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PERIODIC APPROXIMATION OF EXCEPTIONAL LYAPUNOV EXPONENTS FOR SEMI-INVERTIBLE OPERATOR COCYCLES

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Abstract. We prove that for semi-invertible and Hölder continuous linear cocycles A acting on an arbitrary Banach space and defined over a base space that satisfies the Anosov closing property, all exceptional Lyapunov exponents of A with respect to an ergodic invariant measure for base dynamics can be approximated with Lyapunov exponents of A with respect to ergodic measures supported on periodic orbits. Our result is applicable to a wide class of infinite-dimensional dynamical systems.

1. Introduction

Let M be a compact metric space and $f\colon M \to M$ a homeomorphism such that (M, f) satisfies the so-called Anosov closing property, which essentially means that there are many periodic orbits for f in M. Furthermore, let A be a linear cocycle over (M, f) that takes values in the space of all bounded linear operators acting on an arbitrary Banach space $\mathcal{B}$. Finally, let μ be any ergodic f-invariant Borel probability measure on M. The main objective of the present paper is to show that if A is sufficiently regular (as a map on M) and if it satisfies the so-called quasi-compactness property with respect to μ, then all exceptional Lyapunov exponents of A with respect to μ can be approximated by Lyapunov exponents of A with respect to some ergodic f-invariant Borel measure which is supported on a periodic orbit for f.

We emphasize that the assumption that A is quasi-compact with respect to μ is made to ensure that one can apply the most recent versions of the multiplicative ergodic theorem (MET), which in turn give the set of Lyapunov exponents of A with respect to μ. Consequently, the problem of approximating Lyapunov exponents of A with respect to μ becomes well-posed. Starting essentially with the pioneering work of Ruelle [R82] who considered cocycles of operators on a Hilbert space, many authors have been interested in the problem of establishing MET for cocycles of operators acting on Banach spaces. In particular, Mañé [M81] established MET for cocycles of compact and injective operators on a Banach space. His results were generalized by Thieullen [Thi87], who was able to replace the assumption that the operators are compact with a substantially weaker assumption that the cocycle is quasi-compact. More recently, Froyland, Lloyd and Quas [FLQ10, FLQ13], González-Tokman and Quas [GTQ14] and Blumenthal [AB16] were able to remove the assumption present

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in both [M81] and [Thi87] (as well as in more recent works such as [LL10]) that the cocycle consists of injective operators. In addition, they have also been able to relax certain regularity assumptions for the cocycle. Although the present paper addresses the problem of the approximation of Lyapunov exponents for quasi-compact cocycles, we emphasize that our results are new even in a particular case of compact cocycles which are not invertible.

In his seminal paper [Kal11], Kalinin established (as a tool in proving the main result of [Kal11], which is the Livšic theorem) the approximation result described in the first paragraph for cocycles of invertible matrices. This was generalized to cocycles of not necessarily invertible matrices by the first author [Bac]. Furthermore, Kalinin and Sadovskaya [KS] (see also [KS2]) established the approximation result for the largest and smallest Lyapunov exponent of an invertible cocycle acting on an arbitrary Banach space (see Remark 2.7 for details). In the present paper, we go one step further by considering not necessarily invertible cocycles and by establishing the approximation result for all exceptional Lyapunov exponents and not only for the largest one. The importance of our results steems from the fact that in the context of infinite-dimensional dynamics, the invertibility assumption for cocycle is way too restrictive. Indeed, the main motivation for papers [FLQ13, GTQ14] was to establish the version of MET that would enable us to study cocycles of transfer operators that are rarely invertible (or even injective). Furthermore, in the recent paper by Blumenthal and Young [BY17] in which the authors extend many results of the smooth ergodic theory to the case of maps acting on Banach spaces, the derivative cocycle is not assumed to be invertible.

The approach and the arguments in the present paper are inspired by those in [Kal11]. Indeed, when obtaining the approximation property of the largest Lyapunov exponent we follow closely the approach developed in [Kal11] (which in turn inspired arguments in [Bac, KS]). However, the nontrivial adaptation of arguments from [Kal11] occurs when we try to establish the desired approximation property of other Lyapunov exponents. In the classical finite-dimensional case this is done (see [Kal11, Bac]) by using the so-called exterior powers of the cocycle. On the other hand, such a construction doesn't exist in the infinite-dimensional setting. This forced us to adjust the method of estimating the largest Lyapunov exponent devised in [Kal11] to fit the problem of estimating other Lyapunov exponents.

The paper is organized as follows. In Section 2 we introduce terminology, recall basic notions and important results (such as MET) and state the main result of our paper. In Section 3, we introduce the concept of Lyapunov norms for operator cocycles which play an important auxiliary tool in our arguments. In Section 4 we present the proof of our main result. Finally, in Section 5 we discuss various applications of our work in the context of the infinite-dimensional dynamics.

2. Preliminaries

Let (M, d) be a compact metric space, μ a probability measure defined on the Borel subsets of M and $f\colon M \to M$ a μ-preserving homeomorphism. Furthermore, assume also that μ is ergodic.

We say that f satisfies the *Anosov closing property* if there exist $C_1, \varepsilon_0, \theta > 0$ such that if $z \in M$ satisfies $d(f^n(z), z) < \varepsilon_0$ then there exists a periodic point $p \in M$ such that $f^n(p) = p$ and

$$d(f^j(z), f^j(p)) \le C_1 e^{-\theta \min\{j, n-j\}} d(f^n(z), z),$$

for every $j = 0, 1, \ldots, n$. We note that shifts of finite type, basic pieces of Axiom A diffeomorphisms and more generally, hyperbolic homeomorphisms are particular examples of maps satisfying the Anosov closing property. We refer to [KH95, Corollary 6.4.17.] for details.

2.1. Semi-invertible operator cocycles. Let $(\mathcal{B}, \| \cdot \|)$ be a Banach space and let $B(\mathcal{B}, \mathcal{B})$ denote the space of all bounded linear maps from $\mathcal{B}$ to itself. We recall that $B(\mathcal{B}, \mathcal{B})$ is a Banach space with respect to the norm

$$\| T \| = \sup\{\| Tv \| / \| v \|;\ \| v \| \neq 0\}, \quad T \in B(\mathcal{B}, \mathcal{B}).$$

Although we use the same notation for the norms on $\mathcal{B}$ and $B(\mathcal{B}, \mathcal{B})$ this will not cause any confusion. Finally, consider a map $A : M \to B(\mathcal{B}, \mathcal{B})$.

The *semi-invertible operator cocycle* (or just *cocycle* for short) generated by A over f is defined as the map $A\colon \mathbf{N} \times M \to B(\mathcal{B}, \mathcal{B})$ given by

$$A^n(x) := A(n, x) = \begin{cases} A(f^{n-1}(x)) \ldots A(f(x))A(x) & \text{if } n > 0, \\ \text{Id} & \text{if } n = 0 \end{cases} \tag{1}$$

for all $x \in M$. The term 'semi-invertible' refers to the fact that the action of the underlying dynamical system f is assumed to be an invertible transformation while the action on the fibers given by A may fail to be invertible.

2.2. Multiplicative ergodic theorem. We begin by recalling some terminology. Let $B_{\mathcal{B}}(0, 1)$ denote the unit ball in $\mathcal{B}$ centered at 0. For an arbitrary $T \in B(\mathcal{B}, \mathcal{B})$, let $\|T\|_{\text{ic}}$ be the infimum over all $r > 0$ with the property that $T(B_{\mathcal{B}}(0, 1))$ can be covered by finitely many open balls of radius r. It is easy to show that:

$$\|T\|_{\text{ic}} \leq \|T\|, \quad \text{for every } T \in B(\mathcal{B}, \mathcal{B}) \tag{2}$$

and

$$\|T_1 T_2\|_{\text{ic}} \leq \|T_1\|_{\text{ic}} \cdot \|T_2\|_{\text{ic}}, \quad \text{for every } T_1, T_2 \in B(\mathcal{B}, \mathcal{B}). \tag{3}$$

Hence, (3) together with the subadditive ergodic theorem implies that there exists $\kappa(\mu) \in [-\infty, \infty)$ such that

$$\kappa(\mu) = \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)\|_{\text{ic}} \quad \text{for } \mu\text{-a.e. } x \in M.$$

Observe that if A takes values in a family of compact operators on $\mathcal{B}$, we have that $\kappa(\mu) = -\infty$. Indeed, in this case one has that $\|A^n(x)\|_{\text{ic}} = 0$ for each n which readily implies that $\kappa(\mu) = -\infty$.

In addition, by using again the subadditive ergodic theorem together with the subadditivity of the operator norm, we have that there exists $\lambda(\mu) \in [-\infty, \infty)$ such that

$$\lambda(\mu) = \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)\| \quad \text{for } \mu\text{-a.e. } x \in M.$$

Note that (2) implies that $\kappa(\mu) \leq \lambda(\mu)$. We say that the cocycle A is *quasi-compact* with respect to μ if $\kappa(\mu) < \lambda(\mu)$. The following result from [GTQ14, Lemma C.3] gives useful sufficient conditions under which the cocycle is quasi-compact.

Proposition 2.1. *Take $A\colon M \to B(\mathcal{B}, \mathcal{B})$. Let $\mathcal{B}' = (\mathcal{B}', |\cdot|)$ be a Banach space such that $\mathcal{B} \subset \mathcal{B}'$ and with the property that the inclusion $(\mathcal{B}, \|\cdot\|) \hookrightarrow (\mathcal{B}', |\cdot|)$ is compact. Furthermore, suppose that each $A(x)$ can be extended to a bounded operator on $(\mathcal{B}', |\cdot|)$ and that there exist Borel-measurable functions $\alpha, \beta, \gamma\colon M \to (0, \infty)$ such that:*

(1) *for μ-a.e. $x \in M$ and every $v \in \mathcal{B}$,*

$$\|A(x)v\| \le \alpha(x)\|v\| + \beta(x)|v|; \tag{4}$$

(2) *for μ-a.e. $x \in M$,*

$$\|A(x)\| \le \gamma(x); \tag{5}$$

(3)

$$\int \log \alpha \, d\mu < \lambda(\mu) \quad \text{and} \quad \int \log \gamma \, d\mu < \infty. \tag{6}$$

Then, $\kappa(\mu) \le \int \log \alpha \, d\mu$. In particular, A is quasi-compact with respect to μ.

Remark 2.2. In the context of cocycles of transfer operators, i.e. when $A(x)$ is the transfer operator associated to some map T_x for each $x \in M$, the condition (4) is called strong Lasota-Yorke inequality while (5) is called weak Lasota–Yorke inequality. We note that in this setting one has that $\lambda(\mu) = 0$.

For example, when each T_x is a piecewise expanding map on the unit interval $[0,1]$, one can show that under mild assumptions (4), (5) and (6) hold with $(\mathcal{B}, \|\cdot\|) = (BV, \|\cdot\|_{BV})$ and $(\mathcal{B}', |\cdot|) = (L^1, \|\cdot\|_{L^1})$. Here, BV denotes the space of all functions of bounded variation on $[0,1]$ with the corresponding norm $\|\cdot\|_{BV}$ which is defined to be the sum of the L^1 norm of the function and its total variation. We refer to [DFGTV, Section 2.3.1] for a detailed discussion.

Before stating the version of the multiplicative ergodic theorem established in [FLQ13], we recall the notion of μ-continuity. Let Z be an arbitrary Banach space. We say that a map $\Phi\colon M \to Z$ is *μ-continuous* if there exists an increasing sequence $(K_n)_{n\in\mathbf{N}}$ of compact subsets of M satisfying $\mu(\cup_n K_n) = 1$ and such that $\Phi|_{K_n}\colon K_n \to Z$ is continuous for each $n \in \mathbf{N}$.

