DISTORTION THEOREMS, LIPSCHITZ CONTINUITY
AND THEIR APPLICATIONS FOR BLOCH TYPE
MAPPINGS ON BOUNDED SYMMETRIC
DOMAINS IN $\mathbb{C}^n$

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Abstract. Let $B_X$ be a bounded symmetric domain realized as the unit ball of an $n$-dimensional JB$^*$-triple $X = (\mathbb{C}^n, \| \cdot \|_X)$. In this paper, we give a new definition of Bloch type mappings on $B_X$ and give distortion theorems for Bloch type mappings on $B_X$. When $B_X$ is the Euclidean unit ball in $\mathbb{C}^n$, this new definition coincides with that given by Chen and Kalaj or by the author. As a corollary of the distortion theorem, we obtain the lower estimate for the radius of the largest schlicht ball in the image of $f$ centered at $f(0)$ for $\alpha$-Bloch mappings $f$ on $B_X$. Next, as another corollary of the distortion theorem, we show the Lipschitz continuity of $(\det B(z,z))^{1/2n} \| \det Df(z) \|^1/n$ for Bloch type mappings $f$ on $B_X$ with respect to the Bergman distance on $B_X$, where $B(z,z)$ is the Bergman operator on $X$, and use it to give a sufficient condition for the composition operator $C_\varphi$ to be bounded from below on the Bloch type space on $B_X$, where $\varphi$ is a holomorphic self mapping of $B_X$. In the case $B_X = B^n$, we also give a necessary condition for $C_\varphi$ to be bounded from below which is a converse to the above result. Finally, as another application of the Lipschitz continuity, we obtain a result related to the interpolating sequences for the Bloch type space on $B_X$.

1. Introduction

Let $f$ be a Bloch function on the unit disc $U$ in $\mathbb{C}$. Ghatage, Yan and Zheng [6] showed that $(1 - |z|^2)|f'(z)|$ is Lipschitz with respect to the hyperbolic distance on $U$ and used it to give a sufficient condition for the composition operator $C_\varphi$ to be bounded from below on the Bloch space on $U$, where $\varphi$ is a holomorphic self mapping of $U$. Chen and Kalaj [2] generalized the above results to the Euclidean unit ball $B^n$ in $\mathbb{C}^n$. They showed the Lipschitz continuity of $(1 - \|z\|^2)^{1/2n} \| \det Df(z) \|^1/n$ for Bloch type mappings $f$ on $B^n$ with respect to the Bergman distance on $B^n$ and used it to give a sufficient condition for the composition operator $C_\varphi$ to be bounded from below on the Bloch type space on $B^n$, where $\varphi$ is a holomorphic self mapping of $B^n$. In the proof of Lipschitz continuity, the distortion theorem for Bloch type mappings on $B^n$ due to Chen, Pomusamy and Wang [3] played an important role.

Bounded symmetric domains in $\mathbb{C}^n$ can be realized as the unit ball $B_X$ of a JB$^*$-triple $X = (\mathbb{C}^n, \| \cdot \|_X)$. In this paper, we give a new definition of Bloch type mappings on $B_X$ and give distortion theorems for Bloch type mappings on $B_X$ (Theorem 3.1). When $B_X$ is the Euclidean unit ball in $\mathbb{C}^n$, this new definition coincides with that given by Chen and Kalaj [2] and the author [10]. Note that the author

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symmetric domains in $\mathbb{C}^n$ (cf. [5], [13]) gave distortion theorems for other Bloch type mappings on bounded domains. As in [10], [13], as a corollary of the distortion theorem, we obtain the lower estimate for the radius of the largest schlicht ball in the image of $f$ centered at $f(0)$ for $\alpha$-Bloch mappings $f$ on $B_X$ (Theorem 3.2). Next, as another corollary of the distortion theorem, we show the Lipschitz continuity of $(\det B(z, z))^{1/2n} |\det Df(z)|^{1/n}$ for Bloch type mappings $f$ on $B_X$ with respect to the Kobayashi metric (Theorem 4.1), where $B(z, z)$ is the Bergman operator on $X$, and use it to give a sufficient condition for the composition operator $C_\varphi$ to be bounded from below on the Bloch type space on $B_X$ (Theorem 5.1), where $\varphi$ is a holomorphic self mapping of $B_X$. In the case $B_X = B^n$, we also give a necessary condition for $C_\varphi$ to be bounded from below which is a converse to Theorem 5.1 (Theorem 5.2). Finally, as another application of the Lipschitz continuity, we obtain a result related to the interpolating sequences for the Bloch type space on $B_X$ (Corollary 5.3).

2. Preliminaries

Let $X$ and $Y$ be complex Banach spaces. Let $L(X, Y)$ denote the set of continuous linear operators from $X$ into $Y$. Let $I_X$ be the identity in $L(X) = L(X, X)$. For a linear operator $A \in L(X, Y)$, let

$$\|A\|_{X,Y} = \sup \{\|Az\|_Y : \|z\|_X = 1\},$$

where $\| \cdot \|_X$ and $\| \cdot \|_Y$ are the norms on $X$ and $Y$, respectively. In the case $Y = \mathbb{C}^n$ is the Euclidean space, we write $\|A\|_{X,e}$ for $A \in L(X, \mathbb{C}^n)$.

