \| \cdot \|_B \) just defined. A classical source for the theory of Bloch functions is [1]. The closure of the polynomials in the Bloch norm is the little Bloch space \( \mathcal{B}_0 \) which consists of those \( f \in \mathcal{H}ol(\mathbb{D}) \) with the property that
\[
\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.
\]
It is well known that
\[
H^\infty \subsetneq BMOA \subsetneq \cap_{0 < p < \infty} H^p, \quad H^\infty \subsetneq BMOA \subsetneq B, \quad VMOA \subsetneq \mathcal{B}_0 \subsetneq \mathcal{B}.
\]

Anderson, Clunie and Pommerenke [1, p. 36] raised the question of determining the closure of \( H^\infty \) in \( \mathcal{B} \). They remarked that this closure strictly contains \( \mathcal{B}_0 \) but is not identical with \( \mathcal{B} \). The problem is still open. However, Jones gave an unpublished description of the closure of \( BMOA \) in \( \mathcal{B} \) (see [2, Theorem 9]). Given \( f \in \mathcal{B} \) and \( \varepsilon > 0 \), we define
\[
\Omega_\varepsilon(f) = \{ z \in \mathbb{D} : (1 - |z|^2)|f'(z)| \geq \varepsilon \}.
\]
Then a Bloch function \( f \) is in the closure of \( BMOA \) in the Bloch norm if and only if for every \( \varepsilon > 0 \) the Borel measure \( (1 - |z|^2)^{-1} \chi_{\Omega_\varepsilon(f)}(z) dA(z) \) is a Carleson measure in \( \mathbb{D} \). As usual, for a Borel subset \( E \) of \( \mathbb{D} \), \( \chi_E \) denotes the characteristic function of \( E \). A proof of Jones’ description is provided by Ghatage and Zheng [14].

This study has been broadened to determine the closure in the Bloch norm of other subspaces of \( \mathcal{B} \). For simplicity, if \( X \) is a subspace of the Bloch space we shall let \( \mathcal{C}_B(X) \) denote the closure in the Bloch norm of the space \( X \).

Tjani [26] proved that if \( f \in \mathcal{B} \), then \( f \in \mathcal{B}_0 \) if and only if \( \int_{\Omega_\varepsilon(f)} \frac{dA(z)}{(1 - |z|^2)^2} < \infty \) for every \( \varepsilon > 0 \). Since all Besov spaces contain the polynomials and are contained in \( \mathcal{B}_0 \), we have
\[
\mathcal{C}_B(B^p) = \mathcal{B}_0, \quad 1 \leq p < \infty.
\]
This was observed in [29] where the closures in the Bloch norm of other conformally invariant spaces were also described. Bao and Göğüş [5] have recently characterized the closure in the Bloch norm of the space \( \mathcal{D}_\alpha^2 \cap \mathcal{B} \) \((-1 < \alpha \leq 1\)).
Monreal Galán and Nicolau [22] described the closure in the Bloch norm of $\mathcal{B}\cap H^p$, for $1 < p < \infty$. Galanopoulos, Monreal Galán and Pau [13] have extended this result to the whole range $0 < p < \infty$.

Let us fix some notation. Given a Lebesgue measurable subset $\Omega$ of $\mathbb{D}$, we let $A_h(\Omega)$ be the hyperbolic area of $\Omega$, that is,

$$A_h(\Omega) = \int_{\Omega} \frac{dA(z)}{(1-|z|^2)^2}.$$ 

Also, for fixed $a > 1$ and for $\xi \in \partial \mathbb{D}$, we let $\Gamma_a(\xi) = \{z \in \mathbb{D} : |z - \xi| < a(1-|z|)\}$ be the Stolz angle with vertex at $\xi$. Putting [22, Theorem 1] and [13, Theorem 1] together yields the following result.

