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ABSOLUTE LOGICS

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Introduction

An abstract logic, in the sense we use the term here, is a formal language which describes mathematical structures. Its syntax is formulated by defining which objects are sentences; a semantics for the sentences is introduced by defining when a sentence is true in a structure. Often the semantics overrides the syntax: one is more concerned about the expressive power of a logic than about what objects the sentences really are. However, when necessary, a metatheory is introduced to make things clear in this respect, too. The mostly used metatheory is set theory: sentences and structures are sets, and abstract logics are defined by set-theoretical predicates.

In the seventies mathematicians realized that, if set theory is used as a metatheory, set-theoretical methods can be used for deriving model-theoretical results. This approach, initiated by Barwise in [B2], was an object of lively research for several years. However, the research done in this area is far from exhaustive, although practically nothing has been published recently.

The main tools of a set theorist are internal models: various sets or classes which themselves are models of a set theory. Usually it is necessary that certain set theoretical predicates are absolute relative to the constructed models, i.e. have the same meaning in them. To relieve the burden, we have classified predicates by their degree of absoluteness. The basic idea is to restrict the number of internal models taken in account when inspecting whether a predicate is absolute relative to them. Thus you can say, broadly, that a predicate is absolute relative to a theory if it has the same meaning in all models of the theory. Now, if one keeps to standard set theories (i.e. to first-order set theories in a vocabulary containing a single predicate symbol $\in$ only), there are predicates which are not absolute relative to a theory, however strong the theory is. For example, the predicate "x is countable" is such a predicate. The way around this is either to apply other principles to restrict the number of internal models, such as absoluteness relative to ω_1-closed forcing, or to strengthen the language of set theory with new predicates.

A logic is said to be absolute if its syntax is persistent (every object which is a sentence in a set-theoretical universe is a sentence in extensions of the universe), and if its semantics is absolute (the truth of a sentence in a structure is independent of the set-theoretical universe). Absoluteness certainly restricts the expressive power of a logic: one cannot express anything that is not absolute. For example, no logic absolute relative to the set theory of Kripke and Platek can express well-foundedness. In a sense, absoluteness is an abstract notion of "first-order". The semantics of a sentence in a first-order logic depends on the elements a structure has, but the semantics of a sentence of higher order logics depends on the subsets of a structure, as well. Moreover, the borderline between absolute and non-absolute predicates is largely based on the fact that the elements of a set are independent on the set-theoretical universe, but new subsets may be introduced when the universe is extended.

A key fact behind this approach is that the classical way of constructing formal languages tends to produce absolute ones. One starts with atomic sentences and produces new sentences recursively by means of logical operations (conjunction, negation, quantification etc.) Now it happens that the constructed logic is absolute as soon as each logical operation is absolute. This principle will be given an exact formulation in the second part of this work.

The set-theoretical methods are most useful when applied to absolute logics, especially to those absolute relative to a standard set theory. Among the first results is a downwards Löwenheim–Skolem theorem (see [B2]): if a sentence ϕ has a model, it has a model of cardinality at most $|\mathrm{TC}(\phi)|$. This is expectable, since cardinality is not absolute relative to any standard set theory. Moreover, it turned out that $L_{\infty\omega}$ has a special position among absolute logics. Absolute logics have the Karp-property: if two structures satisfy the same sentences of $L_{\infty\omega}$, they satisfy the same sentences of every absolute logic. Additionally, Barwise showed in [B2] that any logic, absolute relative to KP+Inf, is a sublogic of $L_{\infty\omega}$. Of course there are strict extensions of $L_{\infty\omega}$, such as the game quantifier logics, which are absolute for instance relative to ZFC. Through the work of Moschovakis it became known that these logics can be approximated with $L_{\infty\omega}$: each sentence of a game quantifier logic is equivalent to a disjunction of a strict class of $L_{\infty\omega}$-sentences. Burgess in [Bu] further expanded this result to concern every logic, absolute relative to ZFC.

Apart from pure academic interest in the expressive power of absolute logics, the set-theoretical methods are used to show that certain strengthenings of logics are not absolute — and thus, in a sense, hard to define set-theoretically. A general tendency is that absolute logics have weak interpolation properties: they have a strong implicit expressive power but their explicit power is weak. For example, $L_{\infty\omega}$ is absolute relative to KP, but not even $L_{\omega_2\omega}$ has Delta-interpolation in a logic absolute relative to a standard set theory (see [B2]). Moreover, no logic, absolute relative to ZFC and strong enough to express well-foundedness, has the weak Beth interpolation property (see [Bu]). By and large, these results reflect the general difficulty in constructing logics with interpolation properties, for example Δ-closures.

There are also many other aspects of abstract model theory in which the set-theoretical methods have proved fruitful. For a survey of them, see [V2]. In this work we restrict ourselves to the basic problem of finding bounds for the expressive power of logics absolute relative to certain set theories. The main contribution of this work is a new proof method: the applications represent mainly results which have — in one form or another – already been proved by other methods; in some cases we achieve minor strengthenings.

The first part of the work, Sections 1 – 5, is introductory: we present certain concepts, either rarely used or otherwise non-stabilized ones, in order to fix our terminology and to lay a firm foundation for our work. In the second part, in Sections 6 – 8, we discuss and prove the absoluteness of several logics. Its contents fall withing the large category of results which everyone knows but the proofs of which are hard to find or never published.

The third part, Sections 9 – 12, presents the new construction method which is applied in the final part. The construction can be used for translating sentences of an absolute logic into sentences of $L_{\infty\omega}$, a game quantifier logic, or into sentences of the infinitely deep language $M_{\infty\infty}$. The main principle of the construction is as follows. Consider a sentence ϕ of an absolute logic. It is true in a structure $\mathfrak{M}$, whenever a certain set-theoretical internal model exists. We define a game — during which a player, $\exists$, tries to build up a certain kind of set-theoretical structure — in such a manner that the following statements hold:

- If $\exists$ wins the game, i.e. if she has a canonical way of building the set-theoretical structure, the internal model exists which is needed to show that ϕ is true in $\mathfrak{M}$.
- If ϕ is true in $\mathfrak{M}$, then $\exists$ wins the game.

Then we show that this game can be coded as a game sentence, a sentence of $L_{\infty\omega}$, or as a sentence of $M_{\infty\infty}$.

Applications are introduced in the final part of this work in Sections 13 – 17. We start with some classical results by showing that finite logics, such that both their syntax and semantics are absolute relative to KPU (the set theory of Kripke and Platek with urelements), are sublogics of the first-order logic $L_{\omega\omega}$ (see [V2]), and reprove Barwise's result: every logic, absolute relative to KP+Inf, is a sublogic of $L_{\infty\omega}$. In Section 15 we give a new proof for the Burgess's approximation theorem. Meanwhile we sidestep and use our method to show the Craig-interpolation theorem for the countable fragments of $L_{\infty\omega}$.

Since $L_{\infty\omega}$ has a special position among logics, absolute relative to a standard set theory, and recent works on infinitely deep languages have revealed an analogy between both $L_{\infty\omega}$ and $M_{\infty\kappa}$, and $L_{\omega_1\omega}$ and $M_{\kappa^+\kappa}$ for regular cardinals κ, we apply the new method in order to see whether $M_{\infty\kappa}$ has a similar special position. Now we see that logics, absolute relative to such models of set theory which preserve countability, are sublogics of $M^{\text{det}}_{\infty\omega_1}$, a subclass of $M_{\infty\omega_1}$. As a sidestep we give an alternative proof for the separation theorem of Tuuri in [T]: $L_{\lambda^+\kappa}$ has Craig-interpolation in $M_{\lambda^+\lambda}$ if $\lambda = \kappa^{<\kappa}$ and κ is a regular cardinal. In the final section we show that the analogy between $L_{\infty\omega}$ and $M_{\infty\omega}$ fails, if we try to construct an approximation mapping analogous to Burgess's.

I Preliminaries

Our set-theoretical notations are standard, following [J, Kn]. The words "function" and "relation" refer to sets; the words "mapping" and "predicate" are the corresponding terms for classes. Mappings are total, unless otherwise indicated.

If f is a function, $\text{dom}(f)$ and $\text{ran}(f)$ are the domain and range of f, $f\restriction x$ is the function f restricted to domain $\text{dom}(f) \cap x$, and $f''x$ is the range of $f\restriction x$. The notations $\langle\, f(x) \,:\, x \in y\,\rangle$ and $\langle f(x)\rangle_{x\in y}$ stand for a function $x \mapsto f(x)$ the domain of which is y. If α is an ordinal, we denote by $\langle x_i, y_i\rangle_{i<\alpha}$ a function f the domain of which is 2α and for which $f(2i) = x_i$ and $f(2i+1) = y_i$ for each i. The notation $\{ x_i, y_i : i \in I \}$ is to be likewise understood: it is the union of sets $\{ x_i : i \in I \}$ and $\{ y_i : i \in I \}$.

If κ is a cardinal, $\mathcal{P}_\kappa(x)$ is the set of those subsets of x which are of cardinality less than κ. If λ is another cardinal, $\kappa^{<\lambda}$ stands for both the set of functions with domain in λ and range a subset of κ, and for the cardinality of this set. For an ordinal α, α^+ is the least infinite cardinal strictly above α. For the discussion of forcing we use the notation and terminology of [Kn]. We denote by $\mathrm{rank}(x)$ the set-theoretical rank of a set x.

1. Primitive recursive mappings

Though the primitive recursive mappings are a generalization of primitive recursive functions on natural numbers, their principal relevance is not on effectiveness, but they serve as a notion of "easily constructible". This constructibility approach has been present ever since they were introduced. The canonical source on them is [JK], although some details can be found in [D].

1.1. Definition. A mapping is *primitive recursive* (p.r.) if it can be obtained from the initial mappings

(1) $(x_0, \ldots, x_n) \mapsto x_i$, where $0 \leq i \leq n$,
(2) $(x, y) \mapsto \{x, y\}$,
(3) $(x, y) \mapsto x \setminus y$, and
(4) $x \mapsto \omega$

by substitution, union, and primitive recursion as follows:

(5) if f and $g_0, \ldots, g_n$ are p.r., then $\vec{x} \mapsto f(g_0(\vec{x}), \ldots, g_n(\vec{x}))$, is also p.r.,
(6) if f is p.r., then $(\vec{x}, y) \mapsto \bigcup_{z \in y} f(\vec{x}, z)$ is also p.r., and
(7) if f is p.r., then the mapping g for which

$$g(\vec{x}, y) = f(\vec{x}, y, \langle g(\vec{x}, z) : z \in y \rangle)$$

is also p.r.

A predicate is *primitive recursive* if its characteristic mapping is primitive recursive. A set a is *primitive recursive* if the constant mapping $x \mapsto a$ is primitive recursive.

1.2. Definition. Let f be a mapping. A mapping is *primitive recursive in f* if it can be obtained from the initial mappings above and from f by substitution, union, and primitive recursion. A mapping is primitive recursive in a predicate if it is primitive recursive in the characteristic mapping of the predicate. It is similarly defined when a mapping/predicate/set is primitive recursive in a mapping/predicate/set.

Jensen and Karp in [JK] do not consider the initial mapping $x \mapsto \omega$ to be primitive recursive. However, the primitive recursive mappings in our sense are exactly the $\mathrm{Prim}(\omega)$ functions in their sense. Instead of the recursion schema 1.1(7) Devlin uses a stronger recursion schema in his work [D]. We present it in the form of the following lemma, which implies that Devlin's definition equals ours.

1.3. Lemma. *If g and h are p.r., and* $\text{rank}(y) < \text{rank}(x)$ *for every $y \in h(x)$, there exists a p.r. mapping f such that*

$$f(x) = g(x, \langle f(y) : y \in h(x) \rangle).$$

Proof. Let g and h be primitive recursive and assume that $\text{rank}(y) < \text{rank}(x)$ whenever $y \in h(x)$. Define a transitive closure along h by induction: let

$$t(x, 0) = \{x\} \quad \text{and} \quad t(x, n+1) = t(x, n) \cup \bigcup_{y \in t(x,n)} h(y)$$

for each x and $n \in \omega$ and let $\mathrm{TC}_h(x) = \bigcup_{n \in \omega} t(x, n)$. Now TC_h is primitive recursive, $x \in \mathrm{TC}_h(x)$, and $\mathrm{TC}_h(x)$ is *h-transitive*: $h(y) \subseteq \mathrm{TC}_h(x)$ whenever $y \in \mathrm{TC}_h(x)$.

If X is a set, denote by X_α the set of the elements of X having rank less than α. We define a mapping F by induction on ordinals in such a way that whenever X is h-transitive, $F(\alpha, X)$ will be the mapping f restricted to X_α: let

$$\begin{aligned} F(0, X) &= \langle\rangle, \\ F(\alpha+1, X) &= F(\alpha, X) \cup \langle g(x, F(\alpha, X) \restriction h(x)) : x \in X, \text{rank}(x) = \alpha \rangle, \text{ and} \\ F(\gamma, X) &= \bigcup_{\alpha < \gamma} F(\alpha, X), \text{ if } \gamma \text{ is a limit.} \end{aligned}$$

Since rank is primitive recursive, the mapping F is primitive recursive. An easy induction shows that $F(\alpha, X)$ is a function the domain of which is X_α, and, moreover, $F(\alpha, X) \subseteq F(\alpha, Y)$ whenever $X \subseteq Y$ and the set X is h-transitive. Let $f(x) = F(\text{rank}(x) + 1, \mathrm{TC}_h(x))(x)$ for every x. Clearly

$$f(x) = g(x, F(\text{rank}(x), \mathrm{TC}_h(x)) \restriction h(x)),$$

and if $y \in h(x)$, then $\text{rank}(y) < \text{rank}(x)$ and $\mathrm{TC}_h(y) \subseteq \mathrm{TC}_h(x)$, so thus

$$F(\text{rank}(x), \mathrm{TC}_h(x))(y) = F(\text{rank}(y) + 1, \mathrm{TC}_h(x))(y) = f(y).$$

This implies that f is the required mapping. □

In the literature there exist very detailed descriptions and proofs of the kinds of mappings and predicates that are primitive recursive. The proofs of the following facts can be found in [D, JK].

- The p.r. mappings are closed under definition by cases; e.g. if R is a p.r. predicate and f, g are p.r. mappings, the mapping

  $$x \mapsto \begin{cases} f(x) & \text{if } R(x), \text{ and} \\ g(x) & \text{otherwise} \end{cases}$$

  is p.r.

- The p.r. predicates are closed under negation, conjunction, disjunction, and bounded quantification.
- Set-theoretical separation and collection are p.r. operations. In other words, if R is a p.r. predicate and if f is a p.r. mapping, the mapping

$$x \mapsto \{ f(x, y) : y \in x \wedge R(x, y) \}$$

is p.r.
- The predicates "$x \subseteq y$", "x is transitive", "x is ordinal", "x is an ordered pair", "x is a relation", "x is a function", and "x is finite" are p.r.
- The mappings $x \mapsto \cup x$, $(x, y) \mapsto x \cup y$, $(x, y) \mapsto x \cap y$, $(x, y) \mapsto (x, y)$ (ordered pair), $x \mapsto \mathrm{dom}(x)$, $x \mapsto \mathrm{ran}(x)$, $(x, y) \mapsto x \times y$, $(f, x) \mapsto f(x)$, $(f, x) \mapsto f''x$, $(f, x) \mapsto f{\restriction}x$, $x \mapsto \mathrm{TC}(x)$, and $x \mapsto \mathrm{rank}(x)$ are p.r.
- Ordinal arithmetics (sum, product, etc.) is p.r.

The following two properties will become important later.

1.4. Lemma. *(i) If f is p.r., so is the mapping $x \mapsto C_f(x)$, where $C_f(x)$ is the smallest superset of x closed under f.*

(ii) If f is a p.r. mapping, there exists a p.r. mapping b_f, increasing on ordinals, such that for all sets $x_1, \ldots, x_n$

$$\mathrm{rank}(f(x_1, \ldots, x_n)) \leq b_f(\mathrm{rank}(x_1), \ldots, \mathrm{rank}(x_n)).$$

Proof. For the claim (i) define $g(0, x) = x$ and

$$g(m + 1, x) = g(m, x) \cup \{ f(y_1, \ldots, y_n) : y_1, \ldots, y_n \in g(m, x) \}.$$

Now $C_f(x) = \bigcup_{n \in \omega} g(n, x)$. The claim (ii) is shown by induction on the definition of the primitive recursive mappings: it is trivial to find the mappings b_f for the initial mappings and the substitution rule. The union rule is set by equation

$$\mathrm{rank}(\bigcup_{z \in y} f(\vec{x}, z)) = \bigcup_{z \in y} \mathrm{rank}(f(\vec{x}, z)).$$

Finally, the rank of $\langle g(\vec{x}, z) \rangle_{z \in y}$ is the maximum of $\mathrm{rank}(y)+2$ and $\sup\{ \mathrm{rank}(g(\vec{x}, z))+3 : z \in y \}$; thus, if g is defined from f by the primitive recursion scheme 1.1(7), it is enough that b_g satisfies

$$b_g(\vec{\alpha}, \beta) = b_f(\vec{\alpha}, \beta, (\beta + 2) \cup \bigcup_{\gamma \in \beta} (b_g(\vec{\alpha}, \gamma) + 3)).$$

□

2. Definability in set theory

In this work we make use of various set theories, both conventional theories with only a single predicate $\in$ in their vocabulary and expanded theories with auxiliary predicates and constants. The Δ_0-, Σ_1-, and Π_1-formulas are defined as usual, even in expanded vocabularies.

2.1. Definition. Let T be a set theory. A predicate A is Δ_0-*definable* in T (Δ_0^T) if there is a Δ_0-formula B such that $T \vdash A \leftrightarrow B$. The Σ_1^T and Π_1^T predicates are defined likewise. A predicate A is Δ_1-definable in T (Δ_1^T) if it is both Σ_1^T and Π_1^T.

A (partial) mapping F is Δ_0-*definable* in T if there is a Δ_0-formula B such that

$$T \vdash B(\vec{x}, y) \leftrightarrow y = F(\vec{x}) \quad \text{and} \quad T \vdash \forall \vec{x}\, \exists^{\leq 1} y\, B(\vec{x}, y).$$

If, moreover, $T \vdash \forall \vec{x}\, \exists! y\, B(\vec{x}, y)$, we say that the mapping F is *totally* Δ_0-*definable* in T. The definition of a mapping being (totally) Σ_1^T, Π_1^T, or Δ_1^T is achieved similarly. Note that a mapping is totally Σ_1^T exactly when it is totally Δ_1^T.

The weakest conventional set theory we are using is the set theory of Kripke and Platek; it is denoted by KP and has the axioms of extensionality, foundation, pair, union, Δ_0-separation, and Δ_0-collection.

It has a frequently used expanded variant: KPU, the set theory of Kripke and Platek with urelements. In addition to the relation symbol $\in$, its vocabulary has an auxiliary unary predicate symbol U to distinguish urelements from sets. The axioms of KP must be slightly modified to cover the urelements, for instance the axiom of extensionality is

$$\forall x\, \forall y((\neg U(x) \wedge \neg U(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \rightarrow x = y),$$

and the axioms of separation and collection must indicate that the new set is not an urelement. (This implies that an empty set is not an urelement.) Apart from the axioms of KP, KPU contains a new axiom which states "no urelement contains elements".

The theories KPU and KP are nicely treated in [B4]. They are nearly the same theory: KP is equivalent to the theory KPU $\cup \{U = \emptyset\}$. To give reader an idea of the strength of KPU we list some of its elementary properties. The same properties hold for KP, too.

- KPU implies Σ_1-collection and Δ_1-separation.
- The predicates "$x \subseteq y$", "$x = \{y, z\}$", "$x = (y, z)$", "x is an ordered pair", "$x = y \times z$", "x is a relation", "x is a function", and "x is one-to-one function" are Δ_0^{KPU}.
- The predicates "x is transitive", "x is an ordinal", "x is a limit ordinal", "$x \in \omega$", and "$x = \omega$" are Δ_1^{KPU}.
- The mappings dom, ran, $(f, x) \mapsto f{\restriction}x$, and $(f, x) \mapsto f''x$ are totally Σ_1^{KPU}.
- Let R be a predicate which is Δ_1-definable in KPU through a formula ϕ_R. If a predicate S is Δ_1-definable in KPU $\cup \{R \leftrightarrow \phi_R\}$, where R is a new relation symbol, then S is Δ_1-definable in KPU.

