HIGHER INTEGRABILITY OF MINIMIZERS OF DEGENERATE FUNCTIONALS IN CARNOT–CARATHÉODORY SPACES

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Abstract. In this paper, we prove a higher integrability result for the horizontal gradient of a minimizer of a functional of the type

$$I(\Omega, u) = \int_\Omega \sum_{i,j} a_{i,j} X_i u X_j u \, dx$$

whose matrix of the coefficients $A(x) = {}^t A(x)$ satisfies the anisotropic bounds

$$\frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi \rangle \le K(x)|\xi|^2 \quad \forall \xi \in \mathbf{R}^n, \text{ for a.e. } x \in \Omega,$$

where the ellipticity function $K(x) \in A_2 \cap RH_\tau$, τ opportunely related to the homogeneous dimension, and is such that the pair $\left(K, \frac{1}{K}\right) \in A_1$.

1. Introduction

Let Ω be a bounded open set in $\mathbf{R}^n$ and $X = (X_1, \ldots, X_k)$ be a family of vector fields defined in a neighborhood of Ω, with real, C^∞ smooth and globally Lipschitz coefficients satisfying the Hörmander condition.

Let us consider the following functional

$$(1.1) \qquad I(\Omega, u) = \int_\Omega \sum_{i,j=1}^k a_{i,j} X_i u X_j u \, dx$$

where the matrix of the coefficients $A(x) = (a_{ij}(x))_{i,j}$ is symmetric and satisfies the following anisotropic bounds

$$(1.2) \qquad \frac{|\xi|^2}{K(x)} \le \langle A(x)\xi, \xi \rangle \le K(x)|\xi|^2 \quad \forall \xi \in \mathbf{R}^n, \text{ for a.e. } x \in \Omega,$$

with

$$(1.3) \qquad K \colon \Omega \subset \mathbf{R}^n \to [1, +\infty), \quad K \in L^1_{\mathrm{loc}}(\Omega).$$

We suppose that the function K is a weight belonging to the class $A_2 \cap RH_\tau$, with τ related to the homogeneous dimension, and is such that the pair $\left(K, \frac{1}{K}\right) \in A_1$ (see Section 2 for the definitions). Recalling that the Sobolev space $W_X^{1,p}(\Omega)$ is defined as

$$W_X^{1,p}(\Omega) = \{u \in L^p(\Omega) \colon X_j u \in L^p(\Omega), \ j = 1, \ldots, k\},$$

we denote by $H_X(\Omega)$ the set of all $u \in W^{1,1}_{X,\mathrm{loc}}(\Omega)$ such that $u \in L^2(\Omega, K\, dx)$ and u has finite energy, i.e. verifies

$$(1.4) \qquad\qquad \langle AXu, Xu \rangle \in L^1(\Omega).$$

Clearly, $H_X(\Omega)$ is a vector space and, since Ω is bounded, we can make it a normed vector space by taking the norm induced by the bilinear form

$$(1.5) \qquad \alpha(u, \varphi) = \int_\Omega \langle AXu(x), X\varphi(x) \rangle \, dx + \int_\Omega u(x)\varphi(x)\, K(x)\, dx.$$

Define $H_{X,L}(\Omega)$ as the Hilbert space obtained by taking the closure of Lipschitz functions with respect to this norm. To be precise, the elements of $H_{X,L}(\Omega)$ are Cauchy sequences $\{u_k\}$ of Lipschitz functions. It follows from the definition (1.5) and condition (1.2) that u_k converges in $L^2(\Omega, K dx)$ and Xu_k converges in $L^2\left(\Omega, \frac{1}{K} dx\right)$. However, we do not know *a priori* whether an arbitrary $u \in H_X(\Omega)$ is in $H_{X,L}(\Omega)$ unless we make additional assumptions on u, such as assuming $Xu \in L^2\left(\Omega, \frac{1}{K} dx\right)$. Nevertheless, hereafter, when we write $u \in H_{X,L}(\Omega)$ we mean that u is actually a function in $H_X(\Omega) \cap H_{X,L}(\Omega)$ and not a Cauchy sequence.