**Definition 2.1.** (See e.g. [4] and [17]) A complex Banach space $X$ is called a JB*-triple if $X$ is a complex Banach space equipped with a continuous Jordan triple product

$$X \times X \times X \to X \ (x, y, z) \mapsto \{x, y, z\}$$

satisfying

(1) $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,

(2) $\{a, b, \{x, y, z\}\} = \{(a, b, x), y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\},$

(3) $x \Box x \in L(X)$ is a hermitian operator with spectrum $\geq 0$,

(4) $\|\{x, x, x\}\| = \|x\|^3$

for $a, b, x, y, z \in X$, where the box operator $x \Box y : X \to X$ is defined by $x \Box y(\cdot) = \{x, y, \cdot\}$ and $\| \cdot \|$ is the norm on $X$.

An element $u \in X$ which satisfies $\{u, u, u\} = u$ is called a tripotent. Two tripotents $u$ and $v$ which satisfy $u \Box v = 0$ are said to be orthogonal. A tripotent $u$ is said to be maximal if the only tripotent which is orthogonal to $u$ is 0.

For every $x, y \in X$, the Bergman operator $B(x, y) \in L(X)$ is defined by

$$B(x, y)(z) = z - 2(x \Box y)(z) + \{x, y, z\}, \ x \in X.$$ 

When $\|x \Box y\| < 1$, the fractional power $B(x, y)_r \in L(X)$ exists for every $r \in \mathbb{R}$, since the spectrum of $B(x, y)$ lies in $\{\zeta \in \mathbb{C} : |\zeta - 1| < 1\}$ (cf. [14, p. 517]).

Let $B_X$ be the unit ball of $X$. We denote by $H(B_X)$ the family of holomorphic mappings from $B_X$ into $X$. Let $Df(x)$ denote the Fréchet derivative of $f \in H(B_X)$. Also, let $g_a : B_X \to B_X$ be the Möbius transformation induced by $a$, given by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x \Box a)^{-1}(x), \ x \in B_X.$$
Then \( g_a \) is an automorphism of \( B_X \) such that \( g_a^{-1} = g_{-a} \), and \( g_a(-a) = 0 \). We will denote by \( \rho \) the Kobayashi metric on \( B_X \), which is the integral form of the infinitesimal Kobayashi metric. For \( a, b \in B_X \), we have \( \rho(a, b) = \tanh^{-1} \|g_{-a}(b)\| \), where \( g_{-a} \) is the Möbius transformation induced by \(-a\).

In the rest of this paper, let \( B_X \) be the unit ball of an \( n \)-dimensional JB$^*$-triple \( X = (C^n, \| \cdot \|_X) \). Also, let \( h_0 \) be the Bergman metric on \( B_X \) at 0, and let (see \cite{11})

\[
c(B_X) = \frac{1}{2} \sup_{x,y \in B_X} |h_0(x,y)|.
\]

The following lemma is obtained in the proof of \cite[Theorem 3.1]{13}.

**Lemma 2.2.** Let \( \alpha > 0 \), \( c = c(B_X) \) and let

\[
r(t) = t(1 - t^2)^{\alpha} \left( \frac{2\alpha + 1}{2\alpha} \right)^{\alpha} \sqrt{2\alpha + 1}, \quad t \in [0, 1].
\]

Then \( r(t) \) is increasing on \([0, \frac{1}{\sqrt{2\alpha + 1}}]\), decreasing on \([\frac{1}{\sqrt{2\alpha + 1}}, 1]\) and \( r \left( \frac{1}{\sqrt{2\alpha + 1}} \right) = 1 \).

The following result is a Schwarz–Pick lemma on the open unit ball \( B_X \) of the JB$^*$-triple \( X \) \cite{7} (cf. \cite[Theorem 3]{9} and \cite[Corollary 4.7]{12}; see \cite[Lemma 1.1]{8} and \cite{18}, in the case \( B_X = B^n \)).

**Lemma 2.3.** Let \( B_X \) be the unit ball of a finite dimensional JB$^*$-triple \( X \) and let \( \varphi : B_X \rightarrow B_X \) be a holomorphic mapping. Then

\[
|\det D\varphi(z)| \leq \left( \frac{\det B(\varphi(z), \varphi(z))/\det B(z, z)}{2\alpha} \right)^{1/2}, \quad \forall z \in B_X.
\]

The above estimate is sharp and equality at a given point \( z \in B_X \) holds if and only if \( \varphi \) is an automorphism of \( B_X \).

The following lemma was obtained in \cite{10}(see also \cite[Lemma 2.2]{13}).

**Lemma 2.4.** Let \( B_X \) be a bounded symmetric domain realized as the unit ball of an \( n \)-dimensional JB$^*$-triple \( X = (C^n, \| \cdot \|_X) \). Then we have

\[
\det B(z, z) \geq (1 - \|z\|^2)^{2c(B_X)}, \quad z \in B_X.
\]

The equality holds for \( z \neq 0 \) such that \( c = \|z\|/\|z\| \) is a maximal tripotent in \( X \).

**Definition 2.5.** Let \( B_X \) be a bounded symmetric domain realized as the unit ball of an \( n \)-dimensional JB$^*$-triple \( X = (C^n, \| \cdot \|_X) \) and let \( \alpha > 0 \). The prenorm \( \|f\|_{\mathcal{P}(X, \alpha)} \) of \( f \in H(B_X) \) is given by

\[
\|f\|_{\mathcal{P}(X, \alpha)} = \sup_{z \in B_X} D^X_\alpha(f)(z),
\]

where

\[
D^X_\alpha(f)(z) = (\det B(z, z))^{\alpha/2n} |\det Df(z)|^{1/n}.
\]

Let \( \mathcal{B}_{\mathcal{P}(X, \alpha)} \) be the class of all mappings \( f \in H(B_X) \) which satisfy \( \|f\|_{\mathcal{P}(X, \alpha)} < +\infty \). A mapping \( f \in \mathcal{B}_{\mathcal{P}(X, \alpha)} \) will be called a Bloch type mapping on \( B_X \).