**Theorem A.** Let $0 < p < \infty$ and $a > 1$. A Bloch function $f$ is in the closure in the Bloch norm of $\mathcal{B}\cap H^p$ if and only if for every $\varepsilon > 0$ the function $F_\varepsilon(f)$ defined by

$$F_\varepsilon(f)(\xi) = A^{1/2}_h(\Gamma_a(\xi) \cap \Omega_\varepsilon(f)), \quad \xi \in \partial \mathbb{D},$$ 

belongs to $L^p(\partial \mathbb{D})$, that is,

$$\int_{\partial \mathbb{D}} \left( \int_{\Gamma_a(\xi) \cap \Omega_\varepsilon(f)} \frac{dA(z)}{(1-|z|^2)^2} \right)^{p/2} |d\xi| < \infty.$$

It is well known that there exists a positive constant $C$ such that

$$|f(z)| \leq C \|f\|_B \log \frac{2}{1-|z|}, \quad (z \in \mathbb{D}), \quad \text{for every } f \in \mathcal{B},$$

(see [1, p.13]). Then it follows trivially that $\mathcal{B} \subset A^p_\alpha$ whenever $0 < p < \infty$ and $\alpha > -1$. So the question of characterizing $C_B(A^p_\alpha \cap \mathcal{B})$ is trivial:

$$C_B(A^p_\alpha \cap \mathcal{B}) = C_B(\mathcal{B}) = \mathcal{B}, \quad 0 < p < \infty, \quad \alpha > -1.$$ 

The main object of this paper is to characterize the closure in the Bloch norm of the spaces $\mathcal{D}^p_\alpha \cap \mathcal{B}$. As we mentioned above, if $p - 1 < \alpha$ then $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$. Thus, using (1.2) we obtain

$$C_B(\mathcal{D}^p_\alpha \cap \mathcal{B}) = \mathcal{B}, \quad 0 < p < \infty, \quad p - 1 < \alpha.$$ 

If $-1 < \alpha \leq p - 2$ then we have that $\mathcal{D}^p_\alpha \subset \mathcal{D}^p_{p-2} = B^p \subset \mathcal{B}$, and then (1.1) implies that

$$C_B(\mathcal{D}^p_\alpha \cap \mathcal{B}) = C_B(\mathcal{B}^p) \subset C_B(B^p) = \mathcal{B}_0.$$

Now it is clear that the polynomials lie in $\mathcal{D}^p_\alpha$ and then it follows that $\mathcal{B}_0 \subset C_B(\mathcal{D}^p_\alpha)$. Consequently, we have

$$C_B(\mathcal{D}^p_\alpha \cap \mathcal{B}) = \mathcal{B}_0, \quad 0 < p < \infty, \quad \alpha \leq p - 2.$$ 

If remains to consider the case where $p - 2 < \alpha \leq p - 1$ and we shall pay a special attention to the case $\alpha = p - 1$ because the spaces $\mathcal{D}^p_{p-1}$ are the closest ones to Hardy spaces among all the $\mathcal{D}^p_\alpha$-spaces. By the Littlewood–Paley identity, we have $\mathcal{D}^2_1 = H^2$. We have also [21]

$$H^p \subsetneq \mathcal{D}^p_{p-1}, \quad \text{for } 2 < p < \infty,$$

and [11, 27]

$$\mathcal{D}^p_{p-1} \subsetneq H^p, \quad \text{for } 0 < p < 2.$$ 

A number of similarities and differences between the spaces $H^p$ and $\mathcal{D}^p_{p-1}$ are presented in [4, 16, 17, 23, 27]. As in the case of Hardy spaces, there is no inclusion relation between the spaces $\mathcal{D}^p_{p-1}$ and the Bloch space. Despite the fact that there
is no relation of inclusion between $D_{p-1}^p$ and $D_{q-1}^q$ for $p \neq q$ (see [4, 17, 12]), it was observed in [7] that

$$D_{p-1}^p \cap B \subset D_{q-1}^q \cap B, \quad 0 < p < q < \infty.$$  

In the next theorem we give a characterization of the closures in the Bloch norm of the spaces $D_{p-1}^p \cap B \ (1 \leq p < \infty)$. We remark that, contrary to what happens with Hardy spaces, these closures are independent of $p$.

**Theorem 1.** Let $p \in [1, \infty)$ and $f \in B$. Then the following conditions are equivalent.

(i) $f \in C_B(D_{p-1}^p \cap B)$.