Recall the following

Definition 1.1. A function $u \in H_X(\Omega)$ is a local minimizer for $I(\Omega, u)$ if

$$I(\mathrm{supp}\, \varphi, u) \le I(\mathrm{supp}\, \varphi, u + \varphi)$$

for any $\varphi \in H_X(\Omega)$ with compact support.

The aim of this paper is to study the regularity of the local minimizers of $I(\Omega, u)$ which generalizes functionals of the type

$$\mathcal{F}(\Omega, u) = \int_\Omega \sum_{i,j=1}^k a_{i,j} D_i u D_j u \, dx.$$

More precisely we shall prove the following higher integrability result:

Theorem 1.1. *Let $u \in H_{X,L}(\Omega)$ be a local minimizer for the functional (1.1) with the matrix of the coefficients satisfying (1.2). Suppose $K \in A_2(\Omega, \rho) \bigcap RH_\tau(\Omega, \rho)$ with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$ and such that the pair $\left(K, \frac{1}{K}\right) \in A_1(\Omega, \rho)$. Then $Xu \in L^p_{\mathrm{loc}}\left(\Omega, \frac{1}{K} dx\right)$ for some $p > 2$ and there exists a constant C such that, for each ball $B_R \subset\subset \Omega$, it holds*

$$(1.6) \qquad \left(\fint_{B_{\frac{R}{2}}} |Xu|^p \, d\nu \right)^{\frac{1}{p}} \le C \left(\fint_{B_R} |Xu|^2 \, d\nu \right)^{\frac{1}{2}}$$

where $d\nu = \frac{1}{K} dx$.

We point out that the tools used to obtain the higher integrability are the classical ones of the euclidean case. Nevertheless, here we deal with functionals whose integrands have a degeneracy in the coefficients expressed by mean of the assumption (1.2) and, at the same time, a double degeneracy into the geometry due to the presence of a differential operator different from the classical gradient. In particular, this second peculiarity causes a change of metric in $\mathbf{R}^n$ and therefore the use of results valid for arbitrary metric spaces.

The study of the regularity of local minimizers of functionals depending on vector fields of the type

$$(1.7) \qquad \int_\Omega f(Xu)\,dx,$$

as far as we know, has been not largely studied. It seems to us that the literature available on the regularity of solutions of equations is more vast (see for example [17, 3, 7, 5] and references therein).

Anyway, the regularity of local minimizers of integral functionals is interesting by itself since the passage to the Euler–Lagrange equation is not always possible. For example, in [12], assuming that the integrand grows as $|\xi|^p$ for some $p > 1$, and satisfies the following Lipschitz assumption

$$|f(\xi) - f(\eta)| \le \beta|\xi - \eta|^p(|\xi|^{p-1} + |\xi - \eta|^{p-1}),$$

it has been proven that any weak minimizer of (1.7) is actually a local minimizer. Later on, in [13], assuming f satisfying the non-standard growth condition

$$c_1(A(|\xi|) - 1) \le f(\xi) \le c_2(A(|\xi|) + 1)$$

with A suitable N-function, a regularity result in the scale of Campanato spaces for local minimizers has been obtained, as well as an higher integrability result for their horizontal gradients.

We also indicate [11] for higher integrability results for quasi-minima of functionals with standard and non-standard growth conditions like

$$c_1|\xi|^p \le f(\xi) \le c_2|\xi|^q$$

with exponents $1 < p \le q$ opportunely related.

Finally, some results of local Hölder continuity in the context of a metric measure space where the measure is doubling and the space supports a weak Poincaré inequality, also in the case the integrand satisfies the non-standard growth condition above are available (see [16, 1, 18]).

Concerning the case of variational integrals, whose integrands have a degeneracy expressed through Muckenhoupt weights, we mention the pioneering paper by Modica [19] but also the contribution given by Moscariello in [20]. Anyway, they both concern the euclidean setting and not a metric context.

Actually, it is worth pointing out that in [5], the authors give a result of local Hölder continuity for the weak solutions of the p-Laplace equation

$$\mathrm{div}(\langle AXu, Xu\rangle^{\frac{p-2}{2}} AXu) = 0$$

where the ellipticity function is of the type considered in Theorem 1.1 (see also [6] for the euclidean setting).