- Let a mapping F be totally Σ_1-definable in KPU through a formula ϕ_F. If a predicate (or a mapping) is Σ_1-definable in KPU$\cup\{F(\vec{x}) = y \leftrightarrow \phi_F(\vec{x}, y)\}$, it is Σ_1-definable in KPU.
- KPU allows definition by recursion: if G is a Σ_1^{KPU} mapping, there is a Σ_1^{KPU} mapping F such that

$$\mathrm{KPU} \vdash F(\vec{x}, y) = G(\vec{x}, y, \langle\, F(\vec{x}, z) \;:\; z \in TC(y) \,\rangle).$$

If $\mathfrak{A} = (A, E, \ldots)$ and $\mathfrak{A}' = (A', E', \ldots)$ are structures of a vocabulary of an (expanded) set theory such that $\mathfrak{A}$ is a substructure of $\mathfrak{A}'$ and for every $a \in A$

$$\{\, b \in A \;:\; E(b, a) \,\} = \{\, b \in A' \;:\; E'(b, a) \,\},$$

we denote $\mathfrak{A} \subseteq_{\text{end}} \mathfrak{A}'$ and say that the structure $\mathfrak{A}'$ is an *end extension* of $\mathfrak{A}$ (or that the structure $\mathfrak{A}$ is an *initial substructure* of $\mathfrak{A}'$).

If $\mathfrak{A} = (A, E, \ldots)$ is a structure in a vocabulary of an (expanded) set theory, its *well-founded part* $\mathrm{Wf}(\mathfrak{A})$ is the largest well-founded initial substructure.

2.2. Proposition (Truncation lemma [B4]). *If* $\mathfrak{A}$ *is a model of* KPU, *then* $\mathrm{Wf}(\mathfrak{A})$ *is a model of* KPU. *The same holds for* KP. □

From this the reader should not hasten to conclude that KPU and KP are "equal" theories in sense that wherever KP occurs it can be replaced with KPU by only remarking that urelements exist. There are some pitfalls, one of them being the notion of infinity: the predicate "x is finite" is Δ_1-definable in KP but only Σ_1-definable in KPU. As to the first claim, finiteness is Δ_1-definable in KP + Inf (KP with the axiom of infinity), since the mapping $x \mapsto \mathcal{P}_\omega(x)$ is totally Σ_1-definable in KP + Inf. (The same argument shows that finiteness is Δ_1-definable in KPU + Inf, too.) Since KP + $\neg$Inf implies that all sets are finite, we can conclude that "x is finite" is Δ_1^{KP}. On the other hand, finiteness is trivially Σ_1-definable in KPU. In order to see that it is not Π_1-definable, note that by first-order compactness the theory containing KPU and the axioms

- a is a set of urelements,
- $a_i \in a$ for $i \in \omega$,
- c is a finite ordinal (i.e. a successor ordinal having no limit ordinals below itself), and
- f is a one-to-one function from a onto c

has a model $\mathfrak{A}$ in which the set a has a nonstandard finite cardinality c. By the truncation lemma $\mathrm{Wf}(\mathfrak{A})$ is a model of KPU, obviously containing a, but naturally a cannot be finite in it.

Among stronger conventional set theories we mention ZFC. Although the theory becomes much stronger, the number of Δ_1- and Σ_1-definable predicates does not increase dramatically. Some examples of Δ_1^{ZFC} predicates which are not Δ_1^{KP} will be presented when discussing logics.

To close this section, we present a proposition binding together the notion of primitive recursiveness and definability. We say that a set A is *admissible* if it is transitive and $(A, \in)$ is a model of KP. Thus an admissible set containing ω is a model of KP + Inf.

2.3. Proposition ([JK]). *(i) Primitive recursive mappings are totally Σ_1-definable in* KP + Inf.

(ii) A transitive set A is admissible and $\omega \in A$ if and only if A is closed under primitive recursive mappings and $(A, \in)$ satisfies Σ_1-collection scheme. □

3. Expanded set theories

The Σ_1- and Δ_1-definable predicates are our main interest: we want to strengthen the set theory sufficiently to obtain as many Δ_1-definable predicates as possible. As it is known, beyond ZFC there is a multitude of first-order extensions by strong axioms, e.g. the existence of diamonds, boxes, and measurable cardinals. Certain axioms, such as "all sets are constructible", produce new Σ_1-definable predicates (e.g. the well-ordering of the universe). However, certain predicates, such as "x is countable", will never become Δ_1-definable in this way. The way to proceed beyond these limits is to expand the vocabulary of the set theory with new predicates.

To have an example, we add a new constant symbol κ and a binary predicate P. Consider the theory

$$\mathrm{ZFC}(\mathcal{P}_\kappa) = \mathrm{ZFC} \cup \{\kappa \text{ is a cardinal }\} \cup \{P(x,y) \leftrightarrow y = \mathcal{P}_\kappa(x)\}.$$

Now "$|x| < \kappa$" and "$|x| \leq \kappa$" are even Δ_0-definable in ZFC$(\mathcal{P}_\kappa)$. We say that a predicate R is $\Sigma_1(\mathcal{P}_\kappa)$-definable if it is Σ_1-definable in ZFC$(\mathcal{P}_\kappa)$. We similarly define $\Delta_1(\mathcal{P}_\kappa)$-definability. Note that the $\Sigma_1(\mathcal{P}_\kappa)$- and $\Delta_1(\mathcal{P}_\kappa)$-definable predicates are closed under the restricted quantifications "$\forall x \in \mathcal{P}_\kappa(y) \ldots$" and "$\exists x \in \mathcal{P}_\kappa(y) \ldots$".

The problem with this approach is that very little is known about these kinds of expansions. Fortunately something can be achieved using the same means as in conventional set theory. To illustrate this, we next show a reflection property for $\Sigma_1(\mathcal{P}_\kappa)$-formulas analogous to the known Levy reflection property: if λ is an uncountable regular cardinal, $\phi(x)$ is a Σ_1-formula, and $a \in H_\lambda$ is such that $\phi(a)$ holds, then $\phi(a)$ holds in H_κ. See for example [J] for the proof.

Let P be a predicate, A an extensional set, and let c be the Mostowski collapsing function of A. If for every $a \in A$

$$P(a) \iff P(a)^A,$$

we say that the set A *reflects* the predicate P. Moreover, if for every $a \in A$

$$P(a) \iff P(ca),$$

we say that the reflection is *strong*. A set A reflects a mapping F if A is closed under F and A reflects the graph of F. One of the basic results in set theory is that every first-order definable predicate is reflected by a countable set, although not necessarily strongly.

3.1. Lemma. *Let $\phi_0, \dots, \phi_n$ be formulas of an expanded vocabulary $\{\in, \kappa\}$, let κ be a regular infinite cardinal, and let A be a set. There exists $B \supseteq A$ such that $\kappa \in B$, B strongly reflects $\mathcal{P}_\kappa$, B reflects each formula ϕ_i, and $|B| \leq \max\{\kappa^{<\kappa}, |A|^{<\kappa}\}$.*

Proof. We may assume that the sequence $\phi_0, \dots, \phi_n$ contains the axiom of extensionality, and formulas "$x \subseteq y$" and

$$\text{“}x = \mathcal{P}_\kappa(y) \leftrightarrow \forall z\,(z \in x \leftrightarrow z \subseteq y \wedge \exists f\,\exists \alpha < \kappa\,(f\colon x \to \alpha \text{ one-to-one}))\text{”}.$$

Moreover, we may assume that the universal quantifier does not occur in the formulas $\phi_0, \dots, \phi_n$, and that all the subformulas of each formula occur in the sequence. If $\mathbf{C}$ is a class, denote

$$\hat{\mathbf{C}} = \{\, x \in \mathbf{C} \,:\, \forall y \in \mathbf{C}\,(\text{rank}(x) \leq \text{rank}(y))\,\}.$$

For every formula ϕ_i of form $\exists x\, \psi_i(x_1, \dots, x_m, x)$ define mapping

$$H_i(x_1, \dots, x_m) = \hat{\mathbf{C}}, \text{ where } \mathbf{C} = \{\, x \,:\, \psi(x_1, \dots, x_m, x)\,\}.$$

If ϕ_i is not of the above form, let $H_i(x_1, \dots, x_m) = \emptyset$. If $|x| < \kappa$, let $F(x)$ be the set of one-to-one mappings from x to $|x|$; otherwise let $F(x) = \emptyset$.

Given a set X, let $L(X)$ be the union of X, $H_i(x_1, \dots, x_m)$ for $i \leq n$ and $x_1, \dots, x_m \in X$, $F(x)$ for $x \in X$, $\{\, \mathcal{P}_\kappa(x) \,:\, x \in X\,\}$, and $\mathcal{P}_\kappa(X)$. Let C be the least set containing A and $\{0, 1, \dots, \kappa\}$ such that $L(C) = C$. It could be shown that the set C strongly reflects the formula "$\mathcal{P}_\kappa(x) = y$" and that it reflects the formulas $\phi_0, \dots, \phi_n$, but it is too large for our purposes. We use the axiom of choice to pick a small subset of it. Let $\lhd$ be a well-ordering of C. Define for $i \leq n$ functions

$$h_i(x_1, \dots, x_m) = \begin{cases} \{\min_\lhd H_i(x_1, \dots, x_n)\} & \text{if } H_i(x_1, \dots, x_n) \neq \emptyset, \text{ and} \\ \emptyset & \text{otherwise,} \end{cases}$$

and let $f(x)$ similarly pick a finite subset of $F(x)$. These functions are clearly defined for arguments in C, and their values are subsets of C. For a subset X of C, let $K(X)$ be the union of sets X, $h_i(\vec{x})$ for $i \leq n$ and $\vec{x} \in X$, $f(x)$ for $x \in X$, $\{\, \mathcal{P}_\kappa(x) \,:\, x \in X\,\}$, and $\mathcal{P}_\kappa(X)$. Let

$$B_0 = A \cup \{0, 1, \dots, \kappa\}, \quad B_{\alpha+1} = K(B_\alpha), \quad \text{and} \quad B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha,$$

where γ denotes a limit ordinal. Clearly $A \cup \{0, 1, \dots, \kappa\} = B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots \subseteq C$, and, since κ is regular, $K(B_\kappa) = B_\kappa$. Let $B = B_\kappa$.

Claim A: B reflects the formulas $\phi_0, \dots, \phi_n$.

Suppose ϕ_i is of form $\exists x \psi(x_1, \dots, x_m, x)$, and let $x_1, \dots, x_m$ be elements of B. Since $K(B) = B$, $h_i(x_1, \dots, x_m) \subseteq B$, and thus

$$\begin{aligned} \phi_i(x_1, \dots, x_m) &\iff \exists x \in H_i(x_1, \dots, x_m)\, \psi(x_1, \dots, x_m, x) \\ &\iff \exists x \in h_i(x_1, \dots, x_m)\, \psi(x_1, \dots, x_m, x) \\ &\iff \exists x \in B\, \psi(x_1, \dots, x_m, x). \end{aligned}$$

Since the sequence $\phi_0, \ldots, \phi_n$ is closed under subformulas, an easy induction shows that all formulas in the sequence are reflected in B.

Especially the above claim implies that B is extensional. Denote the Mostowski collapsing function of $(B, \in)$ by c. Now $c\kappa = \kappa$,

$$x = \mathcal{P}_\kappa(y) \leftrightarrow \forall z(z \in x \leftrightarrow z \subseteq y \wedge \exists f\, \exists \alpha < \kappa(f\colon x \to \alpha \text{ one-to-one}))$$

is true in B for all elements $x, y \in B$, and whenever $a, b \in B$,

$$a \subseteq b \iff [a \subseteq b]^B \iff ca \subseteq cb,$$

i.e. the relation $\subseteq$ is strongly reflected in B.

Claim B: B strongly reflects the relations "$|x| < \kappa$"and "$y \in \mathcal{P}_\kappa(x)$".

The reflection of "$y \in \mathcal{P}_\kappa(x)$" follows immediately from the reflection of "$|x| < \kappa$", so it is enough to show the latter. Suppose $a \in B$. If $|ca| < \kappa$, then $a \cap B$ is a subset of B having cardinality less than κ. Since $K(B) = B$, $a \cap B$ is in B. By extensionality $a \cap B = a$, and thus $|a| < \kappa$. If $|a| < \kappa$, then $f(a)$ is nonempty, which makes $|a| < \kappa$ in B. Finally, if $|a| < \kappa$ in B, then $|ca| < \kappa$, since the truth of Σ_1-sentences is preserved in extensions.

Claim C: B reflects "$y = \mathcal{P}_\kappa(x)$" strongly.

Suppose a and b are elements of B. If $b = \mathcal{P}_\kappa(a)$, then $[b = \mathcal{P}_\kappa(a)]^B$ follows immediately from the claim B. Suppose $[b = \mathcal{P}_\kappa(a)]^B$. Clearly the claim B implies $cb \subseteq \mathcal{P}_\kappa(ca)$. On the other hand, if x is in $\mathcal{P}_\kappa(ca)$, then x gives rise to a subset y of $B \cap a$ such that $cy = x$ and $|y| < \kappa$. Since $\mathcal{P}_\kappa(B) \subseteq K(B) = B$, y is an element of B. Thus the claim B implies $[y \in \mathcal{P}_\kappa(a) = b]^B$, and so $x = cy \in cb$. Finally, suppose $cb = \mathcal{P}_\kappa(ca)$. Now $d = \mathcal{P}_\kappa(a) \in K(B) = B$, and we have already seen that $cd = \mathcal{P}_\kappa(ca)$. Thus $cb = cd$, and by extensionality of B we must have $b = d$.

Claim D: $|B| \leq \max\{\kappa^{<\kappa}, |A|^{<\kappa}\}$.

If X is an infinite set, obviously

$$|K(X)| \leq |X| + |\mathcal{P}_\kappa(X)| = |X| + |X|^{<\kappa} = |X|^{<\kappa}.$$

Denote $\lambda_i = |B_i|$. Clearly $\lambda_0 = \max\{\kappa, |A|\}$. By induction one can see that $\lambda_\alpha \leq \lambda_0^{<\kappa}$ for each $\alpha \leq \kappa$: the limit step is trivial, and for the successor step note that if $\lambda \geq \kappa$, $(\lambda^{<\kappa})^{<\kappa} = \lambda^{<\kappa}$. Thus $|B| = \lambda_\kappa \leq \lambda_0^{<\kappa}$. □

3.2. Theorem. *If κ is a regular cardinal, ϕ is a $\Sigma_1(\mathcal{P}_\kappa)$-formula, and $\mu > \kappa$ is a cardinal such that $\forall \lambda < \mu(\lambda^{<\kappa} < \mu)$, then H_μ reflects ϕ.*

Proof. It is not hard to see that H_μ reflects $\mathcal{P}_\kappa$ for every $\mu \geq \kappa$. Thus for every $a \in H_\mu$

$$[\phi(a)]^{(H_\mu, \in, \mathcal{P}_\kappa, \kappa)} \to \phi(a).$$

On the other hand, suppose $a \in H_\mu$ is such that $\phi(a)$ is true. Assume $\phi(a) = \exists x \psi(a, x)$, where ψ is $\Delta_0(\mathcal{P}_\kappa)$. Choose b such that $\psi(a, b)$ is true. Let ψ^* be the formula ψ where each occurrence of the symbol P is replaced with a formula defining $\mathcal{P}_\kappa$, and let $A = \mathrm{TC}(\{a, \kappa\}) \cup \{b\}$. By Lemma 3.1 there exists $B \supseteq A$ such that B strongly reflects $\mathcal{P}_\kappa$, $[\psi^*(a, b)]^B$, and $|B| \leq |A|^{<\kappa} < \mu$. Thus cB, the Mostowski collapse of B, is in H_μ, reflects $\mathcal{P}_\kappa$, and $\exists x \psi^*(a, x)$ holds in cB. This implies that $\phi(a)$ holds in $(cB, \in, \mathcal{P}_\kappa, \kappa)$, and since ϕ is a Σ_1-formula, $\phi(a)$ holds in $(H_\mu, \in, \mathcal{P}_\kappa, \kappa)$. □

One could not have a better result, since any H_μ reflecting all $\Sigma_1(\mathcal{P}_\kappa)$-formulas must satisfy $\forall x \exists y (y = \mathcal{P}_\kappa(x))$. If $\lambda < \mu$, then λ is an element of H_μ; thus $\mathcal{P}_\kappa(\lambda)$ is in H_μ and

$$|\mathrm{TC}(\mathcal{P}_\kappa(\lambda))| = |\mathcal{P}_\kappa(\lambda)| + \lambda = \lambda^{<\kappa} + \lambda < \mu.$$

4. Trees and games

If S is a partial ordering and $u \in S$, denote the set of predecessors of u by $\mathrm{pred}_S(u)$ (or $\mathrm{pred}(u)$ if S is clear from the context). The notation $\mathrm{succ}_S(u)$ stands for the set of immediate successors of u. A *chain* of a partial ordering is a linearly ordered subset. Each subset X of a partial ordering S spans an initial segment $\mathrm{init}_S(X)$ and an end segment $\mathrm{end}_S(X)$

$$\begin{aligned} \mathrm{init}_S(X) &= \{\, u \in S : \exists v \in X \, (u \leq v) \,\}, \\ \mathrm{end}_S(X) &= \{\, u \in S : \exists v \in X \, (v \leq u) \,\}. \end{aligned}$$

A *tree* is a partial ordering where sets of predecessors are well-ordered. A minimal element in a tree is a *root*, a maximal element is a *leaf.* A tree has *unique limits* if $u = v$ whenever $\mathrm{pred}(u) = \mathrm{pred}(v)$ and the order type of $\mathrm{pred}(u)$ is a limit ordinal. Unless otherwise indicated, trees have a single root. A maximal chain is called a *branch.* A tree is a *leaftree* if each branch has a leaf (or, equivalently, if each chain has an upper bound). A tree is a κ, λ*-tree* if it has a single root, unique limits, each element has less than κ immediate successors, and each branch is of length less than λ. When the number of successors is inessential, we speak about λ*-trees.* A *path* in a tree is a strictly increasing sequence $\langle u_i \rangle_{i<\xi}$ of elements such that $\mathrm{pred}(u_i) = \{\, u_j : j < i \,\}$ for each i. If T is a tree, $[T]$ is the set of its branches. If u is an element of T, then $T_u = \mathrm{end}_T(\{u\})$ is the subtree consisting of the element u and its successors. If X is a subset of T, denote

$$T[X] = \mathrm{init}_T(X) \cup \mathrm{end}_T(X).$$

The partial ordering $T[X]$ is a tree, which contains the elements in X, their predecessors, and all their successors. The ordinal type of $\mathrm{pred}_T(t)$ for an element t, denoted by $\mathrm{ht}_T(t)$, is called the *height* of t in T. The *height* of a tree T, $\mathrm{ht}(T)$ is the supremum of the ordinal types of its branches.

Trees are usually compared using embeddings: we write $S \leq T$ if there exists a function $f\colon S \to T$ which preserves the tree order (f need not be one-to-one).

If T is a tree, σT is the tree of paths in T, ordered by end extension. An *infimum* $S \otimes T$ of two trees S and T is the tree of pairs (s, t), where the elements $s \in S$ and $t \in T$ have the same height. The tree $S \otimes T$ is ordered by

$$(s,t) \leq (s',t') \iff s \leq s \wedge t \leq t'.$$

A *supremum* $S \oplus T$ of single-rooted trees S and T is constructed by identifying the roots of S and T, i.e.

$$S \oplus T = (S \times \{0\}) \cup (T \times \{1\}) / \sim_r,$$

where $(s,i) \sim_r (t,j)$ if and only if either $(s,i) = (t,j)$ or $i \neq j$ and both s and t are roots. For every tree S and T, it can be shown that $T \leq \sigma T$, $\sigma T \not\leq T$, and $S \otimes T$ and $S \oplus T$ are the infimum and supremum of trees S and T relative to $\leq$.