Theorem 2.3. *Assume that the cocycle $A\colon M \to B(\mathcal{B}, \mathcal{B})$ is μ-continuous and quasi-compact with respect to μ. Then, we have the following:*

(1) *there exists $l = l(\mu) \in [1, \infty]$ and a sequence of numbers $(\lambda_i(\mu))_{i=1}^l$ such that*

$$\lambda(\mu) = \lambda_1(\mu) > \lambda_2(\mu) > \ldots > \lambda_i(\mu) > \ldots > \kappa(\mu).$$

Furthermore, if $l = \infty$ we have that $\lim_{i\to\infty} \lambda_i(\mu) = \kappa(\mu)$;

(2) *there exists a Borel subset $\mathcal{R}^\mu \subset M$ such that $\mu(\mathcal{R}^\mu) = 1$ and for each $x \in \mathcal{R}^\mu$ and $i \in \mathbf{N} \cap [1, l]$, there is a unique and measurable decomposition*

$$\mathcal{B} = \bigoplus_{j=1}^{i} E_j(x) \oplus V_{i+1}(x), \tag{7}$$

where $E_j(x)$ are finite-dimensional subspaces of $\mathcal{B}$ and $A(x)E_j(x) = E_j(f(x))$. Furthermore, $V_{i+1}(x)$ are closed subspaces of $\mathcal{B}$ and $A(x)V_{i+1}(x) \subset V_{i+1}(f(x))$;

(3) *for each $x \in \mathcal{R}^\mu$ and $v \in E_j(x) \setminus \{0\}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| = \lambda_j(\mu).$$

In addition, for every $v \in V_{i+1}(x)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| \le \lambda_{i+1}(\mu).$$

The numbers $\lambda_i(\mu)$ *are called* exceptional Lyapunov exponents *of the cocycle* A *with respect to* μ *and the dimensions* $d_i(\mu) = \dim E_i(x)$ *are called* multiplicities *of* $\lambda_i(\mu)$. *In addition, the decomposition* (7) *is called the* Oseledets splitting. *Finally, the points in* $\mathcal{R}^\mu$ *are called* μ-regular *(or simply* regular*).*

We denote by

$$\gamma_1(\mu) \geq \gamma_2(\mu) \geq \gamma_3(\mu) \geq \ldots$$

the Lyapunov exponents counted with multiplicities of A with respect to the measure μ. This means that $\gamma_i(\mu) = \lambda_1(\mu)$ for $i = 1, \ldots, d_1(\mu)$, $\gamma_i(\mu) = \lambda_2(\mu)$ for $i = d_1(\mu)+1, \ldots, d_1(\mu)+d_2(\mu)$ and so on. When there is no risk of ambiguity, we suppress the index μ from the previous objects. Moreover, when the f-invariant measure μ is supported on the orbit of some periodic point p we simply write $\lambda_i(p)$ and $\gamma_i(p)$ for its Lyapunov exponents and Lyapunov exponents counted with multiplicities, respectively. Furthermore, given $x \in M$ and $v \in \mathcal{B}$ we denote by

$$\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log \| A^n(x)v \|$$

the *Lyapunov exponent of* A *at* x *in the direction* v.

Remark 2.4. Since the arguments in our paper will heavily rely on the measurability of the Oseledets splitting (7), we would like to explain what exactly it means for (7) to be measurable. Let $\mathcal{G}(\mathcal{B})$ denote the set of all closed subspaces F of $\mathcal{B}$ that are complemented, i.e. such that there exists a closed subspace $\tilde{F}$ of $\mathcal{B}$ with the property that $\mathcal{B} = F \oplus \tilde{F}$. We recall that each finite-dimensional subspace F of $\mathcal{B}$ belongs to $\mathcal{G}(\mathcal{B})$. It turns out that one can equip $\mathcal{G}(\mathcal{B})$ with the structure of a metric space (see [BY17, Section 2.1.2]) and thus in particular it makes sense to discuss the measurability of the map that is defined on some measurable space and that takes values in $\mathcal{G}(\mathcal{B})$.

Now we observe that all subspaces of $\mathcal{B}$ that appear in (7) belong to $\mathcal{G}(\mathcal{B})$. Hence, we can associate to (7) the following maps

$$E_1 \colon \mathcal{R}^\mu \to \mathcal{G}(\mathcal{B}), \ldots, E_i \colon \mathcal{R}^\mu \to \mathcal{G}(\mathcal{B}) \quad \text{and} \quad V_{i+1} \colon \mathcal{R}^\mu \to \mathcal{G}(\mathcal{B}). \tag{8}$$

We now say that (7) is measurable if all maps in (8) are measurable. Moreover, those maps are also μ-continuous as a consequence of a deep result by Fremlin [KP84, Theorem 4.1] (see also [BY17, Remark 3.5.]).

2.3. Main result. We say that $A \colon M \to B(\mathcal{B}, \mathcal{B})$ is an α*-Hölder continuous map* if there exists a constant $C_2 > 0$ such that

$$\| A(x) - A(y) \| \leq C_2 d(x, y)^\alpha,$$

for all $x, y \in M$. Clearly, if $A \colon M \to B(\mathcal{B}, \mathcal{B})$ is an α-Hölder continuous map, then A is also μ-continuous and consequently Theorem 2.3 is applicable. We are now in the position to state the main result of our paper.

Theorem 2.5. *Let* $f \colon M \to M$ *be a homeomorphism satisfying the Anosov closing property,* μ *an ergodic* f*-invariant probability measure and* $A \colon M \to B(\mathcal{B}, \mathcal{B})$ *an* α*-Hölder continuous map that is quasi-compact with respect to* μ*. Then, given* $s \in \mathbf{N} \cap [1, l(\mu)]$ *there exists a sequence of periodic points* $(p_k)_{k\in\mathbf{N}}$ *such that*

$$\gamma_i(p_k) \xrightarrow{k\to+\infty} \gamma_i(\mu) \quad \text{for every } i \in \{1, \ldots, d_1(\mu) + \ldots + d_s(\mu)\},$$

where $d_i(\mu) = \dim E_i(x)$.

Remark 2.6. We stress that without the assumption that the cocycle is quasi-compact, it is not always possible to get an approximation result in the spirit of Theorem 2.5 even if the cocycle takes values in the space of bounded and invertible linear operators on a Banach space. Indeed, Kalinin and Sadovskaya [KS, Proposition 1.5] presented an example of a locally constant operator cocycle A over a full shift on two symbols and an ergodic invariant measure μ such that $\lambda_1(\mu) > \sup_{\mu_p} \lambda_1(\mu_p)$, where the supremum is taken over all invariant measures μ_p supported on periodic orbits. Related examples were also constructed by Hurtado [Hur].

Let us discuss in detail the relationship between Theorem 2.5 and various related results in the literature.

Remark 2.7. Observe that whenever $\mathcal{B}$ is finite-dimensional and the cocycle is invertible, we have that $\kappa(\mu) = -\infty$ and that the set of exceptional Lyapunov exponents given by Theorem 2.3 coincides with the set of Lyapunov exponents given by the classical Oseledets multiplicative ergodic theorem. Therefore, in this setting, Theorem 2.5 reduces to [Kal11, Theorem 1.4.].

Recently, the first author [Bac] has generalized [Kal11, Theorem 1.4.] to the case of semi-invertible cocycles of matrices, i.e. $\mathcal{B}$ is again assumed to be finite-dimensional but $A(x)$ doesn't have to be an invertible matrix. In this setting, the family of exceptional Lyapunov exponents can differ from the family of Lyapunov exponents given by the version of the Oseledets multiplicative ergodic theorem established in [FLQ10]. More precisely, let Λ_1 denote the set of exceptional Lyapunov exponents in the sense of Theorem 2.3 and let Λ_2 denote the set of Lyapunov exponents in the sense of [FLQ10]. Then,

$$\Lambda_1 = \begin{cases} \Lambda_2 & \text{if } -\infty \notin \Lambda_2; \\ \Lambda_2 \setminus \{-\infty\} & \text{if } -\infty \in \Lambda_2. \end{cases}$$

Since the main result of [Bac] establishes the desired approximation property of elements in Λ_2 including $-\infty$ (if present), we conclude that Theorem 2.5 provides only a partial generalization of the main result in [Bac].

In addition, Kalinin and Sadovskaya [KS] established the approximation property similar to that in Theorem 2.5 for the largest Lyapunov exponent of an arbitrary invertible Hölder continuous cocycle $A\colon M \to B(\mathcal{B}, \mathcal{B})$. More precisely, they proved that for each $\epsilon > 0$ there exists a periodic point $p \in M$ satisfying $f^k(p) = p$ and such that

$$\left| \lambda_1(\mu) - \frac{1}{k} \log \|A^k(q)\| \right| < \epsilon.$$

However, the above result is weaker than the approximation property for $\lambda_1(\mu) = \gamma_1(\mu)$ established in Theorem 2.5 (see the discussion in [KS] after Remark 1.5.). Moreover, our Theorem 2.5 deals with *all* exceptional Lyapunov exponents (and not only with the largest one) of a *semi-invertible* quasi-compact cocycle acting on a Banach space and thus represents a natural extension of the results from [Kal11, Bac] described above.

Finally, in their recent paper [KS2], Kalinin and Sadovskaya established results similar to those in [Kal11] and [KS] for cocycles over non-uniformly hyperbolic dynamical systems. Although these systems will in general fail to satisfy the Anosov closing property, they will exhibit a similar type of behaviour (provided by the so-called Katok's closing lemma). It turns out that this weaker form of closing property is still sufficient to adapt the arguments from [Kal11, KS] and obtain the desired approximation property of Lyapunov exponents in this setting.

2.4. Examples. We now discuss some concrete examples of non-compact cocycles that satisfy all of our assumptions.

Example 2.8. Assume that $T_1, \dots, T_k \colon [0,1] \to [0,1]$ are piecewise expanding maps such that

$$\delta_i := \operatorname*{ess\,inf}_{x \in [0,1]} |T_i'(x)| > 2 \quad \text{for } i \in \{1, \dots, k\}.$$

Let us denote by L_i the transfer operator associated to T_i. We note that L_i acts on the BV space. Furthemore, let $M = \{1, \dots, k\}^{\mathbf{Z}}$ with the standard topology and consider a two-sided shift $f \colon M \to M$. Note that (M, f) satisfies Anosov closing property. Furthermore, we define a cocycle A on M of operators acting on BV by

$$A(x) = L_{x_0} \quad \text{for } x = (x_n)_{n \in \mathbf{Z}} \in M.$$

It is straightforward to verify that A is Hölder continuous. On the other hand, one can also show (see [DFGTV, Section 2.3.1]) that (4) holds with a constant $\alpha \in (0,1)$ and that in fact A is quasi-compact with respect to any f-invariant ergodic Borel probability measure.

The following example is somewhat of different nature.

Example 2.9. In their recent remarkable paper [BY17], Blumenthal and Young extend various results from smooth ergodic theory to the case of maps acting on Banach spaces. More precisely, let $\mathcal{B}$ be an arbitrary Banach space and consider a C^2 Frechet differentiable map $f \colon \mathcal{B} \to \mathcal{B}$ with the property that there exists an compact, f-invariant set $\mathcal{A} \subset \mathcal{B}$. In addition, the results in [BY17] assume the existence of an ergodic, f-invariant measure μ such that supp $\mu = \mathcal{A}$.

Under the additional assumption that $(\mathcal{A}, f|_{\mathcal{A}})$ satisfies Anosov closing property, the results of the present paper can be used to study the derivative cocycle associated to f which is given by $A(x) = Df(x)$.

3. Lyapunov norm

In order to estimate the growth of the cocycle A along an orbit we introduce the notion of *Lyapunov norm* for quasi-compact operator cocycles and describe some of its properties. This is based on a similar notion introduced in [Bac] in the finite dimensional setting which in turn was based on a similar notion for *invertible cocycles* that goes back to the work of Pesin (see for instance [BP07]).