2. Notation and preliminary results

In order to frame the setting of our problem, we devote this section to some useful notation and preliminaries. First of all, we recall the following iteration lemma (see Lemma 1.1 in [14]):

Lemma 2.1. *Let $f(t)$ be a nonnegative bounded function defined for $0 \le T_0 \le t \le T_1$. Suppose that for $T_0 \le t < s \le T_1$,*

$$f(t) \le A(s - t)^{-\gamma} + B + \theta f(s),$$

where A, B, γ, θ are nonnegative constants and $\theta < 1$. Then there exists a constant c, depending only on γ and θ such that for every $\beta, R, T_0 \leq \beta < R \leq T_1$, one has

$$f(\beta) \leq c[A(R - \beta)^{-\gamma} + B].$$

2.1. Carnot–Carathéodory spaces. Let $X = (X_1, \cdots, X_k)$ be a family of vector fields defined in $\mathbf{R}^n$, with real , C^∞ smooth coefficients. We say that they satisfy the Hörmander's condition if there exists an integer m such that the family of commutators of $X_1, \cdots, X_k$ up to length m

$$X_1, \cdots, X_k, [X_{i_1}, X_{i_2}], \cdots, [X_{i_1}, [X_{i_1}, \cdots X_{i_m}]] \cdots], \quad \forall i_j = 1, 2, \cdots, k$$

spans the tangent space $T_x \mathbf{R}^n$ at every point $x \in \mathbf{R}^n$.

For any real valued Lipschitz continuous function u we define

$$X_j u(x) = \langle X_j(x), \nabla u(x) \rangle, \quad j = 1, 2, \cdots, k,$$

and we call the horizontal gradient of u the vector $Xu = (X_1 u, \cdots, X_k u)$ whose length is given by

$$|Xu| = \Big(\sum_{j=1}^k (X_j u)^2\Big)^{1/2}.$$

An absolutely continuous curve $\gamma \colon [a, b] \to \mathbf{R}^n$ is said to be *admissible*, if there exist functions $c_j \colon [a, b] \to \mathbf{R}$, $j = 1, \cdots, k$, such that

$$\dot{\gamma}(t) = \sum_{j=1}^k c_j(t) X_j(\gamma(t)) \quad \text{and} \quad \sum_{j=1}^k c_j(t)^2 \leq 1.$$

Observe that X_j do not need to be linearly independent and therefore functions c_j do not need to be unique. Define the distance function ρ as

$$\rho(x, y) = \inf\{T > 0 \colon \exists \gamma \colon [0, T] \to \mathbf{R}^n \text{ admissible}, \ \gamma(0) = x, \ \gamma(T) = y\}.$$

If there is not any such a curve, we set $\rho(x, y) = \infty$. The function ρ is called Carnot–Carathéodory distance and, since it is not clear whether one can connect any two points of $\mathbf{R}^n$ by an admissible curve, it's not clear whether ρ is a metric.

The assumption for which the vector fields $X_1, \cdots, X_k$ satisfy the Hörmander condition ensures that ρ is a metric and in this case $(\mathbf{R}^n, \rho)$ is said to be a Carnot–Carathéodory space.

The following theorem, due to Nagel, Stein and Wainger [21], shows that the metric ρ is locally Hölder continuous with respect to the Euclidean metric.

Theorem 2.2. *Let* $X_1, \cdots, X_k$ *be as above. Then for every bounded open set* $\Omega \subset \mathbf{R}^n$ *there are constants* c_1, c_2 *and* $\lambda \in (0, 1]$ *such that*

(2.1) $$c_1|x - y| \leq \rho(x, y) \leq c_2|x - y|^\lambda$$

for every $x, y \in \Omega$.

It follows that the space $(\mathbf{R}^n, \rho)$ is homeomorphic with the Euclidean space $\mathbf{R}^n$ and therefore bounded sets in the Euclidean metric are bounded sets in the metric ρ. The inverse is not always true but it is certainly valid if $X_1, \cdots X_k$ have globally Lipschitz coefficients (see [10]). In the sequel all the distances will be respect to the metric ρ, in particular all the balls will be balls with respect to the Carnot–Carathéodory metric. We shall consider in $(\mathbf{R}^n, \rho)$ the Lebesgue measure $|\cdot|$ and

denote by $\fint_B f(x)\,dx$ the average of the function f on the ball B, i.e.

$$\fint_B f(x)\,dx = \frac{1}{|B|}\int_B f(x)\,dx.$$

Note that the Lebesgue measure locally satisfies the following *doubling condition* (see for example [21]):

Proposition 2.3. *Let Ω be an open, bounded subset of $\mathbf{R}^n$. There exists a constant $C_d \geq 1$, called doubling constant, such that*

$$|B(x_0, 2R)| \leq C_d |B(x_0, R)|$$

provided the center $x_0 \in \Omega$ and the radius $R \leq 5\operatorname{diam}\Omega$.