(ii) For every $\varepsilon > 0$

$$\int_{\Omega_{\varepsilon}(f)} \frac{dA(z)}{1 - |z|^2} < \infty.$$  

(iii) $f \in C_B(H^2 \cap B)$.

As remarked in [22], the equivalence (ii) $\iff$ (iii) follows immediately from the case where $p = 2$ in Theorem A by using Fubini’s theorem. Indeed, using Fubini’s theorem, for $f \in B$, $\varepsilon > 0$, and $a > 1$, we have

$$\int_{\partial D} \int_{\Gamma_{\varepsilon}(f)} \frac{1}{(1 - |z|^2)^2} dA(z) |d\xi|$$

$$= \int_{\partial D} \int_{\Omega_{\varepsilon}(f)} \chi_{\Gamma_{\varepsilon}(f)}(z) \frac{1}{(1 - |z|^2)^2} dA(z) |d\xi|$$

$$= \int_{\Omega_{\varepsilon}(f)} \left( \int_{\partial D} \chi_{\Gamma_{\varepsilon}(f)}(z) |d\xi| \right) \frac{1}{(1 - |z|^2)^2} dA(z)$$

$$\leq \frac{1}{(1 - |z|^2)^2} \int_{\Omega_{\varepsilon}(f)} dA(z) - \frac{1}{1 - |z|^2} \int_{\Omega_{\varepsilon}(f)} dA(z).$$

Bearing in mind that (ii) $\iff$ (iii), Theorem 1 follows from the following one where we give a characterization of $C_B(D_{\alpha}^\infty \cap B)$ whenever $1 \leq p < \infty$ and $p - 2 < \alpha \leq p - 1$.

**Theorem 2.** Suppose that $1 \leq p < \infty$, $p - 2 < \alpha \leq p - 1$, and let $f$ be a Bloch function. Then the following conditions are equivalent.

(i) $f \in C_B(D_{\alpha}^\infty \cap B)$.

(ii) For every $\varepsilon > 0$ we have that

$$\int_{\Omega_{\varepsilon}(f)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} < \infty.$$  

The proof of Theorem 2 will be presented in Section 2. In Section 3 we discuss the case $0 < p < 1$ and we study also the membership of Blaschke products in the spaces $C_B(D_{\alpha}^\infty \cap B)$.

We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta, \ldots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta, \ldots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions $E_1, E_2$ we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant $C$ independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that $E_1$ and $E_2$ are equivalent and we write $E_1 \asymp E_2$.  

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2. Proof of Theorem 2

We start recalling a well known lemma (see [31, Lemma 3.10, p. 55]).

Lemma A. Suppose that \( c \) is real and \( t > -1 \), and set

\[
F(z) = \int_{D} \frac{(1 - |w|^2)^t}{|1 - \overline{w}z|^{2 + t + c}} dA(w), \quad z \in D.
\]

(i) If \( c < 0 \), then \( F(z) \) is a bounded function of \( z \)

(ii) If \( c > 0 \), then \( F(z) \propto (1 - |z|^2)^{-c}, |z| \to 1^- \).

(iii) If \( c = 0 \), then \( F(z) \propto \log \frac{1}{1 - |z|^2}, |z| \to 1^- \).

We shall also need the following representation formula for Bloch functions (see [31, Proposition 4.27 and p. 112]).

Proposition A. Let \( f \) be a Bloch function with \( f(0) = f'(0) = 0 \), then

\[
f(z) = \int_{D} \frac{(1 - |w|^2)f'(w)}{(1 - z\overline{w})^2 w} dA(w), \quad z \in D.
\]