**Remark 2.6.** (i) In \cite[Definition 2.5]{13}, another prenorm \( \|f\|_{0, \alpha} \) is defined as follows:

\[
\|f\|_{0, \alpha} = \sup \left\{ (1 - \|z\|^2)^{\alpha(B_X)/n} |\det Df(z)|^{1/n} : z \in B_X \right\}.
\]

Then, by Lemma 2.4, we have \( \|f\|_{0, \alpha} \leq \|f\|_{\mathcal{P}(X, \alpha)} \).
(ii) When $B_X$ is the Euclidean unit ball $B^n$ in $C^n$, then $c(B^n) = (n + 1)/2$ ([11]) and det $B(z, z) = (1 - \|z\|^2)^{n+1}$ for all $z \in B^n$ ([10], see also [13, Lemma 2.2]). Therefore, if $B_X$ is the Euclidean unit ball $B^n$ in $C^n$, then we have $\|f\|_{P(X,\alpha)} = \|f\|_{P(X,\alpha)}$.

(iii) When $B_X$ is the unit polydisc $U^n$ in $C^n$, we have det $B(z, z) = \Pi_{i=1}^n(1 - \langle z_i \rangle ^2)$. Therefore, we have

$$\|f\|_{P(X,\alpha)} = \sup_{z \in U^n} \Pi_{i=1}^n(1 - \langle z_i \rangle ^2)^{\alpha/n} \det Df(z)^{1/n}.$$  

When $\alpha = 1$, $\|f\|_{P(X,1)}$ coincides with the prenorm $\|f\|_0$ defined by Wang and Liu [19].

As in [13, Lemma 2.6], we obtain the following lemma.

**Lemma 2.7.** Let $B_X$ be a bounded symmetric domain realized as the unit ball of an $n$-dimensional JB$^*$-triple $X = (C^n, \| \cdot \|_X)$ and let $\alpha > 0$.

(i) If $\|f\|_{P(X,\alpha)} < +\infty$, then

$$|\det Df(z)| \leq \frac{\|f\|_{P(X,\alpha)}^n}{(1 - \|z\|^2)^{n+1}} \text{ for } z \in B_X.$$

(ii) If $\|f\|_{P(X,\alpha)} = 1$ and $\det Df(0) = 1$, then $|\det Df(z)| = 1 + o(\|z\|)$.

**Proof.** (i) This is a consequence of the definition of $\|f\|_{P(X,\alpha)}$ and Lemma 2.4. (ii) This follows from (i). $\square$

For $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^*: l_x(x) = \|x\|, \|l_x\| = 1\},$$

where $X^*$ is the dual space of $X$. Then $T(x) \neq \emptyset$ in view of the Hahn–Banach theorem. Let $H(U)$ denote the set of holomorphic functions on the unit disc $U$ in $C$. The following lemma was proved by the author [10] (see also [13, Lemma 2.9]).

**Lemma 2.8.** Let $B_X$ be a bounded symmetric domain realized as the unit ball of an $n$-dimensional JB$^*$-triple $X = (C^n, \| \cdot \|_X)$. Let $u \in \partial B_X$ be fixed and let

$$f_{\psi,l_u}(z) = \left( \int_0^{l_u(z)} \psi(\zeta) d\zeta \right) u + z - l_u(z)u, \quad z \in B_X,$$

where $l_u \in T(u)$ and $\psi \in H(U)$. Then $f_{\psi,l_u} \in H(B_X)$, $f_{\psi,l_u}(0) = 0$ and $\det Df_{\psi,l_u}(z) = \psi(l_u(z))$ for $z \in B_X$.

Let $d_h$ denote the hyperbolic distance on $U$:

$$d_h(x, y) = \frac{1}{2} \log \frac{1 + \frac{|x-y|}{1-\pi y}}{1 - \frac{|x-y|}{1-\pi y}} = \arctanh \left| \frac{x-y}{1-\pi y} \right|, \quad x, y \in U.$$  

Denote by $D_h(a, r)$ the hyperbolic disc in $U$ with center $a$ and radius $r$. Also, let $U(a, r) = \{z \in C: |z - a| < r\}$ for $a \in C$ and $r > 0$. Then the following lemma holds. The relation (i) was proved by Wang and Liu [20, Lemma 1], while the relation (ii) was recently proved by the author [10].

**Lemma 2.9.** Assume that $0 < a < 1$ and $0 \in D_h(a, R)$ if $g \in H(U)$ satisfies $g(0) = \lambda > 0$ and $g(D_h(a, R)) \subset U \left( 0, \frac{\lambda}{a \tanh(R)} \right)$, then

(i) $\Re g(x) \geq G(x)$ for $x \in \left( 0, \frac{a + \tanh(R)}{1 + a \tanh(R)} \right)$ with equality for some $x$ if and only if $g = G$, where $G(\zeta) = \lambda(a - \zeta)/(a(1 - a\zeta))$;
(ii) \(|g(-x)| \leq G(-x)\) for \(x \in \left(0, \frac{\tanh(R) - a}{1 - a \tanh(R)}\right)\) with equality for some \(x\) if and only if \(g = G\).