3.1. Lyapunov norm. Let us use the same notation as in the statement of Theorem 2.3. Given $x \in \mathcal{R}^\mu$, $s \in \mathbf{N} \cap [1, l(\mu)]$, $i \in \{1, \dots, s\}$ and $n \in \mathbf{N}$, we consider the map

$$A^n(f^{-n}(x))_{|E_i(f^{-n}(x))} \colon E_i(f^{-n}(x)) \to E_i(x)$$

which is invertible and let us denote its inverse by $(A^n(f^{-n}(x)))_i^{-1}$. Now, for every $n \in \mathbf{Z}$ we can define the linear map $A_i^n(x) \colon E_i(x) \to E_i(f^n(x))$ by

$$A_i^n(x)u = \begin{cases} A^n(x)_{|E_i(x)} u & \text{if} \quad n \geq 0, \\ (A^{-n}(f^n(x)))_i^{-1} u & \text{if} \quad n < 0. \end{cases}$$

It is easy to verify (see [Bac, p. 4]) that

$$A_i^{m+n}(x) = A_i^n(f^m(x)) A_i^m(x), \quad \text{for every } m, n \in \mathbf{Z}. \tag{9}$$

We are now ready to define the *Lyapunov norm of level s* associated to the operator cocycle A at a regular point $x \in \mathcal{R}^\mu$: we may write each $u \in \mathcal{B}$ uniquely as

$$u = u_1 + \ldots + u_s + u_{s+1},$$

where $u_i \in E_i(x)$ for $i \in \{1, \ldots, s\}$ and $u_{s+1} \in V_{s+1}(x)$. Thus, given $\delta > 0$ we define its *δ-Lyapunov norm of level s* by

$$\| u \|_{x,\delta} = \sum_{i=1}^{s+1} \| u_i \|_{x,\delta,i},$$

where

$$\| u_i \|_{x,\delta,i} = \sum_{n \in \mathbf{Z}} \| A_i^n(x) u_i \| e^{-\lambda_i n - \delta |n|}, \quad i \in \{1, \ldots, s\} \tag{10}$$

and

$$\| u_{s+1} \|_{x,\delta,s+1} = \sum_{n=0}^{+\infty} \| A^n(x) \tilde{u} \| e^{-\tilde{\lambda} n}. \tag{11}$$

Here $\tilde{\lambda}$ is any fixed number smaller than $\lambda_s(\mu)$ with the property that $[\tilde{\lambda}, \lambda_s(\mu)) \cap \Lambda(\mu) = \emptyset$, where $\Lambda(\mu)$ denotes the set of all exceptional Lyapunov exponents of A with respect to μ. Observe that such number $\tilde{\lambda}$ does exist since by Theorem 2.3 elements of $\Lambda(\mu)$ can only accumulate at $\kappa(\mu)$. Moreover, both series (10) and (11) converge. Indeed, this follows readily from the following lemma.

Lemma 3.1. *For every $u \in E_i(x) \setminus \{0\}$,*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \| A_i^n(x) u \| = \lambda_i.$$

Moreover, there exists $\epsilon > 0$ such that for every $\tilde{u} \in V_{s+1}(x)$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log \| A^n(x) \tilde{u} \| < \tilde{\lambda} - \epsilon.$$

Proof. The first assertion is a consequence of [FLQ13, Lemma 20], while the second claim follows easily from the choice of $\tilde{\lambda}$ and the properties of the Oseledets splitting given by Theorem 2.3. □

One can easily verify that $\| \cdot \|_{x,\delta}$ is indeed a norm on $\mathcal{B}$. When there is no risk of ambiguity, we will write $\| \cdot \|_x$ and $\| \cdot \|_{x,i}$ instead of $\| \cdot \|_{x,\delta}$ and $\| \cdot \|_{x,\delta,i}$ respectively, and call it simply *Lyapunov norm*.

Given a bounded linear operator $T \in B(\mathcal{B}, \mathcal{B})$, its Lyapunov norm with respect to $x, y \in \mathcal{R}^\mu$ is defined by

$$\| T \|_{y \leftarrow x} = \sup\{ \| Tu \|_y / \| u \|_x;\ u \in \mathcal{B} \setminus \{0\}\}.$$

3.2. Auxiliary result. In the next section we are going to describe some properties of the Lyapunov norm. In order to do so, we need the following auxiliary result which is a version of Theorem 2 from [DrF] for cocycles acting on Banach spaces.

Proposition 3.2. *Given $x \in \mathcal{R}^\mu$, let us consider the splitting*

$$\mathcal{B} = E_1(x) \oplus \ldots \oplus E_s(x) \oplus V_{s+1}(x).$$

There exists a full μ-measure set $\Omega \subset \mathcal{R}^\mu$ so that for each $\varepsilon > 0$ small enough there are function $C, K \colon M \to (0, +\infty)$ satisfying for every $x \in \Omega$, the following properties:

i) *for each* $1 \le i \le s$, $u \in E_i(x)$ *and* $n \in \mathbf{Z}$,

$$\frac{1}{C(x)} e^{\lambda_i n - \varepsilon |n|} \| u \| \le \| A_i^n(x) u \| \le C(x) e^{\lambda_i n + \varepsilon |n|} \| u \|;$$

ii) *for each* $\tilde{u} \in V_{s+1}(x)$ *and* $n \in \mathbf{N}$,

$$\| A^n(x) u \| \le C(x) e^{(\tilde{\lambda} - \varepsilon) n} \| u \|;$$

iii) $C(f^n(x)) \le C(x) e^{\varepsilon |n|}$ *for every* $n \in \mathbf{Z}$.

iv) $K(f^n(x)) \le K(x) e^{\varepsilon |n|}$ *for every* $n \in \mathbf{Z}$ *and*

$$\|u\| \le K(x) \|u + v\| \quad \text{and} \quad \|v\| \le K(x) \|u + v\|,$$

for $u \in E_1(x) \oplus \ldots \oplus E_s(x)$ *and* $v \in V_{s+1}(x)$.

We will use the following well-known result (see [BY17]).

Theorem 3.3. (John's Theorem) *Let* $E \subset \mathcal{B}$ *be a subspace of dimension* $k \in \mathbf{N}$. *Then, there exists a scalar product* $\langle \cdot, \cdot \rangle_E$ *on* E *that induces norm* $\|\cdot\|_E$ *such that*

$$\|v\|_E \le \|v\| \le \sqrt{k} \|v\|_E, \quad \text{for each } v \in E. \tag{12}$$

Proof of Proposition 3.2. We follow closely the arguments in [BY17, DrF]. Take any $i \in \{1, \ldots, s\}$.

Lemma 3.4. *We have*

$$\limsup_{n \to \infty} \frac{1}{n} \log \|A_i^n(x)\| \le \lambda_i \quad \text{for } \mu\text{-a.e. } x \in X. \tag{13}$$

Proof of the lemma. Let $\langle \cdot, \cdot \rangle_{E_i(x)}$ be a scalar product on $E_i(x)$ given by Theorem 3.3 and let $\|\cdot\|_{E_i(x)}$ denote the associated norm. Let $\{e_1, \ldots, e_t\}$ be an orthonormal basis for $E_i(x)$, $t = \dim E_i(x)$. For each $n \in \mathbf{N}$, choose $v_n \in E_i(x)$ such that $\|v_n\| = 1$ and $\|A_i^n(x)\| = \|A^n(x) v_n\|$. Furthermore, for $n \in \mathbf{N}$, write v_n in the form

$$v_n = \sum_{j=1}^{t} a_{j,n} e_j, \quad \text{for some } a_{j,n} \in \mathbf{R}.$$

We note that it follows from (12) that

$$|a_{j,n}| = |\langle v_n, e_j \rangle_{E_i(x)}| \le \|v_n\|_{E_i(x)} \cdot \|e_j\|_{E_i(x)} \le 1$$

and thus

$$\|A_i^n(x)\| \le \sum_{j=1}^{t} |a_{j,n}| \cdot \|A^n(x) e_j\| \le \sum_{j=1}^{t} \|A^n(x) e_j\|. \tag{14}$$

Since $e_j \in E_i(x)$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|A^n(x) e_j\| = \lambda_i, \quad \text{for } j \in \{1, \ldots, t\}. \tag{15}$$

It remains to observe that (14) and (15) readily imply (13). □

It follows from (13) that for $\epsilon > 0$,

$$D(x) := \sup_{n \ge 0} \{ \|A_i^n(x)\| \cdot e^{-(\lambda_i + \epsilon) n} \} < \infty, \tag{16}$$

for μ a.e. $x \in X$.

Lemma 3.5. *We have*

$$\lim_{n\to\pm\infty}\frac{1}{n}\log D(f^n(x))=0 \quad \text{for } \mu\text{-a.e. } x\in X. \tag{17}$$

Proof of the lemma. For $n\ge 1$, we have

$$\|A_i^n(x)\|\le \|A_i^{n-1}(f(x))\|\cdot\|A_i(x)\|\le \|A_i^{n-1}(f(x))\|\cdot\|A(x)\|.$$

By multiplying the above inequality by $e^{-(\lambda_i+\epsilon)n}$, we obtain

$$e^{-(\lambda_i+\epsilon)n}\|A_i^n(x)\|\le e^{-(\lambda_i+\epsilon)(n-1)}\|A_i^{n-1}(f(x))\|\cdot e^{-(\lambda_i+\epsilon)}\|A(x)\|.$$

Hence,

$$D(x)\le D(f(x))\cdot\max\{e^{-(\lambda_i+\epsilon)}\|A(x)\|,1\}.$$

It follows from the continuity of A and compactness of M that there exists $T>0$ such that

$$\log D(x)-\log D(f(x))\le T. \tag{18}$$

Set

$$\tilde{D}(x)=\log D(x)-\log D(f(x)).$$

We note that

$$\frac{1}{n}\log D(f^n(x))=\frac{1}{n}\log D(x)-\frac{1}{n}\sum_{j=0}^{n-1}\tilde{D}(f^j(x)), \tag{19}$$

for each $x\in X$ and $n\in\mathbf{N}$. Hence, we can apply the Birkhoff ergodic theorem and conclude that there exists $a\in[-\infty,\infty)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\tilde{D}(f^j(x))=a, \tag{20}$$

for μ-a.e. $x\in X$. It follows from (19) and (20) that

$$\lim_{n\to\infty}\frac{1}{n}\log D(f^n(x))=-a.$$

On the other hand, since μ is f-invariant, for any $c>0$ we have that

$$\lim_{n\to\infty}\mu(\{x\in X\colon\ \log D(f^n(x))/n\ge c\})=\lim_{n\to\infty}\mu(\{x\in X:\log D(x)\ge nc\})=0,$$

which immediately implies that $a\ge 0$. Thus,

$$\lim_{n\to\infty}\frac{1}{n}\log D(f^n(x))\le 0.$$

It follows from (16) that $D(x)\ge 1$ for μ a.e. $x\in X$ and therefore we can conclude that (17) holds when $n\to\infty$. One can similarly establish (17) for the case $n\to-\infty$. □

It follows from (17) and [Arn98, Proposition 4.3.3(ii)] that there exists a nonnegative and measurable function C defined on a set of full-measure satisfying inequality in part iii) in the statement of the Lemma and such that $D(x)\le C(x)$, which together with (16) implies that the second inequality in the part i) of the Lemma holds. The proof of ii) is analogous.