Proof of the implication (i) \( \implies \) (ii) in Theorem 2. Take a function \( f \) in the closure in the Bloch norm of \( D^p_\alpha \cap \mathcal{B} \) and \( \varepsilon > 0 \). Then there exists a function \( g \in D^p_\alpha \cap \mathcal{B} \) such that \( \|f - g\|_B < \frac{\varepsilon}{2} \). Clearly, this implies that \( \Omega_\varepsilon(f) \subseteq \Omega_\varepsilon(g) \). Then it follows that

\[
\int_{D} |g'(z)|^p (1 - |z|^2)^\alpha dA(z) \geq \int_{\Omega_\varepsilon(g)} |g'(z)|^p (1 - |z|^2)^\alpha dA(z) \\
= \int_{\Omega_\varepsilon(g)} \frac{|g'(z)|^p (1 - |z|^2)^p}{(1 - |z|^2)^{p-\alpha}} dA(z) \\
\geq \left( \frac{\varepsilon}{2} \right)^p \int_{\Omega_\varepsilon(g)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}}.
\]

Since \( g \in D^p_\alpha \), (ii) follows. \( \square \)

Proof of the implication (ii) \( \implies \) (i) in Theorem 2. Suppose that \( 1 \leq p < \infty \), \( p - 2 < \alpha \leq p - 1 \), and let \( f \) be a Bloch function which satisfies (ii). Assume without loss of generality that \( f(0) = f'(0) = 0 \). Using Proposition A we can write \( f \) as follows

\[
f(z) = \int_{D} \frac{(1 - |w|^2)f'(w)}{(1 - z\overline{w})^2 w} dA(w), \quad z \in D.
\]

Take \( \varepsilon > 0 \). We decompose \( f \) in the following way

\[
f(z) = \int_{\Omega_\varepsilon(f)} \frac{(1 - |w|^2)f'(w)}{(1 - \overline{w}z)^2 w} dA(w) + \int_{D \setminus \Omega_\varepsilon(f)} \frac{(1 - |w|^2)f'(w)}{(1 - \overline{w}z)^2 w} dA(w) \\
= f_1(z) + f_2(z).
\]
For any \( z \in D \), we have
\[
(1 - |z|^2)|f'_2(z)| \leq 2(1 - |z|^2) \int_{D \setminus \Omega_t(f)} \frac{(1 - |w|^2)|f'(w)|}{|1 - \overline{w}z|^3} dA(w)
\]
\[
\leq 2\varepsilon(1 - |z|^2) \int_{D \setminus \Omega_t(f)} \frac{dA(w)}{|1 - \overline{w}z|^3}
\]
\[
\leq 2\varepsilon(1 - |z|^2) \int_{D} \frac{dA(w)}{|1 - \overline{w}z|^3}.
\]
Using Lemma A with \( t = 0 \) and \( c = 1 \), we obtain that \( (1 - |z|^2)|f'_2(z)| \leq C\varepsilon \) where \( C \) is a positive constant. Hence, \( \|f_2\|_B \leq C\varepsilon \). Equivalently, \( f_1 \) is a Bloch function with \( \|f - f_1\|_B \leq C\varepsilon \).

The proof will be finished if we prove that \( f_1 \in \mathcal{D}_\alpha^p \) or, equivalently, that \( f'_1 \in A_\alpha^p \). We have
\[
\int_D (1 - |z|^2)^\alpha |f'_1(z)|^p dA(z) = \int_D (1 - |z|^2)^\alpha |f'_1(z)|^{p-1} |f'_1(z)| dA(z)
\]
\[
= \int_D (1 - |z|^2)^{p-1} |f'_1(z)|^{p-1} (1 - |z|^2)^{\alpha - p + 1} |f'_1(z)| dA(z)
\]
\[
\leq \|f_1\|_B^{p-1} \int_D (1 - |z|^2)^{\alpha - p + 1} |f'_1(z)| dA(z)
\]
\[
\leq \|f_1\|_B^{p-1} \|f\|_B \int_{\Omega_t(f)} \left( \int_D \frac{(1 - |w|^2)|f'(w)|}{|1 - \overline{w}z|^3} dA(w) \right) dA(z)
\]
\[
\leq \|f_1\|_B^{p-1} \|f\|_B \int_{\Omega_t(f)} \left( \int_D \frac{(1 - |z|^2)^{\alpha - p + 1}}{|1 - \overline{w}z|^3} dA(z) \right) dA(w).
\]
Observe that \( \alpha - p + 1 > -1 \) and \( p - \alpha > 0 \). Then, using Lemma A with \( t = \alpha - p + 1 \) and \( c = p - \alpha \) and (ii), we obtain
\[
\int_D (1 - |z|^2)^\alpha |f'_1(z)|^p dA(z) \lesssim \|f_1\|_B^{p-1} \|f\|_B \int_{\Omega_t(f)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} < \infty,
\]
that is, \( f'_1 \in A_\alpha^p \) as desired. \( \square \)

3. The case \( 0 < p < 1 \) and some further results

Putting together (1.3), (1.4) and Theorem 2 we have the following result.