Let Y be a metric space and μ a Borel measure in Y. Assume μ finite on bounded sets and satisfying the doubling condition on every open, bounded subset Ω of Y. If there exists a positive constant C such that

$$(2.2) \qquad \frac{\mu(B)}{\mu(B_0)} \geq C\left(\frac{R}{R_0}\right)^Q$$

for any ball B_0 having center in Ω and radius $R_0 < \operatorname{diam}\Omega$ and any ball B centered in $x_0 \in B_0$ and having radius $R \leq R_0$, we say that Q is a *homogeneous dimension* relative to Ω.

It is well known that the doubling condition implies the existence of such a Q. However, Q is not unique and it may change with Ω. Obviously any $Q' \geq Q$ is also a homogeneous dimension. For a bounded open set Ω containing a family of vector fields satisfying the Hörmander condition, the Carnot–Carathéodory space (Ω, ρ) with the Lebesgue measure has the homogeneous dimension $Q = \log_2 C_d$.

We shall need the following version of Gehring Lemma for metric spaces with a doubling measure μ.

Lemma 2.4. *[11, 24] Let $q_0 \leq q \leq 2Q$ where $q_0 > 1$ is fixed. Assume $f \in L^{r_0}_{\mathrm{loc}}(Y, \mu)$ and $g \in L^q_{\mathrm{loc}}(Y, \mu)$, $r_0 > q$, be nonnegative functions and suppose the existence of constants $b > 1$ and $0 < \vartheta < 1$ such that for every ball $B \subset \sigma B \subset Y$, the following inequality holds*

$$(2.3) \qquad \fint_B g^q\,d\mu \leq b\left[\left(\fint_{\sigma B} g\,d\mu\right)^q + \fint_{\sigma B} f^q\,d\mu\right] + \vartheta \fint_{\sigma B} g^q\,d\mu.$$

Then there exist nonnegative constants $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$ and $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ such that, if $0 \leq \vartheta < \theta_0$, then $g \in L^p_{\mathrm{loc}}(Y, \mu)$ for $q \leq p < q + \varepsilon_0$ and moreover

$$(2.4) \qquad \left(\fint_B g^p\,d\mu\right)^{1/p} \leq C\left[\left(\fint_{\sigma B} g^q\,d\mu\right)^{1/q} + \left(\fint_{\sigma B} f^q\,d\mu\right)^{1/p}\right]$$

for $C = C(b, q_0, Q, C_d, \sigma)$.

Note that we used the notation σB for the ball concentric with B having radius σ-times that of B.

2.2. Muckenhoupt weights. In the following we recall some useful definitions and properties; we refer to [23] or [22] for more details.

Let $\Omega \subset \mathbf{R}^n$ be an open, bounded and connected set. By a weight we mean a positive function in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. We say that a weight w is doubling in Ω if

$$\int_{2B} w(x)\,dx \leq C\int_B w(x)\,dx$$

where the constant C is independent by the ball $B \subset \Omega$. We say that w is locally doubling in Ω if for each compact set $V \subset \Omega$ and $R > 0$ there exists $C_{V,R}$ such that

$$\int_{2B} w(x)\,dx \leq C_{V,R} \int_B w(x)\,dx,$$

where the ball B has center in V and radius $r < R$. Given $1 < p < \infty$ and a weight w, we say $w \in A_p(\Omega, \rho)$ if

$$(2.5) \qquad [w]_{A_p} := \sup_B \left(\fint_B w(x)\,dx \right) \left(\fint_B w(x)^{1-p'}\,dx \right)^{p-1} < \infty$$