Lemma 3.6. *We have*

$$\int_X \log^+\|A_i(x)^{-1}\|\,d\mu(x)<\infty.$$

Proof of the lemma. One can repeat arguments from [DrF, Lemma 4] using $\|\cdot\|_{E_i(x)}$ from Lemma 3.4 instead of the original norm to establish that

$$\int_X \log^+ \|A_i(x)^{-1}\|' \, d\mu(x) < \infty,$$

where

$$\|A_i(x)^{-1}\|' = \sup_{\|v\|_{E_i(x)} \le 1} \|A_i(x)^{-1} v\|_{E_i(x)}.$$

In view of (12), the conclusion of the lemma follows. □

We now prove that the first inequality in i) holds. Let us consider the cocycle $x \mapsto B(x) := A_i(f^{-1}(x))^{-1}$ over f^{-1} that acts on a subbundle $E_i(x)$. Then, $-\lambda_i$ is the only Lyapunov exponent of B. Because of Lemma 3.6, we can apply the first part of the proof to B to conclude that that there exists a function $C\colon M \to (0, \infty)$ such that

$$\|B^n(x)v\| \le C(x) e^{(-\lambda_i + \frac{\epsilon}{2})n}, \quad \text{for } \mu\text{-a.e. } x \in M,\ n \ge 0 \text{ and } v \in E_i(x) \tag{21}$$

and

$$C(f^m(x)) \le C(x) e^{\frac{\epsilon}{2}|m|}, \quad \text{for } \mu\text{-a.e. } x \in M \text{ and } m \in \mathbf{Z}, \tag{22}$$

which readily implies first estimate in i). Finally, the existence of a function K that satisfies assertion iv) follows from [DrF, Lemma 1.]. The proof of Proposition 3.2 is completed. □

3.3. Properties of the Lyapunov norm. Some useful properties of the Lyapunov norm are given in the next proposition.

Proposition 3.7. *Let $x \in \mathcal{R}^\mu$.*

i) *For every $1 \le i \le s$, $u \in E_i(x)$ and $n \in \mathbf{N}$, we have that*

$$e^{(\lambda_i - \delta)n} \| u \|_{x,i} \le \| A^n(x) u \|_{f^n(x),i} \le e^{(\lambda_i + \delta)n} \| u \|_{x,i}; \tag{23}$$

ii) *For every $u \in V_{s+1}(x)$ and $n \in \mathbf{N}$, we have that*

$$\| A^n(x) u \|_{f^n(x),s+1} \le e^{\tilde{\lambda} n} \| u \|_{x,s+1};$$

iii) *For every $\delta > 0$ and $n \in \mathbf{N}$, we have that*

$$\| A^n(x) \|_{f^n(x) \leftarrow x} \le e^{(\lambda_1 + \delta)n}; \tag{24}$$

iv) *For every $\delta > 0$ sufficiently small, there exists a measurable function K_δ : $\mathcal{R}^\mu \to (0, +\infty)$ such that*

$$\| u \| \le \| u \|_x \le K_\delta(x) \| u \| \quad \text{for } x \in \mathcal{R}^\mu \text{ and } u \in \mathcal{B}. \tag{25}$$

Furthermore,

$$K_\delta(x) e^{-\delta n} \le K_\delta(f^n(x)) \le K_\delta(x) e^{\delta n} \quad \text{for } x \in \mathcal{R}^\mu \text{ and } n \in \mathbf{N}. \tag{26}$$

Consequently, for any $B \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ and any two regular points x and y, we have that

$$K_\delta(x)^{-1} \| B \| \le \| B \|_{y \leftarrow x} \le K_\delta(y) \| B \|. \tag{27}$$

Proof. In order to prove i) we observe that for any $u \in E_i(x)$,

$$\begin{aligned}\| A(x) u \|_{f(x),i} &= \sum_{n \in \mathbf{Z}} \| A_i^n(f(x)) A(x) u \| e^{-\lambda_i n - \delta|n|} = \sum_{n \in \mathbf{Z}} \left\| A_i^{n+1}(x) u \right\| e^{-\lambda_i n - \delta|n|} \\ &= \sum_{n \in \mathbf{Z}} \left\| A_i^{n+1}(x) u \right\| e^{-\lambda_i (n+1) - \delta|n+1|} e^{\lambda_i + \delta(|n+1| - |n|)}.\end{aligned}$$

Consequently,

$$e^{(\lambda_i-\delta)}\| u \|_{x,i} \le \| A(x)u \|_{f(x),i} \le e^{(\lambda_i+\delta)}\| u \|_{x,i},$$

which readily implies i). The proof of item ii) is analogous. Indeed, we have that

$$\begin{aligned}\| A^n(x)u \|_{f^n(x),s+1} &= \sum_{k=0}^{+\infty} \| A^k(f^n(x))A^n(x)u \| e^{-\tilde{\lambda}k} \\ &= \sum_{k=0}^{+\infty} \| A^{k+n}(x)u \| e^{-\tilde{\lambda}(k+n)} e^{\tilde{\lambda}n} \le e^{\tilde{\lambda}n}\| u \|_{x,s+1},\end{aligned}$$

for each $u \in V_{s+1}(x)$.

In order to obtain iii), take an arbitrary $u \in \mathcal{B}$ and write it in the form

$$u = u_1 + \ldots + u_s + u_{s+1}, \tag{28}$$

where $u_i \in E_i(x)$ for $i \in \{1, \ldots, s\}$ and $u_{s+1} \in V_{s+1}(x)$. Then, it follows from i) and ii) that

$$\begin{aligned}\| A^n(x)u \|_{f^n(x)} &= \sum_{i=1}^{s+1} \| A^n(x)u_i \|_{f^n(x),i} \le \sum_{i=1}^{s} e^{(\lambda_i+\delta)n}\| u_i \|_{x,i} + e^{\tilde{\lambda}n}\| u_{s+1} \|_{x,s+1} \\ &\le e^{(\lambda_1+\delta)n} \sum_{i=1}^{s+1} \| u_i \|_{x,i} = e^{(\lambda_1+\delta)n}\| u \|_x,\end{aligned}$$

which implies the desired conclusion.

The first inequality of iv) is trivial. In order to prove the second one, take $\varepsilon \in (0, \frac{\delta}{2})$ small enough and let $C\colon \mathcal{R}^\mu \to (0, \infty)$ be the map given by Proposition 3.2 (diminishing $\mathcal{R}^\mu$, if necessary, we may assume $\Omega = \mathcal{R}^\mu$). Thus, for every $1 \le i \le s$, $u \in E_i(x)$ and $n \in \mathbf{Z}$, we have

$$\frac{1}{C(x)} e^{\lambda_i n - \varepsilon|n|}\| u \| \le \| A_i^n(x)u \| \le C(x)e^{\lambda_i n + \varepsilon|n|}\| u \|.$$

Therefore,

$$\begin{aligned}\| u \|_{x,i} &= \sum_{n\in\mathbf{Z}} \| A_i^n(x)u \| e^{-\lambda_i n - \delta|n|} \le \sum_{n\in\mathbf{Z}} \left(C(x)e^{\lambda_i n+\varepsilon|n|}\| u \|\right) e^{-\lambda_i n - \delta|n|} \\ &= C(x) \sum_{n\in\mathbf{Z}} e^{(\varepsilon-\delta)|n|}\| u \|.\end{aligned} \tag{29}$$

On the other hand, for $u \in V_{s+1}(x)$, Proposition 3.2 implies that

$$\| A^n(x)u \| \le C(x)e^{(\tilde{\lambda}-\varepsilon)n}\| u \|,$$

for each $n \in \mathbf{N}$. Thus,

$$\| u \|_{x,s+1} = \sum_{n\ge 0} \| A^n(x)u \| e^{-\tilde{\lambda}n} \le C(x) \sum_{n\ge 0} e^{-\varepsilon n}\| u \|. \tag{30}$$

Set

$$K = \max\left\{ \sum_{n\in\mathbf{Z}} e^{(\varepsilon-\delta)|n|}, \sum_{n\ge 0} e^{-\varepsilon n} \right\}.$$

Take now an arbitrary $u \in \mathcal{B}$ and write it in the form (28), where $u_i \in E_i(x)$ for $i \in \{1, \dots, s\}$ and $u_{s+1} \in V_{s+1}(x)$. Then, it follows from (29) and (30) that

$$\| u \|_x = \sum_{i=1}^{s+1} \| u_i \|_{x,i} \leq KC(x) \sum_{i=1}^{s+1} \| u_i \|.$$

It remains to obtain an upper bound for $\|u_i\|$ in terms of $\|u\|$. This can be achieved by using the map K given by Proposition 3.2. More precisely, let K^1 be the map given by Proposition 3.2 applied for $s = 1$ and sufficiently small $\epsilon > 0$. We then have that

$$\|u_1\| \leq K^1(x)\|u\| \quad \text{and} \quad \|u_2 + \ldots + u_{s+1}\| \leq K^1(x)\|u\| \tag{31}$$

The first inequality in (31) gives a desired bound for $\|u_1\|$. In order to obtain the bound for $\|u_2\|$, we can apply again Proposition 3.2 but now for $s = 2$ (and again for $\epsilon > 0$ sufficiently small) to conclude that there exists K^2 such that

$$\|u_2\| \leq K^2(x)\|u_2 + \ldots + u_{s+1}\| \quad \text{and} \quad \|u_3 + \ldots + u_{s+1}\| \leq K^2(x)\|u_2 + \ldots + u_{s+1}\|. \tag{32}$$

By combining the second inequality in (31) with the first inequality in (32), we conclude that $\|u_2\| \leq K^1(x)K^2(x)\|u\|$. By proceeding, one can establish desired bounds for all $\|u_j\|$, $j = 1, \dots, s+1$ and construct function K_δ. □

For any $N > 0$, let $\mathcal{R}^\mu_{\delta,N}$ be the set of regular points $x \in \mathcal{R}^\mu$ for which $K_\delta(x) \leq N$. Observe that $\mu(\mathcal{R}^\mu_{\delta,N}) \to 1$ as $N \to +\infty$. Moreover, invoking Lusin's theorem together with the μ-continuity of decomposition (7) for $i = s$ (see Remark 2.4), we may assume without loss of generality that this set is compact and that the Lyapunov norm and the Oseledets splitting are continuous when restricted to it.

4. Proof of Theorem 2.5

Let $f\colon M \to M$, $A\colon M \to B(\mathcal{B}, \mathcal{B})$, μ and $s \in \mathbf{N} \cap [1, l(\mu)]$ be given as in the statement of Theorem 2.5. We may assume without loss of generality that μ is not supported on a periodic orbit since otherwise there is nothing to prove. Recall that $d_i(\mu) = \dim(E_i(x))$ and consider $d = 10\prod_{i=1}^{s}(d_i(\mu) + 4)$. Take $\delta_0 > 0$ so that $\delta_0 < \frac{1}{d}\min_{i=1,\dots,s}\{\theta\alpha, (\lambda_i - \lambda_{i+1})\}$ if $l(\mu) \geq 2$ and $\delta_0 < \frac{1}{4}\theta\alpha$ otherwise. Fix $N > 0$ and $\delta \in (0, \delta_0)$.

Let

$$B(\mu) = \left\{ x \in M;\ \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \xrightarrow{n\to\infty} \mu \ \text{ in the weak}^* \text{ topology} \right\}$$

be the *basin* of μ. Since μ is ergodic, $B(\mu)$ has full measure. Choose $x \in B(\mu) \cap \mathcal{R}^\mu_{\delta,N}$ such that $\mu(B(x, \frac{1}{k}) \cap \mathcal{R}^\mu_{\delta,N}) > 0$ for every $k \in \mathbf{N}$, where $B(x, \frac{1}{k})$ denotes the open ball of radius $\frac{1}{k}$ centered at x. By Poincaré's Recurrence Theorem, there exists a sequence $(n_k)_{k\in\mathbf{N}}$ of positive integers so that $n_k \to +\infty$ and $f^{n_k}(x) \in B(x, \frac{1}{k}) \cap \mathcal{R}^\mu_{\delta,N}$ for each $k \in \mathbf{N}$. By the Anosov closing property it follows that, for each k sufficiently large, there exists a periodic point p_k of period n_k such that

$$d(f^j(x), f^j(p_k)) \leq C_1 e^{-\theta \min\{j, n_k - j\}} d(f^{n_k}(x), x) \leq \frac{C_1}{k} e^{-\theta \min\{j, n_k - j\}}, \tag{33}$$

for every $j \in \{0, 1, \ldots, n_k\}$. For each $k \in \mathbf{N}$, let us consider the ergodic *periodic measure* given by

$$\mu_{p_k} = \frac{1}{n_k} \sum_{j=0}^{n_k-1} \delta_{f^j(p_k)}.$$

From the choice of $x \in B(\mu)$ and (33) it follows that the sequence $\{\mu_{p_k}\}_{k\in\mathbf{N}}$ converges to μ in the weak*-topology.