The sum $S + T$ of trees S and T is obtained by placing a copy of T on top of each branch of S. In other words,

$$S + T = S \times \{0\} \cup ([S] \times T) \times \{1\}$$

so ordered that

$$\begin{aligned} (s,0) \leq (t,0) &\iff t \leq s, \\ (s,0) \leq ((b,t),1) &\iff s \in b, \quad \text{and} \\ ((a,s),1) \leq ((b,t),1) &\iff a = b \text{ and } s \leq t. \end{aligned}$$

The product $S \cdot T$ of trees S and T is constructed by replacing each element of T with a copy of S and by replicating the subtrees when necessary. To be exact, the tree $S \cdot T$ consists of tuples (g, s, t), where $s \in S$, $t \in T$, and $g\colon \mathrm{pred}_T(t) \to [S]$. It is ordered by setting $(g,s,t) \leq (g',s',t')$ if and only if either $g = g'$, $t = t'$, and $s \leq s$ or $t < t'$, $g = g' \restriction \mathrm{pred}_T(t)$, and $s \in g'(t)$.

A tree is *well-founded* if its every branch is finite. Thus a well-founded tree is a leaftree. Let T be a well-founded tree. The (unique) function r for which $\mathrm{dom}(r) = T$ and

$$r(x) = \sup\{\, r(y) + 1 \ : \ y \in \mathrm{succ}(x)\,\} \text{ for every } x \in T$$

is called the *rank function* of T. The supremum of its range is called the *ordinal of the tree* T. For each ordinal α the tree B_α of strictly decreasing sequences of ordinals less than α is a well-founded tree having ordinal α. Moreover $|B_\alpha| = |\alpha|$ if α is infinite. In the next two lemmas we show that the mapping which maps a well-founded tree to its ordinal is not primitive recursive but, however, effective enough so that admissible sets are closed under it.

4.1. Lemma. *Let f be a mapping such that $f(T)$ is the ordinal of T whenever T is a well-founded tree. Then f is not p.r.*

Proof. Assume for the purpose of contradiction that there is a p.r. mapping f which maps every well-founded tree to its ordinal. Let g be an increasing p.r. mapping such that for every set x

$$\operatorname{rank}(f(x)) \leq g(\operatorname{rank}(x)).$$

Let $\gamma = g(\omega+\omega)+1$. Since g is Σ_1-definable, γ must be countable. Choose a well-founded tree $T = (\omega, \leq_T)$ such that its ordinal is γ. Now $\operatorname{rank}(T) < \omega + \omega$, but $\operatorname{rank}(f(T)) = \gamma$, which is a contradiction. □

4.2. Lemma. *Let A be an admissible set. If $T \in A$ is a well-founded tree (in the true universe), the rank function of T and the ordinal of T are elements of A.*

Proof. We show by induction on T that for every $x \in T$ there is $r_x \in A$ which is a rank function of T_x. Note that "r is a rank function of T_x" is a Δ_1^{KP}-formula (free variables r, x, T). Let ρ be the rank function of T.

If $\rho(x) = 0$, then clearly $r_x = \{(x, 0)\} \in A$. Suppose $\rho(x) = \alpha$ and $r_y \in M$ for every y for which $\rho(y) < \alpha$. Thus

$$\forall y \in \operatorname{succ}(x)\, \exists r \in A\, (r \text{ is a rank function of } T_y).$$

Since Σ_1-collection and Δ_1-separation schemes hold in an admissible set, the set $R_x = \{\, r_y \ :\ y \in \operatorname{succ}(x) \,\}$ is in A. Since $\rho(y) = r_y(y)$ for each y, this implies that α and $r_x = \bigcup R_x \cup \{(x, \alpha)\}$ are elements of A. The rank function of T is r_x, where x is the root of T, and the ordinal of T is $\bigcup \operatorname{ran}(r_x)$. □

Intuitively, a game for two persons is based on rules which determine the acceptable moves and the winner. Here we name the players $\forall$ and $\exists$, and follow the convention that $\forall$ is male and $\exists$ female. In the following formal definition the rules determining the acceptable moves are presented as a tree of acceptable game positions. The root of the tree is the initial position, and the successors of each position are those positions into which one comes with an acceptable move. A play terminates when a final position, a leaf of the position tree, is reached. The opponent of the mover in the final position wins the play (i.e. the first player unable to move loses).

4.3. Definition. (i) A *game* is a pair (R, m), where R is a leaftree having a single root and unique limits and $m\colon R \to \{\forall, \exists\}$ maps positions to *movers*. The branches of the position tree are called *plays*. A player *wins* a play if his/her opponent is the mover in the final position of the play.

(ii) An initial segment $T \subseteq R$ is a $\forall$-*strategy* ($\exists$-*strategy*) if it is a leaftree and $\operatorname{succ}_R(u) = \operatorname{succ}_T(u)$ for every $u \in T$ such that $m(u) = \exists$ ($m(u) = \forall$). Denote by $\operatorname{strat}_\forall(G)$ ($\operatorname{strat}_\exists(G)$) the set of $\forall$-strategies ($\exists$-strategies) of the game G.

(iii) A $\forall$-strategy ($\exists$-strategy) T is *complete* if its every leaf is a leaf of R.

(iv) A complete $\forall$-strategy ($\exists$-strategy) T is a *winning strategy* of $\forall$ ($\exists$) if $\forall$ ($\exists$) wins every play in $[T]$. We say that $\forall$ ($\exists$) *wins the game* G if $\forall$ ($\exists$) has a winning strategy.

This definition, though rigorous, is technical, so in practice one usually defines games by means of verbal description. To have an example, consider a so-called Ehrenfeucht-Fraïssé game. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures having the same vocabulary, and let T be a tree. The game $\mathrm{EF}_T(\mathfrak{A}, \mathfrak{B})$ is played as follows: for as many ordinals i as possible

- $\forall$ chooses an element $t_i \in T$ such that the sequence $\langle t_j \rangle_{j \leq i}$ is strictly increasing.
- $\forall$ chooses a model ($\mathfrak{A}$ or $\mathfrak{B}$) and an element u_i from the model.
- $\exists$ chooses an element v_i from the other model. If u_i was chosen in $\mathfrak{A}$ and v_i in $\mathfrak{B}$, denote $a_i = u_i$ and $b_i = v_i$; otherwise denote $b_i = u_i$ and $a_i = v_i$. Player $\exists$ must choose v_i such that the function $\{\, (a_j, b_j) \ : \ j \leq i \,\}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

This continues as long as either player is unable to move: the player who is finally unable to move, is the loser.

This description can be turned into a rigorous definition of a game for example in the following way: the position tree of the game $\mathrm{EG}_T(\mathfrak{A}, \mathfrak{B})$ consists of sequences $\langle r_i \rangle_{i<\xi}$, where

- r_{3i} is an element of the tree T such that the sequence $\langle r_{3i} \rangle_{3i<\xi}$ is strictly increasing.
- r_{3i+1} is a pair (x_i, u_i) where x_i is either $\mathfrak{A}$ or $\mathfrak{B}$ and u_i is an element of the model x_i.
- r_{3i+2} is a pair (y_i, v_i) where y_i is the model other than x_i and v_i is an element of the model y_i. Moreover, the partial isomorphism condition must be satisfied.

The mover in position $\langle r_i \rangle_{i<\xi}$ is $\exists$ if $\xi = 3\zeta + 2$ for some ζ, and $\forall$ otherwise.

A *dual* of a game G is the game $\sim G$, where the roles of the players are switched. In other words, if $G = (R, m)$, then $\sim G = (R, m')$, where $m'(u) = \exists \iff m(u) = \forall$ for every position u. Clearly $\exists$ wins G if and only if $\forall$ wins $\sim G$. A game is *determined* if either player has a winning strategy.

A strategy is described as a tree rather than as a function, as what is sometimes called a quasi-strategy. A usual way to present a winning strategy for a player, say $\exists$, is to describe how $\exists$ must make her choices in order to win a play. Consider for example the Ehrenfeucht-Fraïssé game presented above. We assume that a certain number, say ξ, of moves have already been made, i.e. elements t_i, a_i, and b_i for $i < \xi$ have been picked from T, $\mathfrak{A}$, and $\mathfrak{B}$, respectively. We suppose that $\forall$ picks t_ξ and, say, a_ξ in $\mathfrak{A}$. Then we describe which kind of element $\exists$ must pick in $\mathfrak{B}$. This results in a strategy: the strategy consists of those positions $\langle r_i \rangle_{i<\xi}$ where the elements r_{3i+2} satisfy the requirements laid down for the elements picked by $\exists$.

However, this simple construction is not always enough: in some cases $\exists$ needs to maintain an auxiliary construction during the play. The next lemma firmly states that certain auxiliary constructions can be used when describing a strategy.

4.4. Lemma. *Let $G = (R, m)$ be a game. Suppose there is a set A and a tree S of sequences $\langle u_i, a_i \rangle_{i<\xi}$, where $u_i \in R$ and $a_i \in A$ such that the following hold.*

(i) If $\langle u_i, a_i \rangle_{i<\xi}$ is in S, then $\langle u_i \rangle_{i<\xi}$ is a path in R.

(ii) If $\langle u_i, a_i \rangle_{i\leq\xi}$ is in S and $m(u_\xi) = \forall$ ($m(u_\xi) = \exists$), then for every $u_{\xi+1} \in \operatorname{succ}_R(u_\xi)$ there is $a_{\xi+1} \in A$ such that $\langle u_i, a_i \rangle_{i\leq\xi+1} \in S$.

(iii) If $\langle u_i, a_i \rangle_{i\leq\xi}$ is in S, $m(u_\xi) = \exists$ ($m(u_\xi) = \forall$), and $\operatorname{succ}_R(u_\xi) \neq \emptyset$, there is $u_{\xi+1} \in \operatorname{succ}_R(u_\xi)$ and $a_{\xi+1} \in A$ such that $\langle u_i, a_i \rangle_{i\leq\xi+1} \in S$.

(iv) If ξ is limit, $\langle u_i, a_i \rangle_{i<\xi}$ is in S, and $u_\xi = \sup\{ u_i : i < \xi \}$, there is $a_\xi \in A$ such that $\langle u_i, a_i \rangle_{i\leq\xi} \in S$.

(v) If $\langle u_i, a_i \rangle_{i\leq\xi}$ is maximal in S, then $m(u_\xi) = \forall$ ($m(u_\xi) = \exists$).

Then $\exists$ ($\forall$) wins G.

Proof. Let $\lhd$ be a well-ordering of A. Let S^* be the set of those $\langle u_i, a_i \rangle_{i<\xi} \in S$ for which

$$a_i = \min_{\lhd}\{ a_i \in A : \langle u_j, a_j \rangle_{j\leq i} \in S \} \text{ for every } i < \xi.$$

Now, if $\langle u_i \rangle_{i<\xi}$ is a path in R, there is at most one sequence $\langle a_i \rangle_{i<\xi}$ such that $\langle u_i, a_i \rangle_{i<\xi} \in S^*$. Moreover, (i)–(v) hold for S^*, and thus

$$T = \{ u_\xi \in R : \exists \langle u_i \rangle_{i<\xi} \, \exists \langle a_i \rangle_{i\leq\xi} \, \langle u_i, a_i \rangle_{i\leq\xi} \in S^* \}$$

is a winning strategy of $\exists$ ($\forall$) in G. □

To illustrate this, let us briefly discuss a case in which forcing has no effect on a game. Let λ be an ordinal. A game $G = (R, m)$ is *λ-open* (*λ-closed*) if there exists a set $A \subseteq R$ of positions such that the height of each position in A is less than λ and $\exists$ ($\forall$) wins a play $p \in [R]$ exactly when $p \cap A \neq \emptyset$. Note that once this kind of set A exists, there is a canonical way of picking one: the set of such minimal positions $u \in R$ that $\exists$ ($\forall$) wins each play containing the position u. Moreover, note that if the position tree is a λ-tree, the game is both λ-open and λ-closed.

Let $\mathbb{P}$ be a notion of forcing and let $G = (R, m)$ be a game. The first problem we encounter is that G might not be a game in the extended universe: the position tree may have new branches with no final positions. We solve the problem by adding the missing positions. If the game is open or closed and all the new positions are high enough, there is a natural way of attaching a mover to these new positions. Define first a canonical $\mathbb{P}$-name for the extended position tree: let $\tilde{R}$ contain the pairs (v, p) where either $p = 1_{\mathbb{P}}$ and v is a canonical name for an element of R, or p forces v to be a branch of $\check{R}$ having no upper limit. Note that $\tilde{R}$ is a set, since we may assume that the $\mathbb{P}$-names for the branches of R are subsets of $\mathbb{P} \times \check{R}$.

Suppose then that G is open: let $A \subseteq R$ be the canonical set witnessing it. Define $\tilde{m}_o$ to be a canonical $\mathbb{P}$-name for a function from $\tilde{R}$ to players, which maps each position in R in the same way as m and each new position to either $\forall$ or $\exists$ depending on whether the new position is above a position in A or not. To be exact, let $\tilde{m}_o$ be such that

- $1_{\mathbb{P}} \Vdash \tilde{m}_o(\check{u}) = \check{m}(\check{u})$ if u is in R,
- $p \Vdash \tilde{m}_o(b) = \forall$ if p forces b to be a branch of $\check{R}$ without an upper limit and $b \cap \check{A} \neq \emptyset$, and
- $p \Vdash \tilde{m}_o(b) = \exists$ if p forces b to be a branch of $\check{R}$ without an upper limit and $b \cap \check{A} = \emptyset$.

Let $\tilde{G}_o$ be a $\mathbb{P}$-name for the pair $(\tilde{R}, \tilde{m}_o)$. Define $\mathbb{P}$-name $\tilde{G}_c$ for a closed game G in a similar way.

4.5. Lemma. *Let $G = (R, m)$ be a game and let $\mathbb{P}$ be a λ-closed forcing.*

(i) If G is λ-open, $\mathbb{P}$ forces $\tilde{G}_o$ to be a λ-open game. Moreover, $\forall$ wins G if and only if $\mathbb{P}$ forces $\forall$ to win $\tilde{G}_o$.

(ii) If G is λ-closed, $\mathbb{P}$ forces $\tilde{G}_c$ to be a λ-closed game. Moreover, $\exists$ wins G if and only if $\mathbb{P}$ forces $\exists$ to win $\tilde{G}_c$.

Proof. It is enough to show (ii), since (i) is similar. Let $A \subseteq R$ be the canonical set witnessing a game $G = (R, m)$ closed. Clearly $\mathbb{P}$ forces $\tilde{R}$ to be a leaftree, and it is not hard to see that $\mathbb{P}$ forces $\mathrm{dom}(\tilde{m}_c) = \tilde{R}$; thus $\mathbb{P}$ forces $\tilde{G}_c$ to be a game. Since $\mathbb{P}$ is λ-closed, the height of every new position in $\tilde{G}_c$ is at least λ. Thus a condition p forces $\forall$ to win a play b in $\tilde{G}_c$ if and only if the set of those conditions $q \leq p$ which force $b \cap \check{A} \neq \emptyset$ is dense.

Suppose that $\mathbb{P}$ forces $\exists$ to win $\tilde{G}_c$. Informally, $\exists$ wins G as follows. First choose $p_0 \in \mathbb{P}$ and a $\mathbb{P}$-name S such that p_0 forces S to be a winning strategy of $\exists$ in $\tilde{G}_c$. One immediately sees, that p_0 forces $S \cap \check{A} = \emptyset$. Let $u_0 \in R$ be the root of R: now $p_0 \Vdash \check{u}_0 \in S$. Then suppose $p_i \Vdash \check{u}_i \in S$ and $m(u_i) = \forall$. If u_i is a leaf, $\exists$ has won, otherwise suppose that $\forall$ moves into position $u_{i+1} \in \mathrm{succ}(u_i)$. Since p_i forces S to be a winning strategy and $\tilde{m}_c(\check{u}_i) = \forall$, $p_i \Vdash \check{u}_{i+1} \in S$. Choose $p_{i+1} = p_i$. On the other hand, if $p_i \Vdash \check{u}_i \in S$ and $m(u_i) = \exists$, u_i cannot be a leaf, since p_i forces S to be a winning strategy and $\tilde{m}_c(\check{u}_i) = \exists$. Thus there exists $p_{i+1} \leq p_i$ and $u_{i+1} > u_i$ such that $p_{i+1} \Vdash \check{u}_{i+1} \in S$. At limit k let u_k be the limit position of u_i $(i < k)$ and, if $k < \lambda$, let p_k be a lower bound for p_i $(i < k)$. In this way we can play up to height λ in the position tree, thus reaching a position u_λ. But now $u_i \notin A$ for each $i < \lambda$, which implies that $\exists$ must finally win the play, independently of how she continues above height λ.

Using Lemma 4.4 this strategy is formally described as follows. Choose $p_0 \in \mathbb{P}$ and S as above. Let T be the tree of sequences $\langle u_i, p_i \rangle_{i<k}$ such that $\langle u_i \rangle_{i<k}$ is a path in R, $\langle p_i \rangle_{i<\min\{k,\lambda\}}$ is an increasing sequence in $\mathbb{P}$, and p_i forces $\check{u}_i \in S$ for each $i < \min\{k, \lambda\}$. Then T satisfies the conditions of Lemma 4.4, which was shown by the informal proof.

To see the converse, suppose that S is a winning strategy of $\exists$ in G. Define a $\mathbb{P}$-name $\tilde{S}$ for a strategy of $\exists$ in $\tilde{G}_c$: let $\tilde{S}$ contain pairs $(\check{u}, 1_{\mathbb{P}})$ for $u \in S$ and pairs (b, p) whenever p forces b to be a branch of $\check{S}$ without a maximal element. It is not hard to see that $\mathbb{P}$ forces $\tilde{S}$ to be a complete $\exists$-strategy in $\tilde{G}_c$. Moreover, since $\tilde{G}_c$ contains no new positions of height less than λ, $\mathbb{P}$ forces $\tilde{S} \cap \check{A} = \check{S} \cap \check{A} = \emptyset$, which immediately implies that $\tilde{S}$ is a winning strategy of $\exists$ in $\tilde{G}_c$. □

A frequently occurring auxiliary construction is a phantom game. In a phantom game construction $\exists$ wins a game G by playing another game G' in the background: whenever player $\forall$ makes a move in G, she maps this move to certain move or moves of $\forall$ in the game G', uses the winning strategy of $\exists$ there, and again maps moves in the game G' to her moves in G. This will be carefully formulated below.

4.6. Definition. Let $G = (R, m)$ and $G' = (R', m')$ be games. The game G' is *an $\exists$-phantom* of G (or G is a *$\forall$-phantom* of G') if there exist functions

$$f_\forall \colon R \to \mathrm{strat}_\forall(G') \quad \text{and} \quad f_\exists \colon R' \to \mathrm{strat}_\exists(G)$$

such that the following conditions hold:

(i) $u \leq v \Rightarrow f_\forall(u) \subseteq f_\forall(v)$ for every $u, v \in R$, and similarly for $f_\exists$.

(ii) If v is maximal in $f_\forall(u)$ and u is maximal in $f_\exists(v)$, both u and v are final positions.

(iii) If u is a final position of R and $v \in R'$ is such that $v \in f_\forall(u) \wedge u \in f_\exists(v)$, then $f_\forall(u)$ is a complete $\forall$-strategy in the game G' restricted to $R'[\{v\}]$ (i.e. in the game G' restricted to positions compatible with v); and similarly for $f_\exists$.

(iv) If $u \in R$ and $v \in R'$ are final positions such that $u \in f_\exists(v)$, $v \in f_\forall(u)$, and $m'(v) = \forall$, then $m(u) = \forall$.

The functions $f_\forall$ and $f_\exists$ are called *reduction functions*.