where the supremum is taken over all balls $B \subset \Omega$ and p' denotes the Hölder conjugate exponent of p. When $p = 1$, we say that $w \in A_1(\Omega, \rho)$ if there exists a constant $c \geq 1$ such that, for every ball $B \subset \Omega$, it holds

$$\fint_B w(x)\,dx \leq c \operatorname{ess\,inf}_B w.$$

Weights belonging to a class A_p are called Muckenhoupt weights. Moreover, we say that w satisfies a *reverse Hölder inequality* in Ω with exponent $t > 1$, and we write $w \in RH_t(\Omega, \rho)$, if

$$(2.6) \qquad [w]_{RH_t} := \sup_B \frac{\left(\fint_B w(x)^t\,dx \right)^{\frac{1}{t}}}{\fint_B w(x)\,dx} < \infty.$$

Next two Lemmas contain useful properties for the weights defined above.

Lemma 2.5. *If $w \in A_p(\Omega, \rho)$ for some $p > 1$, then*

1) $w^{1-p'} \in A_{p'}(\Omega, \rho)$;
2) $\exists\, t > 1$ *such that* $w \in RH_t(\Omega, \rho)$;
3) $\exists\, \varepsilon > 0$ *such that* $w \in A_{p-\varepsilon}(\Omega, \rho)$.

As in the Euclidean case, assertion 1) is a direct consequence of the definition of A_p weight; assertion 2) is proved in [22] and assertion 3) easily follows from 1) and 2).

Lemma 2.6. 1) *If $w \in RH_t(\Omega, \rho)$ for some $t > 1$, then for any $x_0 \in \Omega$, $r > 0$ and any measurable set $E \subset B(x_0, r) \cap \Omega$, we have*

$$(2.7) \qquad \frac{\int_E w(x)\,dx}{\int_{B(x_0,2r)\cap\Omega} w(x)\,dx} \leq [w]_{RH_t(\Omega,\rho)} \left(\frac{|E|}{|B(x_0,r)\cap\Omega|} \right)^{\frac{1}{t'}}.$$

2) *If $w \in A_p(\Omega, \rho)$ for some $p > 1$, then for any $x_0 \in \Omega$, $r > 0$ and any measurable set $E \subset B(x_0, r) \cap \Omega$, it holds*

$$(2.8) \qquad \frac{|E|}{|B(x_0,r)\cap\Omega|} \leq \left([w]_{RH_t(\Omega,\rho)} \frac{\int_E w(x)\,dx}{\int_{B(x_0,2r)\cap\Omega} w(x)\,dx} \right)^{\frac{1}{p}}$$

and

$$(2.9) \qquad \int_{B(x_0,2r)\cap\Omega} w(x)\,dx \leq C_d^p[w]_{A_p} \int_{B(x_0,r)\cap\Omega} w(x)\,dx,$$

where C_d *is a doubling constant for (Ω, ρ), that is w is doubling.*

The proof of previous Lemma can be found in [9] in the euclidean setting, we suggest [5] for some comments useful in our context.

Finally, we recall that a pair of weights $(w, v) \in A_1(\Omega, \rho)$ provided

$$(2.10) \qquad \fint_B w(x)\, dx \le c \operatorname*{ess\,inf}_B v \quad \text{for all } B \subset \Omega.$$

In the sequel, we will be interested in the following class of weights, already introduced in [8].

Definition 2.7. Given a pair of weights (w, v), $w \le v$ and an exponent $1 < p < \infty$, we say that the pair is *p-admissible* in Ω if

1) v is locally doubling in (Ω, ρ) and $w \in A_p(\Omega, \rho)$;
2) for a given compact set $V \subset \Omega$ there exist $q > p$ and $C \ge 1$ such that , for every ball B centered in V and $0 < r < 1$, it holds

$$r \left(\frac{\int_{rB} v(x)\, dx}{\int_B v(x)\, dx} \right)^{\frac{1}{q}} \le C \left(\frac{\int_{rB} w(x)\, dx}{\int_B w(x)\, dx} \right)^{\frac{1}{p}}.$$

This class of weights are fundamental for our aims since their properties imply the following two-weights Poincaré inequality (see [4] for $X = \nabla$ and [8] for the general case). We point out that the definition of p-admissible weight was given in [15].