**Theorem 3.** Suppose that \( 0 < p < \infty \) and \( \alpha > -1 \).

(i) If \( \alpha \leq p - 2 \), then \( C_B(\mathcal{D}_\alpha^p \cap B) = C_B(\mathcal{D}_\alpha^p) = B_0 \).

(ii) If \( \alpha > p - 1 \), then \( C_B(\mathcal{D}_\alpha^p \cap B) = B \).

(iii) If \( p \geq 1 \) and \( p - 2 < \alpha \leq p - 1 \), then
\[
C_B(\mathcal{D}_\alpha^p \cap B) = \left\{ f \in B : \int_{\Omega_t(f)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} < \infty \text{ for all } \varepsilon > 0 \right\}.
\]

We do not know whether (iii) remains true for \( 0 < p < 1 \). In particular, we do not know whether \( C_B(\mathcal{D}_{p-1}^p \cap B) \) coincides with \( C_B(H^2 \cap B) \) when \( 0 < p < 1 \).

We can prove the following result.

**Theorem 4.** Suppose that \( 0 < p < 1 \), \(-1 < \alpha \leq p - 1 \), and let \( f \) be a Bloch function.

(a) If \( f \in C_B(\mathcal{D}_\alpha^p \cap B) \), then \( \int_{\Omega_t(f)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} < \infty \) for every \( \varepsilon > 0 \).
(b) If there exists $\gamma > 2 - \frac{1 + \alpha}{p}$ such that $\int_{\Omega_{\varepsilon}(f)} \frac{dA(z)}{(1 - |z|^2)^{\gamma}} < \infty$ for every $\varepsilon > 0$, then $f \in C_{B}(D_{\alpha}^p \cap B)$.

Proof. For $f \in B$, we have

$$
\int_{D} (1 - |z|^2)^{\alpha + 1 - p} |f'(z)| dA(z) = \int_{D} (1 - |z|^2)^{\alpha} |f'(z)|^p \left[(1 - |z|^2) |f'(z)|\right]^{1 - p} dA(z)
$$

$$
\leq \|f\|_B^{1 - p} \int_{D} (1 - |z|^2)^{\alpha} |f'(z)|^p dA(z).
$$

Hence, it follows that $D_{\alpha}^p \cap B \subseteq D_{\alpha + 1 - p} \cap B$. Using this, the fact that $-1 < \alpha + 1 - p \leq 0$, and Theorem 2, (a) follows.

We turn to prove (b). Observe that

$$
1 \leq 2 - \frac{1 + \alpha}{p} < 2.
$$

Suppose that $\gamma > 2 - \frac{1 + \alpha}{p}$ and that $\int_{\Omega_{\varepsilon}(f)} \frac{dA(z)}{(1 - |z|^2)^{\gamma}} < \infty$ for every $\varepsilon > 0$. Clearly, we may assume without loss of generality that $\gamma < 2$. Arguing as is the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 2, the fact $f \in C_{B}(D_{\alpha}^p \cap B)$ will follow if we prove that the Bloch function $f_1$ defined by

$$f_1(z) = \int_{\Omega_{\varepsilon}(f)} \frac{(1 - |w|^2) f'(w)}{(1 - \overline{w} z)^2 w} dA(w), \quad z \in D,$$

belongs to the space $D_{\alpha}^p$ or, equivalently, that

$$f_1' \in A_{\alpha}^p. \tag{3.1}$$

We are going to present two proofs of (3.1), the second one has been suggested to us by one of the referees. Observe that $0 < 2 - \gamma < \frac{\alpha + 1}{p}$ and $1 - \gamma > -1$. Then it follows that $A_{1 - \gamma}^1 \subseteq A_{\alpha}^p$ (see [20, p. 703] or [6, Lemma 1.2]). Hence it suffices to show that