The moves are mapped between the games G and G' with the functions $f_\forall$ and $f_\exists$. For example, the function $f_\forall$ maps a position u in the game G to a $\forall$-strategy in G': the strategy tells us which moves of $\forall$ in G' correspond to the moves in u. The condition (ii) prohibits a deadlock: a situation where neither $f_\exists(v)$ tells $\exists$ how to continue in position u in the game G nor $f_\forall(u)$ tells $\forall$ how to continue in G'. The condition (iii) ensures that a phantom game provides one with enough information for playing the game through. Finally, the condition (iv) states that if a play in G and a play in G' are coupled by the reduction functions, the winners are related.

4.7. Lemma. *Let G and G' be games. If G' is an $\exists$-phantom of G, then*

$$\exists \textit{ wins } G' \implies \exists \textit{ wins } G \quad \textit{and} \quad \forall \textit{ wins } G \implies \forall \textit{ wins } G'.$$

Proof. We use Lemma 4.4. It is enough to show the first implication, since the other case is similar. Suppose T is a winning strategy of $\exists$ in G'. Let S be the tree of sequences $\langle u_i, v_i \rangle_{i<\xi}$, where $\langle u_i \rangle_{i<\xi}$ is a path of R, $\langle v_i \rangle_{i<\xi}$ is an increasing sequence of positions in T, and $u_i \in f_\exists(v_i) \wedge v_i \in f_\forall(u_i)$ for every $i < \xi$. We need to show that the requirements of Lemma 4.4 hold.

4.4(i): Trivial.

4.4(ii): Let $\langle u_i, v_i \rangle_{i \leq \xi} \in S$, $m(u_\xi) = \forall$ and $u_{\xi+1} \in \mathrm{succ}_R(u_\xi)$. Since $f_\exists$ is an $\exists$-strategy and (i) holds, one can choose $v_{\xi+1} = v_\xi$.

4.4(iii): Let $\langle u_i, v_i\rangle_{i\leq\xi} \in S$, $m(u_\xi) = \exists$ and $\mathrm{succ}_R(u_\xi) \neq \emptyset$. Let $v_{\xi+1}$ be a maximal element of $f_\forall(u_\xi) \cap T$ above every v_i for $i \leq \xi$. Now (ii) implies $f_\exists(v_{\xi+1}) \cap \mathrm{succ}_R(u_\xi) \neq \emptyset$, and it is enough to choose $u_{\xi+1}$ from this set.

4.4(iv): Let ξ be a limit and suppose $\langle u_i, v_i\rangle_{i<\xi}$ is in S such that $u_\xi = \sup_{i<\xi} u_i$ exists in R. Since $v_\xi = \sup_{i<\xi} v_i$ exists in T, the assumption (i) implies that $\langle u_i, v_i\rangle_{i\leq\xi}$ is in S.

4.4(v): Let $\langle u_i, v_i\rangle_{i<\xi}$ be maximal in S. The claims 4.4 (ii)–(iv) shown above imply that ξ is a successor, say $\xi = i + 1$, and u_i is a final position in R. By (iii) there is a final position v above v_i in $f_\forall(u_i) \cap T$. Now $m'(v)$ must be $\forall$, since T is a winning strategy of $\exists$, and $u_i \in f_\exists(v)$ by (i), so (iv) implies $m(u_i) = \forall$. □

Though precise, the lemma above is technical and in practice often inconvenient. This is why an informal presentation is usually preferable. Consider games G and G'. For simplicity, assume that in the games the players pick elements alternatively from some sets. Suppose $\forall$ starts the game G by picking an element a_0. We then informally describe which kind of element $\forall$ must pick in the phantom game G', i.e. we describe the first move of $\forall$ in the strategy $f_\forall(a_0)$. This element may depend on a_0; let it be b_0. Suppose $\exists$ answers b_1. Then we, similarly, describe the strategy $f_\exists(b_0, b_1)$ by indicating which kind of element $\exists$ must pick in G. This element a_1 may depend on both a_0 and b_0, b_1. This game-playing can be presented in the form of a diagram, see Diagram 1.

G	G'
$\forall : a_0$	
	$\forall : b_0 = f(a_0)$
	$\exists : b_1$
$\exists : a_1 = g(b_0, b_1)$	
$\vdots$	$\vdots$
$\forall : a_\omega$	
	$\forall : b_\omega = f(a_0, \ldots, a_\omega)$
	$\exists : b_{\omega+1}$
$\vdots$	$\vdots$

Diagram 1: Phantom game diagram

On the left side of the diagram there are the moves in the game G, on the right side the moves in G'. A notation like $a_1 = g(b_0, b_1)$ indicates that the element a_1 may depend on b_0 and b_1. Note that the moves of $\forall$ in the game G and the moves of $\exists$ in G' must be freely chosen. This gives rise to the reducing functions: for example, given a position $v = \langle b_i\rangle_{i<\xi}$ in the game G', every such position $u = \langle a_i\rangle_{i<\zeta}$ in the game G which is not too long and which satisfies the described dependencies belongs to the strategy $f_\exists(v)$.

The requirements (i)–(iii) of Lemma 4.7 are usually trivial to check, so one only needs to ensure only that the requirement (iv) holds, namely, that if p and q are plays on the left and right side of the diagram, respectively, then $\exists$ wins p whenever $\exists$ wins q.

The phantom game construction was presented above assuming there is only one phantom game. Actually there can be number of phantom games (games on the right side of the diagram). For example, a game G gets three phantom games G_1, G_2, and G_3 as follows. We define three reduction functions $f_{\forall,1}$, $f_{\forall,2}$, and $f_{\forall,3}$ which reduce the moves of $\forall$ in G to moves of $\forall$ in the phantom games. On the other hand, the reduction function $f_\exists$ has three arguments: a position in each game G_1, G_2, and G_3.

5. Logics

Our logical setting is according to [E], and we use set theory as a metatheory. Vocabularies are single-sorted, and metatheoretically they are sets of symbols. Each symbol contains information of its kind (relation, function, constant), arity, and name. We use no separate variable symbols. In some cases we join vocabularies to two-sorted ones: if σ and τ are vocabularies, $(\sigma; \tau)$ is a two-sorted vocabulary, where the symbols of σ are of sort 0, the symbols of τ are of sort 1, and the sorts are separated (i.e. terms of different sort do not occur in the same atomic formulas).

Structures are pairs (M, F), where M is a nonempty set and F is a function which maps a vocabulary to interpretations. If $\mathfrak{M} = (M, F)$ is a model, Voc$(\mathfrak{M}) =$ dom(F) is the vocabulary of $\mathfrak{M}$. If τ is a vocabulary, Str(τ) is the class of those structures which have the vocabulary τ. By $(\mathfrak{M}, R)$ we denote the structure $\mathfrak{M}$ expanded with a new relation R (or any other interpretation of a symbol) and by $(\mathfrak{A}; \mathfrak{B})$ the obvious two-sorted structure in vocabulary (Voc$(\mathfrak{A})$; Voc$(\mathfrak{B})$) which one gets by putting together the structures $\mathfrak{A}$ and $\mathfrak{B}$.

An abstract logic is a pair $(\mathcal{L}, \models_\mathcal{L})$, where $\mathcal{L}$ is a binary predicate between vocabularies and sentences, usually denoted by "$\phi \in \mathcal{L}(\tau)$", and $\models_\mathcal{L}$ is a ternary predicate between models, vocabularies, and sentences, usually denoted by "$\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$". It is customary to speak about "the logic $\mathcal{L}$" and to shorten the predicates "$\phi \in \mathcal{L}$" and "$\mathfrak{M} \models \phi$" if there is no risk of ambiquity. An abstract logic is assumed to have the following properties:

- Syntactic expansion property: if $\phi \in \mathcal{L}(\tau)$ and $\tau \subseteq \sigma$, then $\phi \in \mathcal{L}(\sigma)$.
- Isomorphism property: if $\mathfrak{M} \cong \mathfrak{N}$ and $\mathfrak{M} \models \phi$, then $\mathfrak{N} \models \phi$.
- Reduction property: if $\phi \in \mathcal{L}(\tau)$ and $\tau \subseteq$ Voc$(\mathfrak{M})$, then

$$\mathfrak{M} \models \phi \iff \mathfrak{M}\restriction\tau \models \phi.$$

A logic $\mathcal{L}$ has the *substitution property* if for every n-ary relation symbol R and for every sentence $\phi \in \mathcal{L}(\tau \cup \{R\})$ and $\psi \in \mathcal{L}(\tau \cup \{x_1, \ldots, x_n\})$ there exists a sentence $\phi^* \in \mathcal{L}(\tau)$ such that for every structure $\mathfrak{M} \in$ Str(τ)

$$\mathfrak{M} \models \phi^* \iff (\mathfrak{M}, \psi^\mathfrak{M}) \models \phi,$$

where $\psi^\mathfrak{M} = \{\, (a_1, \ldots, a_n) \in M^n \,:\, (\mathfrak{M}, a_1, \ldots, a_n) \models \psi \,\}$, and if similar conditions hold for the other kinds of symbols. In this case we denote $\phi^* = \phi[R \mapsto \psi(x_1, \ldots, x_n)]$, leaving out the constant symbols if there is no ambiquity.

The most trivial logic we use is L_{basic} containing only atomic and negated atomic sentences. The logic $L_{\omega\omega}$ extends L_{basic} by conjunction, disjunction, negation, and universal and existential quantification. For an infinite cardinal κ, the logic $L_{\kappa\omega}$ extends $L_{\omega\omega}$ by conjunction and disjunction of less than κ sentences, and $L_{\infty\omega}$ by arbitrary conjunctions and disjunctions. The logic $L_{\infty\kappa}$ extends $L_{\infty\omega}$ by existential and universal quantification over less than κ variables. $L_{\infty\infty}$ allows quantification over any number of variables. All the above logics can be extended using Lindström quantifiers.

The game logic $L_{\infty\mathrm{G}}$ is the logic $L_{\infty\omega}$ extended by conjunctive and disjunctive game sentences of form

$$(\forall x_i\, \exists y_i)_{i<\omega} \bigwedge_{i<\omega} \phi_i(x_0 y_0 \ldots x_i y_i) \quad \text{and} \quad (\exists x_i\, \forall y_i)_{i<\omega} \bigvee_{i<\omega} \phi_i(x_0 y_0 \ldots x_i y_i).$$

The Vaught game logics $V_{\infty\kappa}$ extend the logics $L_{\infty\kappa}$ by conjunctive and disjunctive Vaught sentences

$$\begin{aligned} &\Big(\forall x_i \bigwedge_{a_i \in A} \exists y_i \bigvee_{b_i \in A}\Big)_{i<\kappa} \bigwedge_{i<\kappa} \phi_{a_0 b_0 \ldots a_i b_i}(x_0 y_0 \ldots x_i y_i) \quad \text{and} \\ &\Big(\exists x_i \bigvee_{a_i \in A} \forall y_i \bigwedge_{b_i \in A}\Big)_{i<\kappa} \bigvee_{i<\kappa} \phi_{a_0 b_0 \ldots a_i b_i}(x_0 y_0 \ldots x_i y_i). \end{aligned}$$

The sentences of the logic $M_{\kappa\lambda}$ are pairs (T, L), where T is a κ, λ-leaftree and L is a labelling function the domain of which is T and the values of which satisfy the following conditions:

- If $u \in T$ is a leaf, $L(u)$ is a sentence in the basic logic L_{basic}.
- If $u \in T$ is not a leaf, $L(u)$ is $\bigwedge$, $\bigvee$, or a quantification symbol $\forall x$ or $\exists x$, where x is a constant symbol.
- If $L(u)$ is $\forall x$ or $\exists x$, then u has exactly one successor and there is no $v \in \mathrm{pred}(u)$ such that $L(v)$ is $\forall x$ or $\exists x$.

The vocabulary of a $M_{\kappa\lambda}$-sentence $\phi = (T, L)$ is defined as follows. For each $u \in T$ let $c(u)$ be the set of those constant symbols which occur in $L(v)$ for $v < u$. Each branch $p \in [T]$ has a maximal element u_p and, as mentioned above, $L(u_p)$ is a sentence of L_{basic}. The vocabulary of the sentence ϕ is

$$\mathrm{Voc}(\phi) = \bigcup_{p \in [T]} \mathrm{Voc}(L(u_p)) \setminus c(u_p).$$

If u is a node in the syntax tree T of a $M_{\kappa\lambda}$ sentence ϕ, the sentence $\phi_u = (T_u, L{\restriction}T_u)$ is a *subsentence* of ϕ.

The semantics of the logic $M_{\kappa\lambda}$ is defined by a game. Let $\phi = (T, L)$ be a sentence of $M_{\kappa\lambda}$ and let $\mathfrak{M}$ be a structure such that $\mathrm{Voc}(\phi) \subseteq \mathrm{Voc}(\mathfrak{M})$. Informally the semantic game $\mathrm{S}(\mathfrak{M}, \phi)$ is played as follows: during the game we traverse the syntax tree T from root to leaf. Player $\forall$ makes a move when we are in a node labelled $\forall x$ or $\bigwedge$: if the node is labelled $\forall x$, he picks an element from the model, and in any case he picks a subformula (i.e. a successor in the syntax tree). Player $\exists$ moves similarly in nodes labelled $\exists x$ or $\bigvee$. Finally, when a leaf is reached, $\exists$ wins if the basic sentence, which is the label of the leaf, is true.

Formally the positions of the semantic game are tuples (u, f), where $u \in T$ and f maps the constants of set $c(u)$ to elements of $\mathfrak{M}$. The tree $T_{\mathfrak{M}}$ is ordered by

$$(u, f) \leq (v, g) \iff u \leq v \wedge f \subseteq g.$$

The mover in a non-final position (u, f) is $\forall$ if $L(u)$ is $\bigwedge$ or $\forall x$ for some constant x, and $\exists$ otherwise. Each play p in the semantic game has a final position (u_p, f_p), where u_p is a leaf of T and f_p interprets the constants in $c(u_p)$. Player $\exists$ wins the play p if $\mathfrak{M}(f_p) \models L(u_p)$, where $\mathfrak{M}(f_p)$ stands for the model $\mathfrak{M}$ expanded by interpretations f_p.

Let it be noted here that the semantic game of a $M_{\infty\infty}$-sentence is not always determined. The restriction of $M_{\infty\kappa}$ to those sentences the semantic game of which is always determined is denoted by $M_{\infty\kappa}^{\mathrm{det}}$.

If $\phi = (T, L) \in M_{\infty\infty}$, the *quantifier rank* of ϕ is the tree

$$\mathrm{qr}(\phi) = \{\, u \in T \;:\; L(u) \text{ is a quantification } \}.$$

If T is a tree, $M^{\leq T}$ is the class of those $\phi \in M_{\infty\infty}$ for which $\mathrm{qr}(\phi) \leq T$.

The expressive power of a logic can be approached from two angles. The first approach is to look at single sentences only. Each sentence ϕ of an abstract logic determines a class $\mathrm{Mod}(\phi)$ of structures in which the sentence ϕ is true. For logics $\mathcal{L}$ and $\mathcal{L}'$ we write $\mathcal{L} \leq \mathcal{L}'$ if for each sentence $\phi \in \mathcal{L}$ there is a sentence $\phi' \in \mathcal{L}'$ such that $\mathrm{Mod}(\phi) = \mathrm{Mod}(\phi')$. Moreover, a mapping $t\colon \mathcal{L} \to \mathcal{L}'$ such that for every structure $\mathfrak{M}$

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff \mathfrak{M} \models_{\mathcal{L}'} t(\phi)$$

is called *a translation*. The following chains give trivial examples of translations:

$$L_{basic} \leq L_{\omega\omega} \leq L_{\omega_1\omega} \leq L_{\infty\omega} \leq L_{\infty\omega_1} \quad \text{and} \quad L_{\infty\omega} \leq L_{\infty\mathrm{G}} \leq V_{\infty\omega} \leq V_{\infty\omega_1}.$$

On the other hand, it is known that $L_{\infty\mathrm{G}}$ does not translate into $L_{\infty\infty}$: it is not hard to write a sentence of $L_{\infty\mathrm{G}}$ stating "$(M, \leq)$ is a well-ordering of type $\gamma + \gamma$", but this is not expressible in $L_{\infty\infty}$ (see [M]).

If $\mathbf{A}$ is a class, a mapping $a\colon \mathbf{A} \times \mathcal{L} \to \mathcal{L}'$ is called a *disjunctive approximation* if for every sentence ϕ of $\mathcal{L}$ and for every structure $\mathfrak{M}$

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff \mathfrak{M} \models_{\mathcal{L}'} a(x, \phi) \text{ for some } x \in \mathbf{A}.$$

A *conjunctive approximation* is defined similarly. If the class $\mathbf{A}$ is ordered, we usually try to build the disjunctive approximation in such a way that $a(x, \phi) \Rightarrow a(y, \phi)$ whenever

$y \leq x$. Sometimes the approximation has even the following restriction property: for every *set* X of structures there is $x \in A$ such that, whenever $\mathfrak{M} \in X$,

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff \mathfrak{M} \models_{\mathcal{L}'} a(x, \phi).$$

The idea of approximations originates from classical descriptive set theory in the middle of 1960's. Through the works of Moschovakis the ideas spread to game quantification, and Vaught finally introduced the approximations of $V_{\infty\omega}$ and $L_{\infty \mathrm{G}}$ into $L_{\infty\omega}$ ([Va]).

Another way to compare the strength of various logics is to look at their ability of distinguishing models. For structures $\mathfrak{A}$ and $\mathfrak{B}$ denote $\mathfrak{A} \equiv \mathfrak{B}(\mathcal{L})$, if the structures satisfy the same sentences of logic $\mathcal{L}$. It is clear that if there is a translation from $\mathcal{L}$ to $\mathcal{L}'$, there is an approximation, too. Moreover, if there is an approximation from $\mathcal{L}$ to $\mathcal{L}'$, then

$$\mathfrak{A} \equiv \mathfrak{B}(\mathcal{L}') \implies \mathfrak{A} \equiv \mathfrak{B}(\mathcal{L})$$

for every two structures $\mathfrak{A}$ and $\mathfrak{B}$ having the same vocabulary. We will see later that the converse of the latter implication does not always hold.

The problem of, whether two models can be distinguished with some logic can sometimes be solved using a game of two players. Typically, each logic has its own game, the Ehrenfeucht-Fraïssé game presented in the previous section is a suitable game for the M-languages. Without proof we state the following proposition:

5.1. Proposition ([Kt]). *Let T be a tree. Player $\exists$ wins* $\mathrm{EF}_T(\mathfrak{A}, \mathfrak{B})$ *if and only if* $\mathfrak{A} \equiv \mathfrak{B}(M^{\leq T})$. □

The main bulk of this work is a study of translations and approximations between various kinds of logics. We close the preliminary section with an easy case.

The essential difference between $M_{\infty\omega}$ and $L_{\infty\omega}$ is syntactical. A sentence of $M_{\infty\omega}$ is set theoretically flat: the tree rank of its well-founded syntax tree is not related to the set-theoretical rank of the sentence. On the other hand the set-theoretical rank of a sentence of $L_{\infty\omega}$ always exceeds the tree rank of its syntax tree. We saw earlier that the mapping which gives a tree rank for a well-founded tree is not primitive recursive. Next we show that the same holds for the translation mappings from $M_{\infty\omega}$ to $L_{\infty\omega}$: there exists no primitive recursive one, but the canonical one is effective enough so that admissible sets are closed under it.

5.2. Lemma. *There exists no p.r. translation from $M_{\infty\omega}$ into $L_{\infty\omega}$.*

Proof. Note first that for every ordinal α

$$\exists \text{ wins } \mathrm{EF}_{B_\alpha}(\mathfrak{A}, \mathfrak{B}) \iff \mathfrak{A} \equiv \mathfrak{B}(M^{\leq B_\alpha}) \iff \mathfrak{A} \equiv \mathfrak{B}(L^{\alpha}_{\infty\omega}),$$

where B_α is the canonical well-founded tree having ordinal α and $L^{\alpha}_{\infty\omega}$ is the logic $L_{\infty\omega}$ restricted to formulas of quantifier rank less than or equal to α. The first equivalence above is Proposition 5.1, the latter is a known fact due to Karp [Kp]. From the above it is clear that $(\alpha, \in) \equiv (\beta, \in)\ (L_{\infty\omega})$ if and only if $\alpha = \beta$.