Theorem 2.8. *Let $1 < p < \infty$ and (w, v) be a pair of p-admissible weights in Ω and consider a compact set $V \subset \Omega$. Then there exists $R_0 > 0$ (depending on V and X) such that, if B_R is a ball centered in V and having radius $R < R_0$, then for all $u \in H_{X,L}(B_R)$ it holds*

$$(2.11) \quad \left(\frac{1}{\int_{B_R} v(x)\, dx} \int_{B_R} |u(x) - u_{R,v}|^q v\, dx \right)^{\frac{1}{q}} \le C \cdot R \left(\frac{1}{\int_{B_R} w(x)\, dx} \int_{B_R} |Xu(x)|^p w\, dx \right)^{\frac{1}{p}}$$

where C depends on V, Ω, X, $u_{R,v} = \frac{1}{v(B_R)} \int_{B_R} u(x) v(x)\, dx$ and the constants in the Definition 2.7.

According to Definition 2.7, we give the following

Definition 2.9. We say that the measurable function $K \in L^1_{\text{loc}}(\Omega)$ is *p_0-admissible* if there exists $1 < p_0 < 2$ and $C \ge 1$ such that for every ball B and $0 < r < 1$,

$$(2.12) \qquad r \left(\frac{\int_{rB} K(x)\, dx}{\int_B K(x)\, dx} \right)^{\frac{1}{2}} \le C \left(\frac{\int_{rB} K(x)^{-1}\, dx}{\int_B K(x)^{-1}\, dx} \right)^{\frac{1}{p_0}}.$$

Sometimes (2.12) will be referred to as the Chanillo–Wheeden condition with exponent $1 < p_0 < 2$ and constant $C \ge 1$.

The following sufficient condition for locally integrable functions to be p_0-admissible is analogous to Theorem 4.8 in [5] (see also Proposition 4 in [6]). For the reader's convenience we give the proof below.

Proposition 2.10. *Let $2 > Q - n$. The function K is p_0-admissible provided $K^{-1} \in A_p(\Omega, \rho)$ and $K \in RH_t(\Omega, \rho)$ with $1 < t < \infty$ and $\frac{2}{Q} < p < \infty$ satisfying*

$$(2.13) \qquad \frac{2t'}{n} \le \left(\frac{pQ}{p_0} \right)' - 1.$$

In particular, this is the case if $K \in A_2(\Omega, \rho) \cap RH_\tau(\Omega, \rho)$, $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$

Proof. Condition (2.13) is equivalent to

$$\frac{n}{2t'} \geq \frac{pQ}{p_0} - 1$$

and the assumption $\frac{1}{K} \in A_p$ implies, by virtue of Lemma 2.5, the existence of $\varepsilon > 0$ such that $\frac{1}{K} \in A_{p-\varepsilon}$. Obviously we get

$$(2.14) \qquad \frac{n}{2t'} \geq \frac{pQ}{p_0} - 1 > \frac{(p-\varepsilon)Q}{p_0} - 1$$

which implies

$$\frac{n(t')^{-1}}{2} > (p-\varepsilon)Qp_0^{-1} - 1.$$

Setting

$$q = \frac{n(t')^{-1}}{(p-\varepsilon)Qp_0^{-1} - 1},$$

we deduce that $q > 2$. Then, from the first assertion of Lemma 2.6 , we have for $0 < r < 1$ that

$$(2.15) \qquad r \left(\frac{\int_{rB} K(x)\,dx}{\int_B K(x)\,dx} \right)^{\frac{1}{q}} \leq c\,r \left(\frac{|rB|}{|B|} \right)^{\frac{1}{qt'}}$$

and therefore that

$$(2.16) \qquad r \left(\frac{\int_{rB} K(x)\,dx}{\int_B K(x)\,dx} \right)^{\frac{1}{q}} \leq c\,r^{\frac{n}{qt'}+1} = C\,r^{\frac{(p-\varepsilon)Q}{p_0}}.$$