In order to simplify the proof, we will split it into several lemmas.

Lemma 4.1. *The map*

$$\mu \to \gamma_1(\mu) + \gamma_2(\mu) + \ldots + \gamma_i(\mu)$$

is upper-semicontinuous for every $i \in \{1, \ldots, s\}$.

Proof. Let us fix $i \in \{1, \ldots, s\}$. It follows from [DFGTV, Lemma A.3] that there exists a subadditive sequence $(F_n)_{n\geq 1}$ of functions $F_n \colon M \to \mathbf{R}$ such that

$$\gamma_1(\mu) + \gamma_2(\mu) + \ldots + \gamma_i(\mu) = \inf_{n\in\mathbf{N}} \frac{1}{n} \int_M F_n(q)\, d\mu(q).$$

The desired conclusion can now be obtained by using standard arguments as in [Via14, Lemma 9.1]. □

The following is a simple consequence of Lemma 4.1.

Corollary 4.2. *We have that*

$$\limsup_{k\to+\infty} (\gamma_1(p_k) + \gamma_2(p_k) + \ldots + \gamma_i(p_k)) \leq \gamma_1(\mu) + \gamma_2(\mu) + \ldots + \gamma_i(\mu),$$

for every $i \in \{1, \ldots, d_1(\mu) + \ldots + d_s(\mu)\}$.

4.1. Approximation of the largest Lyapunov exponent. For each $1 \leq j \leq n_k$, let us consider the splitting

$$\mathcal{B} = E_1(f^j(x)) \oplus V_2(f^j(x))$$

and write $u \in \mathcal{B}$ as $u = u_E^j + u_V^j$, where $u_E^j \in E_1(f^j(x))$ and $u_V^j \in V_2(f^j(x))$. Then the *cone* of radius $1 - \gamma > 0$ around $E_1(f^j(x))$ is defined as

$$C_\gamma^{j,1} = \left\{ u_E^j + u_V^j \in E_1(f^j(x)) \oplus V_2(f^j(x));\ \| u_V^j \|_{f^j(x)} \leq (1-\gamma) \| u_E^j \|_{f^j(x)} \right\}.$$

To simplify notation we write $\| \cdot \|_j$ for the Lyapunov norm at the point $f^j(x)$.

Lemma 4.3. *For every* $1 \leq j \leq n_k$ *and* $u \in C_0^{j,1}$,

$$\| (A(f^j(p_k))u)_E^{j+1} \|_{j+1} \geq e^{\lambda_1 - 2\delta} \| u_E^j \|_j. \tag{34}$$

Moreover, for k *sufficiently large there exists* $\gamma \in (0,1)$ *such that*

$$A(f^j(p_k))(C_0^{j,1}) \subset C_\gamma^{j+1,1}. \tag{35}$$

Proof. Given $u \in C_0^{j,1}$ let us consider $v = A(f^j(x))u$. Then, it follows from (23) that $\| v \|_{j+1} \leq e^{\lambda_1+\delta} \| u \|_j$ and moreover that

$$\| v_E^{j+1} \|_{j+1} = \| A(f^j(x)) u_E^j \|_{j+1} \geq e^{\lambda_1 - \delta} \| u_E^j \|_j$$

and

$$\| v_V^{j+1} \|_{j+1} = \| A(f^j(x)) u_V^j \|_{j+1} \leq e^{\lambda_2 + \delta} \| u_V^j \|_j. \tag{36}$$

Let $w = A(f^j(p_k))u$. We now wish to compare the Lyapunov norms of w and its projection onto $E_1(f^{j+1}(x))$ and $V_2(f^{j+1}(x))$ with the respective norms of v. Set $B_j = A(f^j(p_k)) - A(f^j(x))$. Consequently, $w = v + B_j u$ and thus

$$w_E^{j+1} = v_E^{j+1} + (B_j u)_E^{j+1} \quad \text{and} \quad w_V^{j+1} = v_V^{j+1} + (B_j u)_V^{j+1}.$$

Moreover, we have

$$\| B_j \| = \big\| A(f^j(p_k)) - A(f^j(x)) \big\| \le C_2 d(f^j(p_k), f^j(x))^\alpha \le C_1^\alpha C_2 \frac{1}{k^\alpha} e^{-\theta\alpha \min\{j, n_k - j\}},$$

for every $0 \le j \le n_k$. Therefore, invoking (25) and (27) it follows that

$$\| B_j u \|_{j+1} \le \| B_j \|_{f^{j+1}(x) \leftarrow f^{j+1}(x)} \| u \|_{j+1} \le K_\delta(f^{j+1}(x))^2 \| B_j \| \| u \|.$$

Since x and $f^{n_k}(x)$ belong to $\mathcal{R}^\mu_{\delta,N}$, it follows from (26) that

$$K_\delta(f^{j+1}(x)) \le N e^{\delta \min\{j+1, n_k - j - 1\}}.$$

The above inequality together with $\| u \|_j \le 2 \big\| u_E^j \big\|_j$ (recall that $u \in C_0^{j,1}$) implies that

$$\begin{aligned} \| B_j u \|_{j+1} &\le N^2 e^{2\delta \min\{j+1, n_k - j - 1\}} C_1^\alpha C_2 \frac{1}{k^\alpha} e^{-\theta\alpha \min\{j, n_k - j\}} \| u \|_j \\ &\le C_1^\alpha C_2 N^2 \frac{1}{k^\alpha} e^{2\delta \min\{j+1, n_k - j - 1\}} e^{-\theta\alpha \min\{j, n_k - j\}} 2 \big\| u_E^j \big\|_j \\ &\le C \frac{1}{k^\alpha} e^{(2\delta - \theta\alpha) \min\{j, n_k - j\}} \big\| u_E^j \big\|_j, \end{aligned}$$

where $C := 2C_1^\alpha C_2 N^2 > 0$. Thus, since $2\delta - \theta\alpha < 0$, we obtain that

$$\| B_j u \|_{j+1} \le \tilde{C} \frac{1}{k^\alpha} \big\| u_E^j \big\|_j,$$

for some $\tilde{C} > 0$ independent of n_k and j. Consequently,

$$\begin{aligned} \big\| w_E^{j+1} \big\|_{j+1} &\ge \big\| v_E^{j+1} \big\|_{j+1} - \big\| (B_j u)_E^{j+1} \big\|_{j+1} \\ &\ge e^{\lambda_1 - \delta} \big\| u_E^j \big\|_j - \tilde{C} \frac{1}{k^\alpha} \big\| u_E^j \big\|_j \ge e^{\lambda_1 - 2\delta} \big\| u_E^j \big\|_j, \end{aligned}$$

whenever k is sufficiently large which is precisely the inequality (34).

In order to obtain (35), we observe initially that

$$\big\| w_E^{j+1} \big\|_{j+1} \le e^{\lambda_1 + \delta} \big\| u_E^j \big\|_j + \tilde{C} \frac{1}{k^\alpha} \big\| u_E^j \big\|_j \le \hat{C} \big\| u_E^j \big\|_j. \tag{37}$$

On the other hand,

$$\big\| w_E^{j+1} \big\|_{j+1} \ge \big\| v_E^{j+1} \big\|_{j+1} - \| B_j u \|_{j+1}$$

and

$$\big\| w_V^{j+1} \big\|_{j+1} \le \big\| v_V^{j+1} \big\|_{j+1} + \| B_j u \|_{j+1}.$$

Therefore, combining these inequalities and using again that $u \in C_0^{j,1}$, we have that

$$\begin{aligned} \big\| w_E^{j+1} \big\|_{j+1} - \big\| w_V^{j+1} \big\|_{j+1} &\ge \big\| v_E^{j+1} \big\|_{j+1} - \big\| v_V^{j+1} \big\|_{j+1} - 2 \| B_j u \|_{j+1} \\ &\ge e^{\lambda_1 - \delta} \big\| u_E^j \big\|_j - e^{\lambda_2 + \delta} \big\| u_V^j \big\|_j - 2\tilde{C} \frac{1}{k^\alpha} \big\| u_E^j \big\|_j \\ &\ge \left(e^{\lambda_1 - \delta} - e^{\lambda_2 + \delta} - 2\tilde{C} \frac{1}{k^\alpha} \right) \big\| u_E^j \big\|_j. \end{aligned}$$

Taking k large enough so that

$$e^{\lambda_1-\delta}-e^{\lambda_2+\delta}-2\tilde{C}\frac{1}{k^\alpha}>0$$

and applying (37) to the previous inequality, we conclude that there exists $\gamma>0$ such that

$$\| w_E^{j+1} \|_{j+1}-\| w_V^{j+1} \|_{j+1}\geq \gamma \| w_E^{j+1} \|_{j+1}.$$

Hence, $w=A(f^j(p))u\in C_\gamma^{j+1}$ which yields (35). The proof of the lemma is completed. □

As a simple consequence of Lemma 4.3, we obtain the following result.

Corollary 4.4. *For every $k\in \mathbf{N}$ large enough,*

$$\lambda(p_k,u)\geq \lambda_1-3\delta$$

for every $u\in C_0^{0,1}\setminus\{0\}$.

Proof. Recall we are assuming that the Oseledets splitting and the Lyapunov norm are continuous on $\mathcal{R}^\mu_{\delta,N}$. In particular, if k is sufficiently large (and consequently x and $f^{n_k}(x)$ are close) we have that $C_\gamma^{n_k,1}\subset C_0^{0,1}$ and thus by (35),

$$A^{n_k}(p_k)(C_0^{0,1})\subset C_0^{0,1}.$$

Consequently, for any $u\in C_0^{0,1}$ and $m\in\mathbf{N}$ we have $A^{n_km}(p_k)u\in C_0^{0,1}$. Therefore, given $u\in C_0^{0,1}$ and invoking (34) and (35) (together with the fact that the Lyapunov norms at x and $f^{n_k}(x)$ are close whenever $k\gg 0$), we obtain that

$$\begin{aligned}\| A^{n_k}(p_k)u \|_{n_k} &\geq \| (A^{n_k}(p_k)u)_E^{n_k} \|_{n_k}\geq e^{n_k(\lambda_1-2\delta)}\| u_E^0 \|_0\\ &\geq \frac{1}{2}e^{n_k(\lambda_1-2\delta)}\| u \|_0\geq \frac{1}{4}e^{n_k(\lambda_1-2\delta)}\| u \|_{n_k}.\end{aligned}$$

By iterating, we have that

$$\| A^{n_km}(p)u \|_{n_k}\geq \frac{1}{4^m}e^{n_km(\lambda_1-2\delta)}\| u \|_{n_k}\quad \text{for } m\in\mathbf{N}.$$

Consequently,

$$\begin{aligned}\lambda(p_k,u)&\geq \lim_{m\to\infty}\frac{1}{n_km}\log\left(\| A^{n_km}(p)u \|_{n_k}\right)\\ &\geq \lim_{m\to\infty}\frac{1}{n_km}\log\left(\frac{1}{4^m}e^{n_km(\lambda_1-2\delta)}\| u \|_{n_k}\right)\\ &=\lambda_1-2\delta-\frac{\log 4}{n_k}+\frac{1}{n_k}\lim_{m\to\infty}\frac{1}{m}\log\left(\| u \|_{n_k}\right)\geq \lambda_1-3\delta,\end{aligned}$$

for k sufficiently large which proves our claim. □

Let $i_1^k=\max\{i;\, V_i(p_k)\cap C_0^{0,1}\neq\{0\}\}$. Since $V_{i+1}(p_k)\subset V_i(p_k)$ for each $i\in\mathbf{N}$, we note that $V_i(p_k)\cap C_0^{0,1}\neq\{0\}$ for every $i\in\{1,\dots,i_1^k\}$.