$$f_1' \in A_{1 - \gamma}^1. \tag{3.2}$$

We have

$$\int_{D} (1 - |z|^2)^{1 - \gamma} |f_1'(z)| dA(z)$$

$$\leq \int_{D} (1 - |z|^2)^{1 - \gamma} \int_{\Omega_{\varepsilon}(f)} \frac{(1 - |w|^2) |f'(w)|}{|1 - \overline{w} z|^3} dA(w) dA(z)$$

$$\leq \|f\|_B \int_{\Omega_{\varepsilon}(f)} \left(\int_{D} (1 - |z|^2)^{1 - \gamma} \frac{dA(z)}{|1 - \overline{w} z|^3}\right) dA(w)$$

$$\leq \|f\|_B \int_{\Omega_{\varepsilon}(f)} \frac{dA(w)}{(1 - |w|^2)^{\gamma}}.$$
and define $h(z) = (1 - |z|^2)^{\delta}$ ($z \in D$). Using Hölder’s inequality, Fubini’s theorem, the facts that $\frac{\delta}{1-p} - \alpha < 1$, $\alpha + \frac{\delta}{p} > -1$ and $1 - \alpha - \frac{\delta}{p} > 0$, and Lemma A, we obtain

$$\int_D |f'(z)|^p(1 - |z|^2)^\alpha dA(z) \lesssim \int_D \left( \int_{\Omega(f)} \frac{|f'(w)|(1 - |w|^2)^\alpha}{|1 - \overline{w}z|^3} dA(w) \right)^p (1 - |z|^2)^\alpha dA(z)$$

$$\leq \|f\|_B^p \int_D \left( \int_{\Omega(f)} \frac{dA(w)}{|1 - \overline{w}z|^3} \right)^p h(z) h(z)^{-1}(1 - |z|^2)^{\alpha p}(1 - |z|^2)^{\alpha (1-p)} dA(z)$$

$$\lesssim \|f\|_B^p \left( \int_D (1 - |z|^2)^{\alpha + \frac{\delta}{p}} \int_{\Omega(f)} \frac{dA(w)}{|1 - \overline{w}z|^3} dA(z) \right)^{1-p} \left( \int_D (1 - |z|^2)^{\alpha - \frac{\delta}{p}} dA(z) \right)^{1-\alpha p} \lesssim \|f\|_B^p \left( \int_{\Omega(f)} \frac{dA(w)}{(1 - |w|^2)^{1-\alpha - \frac{\delta}{p}}} \right)^p.$$ 

Since $1 - \alpha - \frac{(1-p)(\alpha+1)}{p} = 2 - \alpha + \frac{1}{p}$, (3.1) follows choosing $\delta$ sufficiently close to $(1-p)(\alpha+1)$. 

Our next aim is to give applications of the results that we have obtained so far to study the membership of a Blaschke product in $C_B(D_\alpha \cap B)$ for distinct values of $p$ and $\alpha$. We refer to [9] for the definition, notation, and results about Blaschke products. Since $H^\infty \subset H^2 \cap B$, Theorem 1 trivially implies that

$$H^\infty \subset C_B(D_\alpha \cap B), \quad 1 \leq p < \infty.$$ 

In particular any Blaschke product lies in $C_B(D_\alpha \cap B)$ whenever $1 \leq p < \infty$.

For $0 < p < 2$ the space $H^\infty$ is not included in $D_\alpha$. Rudin [25, Theorem III] proved that there exists a Blaschke product $B$ with $B \notin D_\alpha$. Later on, Vinogradov [27] gave examples of Blaschke products $B$ such that $B \notin D_\alpha$ for all $p$ in $(0, 2)$.