Now, taking into account that the Lebesgue measure satisfies (2.2) and applying the second assertion of Lemma 2.6 , we have

$$r \left(\frac{\int_{rB} K(x)\,dx}{\int_B K(x)\,dx} \right)^{\frac{1}{q}} \leq C \left(\frac{\int_{rB} K(x)^{-1}\,dx}{\int_B K(x)^{-1}\,dx} \right)^{\frac{p-\varepsilon}{p_0 p}} \leq C \left(\frac{\int_{rB} K(x)^{-1}\,dx}{\int_B K(x)^{-1}\,dx} \right)^{\frac{1}{p_0}}$$

since ε is arbitrarily small. Since it is true for every ball B and $q > 2$, we deduce that K is p_0-admissible. $\qquad\square$

3. A two-weights Caccioppoli inequality and the main result

For K satisfying (1.3), it is legitimate to consider the measures

$$d\nu = \frac{1}{K(x)}dx, \quad d\mu = K(x)dx$$

and the Banach spaces $L^p(\Omega, d\nu)$, $L^p(\Omega, d\mu)$ of all p-integrable functions with respect to the measures $d\nu$ and $d\mu$ equipped with the norms

$$||u||_{p,\Omega,d\nu} = \left(\int_\Omega |u|^p d\nu \right)^{\frac{1}{p}}, \quad ||u||_{p,\Omega,d\mu} = \left(\int_\Omega |u|^p d\mu \right)^{\frac{1}{p}}.$$

In the next theorem we prove a two-weights Caccioppoli inequality. We point out that we don't need to assume any condition on the weight K.

Theorem 3.1. *If* $u \in H_X(\Omega)$ *is a local minimizer for* $I(\Omega, u)$ *and* (1.2) *holds with* K *satisfying* (1.3) *then, for any ball* $B_R = B(x_0, R) \subset \Omega$ *we have*

$$(3.1) \qquad \int_{B_{\frac{R}{2}}} |Xu|^2\,d\nu \leq C \int_{B_R} \frac{|u - u_R|^2}{R^2}d\mu.$$

Proof. Let t, s such that $\frac{R}{2} < t < s < R$ and define $\eta(x) = \psi(|x - x_0|)$, where $\psi \in C_0^\infty(B_s)$ is a cut-off function such that $\psi \equiv 1$ on B_t, $|X\psi| \leq \frac{c}{s-t}$. The proof of the existence of a such function can be found, for example, in Lemma 3.6 of [10].

Observe that $\eta \in W_{X,0}^{1,\infty}(B_s)$, the closure of C_0^∞ in $W_X^{1,\infty}$, $\eta(x) = 1$ on B_t and

$$
(3.2) \qquad |X\eta(x)| = \begin{cases} (s-t)^{-1}, & t \leq |x - x_0| \leq s, \\ 0, & \text{otherwise.} \end{cases}
$$

Consider the function

$$
v(x) = u(x) + \eta^2(x)(u_{R,\mu} - u(x)).
$$

For $x \in B_s$, observe that we have

$$
Xv(x) = (1 - \eta^2(x))Xu(x) + 2\eta^2(x)\frac{X\eta(x)}{\eta(x)}(u_{R,\mu} - u(x)).
$$

Since u is a minimizer and (1.2) holds, we have

$$
\int_{B_s} \langle AXu, Xu \rangle \, dx \leq \int_{B_s} \langle AXv, Xv \rangle \, dx
$$
$$
= \int_{B_s} (1 - \eta^2)\langle AXu, Xu \rangle \, dx + \int_{B_s} 2\eta^2 \left\langle A\left(\frac{X\eta}{\eta}(u_{R,\mu} - u)\right), \frac{X\eta}{\eta}(u_{R,\mu} - u)\right\rangle dx
$$
$$
\leq \int_{B_s \setminus B_t} (1 - \eta^2)\langle AXu, Xu \rangle \, dx + c \int_{B_s} \eta^2 \frac{X^2\eta}{\eta^2}(u - u_{R,\mu})^2 K(x) \, dx.
$$

Then, by (3.2), we deduce that

$$
2\int_{B_t} \langle AXu, Xu \rangle \, dx \leq \int_{B_s} \langle AXu, Xu \rangle \, dx + c\frac{1}{(s-t)^2}\int_{B_s}(u - u_{R,\mu})^2 K(x) \, dx.
$$

The assertion follows dividing for two both sides of last inequality and applying Lemma 2.1 and assumption (1.2). $\qquad\square$

Now we are in position to prove our main result on the regularity of minimizers.