Corollary 4.5. *We have that*

$$\lambda_{i_1^k}(p_k)\geq \lambda_1-3\delta.$$

Proof. Take $0\neq u\in V_{i_1^k}(p_k)\cap C_0^{0,1}$. It follows from Corollary 4.4 that $\lambda(p_k,u)\geq \lambda_1-3\delta$. In particular, $\lambda_{i_1^k}(p_k)\geq\lambda_1-3\delta$ as claimed. □

Corollary 4.6. *We have that*

$$\dim(E_1(p_k) \oplus \ldots \oplus E_{i_1^k}(p_k)) = \dim(E_1(x)),$$

for every $k \gg 0$.

Proof. Let $\hat{d}_{i_1^k} = \dim(E_1(p_k) \oplus \ldots \oplus E_{i_1^k}(p_k))$. By Corollary 4.5, we have that $\gamma_i(p_k) \geq \gamma_i(\mu) - 3\delta$ for every $i \in \{1, \ldots, \hat{d}_{i_1^k}\}$. Therefore, it follows from Lemma 4.1 and the choice of δ that $\hat{d}_{i_1^k} \leq d_1(\mu)$. Indeed, suppose $\hat{d}_{i_1^k} > d_1(\mu)$. In particular, $\gamma_{d_1(\mu)+1}(p_k) \geq \lambda_1 - 3\delta$. Thus, on the one hand we have that

$$\sum_{i=1}^{d_1(\mu)+1} \gamma_i(p_k) \geq (d_1(\mu) + 1)(\lambda_1 - 3\delta).$$

On the other hand, by Lemma 4.1 we have that

$$\sum_{i=1}^{d_1(\mu)+1} \gamma_i(p_k) \leq \sum_{i=1}^{d_1(\mu)+1} \gamma_i(\mu) + \delta = d_1(\mu)\lambda_1 + \lambda_2 + \delta,$$

for every $k \gg 0$. Combining these two inequalities we obtain that

$$(3d_1(\mu) + 4)\delta > \lambda_1 - \lambda_2,$$

which yields a contradiction with our choice of δ. Hence, we conclude that $\hat{d}_{i_1^k} \leq d_1(\mu)$.

In order to obtain the reverse inequality, let us suppose that $\hat{d}_{i_1^k} < d_1(\mu)$. Let $\{u_1, \ldots u_{d_1(\mu)}\}$ be a linearly independent subset of $E_1(x)$ and write u_i in the form

$$u_i = u^i_{p_k} + v^i_{p_k} \quad \text{where } u^i_{p_k} \in E_1(p_k) \oplus \ldots \oplus E_{i_1^k}(p_k) \text{ and } v^i_{p_k} \in V_{i_1^k+1}(p_k),$$

for $i = \{1, \ldots, d_1(\mu)\}$. Since $\hat{d}_{i_1^k} < d_1(\mu)$, it follows that $\{u^i_{p_k}\}_{i=1}^{d_1(\mu)}$ is a linearly dependent subset of $E_1(p_k) \oplus \ldots \oplus E_{i_1^k}(p_k)$. Thus, we may assume without loss of generality that

$$u^1_{p_k} = a_2 u^2_{p_k} + \ldots + a_{d_1(\mu)} u^{d_1(\mu)}_{p_k},$$

for some $a_i \in \mathbf{R}$, $i \in \{2, \ldots d_1(\mu)\}$. Consequently, on the one hand we have that

$$0 \neq u_1 - a_2 u_2 - \ldots - a_{d_1(\mu)} u_{d_1(\mu)} \in E_1(x) \subset C_0^{0,1}.$$

On the other hand,

$$0 \neq u_1 - a_2 u_2 - \ldots - a_{d_1(\mu)} u_{d_1(\mu)} = v^1_{p_k} - a_2 v^2_{p_k} - \ldots - a_{d_1(\mu)} v^{d_1(\mu)}_{p_k} \in V_{i_1^k+1}(p_k),$$

contradicting the choice of i_1^k. Thus, $\hat{d}_{i_1^k} = d_1(\mu)$ as claimed. □

Now, as a simple consequence of the previous two corollaries we obtain the following result.

Corollary 4.7.

$$\gamma_i(p_k) \geq \gamma_i(\mu) - 3\delta$$

for every $i = 1, \ldots, d_1(\mu)$ *and* $k \gg 0$.

4.2. Approximation of the second largest Lyapunov exponent. We proceed in a similar manner to that in Subsection 4.1. For each $1 \leq j \leq n_k$, let us consider the splitting $\mathcal{B} = E_1(f^j(x)) \oplus E_2(f^j(x)) \oplus V_3(f^j(x))$. We can write each $u \in \mathcal{B}$ as

$$u = u^j_{E_1} + u^j_{E_2} + u^j_V, \quad \text{where } u^j_{E_i} \in E_i(f^j(x)) \text{ for } i = 1, 2 \text{ and } u^j_V \in V_3(f^j(x)). \tag{38}$$

For $\gamma \in (0,1)$, let us consider the cone $C_\gamma^{j,2}$ defined (in terms of the decomposition in (38)) by

$$C_\gamma^{j,2} = \Big\{ u \in \mathcal{B} \colon \| u_V^j \|_{f^j(x)} \le (1-\gamma) \| u_{E_2}^j \|_{f^j(x)} \Big\}.$$

As before, in order to simplify the notation, we will write $\| \cdot \|_j$ for the Lyapunov norm at the point $f^j(x)$.

Lemma 4.8. *Let $u \in C_0^{j,2} \setminus \{0\}$ for some $j \in \{0, \dots, n_k - 1\}$. Then, either $u \in C_0^{j,1}$ or*

$$\| (A(f^j(p_k))u)_{E_2}^{j+1} \|_{j+1} \ge e^{\lambda_2 - 2\delta} \| u_{E_2}^j \|_j \tag{39}$$

and

$$A(f^j(p_k))u \in C_\gamma^{j+1,2}, \tag{40}$$

for some $\gamma \in (0,1)$ and every k sufficiently large. Moreover, k and γ do not depend on u or j.

Proof. The proof is similar to the proof of Lemma 4.3. Suppose that $u \in C_0^{j,2} \setminus C_0^{j,1}$ since otherwise there is nothing to prove. In particular, $4\| u_{E_2}^j \|_j \ge \| u \|_j$. Indeed, since $u \notin C_0^{j,1}$,

$$\| u_{E_1}^j \|_j < \| u_{E_2}^j + u_V^j \|_j \le \| u_{E_2}^j \|_j + \| u_V^j \|_j \le 2 \| u_{E_2}^j \|_j.$$

Thus,

$$\| u \|_j \le \| u_{E_1}^j \|_j + \| u_{E_2}^j \|_j + \| u_V^j \|_j \le 4 \| u_{E_2}^j \|_j. \tag{41}$$

Let $v = A(f^j(x))u$ and consider $w = A(f^j(p_k))u$. By (23), we have that

$$\| v_{E_2}^{j+1} \|_{j+1} = \| A(f^j(x)) u_{E_2}^j \|_{j+1} \ge e^{\lambda_2 - \delta} \| u_{E_2}^j \|_j$$

and

$$\| v_V^{j+1} \|_{j+1} = \| A(f^j(x)) u_V^j \|_{j+1} \le e^{\lambda_3 + \delta} \| u_V^j \|_j. \tag{42}$$

Moreover, by considering $B_j = A(f^j(p_k)) - A(f^j(x))$ we have (as in the proof of Lemma 4.3) that $w = v + B_j u$ and thus

$$w_{E_1}^{j+1} = v_{E_1}^{j+1} + (B_j u)_{E_1}^{j+1}, \quad w_{E_2}^{j+1} = v_{E_2}^{j+1} + (B_j u)_{E_2}^{j+1} \quad \text{and} \quad w_V^{j+1} = v_V^{j+1} + (B_j u)_V^{j+1}.$$

Therefore, using (41) and proceeding as in Lemma 4.3 we obtain that

$$\| B_j u \|_{j+1} \le \tilde{C} \frac{1}{k^\alpha} \| u_{E_2}^j \|_j,$$

for some $\tilde{C} > 0$ which is independent of n_k and j. Consequently,

$$\begin{aligned} \| w_{E_2}^{j+1} \|_{j+1} &\ge \| v_{E_2}^{j+1} \|_{j+1} - \| (B_j u)_{E_2}^{j+1} \|_{j+1} \\ &\ge e^{\lambda_2 - \delta} \| u_{E_2}^j \|_j - \tilde{C} \frac{1}{k^\alpha} \| u_{E_2}^j \|_j \ge e^{\lambda_2 - 2\delta} \| u_{E_2}^j \|_j, \end{aligned}$$

whenever k is sufficiently large which is precisely inequality (39). In order to obtain (40), we observe initially that

$$\| w_{E_2}^{j+1} \|_{j+1} \le e^{\lambda_2 + \delta} \| u_{E_2}^j \|_j + \tilde{C} \frac{1}{k^\alpha} \| u_{E_2}^j \|_j \le \hat{C} \| u_{E_2}^j \|_j. \tag{43}$$

On the other hand,

$$\| w_{E_2}^{j+1} \|_{j+1} \ge \| v_{E_2}^{j+1} \|_{j+1} - \| B_j u \|_{j+1}$$

and

$$\| w_V^{j+1} \|_{j+1} \leq \| v_V^{j+1} \|_{j+1} + \| B_j u \|_{j+1}.$$

By combining the last two inequalities and using that $u \in C_0^{j,2}$, we have that

$$\begin{aligned} \| w_{E_2}^{j+1} \|_{j+1} - \| w_V^{j+1} \|_{j+1} &\geq \| v_{E_2}^{j+1} \|_{j+1} - \| v_V^{j+1} \|_{j+1} - 2\| B_j u \|_{j+1} \\ &\geq e^{\lambda_2 - \delta} \| u_{E_2}^j \|_j - e^{\lambda_3 + \delta} \| u_V^j \|_j - 2\tilde{C} \frac{1}{k^\alpha} \| u_{E_2}^j \|_j \\ &\geq \left(e^{\lambda_2 - \delta} - e^{\lambda_3 + \delta} - 2\tilde{C} \frac{1}{k^\alpha} \right) \| u_{E_2}^j \|_j. \end{aligned}$$

Taking k large enough so that

$$e^{\lambda_2 - \delta} - e^{\lambda_3 + \delta} - 2\tilde{C} \frac{1}{k^\alpha} > 0$$

and applying (43) to the previous inequality, we conclude that there exists $\gamma > 0$ such that

$$\| w_{E_2}^{j+1} \|_{j+1} - \| w_V^{j+1} \|_{j+1} \geq \gamma \| w_{E_2}^{j+1} \|_{j+1},$$

which implies that $w = A(f^j(p))u \in C_\gamma^{j+1,2}$. Hence, we conclude that (40) holds and the proof of the lemma is completed. □

Corollary 4.9. *For every $k \in \mathbf{N}$ large enough,*

$$\lambda(p_k, u) \geq \lambda_2 - 3\delta$$

for every $u \in C_0^{0,2} \setminus \{0\}$.