Let us say that ordinals α and β are *γ-equivalent*, $\alpha \sim_\gamma \beta$, if there exist ordinals ζ, ξ, and η such that

$$\alpha = \omega^\gamma \cdot \zeta + \eta \quad \text{and} \quad \beta = \omega^\gamma \cdot \xi + \eta,$$

where either $\zeta = \xi = 0$ or $\zeta, \xi > 0$ (ordinal arithmetics). If $\alpha \geq \beta$, denote the (unique) ordinal γ for which $\beta + \gamma = \alpha$ by $\alpha \dot{-} \beta$.

Claim A: If α and β are γ-equivalent, $\delta < \gamma$, and $\alpha' < \alpha$, there exists $\beta' < \beta$ such that

$$\alpha' \sim_\delta \beta' \quad \text{and} \quad \alpha \dot{-} \alpha' \sim_\delta \beta \dot{-} \beta'.$$

Note first that $\eta + \omega^\delta = \omega^\delta$ whenever $\eta < \omega^\delta$. This implies that $\alpha \sim_\delta \alpha \dot{-} \alpha'$, if $\alpha \dot{-} \alpha' \geq \omega^\delta$. Suppose then that $\alpha = \omega^\gamma \cdot \zeta + \eta$ and $\beta = \omega^\gamma \cdot \xi + \eta$, where ζ and ξ are nonzero (the other case is trivial). If $\alpha' = \omega^\gamma \cdot \zeta + \eta'$ for some η', let $\beta' = \omega^\gamma \cdot \xi + \eta'$. Obviously $\alpha' \sim_\gamma \beta'$ and $\alpha \dot{-} \alpha' = \eta \dot{-} \eta' = \beta \dot{-} \beta'$, which implies the claim. On the other hand, if $\alpha' < \omega^\gamma \cdot \zeta$, let ζ' and $\eta' < \omega^\delta$ be such that $\alpha' = \omega^\delta \cdot \zeta' + \eta'$. Choose $\xi' = 0$ if $\zeta' = 0$, and $\xi' = 1$ otherwise. Let $\beta' = \omega^\delta \cdot \xi' + \eta'$. Clearly α' and β' are δ-equivalent. Since

$$\alpha' + \omega^\delta = \omega^\delta \cdot (\zeta' + 1) \leq \omega^\gamma \cdot \zeta \leq \alpha,$$

the ordinals $\alpha \dot{-} \alpha'$ and α are δ-equivalent. Moreover,

$$\beta' + \omega^\delta = \omega^\delta(\xi' + 1) \leq \omega^\delta \cdot 2 \leq \omega^\gamma \leq \beta,$$

and thus $\beta \dot{-} \beta' \sim_\delta \beta$, which completes the proof of the claim A.

Claim B: If ordinals α and β are γ-equivalent, then $(\alpha, \in) \equiv (\beta, \in)\ (L^\gamma_{\infty\omega})$.

By the equivalence stated in the beginning of the proof it is enough to show that $\exists$ wins $\mathrm{EF}_{B_\gamma}((\alpha, \in), (\beta, \in))$. Suppose we are in the middle of a play, the players have chosen n elements from each model, and $\forall$ has chosen a descending chain of ordinals $\gamma > \gamma_0 > \ldots > \gamma_{n-1}$. The chosen elements divide α and β into segments:

$$\alpha = \alpha_0 + \cdots + \alpha_n, \quad \beta = \beta_0 + \cdots + \beta_n.$$

Assume $\exists$ has managed to make such choices that α_i and β_i are γ_{n-1}-equivalent for each i (here $\gamma_{-1} = \gamma$). Suppose $\forall$ picks $\gamma_n < \gamma_{n-1}$ and, say, an element in α. This element belongs to one of the segments α_i and splits it into two parts $\alpha_i = \alpha_i' + \alpha_i''$. Since the segments were originally γ_{n-1}-equivalent, by the claim A $\exists$ manages to pick such an element in β that splits the corresponding segment β_i into two parts $\beta_i = \beta_i' + \beta_i''$ and, moreover, $\alpha_i' \sim_{\gamma_n} \beta_i'$ and $\alpha_i'' \sim_{\gamma_n} \beta_i''$. In this way $\exists$ wins the game.

Claim C: There is no p.r. translation from $M_{\infty\omega}$ to $L_{\infty\omega}$.

For contradiction, suppose that t is a p.r. translation from $M_{\infty\omega}$ into $L_{\infty\omega}$. Let g be a p.r. mapping, increasing on ordinals, such that $\text{rank}(t(x)) \leq g(\text{rank}(x))$ for every set x. Let $\gamma = g(\omega + \omega) + 1$, and let ϕ_γ be a sentence in $M_{\infty\omega}$ which is true in ω^γ and false in $\omega^\gamma \cdot 2$. Since γ is obviously countable, we may assume that the set theoretical rank of ϕ_γ is less than $\omega + \omega$. Now, by the claim B above, the quantifier rank of $t(\phi_\gamma)$ — and thus its set-theoretical rank — must exceed γ, which is a contradiction. □

5.3. Lemma. *There is a translation* $t\colon M_{\infty\omega} \to L_{\infty\omega}$, *primitive recursive in a mapping which maps every well-founded tree to its ordinal.*

Proof. Let $\phi = (T, L)$, where T is the syntax tree and L is the labelling. Clearly the syntax operations of $L_{\infty\omega}$ are p.r.; denote them by $OP_\wedge$, $OP_\vee$, etc. Thus we are able to define a p.r. mapping $s\colon \text{Ord} \times T \to L_{\infty\omega}$ by induction as follows:

- If w is a leaf in T, define $s(0, w) = L(w)$.
- If $s(\alpha, w)$ is defined, define $s(\beta, w) = s(\alpha, w)$ for each $\beta > \alpha$.
- If $s(\alpha, u)$ is defined for every successor u of w, define

$$s(\alpha + 1, w) = OP_{L(w)}\{\, s(\alpha, u) \;:\; u \in \text{succ}(w) \,\}.$$

- Those values $s(\alpha, w)$ that are not defined by the above rules are left undefined.

It is not hard to see that $s(\alpha, w)$ is a sentence equivalent to the subsentence ϕ_w whenever α is at least the tree rank of w in T. Now we can choose $t(\phi) = s(\gamma, root(T))$, where γ is the ordinal of T. □

5.4. Lemma. *Suppose A is admissible and $\phi \in M_{\infty\omega} \cap A$. There is $\phi^* \in L_{\infty\omega} \cap A$ such that $\mathfrak{M} \models \phi \iff \mathfrak{M} \models \phi^*$ for every model $\mathfrak{M}$.*

Proof. Admissible sets are closed under the primitive recursive mapping s defined in the proof of the previous lemma (the initial mapping $x \mapsto \omega$ is not needed for constructing s). Since admissible sets are closed under the mapping which maps well-founded trees to their ordinals, the result follows immediately from the previous lemma. □

5.5. Lemma. *There exists a p.r. translation* $t\colon M_{\omega\omega} \to L_{\omega\omega}$.

Proof. Let $\phi \in M_{\omega\omega}$. The rank of the syntax tree of ϕ is now finite. Thus, if s is the p.r. map of the proof of 5.3, we can choose $t(\phi) = s(\omega, \phi)$. □

II Absoluteness

As it is generally known, we have set-theoretical structures abundantly at hand in the set-theoretical universe. One of the standard tools in set theory has always been playing with these "internal models" for showing certain interesting facts about the set-theoretical universe. When constructing an internal model, we usually want it to share some set-theoretical properties of the real world. This is where the concept of absoluteness comes in.

Basically, absoluteness is a property of a definable predicate relative to a pair of structures: a predicate is absolute relative to the structures if it expresses the same thing in both of them. In the general case it is of course hard to specify what "the same thing" denotes. Thus we only consider pairs $(\mathfrak{A}, \mathfrak{B})$ where $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, and define that a predicate P is *absolute relative to* $(\mathfrak{A}, \mathfrak{B})$ if and only if, for every element a of $\mathfrak{A}$, the predicate $P(a)$ is true in $\mathfrak{A}$ exactly when it is true in $\mathfrak{B}$.

If then $\mathfrak{A} \subseteq \mathfrak{B}$ are models of a set theory T, is there any general way of indicating, which definable predicates are absolute? If one considers such arbitrary models, a result of first-order model theory states that only those definable predicates are absolute which are provably equivalent to an existential formula and to an universal formula. Since this leaves out a large number of interesting predicates, it is a tradition to deal with only those pairs of structures $(\mathfrak{A}, \mathfrak{B})$ where $\mathfrak{B}$ is an end-extension of $\mathfrak{A}$. Thus we define that a predicate is *absolute relative to a set theory* T if it is absolute relative to $(\mathfrak{A}, \mathfrak{B})$ for any pair of models of T such that $\mathfrak{A} \subseteq_{\text{end}} \mathfrak{B}$. These predicates can be syntactically characterized: an application of interpolation by Feferman and Kreisel [FK, F] indicates that a predicate is absolute relative to a theory T if and only if it is Δ_1-definable in T.

For some predicates it is hard to find the theories relative to which they are absolute, but there are other means of restricting the pairs of structures to be taken in account. Since the notion of forcing can be defined even in models of a weak set theory (e.g. KP), it is natural to define a predicate as *absolute relative to forcing* if it is absolute relative to $(\mathfrak{A}, \mathfrak{A}^*)$, where $\mathfrak{A}$ is a model of a set theory and $\mathfrak{A}^*$ its generic extension. Of course we may additionally demand that such forcing has certain special properties, such as ω_1-closedness.

Persistence is defined similarly as absoluteness: a predicate P is *upwards persistent* relative to $(\mathfrak{A}, \mathfrak{B})$ if $P(a)$ holds in $\mathfrak{B}$ whenever it holds in $\mathfrak{A}$. Persistence downwards and persistence relative to a theory is defined analogously.

6. Absolute logics

6.1. Definition. Let T be a set theory. An abstract logic $\mathcal{L}$ is *absolute relative to* T if there is a Σ_1^T-predicate P and a Δ_1^T-predicate Q such that the following claims are provable from T:

(i) If τ is a vocabulary, then

$$\phi \in \mathcal{L}(\tau) \iff P(\tau, \phi).$$

(ii) If τ is a vocabulary, $\phi \in \mathcal{L}(\tau)$, and if $\mathfrak{M} \in \mathrm{Str}(\tau)$, then

$$\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi \iff Q(\mathfrak{M}, \tau, \phi).$$

If the predicates P and Q are Δ_1^T-definable, we say that the logic $\mathcal{L}$ is *first-order relative to* T.

Similarly, for a pair $\mathfrak{A} \subseteq_{\text{end}} \mathfrak{B}$ of models of set theory, a logic $\mathcal{L}$ is *absolute relative to* $(\mathfrak{A}, \mathfrak{B})$ if the predicate "$\phi \in \mathcal{L}(\tau)$" is upwards persistent and the predicate $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$ is absolute relative to $(\mathfrak{A}, \mathfrak{B})$. Moreover, if the syntax predicate is absolute, we say that $\mathcal{L}$ is *first-order* relative to $(\mathfrak{A}, \mathfrak{B})$.

The concept of an absolute logic was first presented by Barwise in [B4]. He chose the syntax of an absolute logic to be Σ_1 instead of Δ_1, since he did not want to rule out certain logics like $L_{\omega_1\omega}$. In order to have a name for the absolute logics with an absolute syntax I call them first-order logics following Burgess in [Bu]. In many contexts there is no practical difference, and one can turn an absolute logic into a first-order logic simply by changing the syntax.

The latter formation, the absoluteness of a logic relative to a pair of structures, is useful when there exists no nice theory relative to which a logic could be absolute. Thus, when we say that a logic is absolute relative to, say, ω_1-closed forcing, we mean that it is absolute relative to $(\mathfrak{A}, \mathfrak{A}^*)$ for every model $\mathfrak{A}$ of ZFC and its generic extension $\mathfrak{A}^*$ in an ω_1-closed forcing.

Since we are dealing with various different set theories, a natural question is, does absoluteness relative to some theory T imply absoluteness relative to some other theory T'. If the theory T' is a consistent extension of T, the answer is clearly yes, no matter whether T' has more predicate or constant symbols. Thus the fragments of ZFC (e.g. KP) or extensions of ZFC (e.g. ZFC($\mathcal{P}_{\omega_1}$)) do not propose any problems.

A more problematic case is the theory KPU. The difficulties arise from the fact that the concept of a structure is different in KPU from the other theories: we demand that the universe of a structure must consist of urelements only. Thus, when giving a set-theoretical definition for a logic, say $L_{\omega\omega}$ in KPU and in ZFC, it is not immediately clear that the definitions really define the same logic.

The "sameness" problem of the logics can be solved by reducing them to the metatheory. Consider first a logic $\mathcal{L}$ in a set theoretical universe V. The logic $\mathcal{L}$ is actually a pair of formulas in a metalanguage, and V is an element of a metauniverse. Suppose that the metauniverse is set-theoretical. Each vocabulary τ in V gives rise to a vocabulary τ^V in the metauniverse, and similarly each structure $\mathfrak{M}$ in V gives rise to a structure $\mathfrak{M}^V$ in the metauniverse. Moreover, a sentence $\phi \in V$ can be turned into a structure $\phi^V = (\mathrm{TC}(\{\phi\}), \in)$ in the metauniverse. We may call these objects τ^V, $\mathfrak{M}^V$, and ϕ^V *metaforms* of the set-theoretical objects τ, $\mathfrak{M}$ and ϕ. The set-theoretical predicates "$\phi \in \mathcal{L}$" and "$\mathfrak{M} \models \phi$" thus naturally turn into predicates between metaforms. Thus, if we have a definition of a logic $\mathcal{L}$ for example in both KP and KPU, we may compare the definitions by examining the predicates between metaforms. To illustrate the method, we will next show that absoluteness relative to KPU implies absoluteness relative to KP.

6.2. Lemma. *If a logic $\mathcal{L}$ is absolute (first order) relative to KPU, it is absolute (first-order) relative to KP.*

Proof. It is enough to show the result for an absolute logic, since the first-order logics are treated similarly. Let P and Q be the Σ_1- and Δ_1-predicates, respectively, which define a logic $\mathcal{L}$ in KPU. We need to find a Σ_1-predicate P^* and a Δ_1-predicate Q^* which define the same logic in KP. The basic idea is to define in KP an internal model $(V^*, \in^*, U^*)$ of KPU, and mappings $\mathfrak{M} \mapsto \mathfrak{M}^*$ and $x \mapsto x^*$ which transfer structures of V into structures in V^* and sets (vocabularies, sentences) into pure sets of V^*.

Let the class of urelements U^* be the class of pairs $(x, 0)$ for $x \in V$. Let V^* be the class of pairs (x, α), where either $\alpha = 0$ or $x \subseteq V^*$ and $\alpha = \sup\{\, \beta : \exists y\, (y, \beta) \in x \,\} + 1$. Let $(y, \beta) \in^* (x, \alpha)$ if and only if $(y, \beta) \in x$ and $\alpha > 0$. The predicate V^* is clearly Δ_1^{KP}, and the predicates $\in^*$ and U^* are Δ_0-definable.

The class $(V^*, \in^*, U^*)$ is now a model of KPU: for example, to see the Δ_0-collection schema, suppose that

$$\forall (y, \beta) \in^* (x, \alpha)\, \exists (z, \gamma)\, [\phi((x, \alpha), (y, \beta), (z, \gamma))]^{V^*}$$

holds with ϕ being a Δ_0-formula. Now $[\phi(\ldots)]^{V^*}$ is Δ_1^{KP}, so by the Δ_1-collection schema there exists $a \in V$ such that

$$\forall (y, \beta) \in^* (x, \alpha)\, \exists (z, \gamma) \in a\, [\phi((x, \alpha), (y, \beta), (z, \gamma))]^{V^*}.$$

By Δ_1-separation $a^* = a \cap V^*$ exists in V, and the required collection $(a^*, \sup \operatorname{ran}(a^*) + 1)$ exists in V^*.

Define embeddings

$$\begin{aligned} &u \colon V \to U^*,\ x \mapsto (x, 0) \quad \text{and} \\ &j \colon V \to V^*, x \mapsto (\{\, j(y) : y \in x \,\}, \operatorname{rank}(x) + 1). \end{aligned}$$

Clearly u is totally Δ_0-definable and j is totally Δ_1-definable in KP. The embedding j makes the pure part of V^* isomorphic with V, and it is used to map vocabularies and sentences. Thus we are able to define the syntax:

$$\phi \in \mathcal{L}^*(\tau) \iff [P(j(\tau), j(\phi))]^{(V^*, \in^*, U^*)}.$$

It is not hard to see that this relation is Σ_1^{KP}.

To embed structures, suppose $\mathfrak{M} = (M, F) \in \operatorname{Str}(\tau)$. We let $\mathfrak{M}^*$ be a pair (M^*, F^*) in V^*, where M^* is the set of $u(x)$ for $x \in M$ in V^* and F^* is defined in the obvious way in order to make $\mathfrak{M}^*$ a $j(\tau)$-structure. Let

$$\mathfrak{M} \models_{\mathcal{L}^*(\tau)} \phi \iff [Q(\mathfrak{M}^*, j(\tau), j(\phi))]^{(V^*, \in^*, U^*)}.$$

This relation is Δ_1^{KP}, and we have no problems in seeing that these relations define a logic $\mathcal{L}^*$ in KPU, which is the same logic as $\mathcal{L}$. $\square$

In the other direction we slightfully strengthen the theory KPU. Recall from Section 2 that the predicate "x is finite" is absolute relative to KP, but not absolute relative to KPU. Moreover, we shall later see that the logic $L_{\omega\omega}(Q_0)$ is first-order relative to KP, but not first-order relative to KPU. The problem in reversing the implication lies in that in KP one cannot have a large set without large ordinals, but in KPU this is possible. Let the *high rank axiom*, HR, state that for every set x of urelements there exists a one-to-one mapping from x to pure sets. This axiom is not provable from KPU: let U be an infinite set of urelements. Then $V_\omega(U)$, the collection of sets of finite rank built from urelements in U, is a model of KPU and contains infinite sets of urelements, but every pure set in it is finite.

6.3. Lemma. *Logic $\mathcal{L}$ is absolute (first-order) relative to KP if and only if it is absolute (first-order) relative to* KPU + HR.

Proof. We have already seen "$\Longleftarrow$". The proof of "$\Longrightarrow$" is similar to what was done above. The internal model V^* of KP is the class of pure sets. It is not hard to see that V^* is Δ_1-definable in KPU. Define logic $\mathcal{L}^*$ using the equivalencies

$$\begin{aligned}
\phi \in \mathcal{L}^*(\tau) &\iff [P(\tau,\phi)]^{V^*} \quad \text{and} \\
\mathfrak{M} \models_{\mathcal{L}^*(\tau)} \phi &\iff \exists f\ \big(f\colon M \to V^* \text{ is bijective } \wedge [Q(f''\mathfrak{M},\tau,\phi)]^{V^*}\big) \\
&\iff \forall f\ \big(f\colon M \to V^* \text{ is bijective } \to [Q(f''\mathfrak{M},\tau,\phi)]^{V^*}\big).
\end{aligned}$$

In order to see the last equivalence we need the high rank axiom. □

7. Absoluteness of L-languages

The sentences of L-languages are inductively built from atomic formulas using logical operations. Since inductive construction is absolute (relative to KPU), the L-languages will be absolute as long as the operations used for building them are absolute. For L-languages it is most natural to define three predicates: unary predicate "$\phi \in \mathcal{L}$" to indicate which objects are sentences, a function Voc which maps a sentence to its vocabulary, and the truth predicate.