3. Distortion theorems

We begin this section with the distortion theorems for Bloch type mappings \(f \in \mathcal{B}_{\mathcal{P}(X, \alpha)}\). The following theorem is a generalization of [3, Theorem 1.2] to any finite dimensional bounded symmetric domain. In [13, Theorem 3.1], the following theorem was obtained under the condition \(\|f\|_{0, \alpha} = 1\) instead of \(\|f\|_{\mathcal{P}(X, \alpha)} = 1\). (When \(\alpha = 1\), see also [1, Theorem 2], [10, Theorem 3.1], [15, Theorem 5], [19, Theorem 3.1] and [20, Theorem 1]).

**Theorem 3.1.** Let \(\mathcal{B}_X\) be a bounded symmetric domain realized as the unit ball of an \(n\)-dimensional JB*-triple \(X = (\mathbb{C}^n, \| \cdot \|_X)\). Let \(\alpha > 0\), \(\lambda \in (0, 1]\), and let \(m(\lambda)\) be the unique root of the equation

\[
t(1 - t^2)^{\alpha c(B_X)} \left( \frac{2\alpha c(B_X) + 1}{2\alpha c(B_X)} \right)^{\alpha c(B_X)} \sqrt{2\alpha c(B_X)} + 1 = \lambda
\]

in \([0, \frac{1}{\sqrt{2\alpha c(B_X) + 1}}]\). Let \(f \in \mathcal{B}_{\mathcal{P}(X, \alpha)}\) be such that \(\det Df(0) = \lambda\) and \(\|f\|_{\mathcal{P}(X, \alpha)} = 1\). Then

(i) \(|\det Df(z)| \geq \Re \det Df(z) \geq \frac{\lambda}{m(\lambda) (1 - m(\lambda)\|z\|)} 2^{\alpha c(B_X) + 1} \alpha c(B_X) + 1\)

(ii) \(|\det Df(z)| \leq \frac{\lambda}{m(\lambda) (1 + m(\lambda)\|z\|)} 2^{\alpha c(B_X) + 1} \alpha c(B_X) + 1\)

for \(\|z\| \leq 1 + \sqrt{2\alpha c(B_X) + 1} \alpha c(B_X) + 1 + m(\lambda)\).

**Proof.** It suffices to use arguments similar to those in the proof of [13, Theorem 3.1] (cf. [3, Theorem 1.2], [10, Theorem 3.1]). For completeness, we give a proof for (i). Let \(c = c(B_X)\). Note that there exists a unique \(m(\lambda) \in \left(0, \frac{1}{\sqrt{2\alpha c + 1}}\right]\) such that

\[
m(\lambda) (1 - m(\lambda)^2)^{\alpha c(B_X)} \left( \frac{2\alpha c + 1}{2\alpha c} \right)^{\alpha c} \sqrt{2\alpha c + 1} = \lambda
\]

by Lemma 2.2. Let \(z \in \mathcal{B}_X \setminus \{0\}\) be fixed and let \(u = z/\|z\|\).

First, we consider the case \(\lambda \in (0, 1)\). In this case, we have \(m(\lambda) \in \left(0, \frac{1}{\sqrt{2\alpha c + 1}}\right]\). Let

\[
g(\zeta) = (1 - m(\lambda)\zeta)^{2\alpha c} \det Df(\zeta u), \quad \zeta \in U.
\]

Then \(g \in H(U)\) and \(g(0) = \lambda \in (0, 1)\). Let \(\zeta \in U\) be such that \(d_h(m(\lambda), \zeta) = \arctanh \frac{1}{\sqrt{2\alpha c + 1}}\). This is equivalent to

\[
\left| \frac{m(\lambda) - \zeta}{1 - m(\lambda)\zeta} \right| = \frac{1}{\sqrt{2\alpha c + 1}}
\]

By the relation \( \|f\|_{\mathcal{P}(X,\alpha)} = 1 \), Lemma 2.7 (i), (3.2), (3.3) and (3.4), we have
\[
|g(\zeta)| \leq \frac{|1 - m(\lambda)\zeta|^{2\alpha c}}{(1 - |\zeta|^2)^{\alpha c}} \|f\|_{\mathcal{P}(X,\alpha)} = \left( \frac{1 - m(\lambda)^2}{1 - |m(\lambda) - \zeta|^2} \right)^{\alpha c} = (1 - m(\lambda)^2)^{\alpha c} \left( \frac{2\alpha c + 1}{2\alpha c} \right)^{\alpha c} = \frac{\lambda}{m(\lambda)\sqrt{2\alpha c + 1}}.
\]
This implies that
\[
g\left( D_h \left( m(\lambda), \arctanh \frac{1}{\sqrt{2\alpha c + 1}} \right) \right) \subset U \left( 0, \frac{\lambda}{m(\lambda)\sqrt{2\alpha c + 1}} \right).
\]
In view of Lemma 2.9 (i), we obtain that
\[
\Re g(\|z\|) \geq \frac{\lambda}{m(\lambda)} \frac{m(\lambda) - \|z\|}{1 - m(\lambda)\|z\|}
\]
for \( \|z\| \in \left( 0, \frac{1 + \sqrt{2\alpha c + 1}m(\lambda)}{\sqrt{2\alpha c + 1} + m(\lambda)} \right) \). By (3.3), we have
\[
\Re \det Df(z) \geq \frac{\lambda}{m(\lambda)} \frac{m(\lambda) - \|z\|}{1 - m(\lambda)\|z\|}^{2\alpha c + 1},
\]
for \( \|z\| \in \left( 0, \frac{1 + \sqrt{2\alpha c + 1}m(\lambda)}{\sqrt{2\alpha c + 1} + m(\lambda)} \right) \), i.e., we obtain the inequality (3.1) for \( \lambda \in (0, 1) \), as desired.