Proof. Let $k \in \mathbf{N}$ be large enough so that $C_\gamma^{n_k,2} \subset C_0^{0,2}$ (recall we are assuming the Oseledets splitting and the Lyapunov norm are continuous on $\mathcal{R}_{\delta,N}^\mu$ and that $\lim_{k \to +\infty} d(x, f^{n_k}(x)) = 0$). Thus, it follows from Lemma 4.8 that given $u \in C_0^{0,2} \setminus \{0\}$, either there exist $m \in \mathbf{N}$ and $j \in \{0, 1, \ldots, n_k - 1\}$ so that

$$A^{n_k m + j}(p_k) u \in C_0^{j,1}$$

or

$$A^{n_k m + j}(p_k) u \in C_0^{j,2} \setminus C_0^{j,1},$$

for every $m \in \mathbf{N}$ and every $j \in \{0, 1, \ldots, n_k - 1\}$. In the first case, it follows from Lemma 4.3 and Corollary 4.4 that

$$\lambda(p_k, u) \geq \lambda_1 - 3\delta \geq \lambda_2 - 3\delta,$$

which gives the desired conclusion.

Suppose now that we are in the second case. By recalling (39), (40) and (41) together with the fact that the Lyapunov norms at x and $f^{n_k}(x)$ are close whenever $k \gg 0$, we obtain that

$$\begin{aligned} \| A^{n_k}(p_k) u \|_{n_k} &\geq \| (A^{n_k}(p_k) u)_{E_2}^{n_k} \|_{n_k} \geq e^{n_k(\lambda_2 - 2\delta)} \| u_{E_2}^0 \|_0 \\ &\geq \frac{1}{4} e^{n_k(\lambda_2 - 2\delta)} \| u \|_0 \geq \frac{1}{8} e^{n_k(\lambda_2 - 2\delta)} \| u \|_{n_k}. \end{aligned}$$

By iterating, we conclude that

$$\| A^{n_k m}(p) u \|_{n_k} \geq \frac{1}{8^m} e^{n_k m(\lambda_2 - 2\delta)} \| u \|_{n_k}.$$

Consequently,

$$\begin{aligned}\lambda(p_k,u) &\geq \lim_{m\to\infty}\frac{1}{n_k m}\log\left(\| A^{n_k m}(p)u\|_{n_k}\right)\\ &\geq \lim_{m\to\infty}\frac{1}{n_k m}\log\left(\frac{1}{8^m}e^{n_k m(\lambda_2-2\delta)}\| u\|_{n_k}\right)\\ &= \lambda_2-2\delta-\frac{\log 8}{n_k}+\frac{1}{n_k}\lim_{m\to\infty}\frac{1}{m}\log\left(\| u\|_{n_k}\right)\geq \lambda_2-3\delta,\end{aligned}$$

for any k sufficiently large which proves our claim. □

Let $i_2^k=\max\{i;\, V_i(p_k)\cap C_0^{0,2}\neq\{0\}\}$.

Corollary 4.10. *We have that*

$$\dim(E_1(p_k)\oplus\ldots\oplus E_{i_2^k}(p_k))\geq \dim(E_1(x)\oplus E_2(x)),$$

for every $k\gg 0$.

Proof. The proof is analogous to the second part of the proof of Corollary 4.6. □

Corollary 4.11. *We have that*

$$\gamma_i(p_k)\geq \gamma_i(\mu)-3\delta, \tag{44}$$

for every $i\in\{1,\ldots,d_1(\mu)+d_2(\mu)\}$ and $k\gg 0$.

Proof. We first note that it follows from Corollary 4.7 that (44) holds for every $i\in\{1,\ldots,d_1(\mu)\}$ and $k\gg 0$. Now, on the one hand we have that

$$\gamma_i(\mu)=\lambda_2,\quad \text{for every } i\in\{d_1(\mu)+1,\ldots,d_1(\mu)+d_2(\mu)\}.$$

On the other hand, Corollary 4.9 implies that

$$\lambda(p_k,u)\geq\lambda_2-3\delta\quad\text{for every } u\in V_{i_2^k}\cap C_0^{0,2}\setminus\{0\} \text{ and } k\gg 0,$$

which implies that $\lambda_{i_2^k}(p_k)\geq\lambda_2-3\delta$. By Corollary 4.10, we have that

$$\gamma_i(p_k)\geq\lambda_{i_2^k}(p_k)\quad\text{for every } i\in\{d_1(\mu)+1,\ldots,d_1(\mu)+d_2(\mu)\},$$

which yields the desired conclusion. □

Corollary 4.12. *We have that*

$$\dim(E_1(p_k)\oplus\ldots\oplus E_{i_2^k}(p_k))= \dim(E_1(x)\oplus E_2(x)),$$

for every $k\gg 0$.

Proof. In a view of Corollary 4.10, it is sufficient to prove that

$$\dim(E_1(p_k)\oplus\ldots\oplus E_{i_2^k}(p_k))\leq \dim(E_1(x)\oplus E_2(x)),\quad \text{for } k\gg 0. \tag{45}$$

In order to establish (45), we adapt the arguments from the proof of Corollary 4.6. Suppose that (45) doesn't hold, i.e. that $\dim(E_1(p_k)\oplus\ldots\oplus E_{i_2^k}(p_k))>d_1(\mu)+d_2(\mu)$. In particular,

$$\gamma_{d_1(\mu)+d_2(\mu)+1}(p_k)\geq\lambda_2-3\delta.$$

Thus, on the one hand we have that

$$\sum_{i=1}^{d_1(\mu)+d_2(\mu)+1}\gamma_i(p_k)\geq d_1(\mu)(\lambda_1-3\delta)+(d_2(\mu)+1)(\lambda_2-3\delta).$$

On the other hand, Lemma 4.1 implies that

$$\sum_{i=1}^{d_1(\mu)+d_2(\mu)+1} \gamma_i(p_k) \le \sum_{i=1}^{d_1(\mu)+d_2(\mu)+1} \gamma_i(\mu) + \delta = d_1(\mu)\lambda_1 + d_2(\mu)\lambda_2 + \lambda_3 + \delta,$$

for every $k \gg 0$. By combining the last two inequalities, we obtain that

$$(3d_1(\mu) + 3d_2(\mu) + 4)\delta > \lambda_2 - \lambda_3,$$

which yields a contradiction with our choice of δ. We conclude that (45) holds and the proof is completed. □

4.3. Conclusion of the proof of Theorem 2.5. More generally, for each $1 \le j \le n_k$ and $h \in \{1, \dots, s\}$, let us consider the splitting

$$\mathcal{B} = E_1(f^j(x)) \oplus \ldots \oplus E_h(f^j(x)) \oplus V_{h+1}(f^j(x)).$$

We can write each $u \in \mathcal{B}$ in the form

$$u = u^j_{E_1} + \ldots + u^j_{E_h} + u^j_V,$$

where $u^j_{E_i} \in E_i(f^j(x))$ for $i \in \{1, \dots, h\}$ and $u^j_V \in V_{h+1}(f^j(x))$. In addition, we can consider cones

$$C^{j,h}_\gamma = \left\{ u \in \mathcal{B} \colon \| u^j_V \|_{f^j(x)} \le (1-\gamma) \| u^j_{E_h} \|_{f^j(x)} \right\},$$

where $\gamma \in (0,1)$ and the corresponding numbers $i^k_h = \max\{i;\ V_i(p_k) \cap C^{0,h}_0 \ne \{0\}\}$. By repeating the previous arguments (with straightforward adjustments), we conclude that

$$\gamma_i(p_k) \ge \gamma_i(\mu) - 3\delta,$$

for every $i \in \{1, \dots, d_1(\mu) + \ldots + d_s(\mu)\}$ and $k \gg 0$. This together with Corollary 4.2 implies the conclusion of Theorem 2.5.

5. Applications

In this section we discuss some applications of the main result of our paper. We shall mostly restrict our attention to the case of compact cocycles in order to avoid dealing with technicalities.

5.1. Uniform hyperbolicity via nonvanishing of Lyapunov exponents. We begin by recalling that the cocycle A is said to be *uniformly hyperbolic* if there exist a family of projections $P(x)$, $x \in M$ on $\mathcal{B}$ and constants $D, \lambda > 0$ such that:

(1) for each $x \in M$, we have

$$A(x)P(x) = P(f(x))A(x) \tag{46}$$

and that the map

$$A(x) \mid \operatorname{Ker} P(x) \colon \operatorname{Ker} P(x) \to \operatorname{Ker}(P(x)) \quad \text{is invertible;} \tag{47}$$

(2) for each $x \in M$ and $n \ge 0$,

$$\|A^n(x)P(x)\| \le De^{-\lambda n} \tag{48}$$

and

$$\|A^{-n}(x)(\operatorname{Id} - P(x))\| \le De^{-\lambda n}, \tag{49}$$

where

$$A^{-n}(x) = (A^n(f^{-n}(x)) \mid \operatorname{Ker} P(f^{-n}(x)))^{-1} \colon \operatorname{Ker} P(x) \to \operatorname{Ker} P(f^{-n}(x)).$$

We note that the condition (49) can be replaced by the requirement that

$$\|A^n(x)v\| \ge \frac{1}{D} e^{\lambda n} \|v\| \quad \text{for } n \ge 0 \text{ and } v \in \operatorname{Ker} P(x) \setminus \{0\}.$$

Let $\mathcal{E}(f)$ denote the set of all f-invariant Borel probability measures on M which are ergodic. Furthermore, let $\mathcal{E}_{per}(f)$ denote those measures in $\mathcal{E}(f)$ whose support is an f-periodic orbit.

Theorem 5.1. *Assume that $A\colon M \to B(\mathcal{B}, \mathcal{B})$ is an α-Hölder continuous cocycle that takes values in a family of compact operators on $\mathcal{B}$. Furthermore, suppose that there exist a family of projections $P(x)$, $x \in M$ and $\delta > 0$ such that:*

1. *$x \mapsto P(x)$ is a continuous map from M to $B(\mathcal{B}, \mathcal{B})$;*
2. *(46) and (47) hold for each $x \in M$;*
3. *for any $\mu \in \mathcal{E}_{per}(f)$, we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| \le -\delta \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)w\| \ge \delta, \tag{50}$$

for μ-a.e. $x \in M$ and every $v \in \operatorname{Im} P(x)$, $w \in \operatorname{Ker} P(x)$.

Then, the cocycle A is uniformly hyperbolic.

Proof. We define a sequence of maps $F_n\colon M \to \mathbf{R} \cup \{-\infty\}$, $n \ge 0$ by

$$F_n(x) = \log\|A^n(x)P(x)\|, \quad x \in M.$$

Lemma 5.2. *The sequence $(F_n)_{n\ge 0}$ is subadditive, i.e.*

$$F_{n+m}(x) \le F_n(f^m(x)) + F_m(x) \quad \text{for every } n, m \ge 0 \text{ and } x \in M.$$

Proof of the lemma. By (46), we have that

$$\begin{aligned}\|A^{n+m}(x)P(x)\| &= \|A^n(f^m(x))A^m(x)P(x)^2\| \\ &= \|A^n(f^m(x))P(f^m(x))A^m(x)P(x)\| \\ &\le \|A^n(f^m(x))P(f^m(x)\| \cdot \|A^m(x)P(x)\|,\end{aligned}$$

for each $x \in M$ and $n, m \ge 0$. This readily implies the desired conclusion. □

Since both $x \mapsto A(x)$ and $x \mapsto P(x)$ are continuous, we have that F_n is a continuous map for each $n \ge 0$. In particular, F_1 is integrable with respect to any $\mu \in \mathcal{E}(f)$. Hence, it follows from Lemma 5.2 and Kingman's subadditive ergodic theorem that for each $\mu \in \mathcal{E}(f)$, there exists $\Lambda(\mu) \in [-\infty, \infty)$ such that

$$\Lambda(\mu) = \lim_{n\to\infty} \frac{F_n(x)}{n} \quad \text{for } \mu\text{-a.e. } x \in M.$$

Lemma 5.3. *We have that $\Lambda(\mu)$ is either $-\infty$ or a Lyapunov exponent of the cocycle A with respect to μ.*

Proof of the lemma. Assume that $\Lambda(\mu) \ne -\infty$ since otherwise there is nothing to prove. Let $\lambda_1 > \lambda_2 > \dots$ denote (distinct) Lyapunov exponents of A with respect to μ. Assuming that $\Lambda(\mu)$ is not a Lyapunov exponent of A with respect to μ, we can find i such that $\Lambda(\mu) \in (\lambda_{i+1}, \lambda_i)$. In particular,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| &= \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)P(x)v\| \\ &\le \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)P(x)\| = \Lambda(\mu) < \lambda_i,\end{aligned} \tag{51}$$

for μ-a.e. $x \in M$ and $v \in \operatorname{Im} P(x) \setminus \{0\}$. On the other hand, it follows from Theorem 2.3 that for μ-a.e. $x \in M$ and every $v \in \operatorname{Im} P(x)$, there exists $j \in \mathbf{N}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| = \lambda_j,$$

which together with (51) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| \le \lambda_{i+1}. \tag{52}$$

By (52) and [FLQ13, Proposition 14.], we have that $\Lambda(\mu) \le \lambda_{i+1}$ which yields a contradiction. □

It follows from (50) that all Lyapunov exponents of A with respect to $\mu \in \mathcal{E}_{per}(f)$ belong to $\mathbf{R} \setminus (-\delta, \delta)$. This together with Theorem 2.5 and Lemma 5.3 implies that $\Lambda(\mu) \le -\delta$ for $\mu \in \mathcal{E}(f)$. Using [S98, Theorem 1], we obtain that

$$\lim_{n\to\infty} \frac{\max_{x\in M} F_n(x)}{n} \le -\delta,$$

which readily implies (48). One can similarly establish (49). Hence, A is uniformly hyperbolic. □

One can also establish the version of Theorem 5.1 for quasi-compact cocycles although under additional assumption that $\kappa(\mu) < \Lambda(\mu)$ for each $\mu \in \mathcal{E}(f)$.