On the other hand, Rudin also proved in [25] that if a sequence $\{a_n\} \subset D$ satisfies the condition

$$\sum (1 - |a_n|) \log \frac{1}{1-|a_n|} < \infty$$

then the Blaschke product whose sequence of zeros is $\{a_n\}$ belongs to $D_\alpha$ (and, consequently to $D_p$ for all $p \geq 1$). The converse of this is not true. Indeed, a result of Vinogradov [27, Theorem 2.9, p. 3814] implies that a Blaschke product with zeros in a Stolz angle lies in $D_p$ for all $p$.

Protas proved in [24, Theorem 1] that if $0 < s < 1$ and the sequence $\{a_n\}$ of the zeros of the Blaschke product $B$ satisfies the condition $\sum (1 - |a_n|^2)^s < \infty$, then $B' \in A_1^{s-1}$. Using again [6, Lemma 1.2] we see that $A_1^{s-1} \subset A_p^{s-1}$ for all $p \in (0, 1)$, whenever $0 < s < 1$. Then we deduce the following:

If the sequence $\{a_n\}$ of the zeros of the Blaschke product $B$ satisfies the condition $\sum (1 - |a_n|^2)^s < \infty$ for some $s < 1$, then $B \in \cap_{0<p<\infty} D_p$.

Let us summarize these facts in the following theorem.

**Theorem 5.** Let $B$ be a Blaschke product and let $\{a_n\}$ be its sequence of zeros.

(i) $B \in C_B(D_p \cap B)$ whenever $1 \leq p < \infty$.

(ii) If $\sum (1 - |a_n|) \log \frac{1}{1-|a_n|} < \infty$, then $B \in \cap_{1<p<\infty} D_p$.

(iii) If $\sum (1 - |a_n|^2)^s < \infty$ for some $s < 1$, then $B \in \cap_{0<p<\infty} D_p$.
Suppose that $1 \leq \gamma < 2$ and let $B$ be the Blaschke product whose sequence of zeros is $\{a_n\}$. Take $\varepsilon > 0$. We have
\[ |B'(z)| \leq \sum \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}, \quad z \in \mathbb{D}, \]
and hence
\[ z \in \Omega_\varepsilon(B) \implies 1 \leq \frac{1}{\varepsilon} (1 - |z|^2) \sum \frac{1 - |a_n|^2}{|1 - \overline{a_n} z|^2}. \]
Then it follows that
\[ \int_{\Omega_\varepsilon(B)} \frac{dA(z)}{(1 - |z|^2)^\gamma} \leq \frac{1}{\varepsilon} \sum (1 - |a_n|^2) \int_{\Omega_\varepsilon(B)} \frac{(1 - |z|^2)^{1 - \gamma}}{|1 - \overline{a_n} z|^2} dA(z) \]
\[ \leq \frac{1}{\varepsilon} \sum (1 - |a_n|^2) \int_{\mathbb{D}} \frac{(1 - |z|^2)^{1 - \gamma}}{|1 - \overline{a_n} z|^2} dA(z). \]
Now, using Lemma A with $t = 1 - \gamma$ and $c = \gamma - 1$, we obtain
\[ \int_{\Omega_\varepsilon(B)} \frac{dA(z)}{1 - |z|^2} \leq \frac{1}{\varepsilon} \sum (1 - |a_n|^2) \log \frac{1}{1 - |a_n|^2} \]
and
\[ \int_{\Omega_\varepsilon(B)} \frac{dA(z)}{(1 - |z|^2)^\gamma} \leq \frac{1}{\varepsilon} \sum (1 - |a_n|^2)^{2 - \gamma}, \quad \text{if } 1 < \gamma < 2. \]

Using these inequalities and Theorem 1 and Theorem 4 with $\alpha = p - 1$, we obtain results which are weaker than those stated in Theorem 5. However, using (3.7) and Theorem 4 in the case $\alpha < p - 1$, we obtain the following result.

**Theorem 6.** Let $B$ be the Blaschke product whose sequence of zeros is $\{a_n\}$. If $1 \leq p < \infty$, $p - 2 < \alpha < p - 1$, and $\sum (1 - |a_n|^2)^{2 - (p - \alpha)} < \infty$, then $B \in \mathcal{C}_B(\mathcal{D}_p^\alpha \cap \mathcal{B})$.