For example, the first-order logic $L_{\omega\omega}$ is built from atomic sentences by negation, conjunction, disjunction, and quantification. Thus its syntax is generated by a set of rules such as

- atomic sentences are sentences of $L_{\omega\omega}$,
- if ϕ and ψ are sentences of $L_{\omega\omega}$, then $\phi \wedge \psi$ is a sentence of $L_{\omega\omega}$,

and so on. A semantics is defined recursively: for example, the truth of $\phi\wedge\psi$ in a structure $\mathfrak{M}$ depends on the truth of ϕ in $\mathfrak{M}$ and the truth of ψ in $\mathfrak{M}$. The next definition gives an abstract formulation of this kind of language definition.

7.1. Definition. (1) A *syntax rule* is a pair of mappings (f, v) such that the following conditions hold:

(i) f is a partial binary one-to-one mapping.
(ii) If $f(x,y) = z$, then $x \subseteq \mathrm{TC}(z)$.
(iii) v is a binary mapping and $\mathrm{ran}(v)$ is a class of vocabularies.

(2) A syntax rule (f, v) *spans a language* $\mathcal{L}$ by recursion

$$\phi \in \mathcal{L} \iff \exists X\, \exists y\, (X \subseteq \mathcal{L} \wedge \phi = f(X, y)).$$

The vocabularies of these sentences are given by mapping Voc, which is defined by recursion: if $\phi = f(X, y)$ is in $\mathcal{L}$, then

$$\mathrm{Voc}(\phi) = v(\mathrm{Voc}{\restriction}X, y).$$

(3) A *semantics rule* attached to a syntax rule (f, v) is a pair (q, Q), where q is a mapping and Q is a relation such that the following conditions hold:
(i) If the vocabulary of a structure $\mathfrak{M}$ contains the vocabulary of $\phi = f(X, y)$, then $q(\mathfrak{M}, \phi)$ is a set of pairs $(\mathfrak{N}, \psi)$, where $\psi \in X$ and $\mathrm{Voc}(\psi) \subseteq \mathrm{Voc}(\mathfrak{M})$.
(ii) If $\mathfrak{M} \cong \mathfrak{M}'$, then for every $\phi \in \mathcal{L}$ there exists a bijection $f\colon q(\mathfrak{M}, \phi) \to q(\mathfrak{M}', \phi)$ such that if $(\mathfrak{N}', \psi') = f(\mathfrak{N}, \psi)$, then $\psi = \psi'$ and $\mathfrak{N} \cong \mathfrak{N}'$. Moreover, if $t\colon q(\mathfrak{M}, \phi) \to 2$, then

$$Q(\mathfrak{M}, \phi, t) \iff Q(\mathfrak{M}', \phi, t \circ f^{-1}).$$

(4) A semantics rule defines a semantics for the language $\mathcal{L}$ by recursion: if $\phi = f(X, y)$ and $\mathrm{Voc}(\mathfrak{M}) \supseteq \mathrm{Voc}(\phi)$, then $\mathfrak{M} \models_{\mathcal{L}} \phi$ if and only if

$$Q(\mathfrak{M}, \phi, t), \text{ where } t\colon q(\mathfrak{M}, \phi) \to 2, (\mathfrak{N}, \psi) \mapsto \begin{cases} 1 & \text{if } \mathfrak{N} \models_{\mathcal{L}} \psi, \text{ and} \\ 0 & \text{if } \mathfrak{N} \not\models_{\mathcal{L}} \psi. \end{cases}$$

(5) A syntax rule and a semantics rule together form a *language rule*.

The mapping f of a syntax rule maps a set x of already existing sentences and an auxiliary set-theoretical object y to a new sentence $f(x, y)$. Since (ii) in the definition of the syntax rule implies that

$$f(X, y) = \phi \implies \mathrm{rank}(\psi) < \mathrm{rank}(\phi) \text{ for every } \psi \in X,$$

the recursive definition of the language $\mathcal{L}$ in (2) is valid, and the definition gives rise to a natural well-founded order on $\mathcal{L}$: "ψ is an immediate subsentence of ϕ" if there exist X and y such that $\psi \in X$ and $\phi = f(X, y)$. The vocabulary mapping Voc is defined recursively along this subsentence relation.

The semantics is defined in a similar way recursively along the subsentence relation: the truth of a sentence ϕ in a structure $\mathfrak{M}$ is determined, once we know the truth of certain immediate subsentences of ϕ in certain other structures. The mapping q gives the subsentence-structure pairs, and the predicate Q indicates how their semantics influences the semantics of ϕ. Finally, the condition (ii) in the semantic rule ensures that the language rule really spans a logic: it implies that the logic is closed under isomorphism of structures.

7.2. Example. Let us see how first-order logic fits in this frame. We start by defining language rules for single logical operations. Atomic sentences are produced with a rule satisfying

$$\begin{array}{ll} f(\emptyset, \phi) = \phi \text{ when } \phi \text{ is atomic}, & v(\emptyset, \phi) = \text{the vocabulary of } \phi, \\ q(\mathfrak{M}, \phi) = \emptyset, & Q(\mathfrak{M}, \phi, t) \iff \phi \text{ is true in } \mathfrak{M}. \end{array}$$

The other logical operations are not much harder: for example, negation is defined by a rule satisfying

$$\begin{array}{ll} f(\{\phi\}, \neg) = \neg\phi, & v(\phi \mapsto \tau, \neg) = \tau, \\ q(\mathfrak{M}, \neg\phi) = \{(\mathfrak{M}, \phi)\}, & Q(\mathfrak{M}, \neg\phi, t) \iff t(\mathfrak{M}, \phi) = 0, \end{array}$$

and existential quantification by a rule satisfying

$$\begin{array}{ll} f(\{\phi\}, \exists x) = \exists x\phi, & v(\phi \mapsto \tau, \exists x) = \tau \setminus \{x\}, \\ q(\mathfrak{M}, \exists x\phi) = \{\, ((\mathfrak{M}, a), \phi) \,:\, a \in M \,\}, & Q(\mathfrak{M}, \exists x\phi, t) \iff 1 \in \operatorname{ran}(t). \end{array}$$

These pieces of language rules can easily be combined into a single rule.

7.3. Definition. Let T be a set theory. A language rule is *absolute relative to* T if the mappings f and v of the syntax rule, and the mapping q of the semantic rule are Σ_1-definable in T, the relation Q of the semantic rule is Δ_1-definable in T, and if

$$T \vdash (f, v) \text{ is a syntax rule and } (q, Q) \text{ is a semantic rule .}$$

Moreover, if the syntactical mappings f and v, and the predicates "$(x, y) \in \operatorname{dom}(f)$" and "$\phi \in \operatorname{ran}(f)$" are Δ_1-definable in T, we say that the language rule is *first-order relative to* T.

7.4. Lemma. *Let* $T \supseteq$ KP *(or* $T \supseteq$ KPU*). If a logic* $\mathcal{L}$ *is spanned by a rule which is absolute relative to* T *(first-order relative to* T*), then* $\mathcal{L}$ *is absolute relative to* T *(first-order relative to* T*).*

Proof. Syntax: Consider first the case of an absolute language rule. To see that the predicate "$\phi \in \mathcal{L}$" is Σ_1-definable in T, note that $\phi \in \mathcal{L}$ if and only if there is a set S consisting of pairs (ψ, y) such that

(i) (ϕ, y) is in S for some y, and

(ii) if (ψ, y) is in S, there exists X such that

$$\psi = f(X, y) \quad \text{and} \quad \forall \eta \in X \, \exists z (\eta, z) \in S.$$

Similarly, $\operatorname{Voc}(\phi) = \tau$ if and only if there is a function u such that

(i) $u(\phi) = \tau$, and

(ii) for each ψ in dom(u) there exists X and y such that

$$f(X,y)=\psi, \quad X\subseteq \mathrm{dom}(u), \quad \text{and} \quad u(\psi)=v(u{\restriction}X,y).$$

Thus Voc is Σ_1-definable in T. Now

$$\phi\in\mathcal{L}(\tau) \iff \phi\in\mathcal{L}\wedge \mathrm{Voc}(\phi)\subseteq\tau.$$

In the case of a first-order syntax rule, the predicate $\phi\in\mathcal{L}$ is Δ_1^T, since it is defined by Δ_1-recursion:

$$\begin{aligned}\phi\in\mathcal{L} &\iff \phi\in\mathrm{ran}(f)\wedge\forall x\,\forall y(f(x,y)=\phi\rightarrow x\subseteq\mathcal{L})\\ &\iff \phi\in\mathrm{ran}(f)\wedge\exists x\,\exists y(f(x,y)=\phi\wedge x\subseteq\mathcal{L}).\end{aligned}$$

In this case the mapping Voc is totally Σ_1-definable, and thus $\phi\in\mathcal{L}(\tau)$ becomes a Δ_1^T-predicate.

Semantics: Denote $P(\mathfrak{M},\phi,t)$ if and only if t is a mapping from pairs $(\mathfrak{N},\psi)$ to $\{0,1\}$ such that the following conditions hold:

(i) $(\mathfrak{M},\phi)$ is in the domain of t.

(ii) If $(\mathfrak{N},\psi)$ is in the domain of t, then $q(\mathfrak{N},\psi)$ is a subset of the domain of t.

(iii) $t(\mathfrak{N},\psi)=1$ if and only if $Q(\mathfrak{N},\psi,t{\restriction}q(\mathfrak{N},\psi))$.

By recursion on the structure of the sentences in $\mathcal{L}$ one can easily define a mapping $(\phi,\mathfrak{M})\mapsto t_{\phi,\mathfrak{M}}$ such that $P(\mathfrak{M},\phi,t_{\phi,\mathfrak{M}})$ whenever $\phi\in\mathcal{L}$ and $\mathfrak{M}\in\mathrm{Str}(\mathrm{Voc}(\phi))$. Suppose, namely, that $\phi=f(X,y)$, and suppose $t_{\psi,\mathfrak{N}}$ is defined for every $\psi\in X$ and $\mathfrak{N}\in\mathrm{Str}(\mathrm{Voc}(\psi))$. Let

$$t_{\phi,\mathfrak{M}}=\{((\mathfrak{M},\phi),i)\}\cup\bigcup\{\,t_{\psi,\mathfrak{N}}\,:\,(\mathfrak{N},\psi)\in q(\mathfrak{M},\phi)\,\},$$

where $i\in\{0,1\}$ is picked with Q in the obvious way. Now an easy induction on the structure of ϕ shows that if $\mathrm{Voc}(\phi)\subseteq\mathrm{Voc}(\mathfrak{M})$ and $P(\mathfrak{M},\phi,t)$, then $\mathfrak{M}\models\phi$ if and only if $t(\mathfrak{M},\phi)=1$. Thus we conclude: if $\phi\in\mathcal{L}$ and $\mathrm{Voc}(\phi)\subseteq\mathrm{Voc}(\mathfrak{M})$, then

$$\begin{aligned}\mathfrak{M}\models\phi &\iff \exists t\,\big(P(\mathfrak{M},\phi,t)\wedge t(\mathfrak{M},\phi)=1\big)\\ &\iff \forall t\,\big(P(\mathfrak{M},\phi,t)\rightarrow t(\mathfrak{M},\phi)=1\big).\end{aligned}$$

Everything stated above is provable from T, and the predicate P is clearly Σ_1^T. □

This theorem can be applied for showing some logics as well as certain closure operations to be absolute. As Example 7.2 suggests, it is convenient first to define rules for single logical operations, and then to combine these rules into rules of languages. We say that language rules (f,v,q,Q) and (f',v',q',Q') are *compatible* if $f\cup f'$ is a one-to-one mapping, and whenever $\phi=f(X,y)=f'(X,y)$, then $v(t,y)=v'(t,y)$, $q(\mathfrak{M},\phi)=q'(\mathfrak{M},\phi)$, and $Q(\mathfrak{M},\phi,t)\iff Q'(\mathfrak{M},\phi,t)$.

7.5. Lemma. *Suppose $R = (f, v, q, Q)$ and $R' = (f', v', q', Q')$ are compatible language rules. There exists a language rule $R^* = (f^*, \dots)$ such that $f^* = f \cup f'$, and logics $\mathcal{L}$ and $\mathcal{L}'$ defined by R and R', respectively, are sublogics of $\mathcal{L}^*$, the logic defined by R^*. Moreover, if R and R' are absolute (first-order), then so is R^*.*

Proof. Define the rule R^* as follows: let $f^* = f \cup f'$, and

$$v^*(u, y) = \begin{cases} v(u, y) & \text{if } (\mathrm{dom}(u), y) \in \mathrm{dom}(f), \\ v'(u, y) & \text{if } (\mathrm{dom}(u), y) \in \mathrm{dom}(f'), \text{ and} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Define q^* and Q^* in a similar way. □

Next discuss some rules. Usually there are no problems in defining the individual rules to be pairwise compatible. However, we concentrate here on the logical contents of the rules without stressing compatibility. Though we occasionally define incompatible rules, even they can be made compatible by minor changes. So there is no objection to assuming that all rules used for constructing a logic are compatible.

7.6. Example. Let $\mathcal{L}$ be a logic. The trivial language rule defined by

$$\begin{array}{ll} f(\emptyset, \phi) = \phi \iff \phi \in \mathcal{L}, & v(\emptyset, \phi) = \mathrm{Voc}(\phi), \\ q(\emptyset, y) = \emptyset, & Q(\mathfrak{M}, \phi, t) \iff \mathfrak{M} \models_{\mathcal{L}} \phi \end{array}$$

is absolute relative to T (first-order relative to T) if and only if $\mathcal{L}$ is absolute relative to T (first-order relative to T).

7.7. Example. The following rules are first-order relative to KPU: atomic sentences, negation, conjunction (of two sentences), infinite conjunction (of arbitrary size), and existential quantification. (We have already presented some of these in Example 7.2.) Thus the logics $L_{\omega\omega}$ and $L_{\infty\omega}$ are first-order relative to KPU.

7.8. Example. Countable conjunction is absolute relative to KPU, since it can be defined by the following rule:

$$\begin{array}{l} f(X, \emptyset) = \bigwedge X \text{ when } X \text{ is countable}, \\ v(\langle \tau_\psi \rangle_{\psi \in X}, \emptyset) = \bigcup_{\psi \in X} \tau_\psi, \\ q(\mathfrak{M}, \bigwedge X) = \{ (\mathfrak{M}, \psi) : \psi \in X \}, \\ Q(\mathfrak{M}, \bigwedge X, t) \iff t(\mathfrak{M}, \psi) = 1 \text{ for every } \psi \in X. \end{array}$$

Thus the logic $L_{\omega_1\omega}$ is absolute relative to KPU.

7.9. Lemma. *Let $T \supseteq$ KP or $T \supseteq$ KPU. A Lindström quantifier Q rule is first-order relative to T if and only if the class defining Q is Δ_1-definable in T.*

Proof. To simplify notations, consider a quantifier Q of type $(1,1)$, i.e. a quantifier binding two sentences and one constant in each sentence. (The general case is similar.) As we know, the semantics of a sentence $Qxy\phi(x)\psi(y)$ is defined by the equation

$$\mathfrak{M} \models Qxy\phi(x)\psi(y) \iff (M, \phi^{\mathfrak{M}}, \psi^{\mathfrak{M}}) \in \mathcal{K}_Q,$$

where $\phi^{\mathfrak{M}} = \{\, a \in M \;:\; (\mathfrak{M}, a) \models \phi \,\}$, and $\psi^{\mathfrak{M}}$ is similar. Let now

$$\begin{aligned}
&f(X, y) = z \iff X = \{\phi, \psi\},\ y = \langle(\phi, c), (\psi, d)\rangle, \text{ and } z = Qcd\phi\psi,\\
&v(t, y) = (t(\phi) \setminus \{c\}) \cup (t(\psi) \setminus \{d\}) \text{ when } y = \langle(\phi, c), (\psi, d)\rangle,\\
&q(\mathfrak{M}, Qcd\phi\psi) = \{\, ((\mathfrak{M}, a), \phi), ((\mathfrak{M}, a), \psi) \;:\; a \in M \,\}, \text{ and}\\
&Q(\mathfrak{M}, Qcd\phi\psi, t) \iff (M, \phi^{\mathfrak{M},t}, \psi^{\mathfrak{M},t}) \in \mathcal{K}_Q,
\end{aligned}$$

where $\phi^{\mathfrak{M},t} = \{\, a \in M \;:\; t((\mathfrak{M}, a), \phi) = 1 \,\}$ and $\psi^{\mathfrak{M},t}$ is similar. The mappings f, v, and q are clearly Δ_0^{KP}. To see that Q is Δ_1^T, note that

$$\begin{aligned}
&(M, \phi^{\mathfrak{M},t}, \psi^{\mathfrak{M},t}) \in \mathcal{K}_Q\\
&\iff \exists R\, \exists S\, \big(R = \phi^{\mathfrak{M},t} \wedge S = \psi^{\mathfrak{M},t} \wedge (M, R, S) \in \mathcal{K}_Q\big)\\
&\iff \forall R\, \forall S\, \big(R = \phi^{\mathfrak{M},t} \wedge S = \psi^{\mathfrak{M},t} \rightarrow (M, R, S) \in \mathcal{K}_Q\big).
\end{aligned}$$

□

7.10. Example. First consider the well-foundedness quantifier

$$\mathrm{WF} = \{\, (A, \leq) \;:\; \leq \text{ is well-founded } \}.$$

It is known that

$$\begin{aligned}
(A, \leq) \text{ is well-founded } &\iff \exists f \colon A \rightarrow \text{Ordinals order-preserving}\\
&\iff \forall X \subseteq A\, (X \text{ contains a minimal element}).
\end{aligned}$$

Moreover, this can be shown in KP + Σ_1-sep. Thus the quantifier WF and the logics $L_{\omega\omega}(\mathrm{WF})$ and $L_{\infty\omega}(\mathrm{WF})$ are first-order relative to KP + Σ_1-sep.

Then consider the quantifiers $Q_\alpha x \ldots$ "there exists at least $\aleph_\alpha$ elements x such that $\ldots$". The quantifier Q_0 is first-order relative to KP, since $(M, A) \in Q_0$ if and only if A is not finite. Similarly, the quantifier Q_1 is first-order relative to KP(Cbl) $=$ KP $\cup \{C(x) \leftrightarrow x \text{ is countable}\}$. Thus the logic $L_{\omega\omega}(Q_0)$ is first-order relative to KP, and the logics $L_{\omega\omega}(Q_1)$ and $L_{\infty\omega}(Q_1)$ are first-order relative to KP(Cbl).

7.11. Example. Let κ be an uncountable cardinal. The quantifier $\exists(x_i)_{i\in I}$, where $|I| < \kappa$, is first-order relative to

$$\mathrm{KP}(\mathcal{P}_\kappa) = \mathrm{KP} \cup \{\kappa \text{ is a cardinal }\} \cup \{P(x, y) \leftrightarrow y = \mathcal{P}_\kappa(x)\} \cup \{\forall x \exists y P(x, y)\}.$$

Let the language rule for this quantifier be

$$\begin{aligned}
&f(X,y)=z \iff X=\{\phi\},\ y\subseteq \text{Constants},\ |y|<\kappa, \text{ and } z=\exists y\phi,\\
&v(t,y)=t(\phi)\setminus y,\\
&q(\mathfrak{M},\exists\{c_i\}_{i\in I}\phi)=\{\,((\mathfrak{M},a_i)_{i\in I},\phi)\ :\ \forall i\in I\,(a_i\in M)\,\},\\
&Q(\mathfrak{M},\exists\{c_i\}_{i\in I}\phi,t) \iff 1\in \operatorname{ran}(t).
\end{aligned}$$

Since $|y|<\kappa \iff y\in\mathcal{P}_\kappa(y)$, the mapping f is Δ_0-definable in KP$(\mathcal{P}_\kappa)$. The mapping v is clearly Δ_0^{KP}. The mapping q can be defined with p.r. functions and $\mathcal{P}_\kappa$, from which it follows that both q and Q are Δ_1-definable in KP$(\mathcal{P}_\kappa)$. Thus the logic $L_{\infty\kappa}$ is first-order relative to KP$(\mathcal{P}_\kappa)$.