Next, we consider the case \( \lambda = 1 \). Let
\[
(3.5) \quad h(\zeta) = (1 - m(1)T(\zeta))^{2\alpha c} \det Df(T(\zeta)u),
\]
where
\[
T(\zeta) = \frac{m(1)(1 - \zeta)}{1 - m(1)^2\zeta}.
\]
Since \( T(1) = 0 \) and
\[
(3.6) \quad T(U) = \{w: |1 - m(1)w|^2 < 1 - |w|^2\} \subset U,
\]
the function \( h \) is holomorphic on \( U \cup \{1\} \). Moreover, by (3.5), (3.6), Lemma 2.7 (i), and the relation \( \|f\|_{\mathcal{P}(X,\alpha)} = 1 \), we have
\[
|h(\zeta)| = |1 - m(1)T(\zeta)|^{2\alpha c} |\det Df(T(\zeta)u)| \leq (1 - |T(\zeta)|^2)^{\alpha c} |\det Df(T(\zeta)u)| \leq \|f\|_{\mathcal{P}(X,\alpha)} = 1
\]
for all \( \zeta \in U \). Since
\[
h'(1) = 2\alpha c(-m(1))T'(1) = \frac{2\alpha cm(1)^2}{1 - m(1)^2} = 1
\]
by Lemma 2.7 (ii) and the relation \( m(1) = 1/\sqrt{2\alpha c + 1} \), we have \( h(U) \subset U \) by the maximum principle. Since \( h(1) = 1 \), in view of a classical version of Julia’s Lemma [16, p. 327], \( h \) maps \( \Delta(1, r), r > 0 \) of \( U \) into itself, where \( \Delta(1, r) \) is a horodisc in \( U \), that is,
\[
\Delta(1, r) = \left\{ z \in U: \frac{|1 - z|^2}{1 - |z|^2} < r \right\} = U \left( \frac{1}{1 + r}, \frac{r}{1 + r} \right).
\]
Therefore, we have
\[
\Re h(x) \geq x, \quad \forall x \in [-1, 1].
\]
By (3.5), we have
\[ \Re \det Df(z) \geq \frac{1}{m(1)} \frac{m(1) - \|z\|}{(1 - m(1)\|z\|)^{2\alpha c + 1}} \]
for all \( \|z\| \in [0, (1 + \sqrt{2\alpha c + 1}m(1))/(\sqrt{2\alpha c + 1} + m(1))] \), i.e., we obtain the inequality (3.1) for \( \lambda = 1 \), as desired.

(ii) The case \( \lambda = 1 \) is trivial. The case \( \lambda \in (0, 1) \) can be proved by using arguments similar those in (i). For the details, see the proof of [13, Theorem 3.1]. This completes the proof. \( \square \)

For \( f \in H(B_X) \), a Euclidean ball with center \( f(p) \) such that \( f \) maps an open subset of \( B_X \) containing \( p \) biholomorphically onto this ball is called a schlicht ball of \( f \) centered at \( f(p) \). For a point \( p \in B_X \), let \( r(p, f) \) denote the radius of the largest schlicht ball of \( f \) centered at \( f(p) \).

A mapping \( f \in H(B_X) \) is called an \( \alpha \)-Bloch mapping if
\[ \|f\|_\alpha < +\infty, \]
where \( \|f\|_\alpha \) denotes the \( \alpha \)-Bloch semi-norm of \( f \) defined by
\[ \|f\|_\alpha = \sup_{z \in B_X} \|Df(z)B(z, z)^{\alpha/2}\|_{X,e}. \]

As a corollary of Theorem 3.1, we obtain the following lower estimate for the radius of the largest schlicht ball in the image of \( f \) centered at \( f(0) \) for \( \alpha \)-Bloch mappings \( f \in H(B_X) \). The following theorem is a generalization of [3, Theorem 1.3] to the unit ball of a finite dimensional JB\( ^* \)-triple. In [13, Theorem 4.5], the following theorem was obtained under the condition \( \|f\|_{0,\alpha} = 1 \) instead of \( \|f\|_{P(X,\alpha)} = 1 \). When \( \alpha = 1 \), it reduces to the result for Bloch mappings on bounded symmetric domains in \( C^n \) [10, Theorem 4.4] (see also [15, Theorem 6], [19, Theorem 3.3], [20, Theorem 2]). We omit the proof, since it suffices to use the arguments similar to those in the proof of [13, Theorem 4.5].

**Theorem 3.2.** Let \( B_X \) be a bounded symmetric domain realized as the unit ball of an \( n \)-dimensional JB\( ^* \)-triple \( X = (C^n, \| \cdot \|_X) \) such that \( B_X \supset B^n \), where \( B^n \) is the Euclidean unit ball of \( C^n \). Let \( \alpha > 0 \) and \( K > 0 \). If \( f \) is an \( \alpha \)-Bloch mapping on \( B_X \) such that \( \|f\|_\alpha \leq K \), \( \|f\|_{P(X,\alpha)} = 1 \) and \( \det Df(0) = \lambda \in (0, 1] \), then
\[ r(0, f) \geq K^{1-n} \frac{\lambda}{m(\lambda)} \int_0^{m(\lambda)} \frac{(1 - t^2)^{\alpha(n-1)}(m(\lambda) - t)}{(1 - m(\lambda)t)^{2\alpha c(B_X)+1}} dt, \]
where \( m(\lambda) \) is the unique root of the equation
\[ t(1 - t^2)^{\alpha c(B_X)} \left( \frac{2\alpha c(B_X) + 1}{2\alpha c(B_X)} \right)^{\alpha c(B_X)} \sqrt{2\alpha c(B_X) + 1} = \lambda \]
in the interval \( \left[ 0, \frac{1}{\sqrt{2\alpha c(B_X)+1}} \right] \).