Remark 5.4. We emphasize that the first results in the spirit of Theorem 5.1 are due to Cao [C03]. More precisely, in the particular case of the derivative cocycle $A(x) = Df(x)$ associated to some smooth diffeomorphism f on a compact Riemmanian manifold M, Cao proved that the existence of a continuous and Df-invariant splitting

$$T_x M = E^s_x \oplus E^u_x \quad \text{for } x \in M,$$

together with an assumption that for each $\mu \in \mathcal{E}(f)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)v\| < 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log\|A^n(x)w\| < 0, \tag{53}$$

for μ-a.e. $x \in M$ and every $v \in E^s_x$, $w \in E^u_x$, implies that the cocycle A is uniformly hyperbolic. Hence, in the statement of Theorem 5.1 we have required that (50) holds for $\mu \in \mathcal{E}_{per}(f)$, while Cao requires that (53) holds for any $\mu \in \mathcal{E}(f)$, although without any type of uniform estimates for Lyapunov exponents as we have in (50).

The importance of this type of results steems from the fact that nonvanishing of Lyapunov exponents corresponds (in general) to a weaker concept of *nonuniform* hyperbolicity (see [BP07] for detailed discussion). Therefore, it is interesting to see under which additional assumptions, nonvanishing of Lyapunov exponents implies the existence of *uniform* hyperbolic behaviour. For some more recent results in this direction and further references, we refer to [HPS14].

5.2. Sacker–Sell spectrum. Let us assume that M is compact and connected metric space and that $f\colon M \to M$ is a continuous map. Furthermore, let A be a continuous cocycle over (M, f) of compact and injective (although not necessarily invertible) operators on $\mathcal{B}$. For each $\lambda \in \mathbf{R}$, we can define a new cocycle A_λ by

$$A_\lambda(x) = e^{-\lambda} A(x), \quad x \in M.$$

Finally, set

$$\Sigma = \{\lambda \in \mathbf{R} \colon A_\lambda \text{ is not uniformly hyperbolic}\}.$$

The set Σ is called the *Sacker–Sell spectrum* of A. It was proved by Magalhães [LM87] (building on the original work of Sacker and Sell [SS78] for cocycles acting on a finite-dimensional space) that if f has a periodic orbit, we have that:

(1) $\Sigma \subset \mathbf{R}$ is closed;
(2) $\Sigma = \emptyset$ or $\Sigma(\Lambda) = \bigcup_{i=1}^{k}[a_i, b_i]$ for some

$$b_1 \geq a_1 > b_2 \geq a_2 > \ldots > b_k \geq a_k,$$

or $\Sigma(\Lambda) = \bigcup_{i=1}^{\infty}[a_i, b_i]$ for some

$$b_1 \geq a_1 > b_2 \geq a_2 > \ldots > b_i \geq a_i > \ldots \quad \text{such that } \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = -\infty.$$

The following result is due to Schreiber [S98].

Theorem 5.5. *For each i, there exist $\mu_1, \mu_2 \in \mathcal{E}(f)$ such that a_1 is the Lyapunov exponent of A with respect μ_1 and b_1 is the Lyapunov exponent of A with respect μ_2.*

We note that for finite-dimensional and invertible cocycles, Theorem 5.5 was first established by Johnson, Palmer and Sell [JPS87]. Let $L(\mu)$ denote the set of all finite Lyapunov exponents of A with respect to μ.

Corollary 5.6. *Assume further that A is an α-Hölder cocycle such that $\Sigma \neq \emptyset$ and that f satisfies Anosov closing property. Then,*

$$\partial\Sigma \subset \overline{\bigcup_{\mu\in\mathcal{E}_{per}(f)} L(\mu)} \quad \text{and} \quad \overline{\bigcup_{\mu\in\mathcal{E}(f)} L(\mu)} \subset \Sigma.$$

Proof. The first inclusion is a direct consequence of Theorems 2.5 and 5.5. The second inclusion is proved in [LM87]. □

We are hopeful that Corollary 5.6 could be useful in numerical estimations of Σ since it recognizes boundary points of Σ as accumulation points of Lyapunov exponents along periodic orbits (which are easy to estimate).

5.3. Spectral radius and growth of the cocycle. In this subsection, $\rho(C)$ will denote the spectral radius of an operator $C \in B(\mathcal{B}, \mathcal{B})$. Furthermore, let us again consider compact, injective and continuous cocycle A. The following result is a particular case of [IM12, Theorem 1.4.].

Theorem 5.7. *For any $\mu \in \mathcal{E}(f)$, we have that*

$$\limsup_{n\to\infty} \frac{1}{n}\log\rho(A^n(x)) = \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)\| = \lambda_1(\mu) \quad \textit{for } \mu\textit{-a.e. } x \in M.$$

We now prove the following result.

Theorem 5.8. *Assume that A is an α-Hölder cocycle and that f satisfies Anosov closing property. Then,*

$$\lim_{n\to\infty}\max_{x\in M}\|A^n(x)\|^{1/n} = \sup_{(x,p)\in M\times\mathbf{N}:\ f^p(x)=x} \rho(A^p(x))^{1/p}.$$

Proof. It follows from [S98, Theorem 1.] and Theorem 2.5 that

$$\lim_{n\to\infty}\frac{1}{n}\max_{x\in M}\log\|A^n(x)\| = \sup_{\mu\in\mathcal{E}(f)}\lambda_1(\mu) = \sup_{\mu\in\mathcal{E}_{per}(f)}\lambda_1(\mu). \tag{54}$$

Assume that $\mu \in \mathcal{E}_{per}(f)$ is supported on an periodic orbit of a point $x \in M$ with period p. Then, it follows from Theorem 5.7 that

$$\begin{aligned}\lambda_1(\mu) &= \lim_{n\to\infty} \frac{1}{np}\log\|A^{np}(x)\| = \limsup_{n\to\infty} \frac{1}{np}\log\rho(A^{np}(x)) \\ &= \limsup_{n\to\infty} \frac{1}{np}\log\rho((A^p(x))^n) = \limsup_{n\to\infty} \frac{1}{np}\log(\rho(A^p(x)))^n = \frac{1}{p}\log\rho(A^p(x)).\end{aligned}$$

Hence, (54) implies that

$$\lim_{n\to\infty} \frac{1}{n}\max_{x\in M}\log\|A^n(x)\| = \sup_{(x,p)\in M\times\mathbf{N}: f^p(x)=x} \log\rho(A^p(x))^{1/p}.$$

Therefore,

$$\lim_{n\to\infty} \max_{x\in M}\log\|A^n(x)\|^{1/n} = \sup_{(x,p)\in M\times\mathbf{N}: f^p(x)=x} \log\rho(A^p(x))^{1/p},$$

which readily yields the desired result. □

The above result is interesting since it connects two quantities that exhibit different behaviour under the action of the cocycle: operator norm which is subadditive and spectral radius which behaves quite badly with respect to composition of operators.

5.4. Conjugacy between cocycles and Lyapunov exponents. Assume now that for $i = 1, 2$ we are given a cocycle A_i of operators acting on $\mathcal{B}_i$ and over a base space (M_i, f_i). We say that A_1 and A_2 are *conjugated* if there exists an invertible map $h\colon M_1 \to M_2$ and a family of invertible bounded linear operators $L(x)\colon \mathcal{B}_1 \to \mathcal{B}_2$, $x \in M_1$ such that:

(1)

$$h \circ f_1 = f_2 \circ h; \tag{55}$$

(2) we have

$$A_1(x) = L(f_1(x))^{-1}A_2(h(x))L(x), \quad \text{for each } x \in M_1. \tag{56}$$

Remark 5.9. In the context of smooth dynamics, this notion corresponds to the classical notion of conjugacy. Indeed, if M_1, M_2 are smooth compact Riemmanian manifolds and f_1, f_2 are smooth diffeomorphisms, then if a differentiable map h satisfies (55), one can easily conclude that (56) holds with

$$A(x) = Df_1(x), \quad B(x) = Df_2(x) \quad \text{and} \quad L(x) = Dh(x).$$

Observe that it follows easily from (56) that

$$A_1^n(x) = L(f_1^n(x))^{-1}A_2^n(h(x))L(x), \quad \text{for } x \in M_1 \text{ and } n \in \mathbf{N}. \tag{57}$$

Theorem 5.10. *Suppose that:*

1. $A_1\colon M_1 \to B(\mathcal{B}_1, \mathcal{B}_1)$ *and* $A_2\colon M_2 \to B(\mathcal{B}_2, \mathcal{B}_2)$ *are cocycles such that* $A_1(x)$ *is a compact operator for each* $x \in M_1$*;*
2. (M_1, f_1) *satisfies the Anosov closing property;*
3. A_1 *is an* α*-Hölder cocycle;*
4. A_2 *is uniformly hyperbolic;*
5. A_1 *and* A_2 *are conjugated.*

Then, all Lyapunov exponents of A_1 *are uniformly bounded away from zero.*

Proof. Observe that it follows from (56) that $A_2(x)$ is a compact operator for each $x \in M_2$. In addition, observe that x is a periodic point with period p for f_1 if and only if $h(x)$ is a periodic point with period p for f_2. Furthermore, in this case it follows from (57) that

$$A_1^{np}(x) = L(x)^{-1} A_2^{np}(h(x)) L(x), \quad \text{for } n \in \mathbf{N}. \tag{58}$$

By (58), Lyapunov exponents of A_1 with respect to a measure which is supported on the orbit of x are the same as Lyapunov exponents of A_2 with respect to a measure which is supported on the orbit of $h(x)$. Hence, since A_2 is uniformly hyperbolic, we have that all Lyapunov exponents of A_1 with respect to invariant measures supported on periodic orbits are uniformly bounded away from zero. Then, Theorem 2.5 implies that the same holds for all Lyapunov exponents. □

Remark 5.11. We emphasize that we haven't assumed any type of information regarding the asymptotic behaviour of maps $x \mapsto \|L(x)\|$ and $x \mapsto \|L(x)^{-1}\|$. If we were to assume that those maps are tempered with respect to any invariant measure for f_1, we could conclude (see [BP07]) that Lyapunov exponents of cocycles A_1 and A_2 are the same and therefore the conclusion of Theorem 5.10 would hold trivially.

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