7.12. Example. The game quantifiers

$$\big(\forall x_n\exists y_n\big)_{n<\omega}\bigwedge_{n<\omega}\phi_n \quad\text{and}\quad \big(\forall x_n\bigwedge_{i_n\in I}\exists y_n\bigvee_{j_n\in I}\big)_{n<\omega}\bigwedge_{n<\omega}\phi_{i_0j_0\ldots i_nj_n}$$

are first-order relative to KP $+\ \Sigma_1$-sep $+$ DC, where Σ_1-sep stands for the Σ_1-separation axiom and DC for the axiom of dependent choices.

It is enough to consider the Vaught game quantification, since the other one is similar. The functions are easy to define: let $f(X,y)=z$ if and only if $X=\{\,\phi_s\ :\ s\in I^{<\omega}\,\}$, y maps $I^{<\omega}$ to pairs (c_s,ϕ_s), where c_s is the sequence of constants which the game quantifier binds in ϕ_s, and z is the game sentence. If $t\colon\phi_s\mapsto\tau_s$ and y is as above, let $v(t,y)$ be the union of vocabularies $\tau_s\setminus\operatorname{ran}(c_s)$ for $s\in I^{<\omega}$. Finally, if z is the game sentence, let $q(\mathfrak{M},z)$ be the set of pairs $((\mathfrak{M},a_0,b_0,\ldots,a_n,b_n),\psi_s)$, where $s\in I^{<\omega}$, $n=\operatorname{len}(s)$, and $a_0,\ldots,b_n$ are elements of $\mathfrak{M}$.

If $A\subseteq q(\mathfrak{M},z)$ and $t\colon q(\mathfrak{M},z)\to\{0,1\}$, let A_t be the following tree of positions in the semantic game of the game sentences: let a position u be in A_t if and only if there is $v=\langle a_k,i_k,b_k,j_k\rangle_{k<n}\supseteq u$ such that $t((\mathfrak{M},a_0,b_0,\ldots,a_k,b_k),\phi_{i_0,j_0,\ldots,i_k,j_k})=1$ for every $k<n$. Now

$$\begin{aligned}
Q(\mathfrak{M},z,t) &\iff \exists A\subseteq q(\mathfrak{M},z)\,(A_t \text{ is a winning strategy of } \exists)\\
&\iff \neg\exists B\subseteq q(\mathfrak{M},z)\,(B_t \text{ is a winning strategy of } \forall).
\end{aligned}$$

This equivalence is provable in KP $+\ \Sigma_1$-sep $+$ DC. We give more details later, when a more general game quantifier is discussed.

Of course one can add Lindström quantifiers to the game quantifier logics as well, producing such logics as for example $V_{\infty\omega}(Q_1)$, which is first-order relative to e.g. ZFC$(\mathcal{P}_{\omega_1})$. An interesting question is, what happens if we introduce a generalized quantifier to the game prefix. For simplicity, consider quantifiers of type (1) only. (These bind a single constant in a single sentence.) A quantifier Q is *monotone* if (M,R) is in Q whenever (M,R') is in Q for some subset R' of R. The *dual* $\check{Q}$ of a quantifier Q is defined by

$$(M,R)\in\check{Q} \iff (M,M\setminus R)\notin Q,$$

and clearly the dual of a monotone quantifier is monotone. It is not hard to see that a quantification with a monotone quantifier Q is equivalent to a second-order quantification as follows:

$$\mathfrak{M} \models Qx\phi(x) \iff \exists R \subseteq M\,\forall a \in R\,(\mathfrak{M}, a) \models \phi.$$

Using this characterization we are able to define a game quantifier

$$\big(\forall x_n \bigwedge_{i_n \in I} Qu_n\, \exists y_n \bigvee_{j_n \in I} \check{Q}v_n\big)_{n<\omega} \bigwedge_{n<\omega} \phi_{i_0 j_0 \ldots i_n j_n}$$

with the following semantics: the sentence is true in $\mathfrak{M}$ if $\exists$ wins the game, where for $n < \omega$

- $\forall$ picks $x_n \in M$ and $i_n \in I$,
- $\exists$ picks $R_n \in Q$,
- $\forall$ picks $u_n \in R_n$,
- $\exists$ picks $y_n \in M$, $j_n \in I$, and $S_n \in \check{Q}$, and
- $\forall$ picks $v_n \in S_n$,

and where $\exists$ wins a play if each sentence $\phi_{i_0 \ldots j_n}$ is true in $(\mathfrak{M}, x_0, \ldots)$.

7.13. Lemma. *The game quantifier defined above and its dual are first-order relative to theory $T \supseteq \mathrm{KP} + \Sigma_1$-sep $+ \mathrm{AC}$ whenever the quantifier Q is Δ_1-definable and provably monotone in T.*

Proof. The syntax rule for the game quantifier is defined similarly to the previous example, and there is no problem in defining the semantic mapping q either. The predicate Q can be based on the fact that the game sentence

$$\big(\forall x_n \bigwedge_{i_n \in I} Qu_n\, \exists y_n \bigvee_{j_n \in I} \check{Q}v_n\big)_{n<\omega} \bigwedge_{n<\omega} \phi_{i_0 j_0 \ldots i_n j_n}$$

has a negation

$$\big(\exists x_n \bigvee_{i_n \in I} \check{Q}u_n\, \forall y_n \bigwedge_{j_n \in I} Qv_n\big)_{n<\omega} \bigvee_{n<\omega} \neg\phi_{i_0 j_0 \ldots i_n j_n}.$$

To be more exact, we can base the predicate Q on the following equivalence:

$$\mathfrak{M} \models \big(\forall x_n \bigwedge_{i_n \in I} Qu_n\, \exists y_n \bigvee_{j_n \in I} \check{Q}v_n\big)_{n<\omega} \bigwedge_{n<\omega} \phi_{i_0 j_0 \ldots i_n j_n}$$

$\iff$ there exists a tree S of sequences such that for every $w \in S$

$$\begin{aligned}
\mathrm{len}(w) &= 6n &&\implies \{\, x_n \in M : w^\wedge\langle x_n\rangle \in S \,\} = M,\\
\mathrm{len}(w) &= 6n+1 &&\implies \{\, i_n \in I : w^\wedge\langle i_n\rangle \in S \,\} = I,\\
\mathrm{len}(w) &= 6n+2 &&\implies \{\, u_n \in M : w^\wedge\langle u_n\rangle \in S \,\} \in Q,\\
\mathrm{len}(w) &= 6n+3 &&\implies \{\, y_n \in M : w^\wedge\langle y_n\rangle \in S \,\} \neq \emptyset,
\end{aligned}$$

$$\begin{aligned}
&\operatorname{len}(w) = 6n+4 \implies \{\, j_n \in I \,:\, w^\wedge\langle j_n\rangle \in S \,\} \neq \emptyset,\\
&\operatorname{len}(w) = 6n+5 \implies \{\, v_n \in M \,:\, w^\wedge\langle v_n\rangle \in S \,\} \in \check{Q}, \text{ and}\\
&\operatorname{len}(w) = 6n+6 \implies \phi_{i_0\ldots j_n}(x_0\ldots v_n) \text{ is true in } \mathfrak{M}
\end{aligned}$$

$\iff$ there exists no well-founded tree T of sequences such that for every $w \in T$

$$\begin{aligned}
&\operatorname{len}(w) = 0 && \implies w \text{ is not leaf},\\
&\operatorname{len}(w) = 6n \text{ and } w \text{ is not a leaf} && \implies \{\, x_n \in M \,:\, w^\wedge\langle x_n\rangle \in T \,\} \neq \emptyset,\\
&\operatorname{len}(w) = 6n+1 && \implies \{\, i_n \in I \,:\, w^\wedge\langle i_n\rangle \in T \,\} \neq \emptyset,\\
&\vdots\\
&\operatorname{len}(w) = 6n+5 && \implies \{\, v_n \in M \,:\, w^\wedge\langle v_n\rangle \in T \,\} \in Q, \text{ and}\\
&\operatorname{len}(w) = 6n+6 \text{ and } w \text{ is leaf} && \implies \phi_{i_0\ldots j_n}(x_0\ldots v_n) \text{ is false in } \mathfrak{M}
\end{aligned}$$

$$\iff \mathfrak{M} \not\models \Big(\exists x_n \bigvee_{i_n\in I} \check{Q}u_n \forall y_n \bigwedge_{j_n\in I} Qv_n\Big)_{n<\omega} \bigvee_{n<\omega} \neg\phi_{i_0j_0\ldots i_nj_n}$$

The first equivalence is easy. The tree S, if it exists, is nearly a winning strategy of $\exists$ in the semantic game of the game quantifier: when playing the quantifier Qu_n, $\exists$ picks the set $\{\, u \in M \,:\, w^\wedge\langle u\rangle \in S \,\}$. On the other hand, a winning strategy of $\exists$ gives rise to a tree S; however, one needs the axiom of choice to pick $\operatorname{succ}(w)$ for a sequence w of length $6n+2$ or $6n+5$.

The last equivalence is not much harder: similarly the tree T, if it exists, gives rise to an $\exists$-strategy in the semantic game of the dual game quantifier. Any complete extension of this strategy is a winning strategy of $\exists$. On the other hand, a winning strategy of $\exists$ gives rise to a tree T satisfying the requirements; however, one needs the axiom of dependent choices to show that T is well-founded. Thus we need to show the middle equivalence.

Suppose first, for contradiction, that both of the trees S and T exist. Now $S \cap T$ is a well-founded tree and thus contains a maximal element w. Since $A \cap A' \neq \emptyset$ whenever $(M, A) \in Q$ and $(M, A') \in \check{Q}$, the length of w is $6n+6$ for some n and w is maximal in T. Thus w provides a sentence ϕ_s and a sequence $\langle x_0, \ldots, v_n\rangle$ such that ϕ_s is both true and false in $(\mathfrak{M}, x_0, \ldots)$, and so both trees cannot exist.

The fact that either of the trees S and T must exist is shown similarly as the determinacy of open games. For a sequence w let $P(w)$ be a predicate indicating that there exists such a well-founded tree of sequences extending w that satisfies the conditions similar to the latter equivalent above. Suppose the tree T does not exist. Let S be the tree of those sequences w which, and the initial segments of which, do not satisfy P. One can inductively see on the length of w that S satisfies the requirements of the former equivalent.

Finally, the above equivalences are provable in the theory T. $\square$

The lemma implies, for example, that the logic $V_{\infty\omega}[Q_\alpha]$, the logic $L_{\infty\omega}$ augmented with the above game quantifier for generalized quantifier Q_α, is first-order relative to ZFC($\mathcal{P}_{\aleph_\alpha}$).

7.14. Example. Let κ be an uncountable regular cardinal. The long game quantifiers

$$(\forall x_n \bigwedge_{i_n \in I} \exists y_n \bigvee_{j_n \in I})_{n<\kappa} \bigwedge_{n<\kappa} \phi_{i_0 j_0 \dots i_n j_n} \quad \text{and} \quad (\exists x_n \bigvee_{i_n \in I} \forall y_n \bigwedge_{j_n \in I})_{n<\kappa} \bigvee_{n<\kappa} \phi_{i_0 j_0 \dots i_n j_n}$$

together with the first-order operations make up the logic $V_{\infty\kappa}$. The semantical games attached to these sentences are not necessarily determined, and thus the same approach as in the case of game sentences of length ω is not available. However, their syntax is clearly Δ_1-definable in ZFC($\mathcal{P}_\kappa$), and since their semantical game is either κ-open or κ-closed, the semantics is absolute relative to κ-closed forcing. Thus we may conclude: $V_{\infty\kappa}$ is first-order relative to κ-closed forcing.

7.15. Example. Consider "almost all" quantifier (aa), the semantics of which is so defined that $\mathfrak{M} \models (aa)S\phi(S)$ if and only if the set $\{ S \in \mathcal{P}_{\omega_1}(M) : (\mathfrak{M}, S) \models \phi \}$ contains a closed unbounded set.

This quantifier is first-order relative to ω_1-closed forcing: its syntax is clearly absolute, as is the semantical mapping q, which maps $(\mathfrak{M}, (aa)S\phi(s))$ to the set of pairs $((\mathfrak{M}, S), \phi)$ for $S \in \mathcal{P}_{\omega_1}(M)$. The obvious predicate

$$Q(\mathfrak{M}, (aa)S\phi(s), t) \iff \{ S \in \mathcal{P}_{\omega_1}(M) : t(\mathfrak{M}, S) = 1 \} \text{ contains a cub}$$

is clearly absolute relative to proper forcing (i.e. relative to forcing which preserves the stationary subsets of $\mathcal{P}_{\omega_1}(\lambda)$ for each cardinal λ), and thus relative to ω_1-closed forcing. We conclude that the logics $L_{\omega\omega}(aa)$ and $L_{\infty\omega}(aa)$ are first-order relative to ω_1-closed forcing.

In fact we could show that $L_{\omega\omega}(aa)$ and $L_{\infty\omega}(aa)$ were first-order relative to proper forcing. However, since $\mathcal{P}_{\omega_1}$ is not necessarily preserved in proper forcing, the semantic mapping q fails to be upwards persistent. Thus a somewhat more refined argument is needed than is presented in the proof of 7.4.

The logic $L_{\omega\omega}(aa)$ has a fragment L_{pos}, where the (aa)-quantifier is only allowed to quantify over those predicate symbols which occur positively in ϕ. We will next see that it is first-order relative to ZFC($\mathcal{P}_{\omega_1}$). Define the syntax mapping f in such a way that the syntax of each sentence consists of a triple (ϕ, p, n), where ϕ is the sentence itself and p and n are the sets of unary relation symbols occurring positively and negatively in it, respectively. Then it is not hard to see that the syntax is Δ_1-definable. For the semantics note that if S occurs positively in ϕ,

$$\mathfrak{M} \models (aa)S\phi(S) \iff \exists S \in \mathcal{P}_{\omega_1}(M)\, \mathfrak{M} \models \phi(S).$$

Thus the semantics is Δ_1-definable in ZFC($\mathcal{P}_{\omega_1}$).

Collect finally all the absoluteness results presented in this section under a single heading:

7.16. Theorem. *(1) The logics $L_{\omega\omega}$ and $L_{\infty\omega}$ are first-order relative to* KPU.

(2) *The logic* $L_{\omega_1\omega}$ *is absolute relative to* KPU.
(3) *The logic* $L_{\omega\omega}(Q_0)$ *is first-order relative to* KP.
(4) *The logics* $L_{\infty G}$ *and* $V_{\infty\omega}$ *are first-order relative to* KP $+ \Sigma_1$-sep $+$ DC.
(5) *The logics* $L_{\omega\omega}(Q_1)$ *and* $L_{\infty\omega}(Q_1)$ *are first-order relative to* KP(Cbl).
(6) *The logics* $V_{\infty\omega}[Q_1]$, L_{pos}, *and* $L_{\infty\omega_1}$ *are first-order relative to* ZFC$(\mathcal{P}_{\omega_1})$.
(7) *The logic* $L_{\infty\kappa}$ *is first-order relative to* KP$(\mathcal{P}_\kappa)$.
(8) *The logic* $V_{\infty\omega}[Q_\alpha]$ *is first-order relative to* ZFC$(\mathcal{P}_{\aleph_\alpha})$.
(9) *The logics* $L_{\omega\omega}(aa)$, $L_{\infty\omega}(aa)$, *and* $V_{\infty\omega_1}$ *are first-order relative to* ω_1*-closed forcing.* □

8. Absoluteness of M-languages

Let κ be a regular cardinal in this section. Since the sentences of M-languages are not inductively constructed, we cannot investigate their absoluteness by methods given in the previous chapter. As such they mostly resemble the game quantifiers. Next we show that the logic $M_{\infty\kappa}$ is first-order relative to κ-closed forcing. Recall that syntactically the sentences of $M_{\infty\kappa}$ are pairs (T, L), where T is a κ-leaftree and L is a labelling.

8.1. Lemma. *If T is a κ-tree and $\mathbb{P}$ is a κ-closed forcing, then $\mathbb{P}$ forces $\check{T}$ to be a κ-tree.*

Proof. Let T be a tree and suppose $p \in \mathbb{P}$ is such a condition that forces $\check{T}$ not to be a κ-tree. Thus there is a $\mathbb{P}$-name b and $p_0 \geq p$, which forces b to be a path of length κ in T. Using κ-closedness at limits, one is able inductively to construct $t_\alpha \in T$ and p_α for $\alpha < \kappa$ in a such way that

$$p_\alpha \Vdash b(\check{\beta}) = \check{t}_\beta \text{ for every } \beta < \alpha.$$

Now $\langle t_\alpha \rangle_{\alpha<\kappa}$ is a path in T, and thus T is not a κ-tree. □

8.2. Theorem. $M_{\infty\kappa}$ *is first-order relative to κ-closed forcing.*

Proof. Syntax: By the above lemma the predicate "T is a κ-tree" is absolute relative to κ-closed forcing. Since κ-closed forcing introduces no new sets of cardinality less than κ, the predicate "T is a κ-leaftree" is absolute relative to κ-closed forcing. The absoluteness of the labelling properties is trivial.

Semantics: Recall first that $\mathfrak{M} \models \phi$ if and only if $\exists$ has a winning strategy in the semantic game S$(\mathfrak{M}, \phi)$. Since the syntax tree of ϕ is a κ-leaftree, the semantic game is both κ-closed and κ-open, and, moreover, if we denote $G =$ S$(\mathfrak{M}, \phi)$, a κ-closed forcing forces both the canonical closed game $\tilde{G}_c$ and the canonical open game $\tilde{G}_o$ to equal S$(\check{\mathfrak{M}}, \check{\phi})$. Thus the absoluteness of semantics follows from Lemma 4.5. □

In the above result we use the κ-closedness of forcing for showing the absoluteness of both syntax and semantics. In fact, the syntax of $M_{\infty\omega_1}$ is not Σ_1-definable in ZFC$(\mathcal{P}_{\omega_1})$

and it is not even upwards persistent relative to ω-distributed forcing (such a forcing that an intersection of countably many dense initial segments of $\mathbb{P}$ is dense). To see it, take a bistationary set $A \subseteq \omega_1$, and let T be the tree of closed subsets of A, ordered by end extension. Since A contains no closed unbounded set, T is a κ-leaftree. Label T trivially: let $L(u) = \bigvee$ if u is not a leaf, and a true atomic sentence otherwise. This makes (T, L) a sentence of $M^{\mathrm{det}}_{\infty\kappa}$. However, we are able to force a closed unbounded subset into A with an ω-distributed forcing (see [BHK]), and thus (T, L) is no longer a sentence in $M_{\infty\omega_1}$ in the generic extension of the universe.

A similar counterexample can be provided for the semantics: for a set $A \subseteq \omega_1$, denote by $\Phi(A)$ the linear order

$$\sum_{\alpha<\omega_1} \eta_\alpha, \text{ where } \eta_\alpha = \begin{cases} 1+\mathbb{Q} & \text{if } \alpha \in A, \text{ and} \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

By works of Convay it is known that $\Phi(A) \cong \Phi(B)$ exactly when $A \Delta B$ is non-stationary. Moreover,

$$\Phi(A) \models \forall x_0\, \exists x_1\, \forall x_2\, \exists x_3 \, \ldots \, \exists x_\omega \, (x_\omega = \sup\{x_0, x_1, \ldots\})$$

if and only if A contains a closed unbounded subset, and if A is non-stationary, we know that $\forall$ wins the semantic game. However, if A is bistationary, we are able both to force it to be non-stationary and to force it to contain a closed unbounded set. Thus the semantics of this sentence of $M_{\infty\omega_1}$ is not absolute relative to ω-distributed forcing nor ZFC$(\mathcal{P}_{\omega_1})$.

On the other hand, the following lemma shows that the semantics of determined $M_{\infty\kappa}$-sentences is absolute relative to ZFC$(\mathcal{P}_\kappa)$.