4. Lipschitz continuity

Chen and Kalaj [2, Theorem 1] obtained the Lipschitz continuity of
\[ (1 - \|z\|^2)^{\frac{\alpha c}{2\alpha c + 1}} |\det Df(z)|^{1/n} \]
for Bloch type mappings $f$ on the Euclidean unit ball in $\mathbb{C}^n$ (see [6, Theorem 1] in the case $n = 1$). The following theorem is a generalization to Bloch type mappings on bounded symmetric domains in $\mathbb{C}^n$.

**Theorem 4.1.** Let $B_X$ be a bounded symmetric domain realized as the unit ball of an $n$-dimensional JB*-triple $X = (\mathbb{C}^n, \| \cdot \|_X)$. Let $f \in \mathcal{B}_{P(X,1)}$. Then we have

$$|D_f^{X,1}(z_1) - D_f^{X,1}(z_2)| \leq M(X)\|f\|_{P(X,1)}[\tanh \rho(z_1, z_2)]^{\frac{1}{\alpha}},$$

for $z_1, z_2 \in B_X$, where

$$M(X) = \left(2c(B_X) + 1\right) \left(\frac{2c(B_X) + 1}{2c(B_X)}\right)^{\frac{c(B_X)}{n}}.$$

**Proof.** Let $z_1, z_2 \in B_X$ be fixed. We may assume that $\|f\|_{P(X,1)} = 1$ and $D_f^{X,1}(z_2) \leq D_f^{X,1}(z_1)$. Let $w = g_{z_1}^{-1}(z_2)$. Then, by the invariance of $\rho$ by automorphisms of $B_X$, we have

$$\tanh \rho(z_1, z_2) = \tanh \rho(g_{z_1}^{-1}(z_1), g_{z_1}^{-1}(z_2)) = \tanh \rho(0, w) = \|w\|.$$  

Therefore, it suffices to show that

$$(4.1) \quad D_f^{X,1}(z_1) - D_f^{X,1}(z_2) \leq M(X)\|w\|^{\frac{1}{\alpha}}.$$  

Let $F = f \circ g_{z_1}$. Since

$$|\det Dg_{z_1}(z)| = \left|\frac{|\det B(g_{z_1}(z), g_{z_1}(z))|^{1/2}}{\det B(z, z)}\right|,$$

by Lemma 2.3, we have

$$D_F^{X,1}(0) = |\det DF(0)|^{\frac{1}{\alpha}} = |\det Df(z_1)|^{\frac{1}{\alpha}}|det B(z_1, z_1)|^{\frac{1}{\alpha}} = D_f^{X,1}(z_1)$$  

and

$$D_F^{X,1}(w) = |\det DF(w)|^{\frac{1}{\alpha}}|det B(w, w)|^{\frac{1}{\alpha}} = |\det Df(z_2)|^{\frac{1}{\alpha}}|det B(z_2, z_2)|^{\frac{1}{\alpha}} = D_f^{X,1}(z_2).$$

By using similar calculations, we also have $\|F\|_{P(X,1)} = 1$.

If $|\det DF(0)| = 0$, then by the inequality $D_f^{X,1}(z_2) \leq D_f^{X,1}(z_1)$, we have $D_f^{X,1}(z_2) = 0$. Therefore, the inequality (4.1) holds. So, we may assume that $\det DF(0) = \lambda e^{i\theta}$ for some $\lambda \in (0, 1]$ and $\theta \in [0, 2\pi]$. Let $c = c(B_X)$ and let $m(\lambda)$ be the constant in Theorem 3.1 for $\alpha = 1$. Then by (3.2), we have

$$(4.2) \quad m(\lambda)(1 - m(\lambda)^2)^cM(X)^n = \lambda.$$  

Case 1. Assume that $\|w\| \leq m(\lambda)$. Then, we have

$$(1 - \|w\|^2)^{c/n} \geq (1 - \|w\|m(\lambda))^{(2c+1)/n}$$

and

$$m(\lambda)^{1/n} - \|w\|^{1/n} \leq (m(\lambda) - \|w\|)^{1/n}.$$  

These inequalities imply that

$$(4.3) \quad m(\lambda)^{1/n} - \frac{(1 - \|w\|^2)^{c/n}(m(\lambda) - \|w\|)^{1/n}}{(1 - m(\lambda)\|w\|)^{(2c+1)/n}} \leq \|w\|^{1/n}.$$  

Since $m(\lambda) \leq 1$, we have

$$\|w\| \leq m(\lambda) \leq \frac{1 + \sqrt{2c + 1}m(\lambda)}{\sqrt{2c + 1} + m(\lambda)}.$$
Also, \(|\det DF(0)| = \lambda \leq 1\). Then, applying Theorem 3.1 to \(e^{-i\theta/n}F\), we have

\[ (4.4) \quad \Re(e^{-i\theta} \det DF(w)) \geq \frac{\lambda}{m(\lambda)} \frac{m(\lambda) - \|w\|}{(1 - m(\lambda)\|w\|^{2c+1}). \]