Restricting ourselves to interpolating Blaschke products (that is, Blaschke products whose sequences of zeros are universal interpolation sequences [9, Chapter 9]), we have the following result.

**Theorem 7.** Let $B$ be an interpolating Blaschke product whose sequence of zeros is $\{a_n\}_{n=1}^\infty$. Suppose that $1 \leq p < \infty$ and $p - 2 < \alpha < p - 1$. Then the following conditions are equivalent.

(i) $\sum (1 - |a_n|^2)^{2 - (p - \alpha)} < \infty$.

(ii) $B \in \mathcal{C}_B(\mathcal{D}_p^\alpha \cap \mathcal{B})$.

We remark that this was proved in [5] for the case where $p = 2$ and $0 < \alpha < 1$.

**Proof of Theorem 7.** The implication (i) $\implies$ (ii) follows trivially from Theorem 6. To prove the other implication, suppose that $B \in \mathcal{C}_B(\mathcal{D}_p^\alpha \cap \mathcal{B})$. By Theorem 3 we have
\[ \int_{\Omega_{\varepsilon}(f)} \frac{dA(z)}{(1 - |z|^2)^{p - \alpha}} < \infty. \]

Since $B$ is an interpolating Blaschke product, the sequence $\{a_n\}$ is uniformly separated, that is, there exists $\delta > 0$ such that
\[ \inf_{m \geq 1} \prod_{n=1, n \neq m}^\infty g(a_n, a_m) \geq \delta. \]
Here \( \rho \) denotes the pseudo-hyperbolic distance:

\[
\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|, \quad z, w \in \mathbb{D}.
\]

Also, for \( a \in \mathbb{D} \) and \( 0 < r < 1 \), \( \Delta(a, r) \) will denote the pseudo-hyperbolic disc of center \( a \) and radius \( r \):

\[
\Delta(a, r) = \{ z \in \mathbb{D} : \rho(z, a) < r \}.
\]

Using Lemma 3.5 of [18] we see that there exist \( \varepsilon > 0 \) and \( \beta \in (0, 1) \) such that the discs \( \{ \Delta(a_n, \beta) : n = 1, 2, 3, \ldots \} \) are pairwise disjoint and so that

\[
|B'(z)| \geq \frac{\varepsilon}{1 - |a_n|^2}, \quad z \in \Delta(a_n, \beta), \quad n = 1, 2, 3, \ldots.
\]

This implies that

\[
(3.9) \quad \bigcup_{n=1}^{\infty} \Delta(a_n, \beta) \subset \Omega_{\varepsilon}(B).
\]

Using the fact that the discs \( \{ \Delta(a_n, \beta) \} \) are pairwise disjoint and (3.9), we obtain

\[
(3.10) \quad \sum_{n=1}^{\infty} \int_{\Delta(a_n, \beta)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} = \int_{\bigcup_{n=1}^{\infty} \Delta(a_n, \beta)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}} \leq \int_{\Omega_{\varepsilon}(B)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}}.
\]

Now, (see [31, p. 69]) it is well known that

\[
(1 - |z|^2) \asymp (1 - |a_n|^2), \quad \text{as long as } z \in \Delta(a_n, \beta),
\]

and that the area \( A(\Delta(a_n, \beta)) \) of \( \Delta(a_n, \beta) \) satisfies \( A(\Delta(a_n, \beta)) \asymp (1 - |a_n|^2)^2 \). These two facts imply that

\[
\sum_{n=1}^{\infty} (1 - |a_n|^2)^{2-(p-\alpha)} \asymp \sum_{n=1}^{\infty} \int_{\Delta(a_n, \beta)} \frac{dA(z)}{(1 - |z|^2)^{p-\alpha}}.
\]

This, together with (3.10) and (3.8), implies that \( \sum_{n=1}^{\infty} (1 - |a_n|^2)^{2-(p-\alpha)} < \infty \). \( \square \)

Acknowledgements. We wish to express our gratitude to the referees for their helpful comments and suggestions which have lead us to improve the paper.

References


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Received 18 March 2018 • Accepted 24 July 2018