Proof of Theorem 1.1. Let $B_R = B(x_0, R) \subset \Omega$. From Theorem 3.1 we immediately have that

$$
(3.3) \qquad \int_{B_{\frac{R}{2}}} |Xu|^2 \, d\nu \leq C \int_{B_R} \frac{|u - u_{R,\mu}|^2}{R^2} \, d\mu.
$$

Observe now that assuming $K \in A_2$, by the second assertion of Lemma 2.6, we have that K is doubling.

On the other hand, the assumption $K \in A_2 \cap RH_\tau$, $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$ implies, thanks to Proposition 2.10, that K is p_0-admissible, where p_0 is related to τ by mean of (2.13), that is

$$
\frac{2\tau'}{n} \leq \left(\frac{2Q}{p_0}\right)' - 1.
$$

Therefore we have

$$
(3.4) \qquad r\left(\frac{\int_{rB} K(x) \, dx}{\int_B K(x) \, dx}\right)^{\frac{1}{2}} \leq C\left(\frac{\int_{rB} K(x)^{-1} \, dx}{\int_B K(x)^{-1} \, dx}\right)^{\frac{1}{p_0}}.
$$

It follows that the pair (K^{-1}, K) verifies the two conditions of Definition 2.7 and hence is a pair of weights p_0-admissible.

At this point we are legitimate to apply Theorem 2.8 having

$$\left(\frac{1}{\int_{B_R} K(x)\,dx}\int_{B_R}|u(x)-u_{R,\mu}|^2 d\mu\right)^{\frac{1}{2}} \leq C\cdot R\left(\frac{1}{\int_{B_R} K^{-1}(x)\,dx}\int_{B_R}|Xu(x)|^{p_0}d\nu\right)^{\frac{1}{p_0}}.$$

Combining estimates (3.3) with the last inequality and recalling that the weight K is doubling, we get

$$\frac{1}{\int_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}\int_{B_{\frac{R}{2}}}|Xu|^2\,d\nu \leq \frac{C}{\int_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}\int_{B_R}\frac{|u-u_{R,\mu}|^2}{R^2}\,d\mu$$

$$= C\frac{\int_{B_R}K(x)\,dx}{\int_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}\left(\frac{1}{\int_{B_R}K(x)\,dx}\int_{B_R}\frac{|u-u_{R,\mu}|^2}{R^2}d\mu\right)$$

(3.5)
$$\leq C\frac{\int_{B_R}K(x)\,dx}{\int_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}|Xu|^{p_0}\,d\nu\right)^{\frac{2}{p_0}}$$

$$\leq C_1\frac{\fint_{B_{\frac{R}{2}}}K(x)\,dx}{\fint_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}|Xu|^{p_0}\,d\nu\right)^{\frac{2}{p_0}}$$

$$= C_2\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}|Xu|^{p_0}\,d\nu\right)^{\frac{2}{p_0}}$$

where

(3.6)
$$C_2 = C_1\frac{\fint_{B_{\frac{R}{2}}}K(x)\,dx}{\fint_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx}.$$

Now observe that

(3.7)
$$\fint_{B_{\frac{R}{2}}}K(x)\,dx \leq c\,\mathrm{ess}\inf_{B_{\frac{R}{2}}}\frac{1}{K}\quad\text{for all }B_{\frac{R}{2}}\subset\Omega$$

thanks to the assumption that the pair $\left(K,\frac{1}{K}\right)\in A_1$. On the other hand, we obviously have

$$\mathrm{ess}\inf_{B_{\frac{R}{2}}}\frac{1}{K}\leq\frac{1}{K}\quad\text{a.e. in }B_{\frac{R}{2}}$$

and therefore

$$\fint_{B_{\frac{R}{2}}}K(x)\,dx \leq c\fint_{B_{\frac{R}{2}}}\frac{1}{K(x)}\,dx\quad\text{for all }B_{\frac{R}{2}}\subset\Omega.$$

It follows that the constant C_2 can be estimated by a constant independent of $B_{\frac{R}{2}}$, μ, ν. Now, from (3.5), applying Lemma 2.4 with $g=|Xu|^{p_0}$, $q=\frac{2}{p_0}$, $f=0$ and $\vartheta=0$ we easily deduce the result. $\qquad\square$

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