Therefore, by (4.2), (4.3), (4.4) and Lemma 2.4, we have

\[ D_f^{X,1}(z_1) - D_f^{X,1}(z_2) = |\det DF(0)|^{1/n} - |\det DF(w)|^{1/n} \left| \det B(w, w)^{1/n} \right| \]
\[ \leq \left( \frac{\lambda}{m(\lambda)} \right)^{\frac{1}{n}} \left( m(\lambda)^{1/n} - \frac{(1 - \|w\|^2)^{c/n}(m(\lambda) - \|w\|)^{1/n}}{(1 - m(\lambda)\|w\|)^{(2c+1)/n}} \right) \]
\[ \leq \left( \frac{\lambda}{m(\lambda)} \right)^{\frac{1}{n}} \|w\|^{\frac{1}{n}} = M(X)(1 - m(\lambda)^{2} M(X) \leq M(X)\|w\|^{\frac{1}{n}}. \]

**Case 2.** Assume that \(\|w\| > m(\lambda)\). Then, by (4.2), we have

\[ D_f^{X,1}(z_1) - D_f^{X,1}(z_2) \leq |\det DF(0)|^{1/n} = \lambda^{1/n} \]
\[ = m(\lambda)^{\frac{1}{n}}(1 - m(\lambda)^{2} M(X) \leq M(X)\|w\|^{\frac{1}{n}}. \]

This completes the proof. \(\square\)

**5. Applications of the Lipschitz continuity**

As an application of Theorem 4.1, we first show a result related to a lower bound for the composition operator between the Bloch type spaces on bounded symmetric domains in \(\mathbb{C}^n\). The following theorem is a generalization of [2, Theorem 3] to bounded symmetric domains in \(\mathbb{C}^n\) (see [6, Theorem 2] in the case \(n = 1\)).

For a holomorphic mapping \(\varphi : B_X \to B_X\), let

\[ \tau_{\varphi}(z) = \frac{\det B(z, z)^{\frac{1}{n}}}{\det B(\varphi(z), \varphi(z))^{\frac{1}{n}}} |\det D\varphi(z)|^{\frac{1}{n}}, \]

where \(B(z, z)\) is the Bergman operator and \(n\) is the dimension of \(X\). By Lemma 2.3, we always have \(0 \leq \tau_{\varphi}(z) \leq 1\).

**Theorem 5.1.** Let \(B_X\) be a bounded symmetric domain realized as the unit ball of an \(n\)-dimensional JB*-triple \(X = (\mathbb{C}^n, \|\cdot\|_X)\). Let \(\varphi\) be a holomorphic mapping of \(B_X\) into \(B_X\). Assume that there are positive constants \(r, \varepsilon\) with \(0 < r < M(X)^{-1}\) such that, for each \(w \in B_X\), there exists a point \(z_w \in B_X\) such that \(\tanh \rho(z_w, w) < r^n\) and \(\tau_{\varphi}(z_w) > \varepsilon\). Then, we have

\[ \|C_{\varphi}(f)\|_{P(X,1)} \geq \frac{(1 - rM(X))\varepsilon}{2} \|f\|_{P(X,1)}, \quad f \in B_{P(X,1)}. \]

**Proof.** We may assume that \(\|f\|_{P(X,1)} = 1\). Then, there exists a point \(w \in B_X\) such that

\[ D_f^{X,1}(w) > 1 - \sigma, \]

where

\[ \sigma = \frac{1 - rM(X)}{2} \in (0, 1/2). \]

By the assumption, there exists a point \(z_w \in B_X\) such that

\[ [\tanh \rho(\varphi(z_w), w)]^{\frac{1}{n}} < r \]

Therefore, by (4.2), (4.3), (4.4) and Lemma 2.4, we have

\[ D_f^{X,1}(z_1) - D_f^{X,1}(z_2) = |\det DF(0)|^{1/n} - |\det DF(w)|^{1/n} \left| \det B(w, w)^{1/n} \right| \]
\[ \leq \left( \frac{\lambda}{m(\lambda)} \right)^{\frac{1}{n}} \left( m(\lambda)^{1/n} - \frac{(1 - \|w\|^2)^{c/n}(m(\lambda) - \|w\|)^{1/n}}{(1 - m(\lambda)\|w\|)^{(2c+1)/n}} \right) \]
\[ \leq \left( \frac{\lambda}{m(\lambda)} \right)^{\frac{1}{n}} \|w\|^{\frac{1}{n}} = M(X)(1 - m(\lambda)^{2} M(X) \leq M(X)\|w\|^{\frac{1}{n}}. \]
and \( \tau_{\varphi}(z_w) > \varepsilon \). Therefore, by applying Theorem 4.1, we have
\[
D_{f}^{X,1}(\varphi(z_w)) \geq D_{f}^{X,1}(w) - M(X)[\tanh(\varphi(z_w), w)]^{1/2} \\
\geq 1 - \sigma - r M(X) = \frac{1 - r M(X)}{2} > 0.
\]
Thus, we have
\[
\|C_{\varphi}(f)\|_{P(X,1)} \geq |\det Df(\varphi(z_w))| \det D\varphi(z_w) B(z_w, z_w)^{1/2} |^{1/2} \\
= D_{f}^{X,1}(\varphi(z_w)) \tau_{\varphi}(z_w) > \frac{(1 - r M(X)) \varepsilon}{2}.
\]
This completes the proof. \( \square \)

In the case \( B_X \) is the Euclidean unit ball \( B^n \) of \( \mathbb{C}^n \), we obtain the following theorem which is a converse to Theorem 5.1. We prepare some notations on \( B^n \). Fix \( a \in B^n \). Let
\[
P_a(z) = \begin{cases} \frac{(z,a)}{(a,a)} a, & a \neq 0, \\ 0, & a = 0. \end{cases}
\]
and let \( Q_a = I - P_a \). Put \( s_a = (1 - \|a\|^2)^{1/2} \) and define the automorphism \( \varphi_a \) of \( B^n \) by
\[
\varphi_a(z) = a - P_a(z) - s_a Q_a(z) \quad \frac{1 - \langle z, a \rangle}{1 - \langle a, a \rangle}.
\]
Also, we denote \( \|\cdot\|_{P(X,1)} \) by \( \|\cdot\|_{P(n,1)} \). Assume that there exists a constant \( k \in (0, 1) \) such that
\[
\|C_{\varphi}(f)\|_{P(n,1)} \geq k\|f\|_{P(n,1)}, \quad f \in B_{P(n,1)}.
\]
Then there exist positive constants \( r, \varepsilon \) with \( R < r < 1 \) such that, for each \( w \in B^n \), there exists a point \( z_w \in B^n \) such that \( \tanh(\varphi(z_w), w) < r^n \) and \( \tau_{\varphi}(z_w) > \varepsilon \), where
\[
R = \left(1 - k \frac{2^n}{n+1}\right) \frac{1}{2^n}.
\]

**Proof.** Let \( w \in B^n \) be fixed. Let \( f_{\psi,l,u} \in H(B^n) \) be as in Lemma 2.8, where
\[
\psi(\zeta) = \frac{(1 - \|w\|^2)^{\frac{n+1}{2}}}{(1 - \|w\|\zeta)^{\frac{n+1}{2}}}, \quad \zeta \in \mathbb{U},
\]
and \( u = \frac{w}{\|w\|} \) for \( w \in B_n \setminus \{0\} \) and \( u \in \partial B^n \) is arbitrary for \( w = 0 \). By Lemma 2.8, we have
\[
D_{f_{\psi,l,u}}^{n,1}(z) = |\det Df_{\psi,l,u}(z)|^{1/2} (1 - \|z\|^2)^{\frac{n+1}{2}} = \left(\frac{(1 - \|w\|^2)(1 - \|z\|^2)}{|1 - \langle z, w \rangle|^2}\right)^{\frac{n+1}{2n}},
\]
where \( z \in B^n \). Therefore, we have \( \|f_{\psi,l,u}\|_{P(n,1)} = 1 \). By the assumption, we have
\[
k = k\|f_{\psi,l,u}\|_{P(n,1)} \leq \|C_{\varphi}(f_{\psi,l,u})\|_{P(n,1)}.
\]
Then, for any \( c \in (0, 1) \), there exists \( z_w \in B^n \) such that
\[
D_{f_{\psi,l,u}}^{n,1}(z_w) \geq ck.
\]
Since
\[
D_{f_{\psi,l,u \circ \varphi}}^{n,1}(z_w) = \tau_{\varphi}(z_w) D_{f_{\psi,l,u}}^{n,1}(\varphi(z_w)) = \tau_{\varphi}(z_w) (1 - \|\varphi(w)\|^{2})^\frac{n+1}{2n},
\]
we have

\[ ck \leq \tau_\varphi(z_w)(1 - \|\varphi_w(\varphi(z_w))\|_2) \frac{n+1}{2n}. \]

Since \( \tau_\varphi(z_w) \leq 1 \), we have

\[ ck \leq \tau_\varphi(z_w) \]

and

\[ ck \leq (1 - \|\varphi_w(\varphi(z_w))\|_2) \frac{n+1}{2n}. \]

Let

\[ r = \left(1 - (ck)^{\frac{n}{n+1}}\right)^{\frac{1}{n}} \]

and let \( \varepsilon = ck \). Then we have \( r \in (R, 1) \),

\[ \tanh \rho(\varphi(z_w), w) = \|\varphi_w(\varphi(z_w))\| \leq r^n \]

and \( \tau_\varphi(z_w) \geq \varepsilon \). This completes the proof. \( \Box \)

As another application of Theorem 4.1, we obtain the following result related to the interpolating sequences for the Bloch type space on bounded symmetric domains in \( \mathbb{C}^n \). A sequence \( \{z_\nu\} \) in \( B_X \) is said to be separated in the Kobayashi metric if there exists a constant \( \varepsilon > 0 \) such that \( \rho(z_\nu, z_\mu) \geq \varepsilon \) whenever \( \nu \neq \mu \). The following corollary generalizes [6, Corollary 1]. We omit the proof, since it suffices to use arguments similar to those in the proof of [6, Corollary 1].

**Corollary 5.3.** Let \( B_X \) be a bounded symmetric domain realized as the unit ball of an \( n \)-dimensional JB*-triple \( X = (\mathbb{C}^n, \|\cdot\|_X) \). If a sequence \( \{z_\nu\} \subset B_X \) satisfies the property that the map \( S : B_{P(X,1)} \to \ell^\infty \) defined by \( S(f) = \{\det Df(z_\nu) \det B(z_\nu, z_\nu)\} \) is onto, then \( \{z_\nu\} \) is separated in the Kobayashi metric.

**References**


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