8.3. Lemma. *Let $A \subseteq B$ be transitive models of a set theory such that $\mathcal{P}_\kappa$ is absolute relative to (A, B). Suppose $\phi \in A$ is such that $[\phi \in M^{\mathrm{det}}_{\infty\kappa}]^A$. Then for each structure $\mathfrak{M} \in A$*

$$[\mathfrak{M} \models \phi]^A \iff [\mathfrak{M} \models \phi]^B.$$

Proof. Since the position tree of the semantic game $\mathrm{S}(\mathfrak{M}, \phi)$ is a κ-tree, the expression "$G = \mathrm{S}(\mathfrak{M}, \phi)$" is absolute relative to (A, B). Thus, if $S \in A$ is a winning strategy of $\exists$ in the semantic game $\mathrm{S}(\mathfrak{M}, \phi)$ in A, it is a winning strategy of $\exists$ in B as well. The result follows from the determinacy of ϕ in A. □

We shall see later that there is no logic, absolute relative to ZFC$(\mathcal{P}_\kappa)$, which would have the same expressive power as $M^{\mathrm{det}}_{\infty\kappa}$.

III Model extending

Usually the absoluteness of a logic $\mathcal{L}$ is used in the following way. Suppose ϕ is a sentence in a logic $\mathcal{L}$, absolute relative to a true set theory T. Let $\mathfrak{M}$ be a structure. The absoluteness of $\mathcal{L}$ implies that $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$ if and only if

> there is a well-founded and extensional model $\mathfrak{A} = (A, E, \ldots)$ of theory T and elements t, p, m in A such that the formula $m \models_{\mathcal{L}(t)} p$ is true in $\mathfrak{A}$, t collapses into τ, p collapses into ϕ, and m collapses into a model which is isomorphic to $\mathfrak{M}$.

Thus, in a suitable language $\mathcal{L}'$, we are able to write a sentence ψ_ϕ such that $\mathfrak{M} \models_{\mathcal{L}} \phi$ turns into an RPC($\mathcal{L}'$) expression "$(\mathfrak{M}; \mathfrak{A}) \models \psi_\phi$ for some $\mathfrak{A}$". Then we hope that by examining this RPC-expression we will be able to derive some properties of the logic $\mathcal{L}$.

For example, consider the proof of the downward Löwenheim–Skolem theorem (see [B2]): if $\mathcal{L}$ is absolute relative to a true set theory in standard vocabulary and if a sentence $\phi \in \mathcal{L}$ has a model, it has a model of cardinality less than $|\mathrm{TC}(\phi)|^+$. To prove the theorem, take first a model $\mathfrak{M}$ for ϕ. Let then A be a transitive set such that $\mathfrak{M}$ and ϕ are elements of A and $\mathfrak{M} \models_{\mathcal{L}} \phi$ is true in A. If B is a small elementary submodel of A such that $\mathrm{TC}(\{\phi\}) \subseteq B$ and $\mathfrak{M} \in B$, the restriction $\mathfrak{M}\restriction(M \cap B)$ is a small model of ϕ. In this case the sentence ψ_ϕ expressed very little about the set theoretical part $\mathfrak{A}$ and nothing about the model $\mathfrak{M}$, but enough to derive the result.

In addition to the RPC-expression a Δ-expression is possible: starting from the Π_1-form of the truth predicate $\models_{\mathcal{L}}$ in the same way as above one gets

$$\mathfrak{M} \not\models_{\mathcal{L}} \phi \iff (\mathfrak{M}; \mathfrak{A}) \models \psi'_\phi \text{ for some } \mathfrak{A}.$$

Moreover, if the logic $\mathcal{L}'$ allows Δ-interpolation, we may be able to translate the sentences in $\mathcal{L}$ into sentences in $\mathcal{L}'$. An example of this approach is provided by Barwise in [B2]: if a logic $\mathcal{L}$ is absolute relative to KP, then $\mathcal{L} \leq L_{\infty\omega}$. The first thing to note is that one needs to show the result in a countable admissible fragment; thus the interpolation is available. Barwise first turns the expression $\mathfrak{M} \models_{\mathcal{L}} \phi$ into a $\Delta(L_{\infty\omega})$-expression. Because the models of KP satisfy the Truncation lemma 2.2, he does not need to express "$\mathfrak{A}$ is well-founded". Then he uses interpolation to remove the extra sort.

Interpolation is not the only known way of removing the set-theoretical extra sort. In some cases one can reduce the question of the existence of a set-theoretical expansion to the question of the existence of an ordinal, thus getting approximation results:

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff \mathfrak{M} \models \psi_\alpha \text{ for some ordinal } \alpha.$$

An example of this approach is given by Burgess in [Bu], where he shows that every logic, absolute relative to ZFC, can be approximated with $L_{\infty\omega}$.

This part deals with a new method of removing the set-theoretical sort. The main idea is based on games and Kueker approximations.

9. Kueker approximations

Let S be a set. The *S-approximation* of a set x is

$$x^S = \{\, y^S \,:\, y \in x \cap S \,\}.$$

The S-approximation of an urelement a is $a^S = a$.

Consider then a structure $\mathfrak{M} \in \mathrm{Str}(\tau)$. It is a pair (M, F) where M is a set and F maps the symbols of the vocabulary τ to their interpretations. Let $N \subseteq M$ be nonempty and let S be a set. The (N, S)-approximation of the model $\mathfrak{M}$ is

$$\mathfrak{M}^{NS} = (N, F^{NS}),$$

where $F^{NS} = \{\, (r^S, F(r){\restriction}N) \,:\, r \in \tau \cap S \,\}$. In other words, F^{NS} maps the symbols in τ^S to the corresponding interpretations restricted to N.

The key idea in the S-approximations is that they are similar to the original sets but smaller: if S is countable, each S-approximation is countable. The same holds for the (N, S)-approximations of a model $\mathfrak{M} \in \mathrm{Str}(\tau)$: almost every time $\mathfrak{M}^{NS}$ is a submodel of $\mathfrak{M}$ having a restricted vocabulary τ^S.

To get a better idea of what the notion "almost every time" in this context means, we briefly consider the case where the set S is countable. Let P be an n-ary predicate of set theory and let $x_1, \ldots, x_n$ be sets. We say $P(x_1^S, \ldots, x_n^S)$ holds *almost everywhere (a.e.)* if for any transitive set A containing $x_1, \ldots, x_n$ the set

$$\{\, S \in \mathcal{P}_{\omega_1}(A) \,:\, P(x_1^S, \ldots, x_n^S) \,\}$$

contains a closed unbounded set. (This definition is independent of the choice of the set A.) Moreover, if P is a Σ_1^{KP}-predicate and $P(x)$ holds, then $P(x^s)$ holds almost everywhere. Finally, using the game formulation of the countable closed unbounded filter one can see that for every predicate P and set x

$$P(x^S)\ a.e. \iff \forall s_0 \in \mathrm{TC}(x)\, \exists s_1 \in \mathrm{TC}(x)\, \forall s_2 \in \mathrm{TC}(x) \ldots\ P(x^{\{s_0, s_1, \ldots\}}).$$

For more details about the S-approximations see [K1, K2, B3].

The above results give rise to the following construction. Let $\mathcal{L}$ be absolute relative to some true set theory $T \supseteq \mathrm{KP}$, $\phi \in \mathcal{L}(\tau)$, and $\mathfrak{M} \in \mathrm{Str}(\tau)$. Consider the following game, later to be called *expansion game*. At each of his moves, player $\forall$ picks an element x_i from the model and an element $s_i \in \mathrm{TC}(\tau, \phi)$. Similarly, at her moves player $\exists$ picks elements y_i and t_i from the model and $\mathrm{TC}(\tau, \phi)$, respectively, and an element a_i from a given fixed set. The sets $X = \{x_0, y_0, \ldots\}$ and $S = \{s_0, t_0, \ldots\}$ constructed during the game define Kueker approximations $\mathfrak{M}^{XS}$, τ^S, and ϕ^S for the model, vocabulary, and the formula. The purpose of player $\exists$ is to play in such a manner that finally

$$\mathfrak{M}^{XS} \in \mathrm{Str}(\tau^S), \quad \phi^S \in \mathcal{L}(\tau^S) \quad \text{and} \quad \mathfrak{M}^{XS} \models_{\mathcal{L}(\tau^S)} \phi^S.$$

The extra moves $a_0, a_1, \ldots$ of $\exists$ are used to construct evidence of this fact. The basic idea (though the details vary) is that whenever $\exists$ wins a play, the sequence $a_0, a_1, \ldots$ defines a set-theoretical structure (A, E, m, t, p) in which $p \in \mathcal{L}(t)$, $m \in \mathrm{Str}(t)$, and $m \models_{\mathcal{L}(t)} p$ hold, the elements t and p collapse into τ^S and ϕ^S, respectively, and m collapses into an isomorphic copy of $\mathfrak{M}^{XS}$. Now, the above propositions give us a good reason to believe (what is more or less true depending on the logic) that $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$ holds exactly when player $\exists$ has a winning strategy in this game. This game playing can be turned into a game sentence in such a way that $\exists$ wins the game exactly when the game sentence is true in the model $\mathfrak{M}$.

10. Expanders

In the end of the previous section we sketched a game during which a player makes a sequence of choices determining a structure. This will be achieved with choices determining a theory in a certain *expansion language* and the theory describing the structure. In this section we define "generic" theories (i.e. theories depending on parameters) which are used for describing expansions. These "theories" are called *expanders*.

Suppose τ is any vocabulary (for the base structure) and let σ be a finite vocabulary without any function symbols (for the expanding part). Let λ be an infinite cardinal (the cardinality of the expanding part), and for each $\alpha \leq \lambda$ let $\tau_\alpha = \tau \cup \{ x_i : i < \alpha \}$ and $\sigma_\alpha = \sigma \cup \{ c_i : i < \alpha \}$, where $x_0, x_1, \ldots$ and $c_0, c_1, \ldots$ are new constant symbols. The *expansion language* $\mathcal{L}_{exp}(\tau, \sigma, \alpha)$ in the two-sorted vocabulary $(\tau_\alpha; \sigma_\alpha)$ consists of *initial sentences* in $L_{basic}(\tau_\alpha)$ and $L_{\omega\omega}(\sigma_\alpha)$, these being of a different sort, and is closed under propositional operations (negation, finite conjunction, finite disjunction).

10.1. Definition. Let τ, σ, and λ be as above. A *λ-expander $\mathcal{F}$* is a function such that $\mathrm{dom}(\mathcal{F}) = I^{<\lambda}$ for some set I and $\mathcal{F}(w)$ is a sentence in $\mathcal{L}_{exp}(\tau, \sigma, \mathrm{len}(w))$ for each $w \in \mathrm{dom}(F)$.

Call a sequence $a\colon \lambda \to I$ an *$\mathcal{F}$-branch.* It defines a theory $\mathrm{Th}_{\mathcal{F}}(a) = \bigcup_{i<\lambda} \mathcal{F}(a{\restriction}i)$. If $\mathfrak{M} \in \mathrm{Str}(\tau)$, an *$\mathcal{F}$-expansion of $\mathfrak{M}$ over a* is a model $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0, \ldots)$ of $\mathrm{Th}_{\mathcal{F}}(a)$ such that every element of $\mathfrak{A}$ is an interpretation of a constant symbol.

The *branching cardinal* $\kappa_{\mathcal{F}}$ of an expander $\mathcal{F}$ is the maximum of $|I|^+$ and λ.

Recall again the sketch in the previous section: during a game, given a structure $\mathfrak{M}$, we construct an expansion $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, \ldots)$ with certain properties. We intend to define an expander $\mathcal{F}$ such that every $\mathcal{F}$-expansion of $\mathfrak{M}$ is one and to play the expansion game broadly as follows. For even ordinals $i < \lambda$ player $\forall$ picks x_i in $\mathfrak{M}$ and a_i in I, and $\exists$ acts similarly for odd ordinals. Thus we get a structure $\mathfrak{M}_x = (\mathfrak{M}, x_0, \ldots)$ and a theory $\mathrm{Th}_{\mathcal{F}}(a)$, and are able to ask whether $\mathfrak{M}_x$ can be expanded into an $\mathcal{F}$-expansion over a. We now distinguish two games: an *expansion game*, where the purpose of $\exists$ is to play in such a way that the $\mathcal{F}$-expansion exists, and a *co-expansion game*, where this role is reserved for $\forall$. Before exactly defining the games (making them slightly more complicated) we take an example.

10.2. Example. As our sketch reveals, we are mainly interested in set-theoretical expansions $(\mathfrak{M}, \ldots; A, E, t, p, m, \ldots)$ of a structure $\mathfrak{M}$, especially in those where the constants t, p, and m collapse into τ^S, ϕ^S, and $\mathfrak{M}^{XS}$, where τ is the vocabulary of $\mathfrak{M}$, ϕ is a sentence, and the sets X and S are picked during the game. Now we construct an expander which gives such expansions.

Let τ be a vocabulary, let ϕ be a set (e.g. a sentence in a logic), let λ be an infinite cardinal, and let $\sigma = \{\in, t, p, m\}$, where t, p, m are constant symbols. Define an expander $\mathcal{F} = \mathcal{F}(\tau, \phi, \lambda)$ as follows. Recall that by σ_λ we denote the vocabulary σ expanded with λ constant symbols $c_0, c_1, \ldots$. Now give alias names to the constants with odd indices: let $a_i = c_{4i+1}$ and $b_i = c_{4i+3}$ for $i < \lambda$. Give other alias names to all the constants (i.e. terms) in σ_λ: let $t_0 = t$, $t_1 = p$, $t_2 = m$, and $t_{3+i} = c_i$ for each i.

Choose $I = \mathrm{TC}(\tau, \phi) \times 2 \times 2$. Define $\mathcal{F}$ in such a way that if $a = \langle (z_i, \alpha_i, \beta_i) \rangle_{i<\lambda}$ is an $\mathcal{F}$-branch, $\mathrm{Th}_{\mathcal{F}}(a)$ contains the following sentences:

(1) $\in$ is extensional.

(2) t is a vocabulary.

(3) $\left.\begin{array}{ll} a_i \in a_j & \text{if } z_i \in z_j \\ a_i \notin a_j & \text{if } z_i \notin z_j \\ a_i = a_j & \text{if } z_i = z_j \\ a_i \neq a_j & \text{if } z_i \neq z_j \end{array}\right\}$ for every $i, j < \lambda$.

(4) $\left.\begin{array}{ll} a_i \in t & \text{if } z_i \in \tau \\ a_i \notin t & \text{if } z_i \notin \tau \end{array}\right\}$ for every $i < \lambda$.

(5) $\left.\begin{array}{ll} a_i \in p & \text{if } z_i \in \phi \\ a_i \notin p & \text{if } z_i \notin \phi \end{array}\right\}$ for every $i < \lambda$.

(6) $\left.\begin{array}{ll} t_i \notin a_j \wedge t_i \notin t \wedge t_i \notin p & \text{if } \alpha_{2i+1} = 0 \\ t_i = a_{2i+1} & \text{if } \alpha_{2i+1} = 1 \end{array}\right\}$ for every $i, j \in \lambda$.

(7) $m \in \mathrm{Str}(t)$.

(8) $b_i \in \mathrm{dom}(m)$ for every $i \in \lambda$.

(9) $b_i = b_j \leftrightarrow x_i = x_j$ for every $i, j \in \lambda$.

(10) $\left.\begin{array}{ll} t_i \notin \mathrm{dom}(m) & \text{if } \beta_{2i+1} = 0 \\ t_i = b_{2i+1} & \text{if } \beta_{2i+1} = 1 \end{array}\right\}$ for every $i \in \lambda$.

(11) ${a_i}^m(b_{j_1}, \ldots, b_{j_k}) \leftrightarrow R(x_{j_1}, \ldots, x_{j_k})$ whenever z_i is a k-ary relation symbol $R \in \tau$, $i, j_1, \ldots, j_k \in \lambda$, and similarly for the other symbols in τ. Here ${a_i}^m$ stands for the interpretation of a symbol a_i in structure m.

Note that intentionally the theory depends on α_i and β_i only for odd ordinals i.

This expander $\mathcal{F}(\tau, \phi, \lambda)$ is intended to be used in an expansion game where $\exists$ tries to build up the expansion. It has a counterpart $\mathcal{F}'(\tau, \phi, \lambda)$ intended to be used in a co-expansion game. It is similar to $\mathcal{F}$, except that a_i and b_i refer to constants c_{4i} and c_{4i+2} instead of c_{4i+1} and c_{4i+3}, respectively, and the sentences (6) and (10) depend on α_{2i} and β_{2i} instead of α_{2i+1} and β_{2i+1}, respectively.

Before we can look at the $\mathcal{F}(\tau, \phi, \lambda)$-expansions, we need a couple of set-theoretical concepts. As we know, the well-founded part $\mathrm{Wf}(A, E)$ of an extensional structure (A, E) can collapse with a Mostowski collapsing function, and partial collapses are possible for elements in the non-well-founded part. For example, if a is an element of A, it gives rise to an unique subset $\{ b \in A : bEa \}$ of A, the *set interpretation* of a. Suppose then that $a \in A$ is a structure in A, i.e. a is an ordered pair (m, f), where f maps some symbols to interpretations. Moreover, suppose that $\mathrm{dom}(f)$ is in the well-founded part of (A, E) and collapses into a vocabulary τ. Then a gives rise to an unique *structural interpretation*: a structure in $\mathrm{Str}(\tau)$, the universe of which is the set interpretation of m and where the interpretations of symbols are defined by f.

10.3. Lemma. *Let τ be a vocabulary, $\mathfrak{M} \in \mathrm{Str}(\tau)$, and let ϕ be a set. Suppose $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0, \ldots)$ is an $\mathcal{F}(\tau, \phi, \lambda)$-expansion over $\langle (z_i, \alpha_i, \beta_i) \rangle_{i<\lambda}$. Let $X = \{x_0, x_1, \ldots\}$ and $S = \{z_0, z_1, \ldots\}$. Then $t^{\mathfrak{A}}$ and $p^{\mathfrak{A}}$ are in the well-founded part of $\mathfrak{A}$, and they collapse into τ^S and ϕ^S, respectively. Moreover, the structural interpretation of m in $\mathfrak{A}$ is isomorphic to $\mathfrak{M}^{XS}$. The same holds for $\mathcal{F}'(\tau, \phi, \lambda)$-expansions.*

Proof. Since the only difference between an $\mathcal{F}(\tau, \phi, \lambda)$- and an $\mathcal{F}'(\tau, \phi, \lambda)$-expansion is in certain constant names, we only need to show the result for an $\mathcal{F}(\tau, \phi, \lambda)$-expansion. Recall first that the elements of A have names t_i for $i < \lambda$. Since (A, E) is extensional by (1), its well-founded part $\mathrm{Wf}(A, E)$ collapses. The sentences (3)–(6) imply that $a_0^{\mathfrak{A}}, a_1^{\mathfrak{A}}, \ldots, t^{\mathfrak{A}}$, and $p^{\mathfrak{A}}$ are in $\mathrm{Wf}(A, E)$, and they collapse into $z_0^S, z_1^S, \ldots, \tau^S$, and ϕ^S, respectively. Thus, since (2) is a Δ_0-expression, τ^S is a vocabulary. The sentences (7)–(10) imply that m is a structure in (A, E) with vocabulary t and universe $\{b_0, b_1, \ldots\}$. Finally, the sentence (11) implies that the structural interpretation of m in $\mathfrak{A}$ is isomorphic to $\mathfrak{M}^{XS}$. □

Let $\mathcal{F}$ be a λ-expander for vocabulary $(\tau; \sigma)$. Now fix an enumeration $\mathbf{\Phi}_{\mathcal{F}} = \langle \phi_i \rangle_{i<\lambda}$ of all the sentences in $L_{\omega\omega}(\sigma_\lambda)$ such that each sentence ϕ_i is in $L_{\omega\omega}(\sigma_i)$. Since σ is finite, this enumeration exists, and we may assume $\mathcal{F} \mapsto \mathbf{\Phi}_{\mathcal{F}}$ to be primitive recursive. (The latter statement holds, since we may assume σ to be hereditarily finite and thus a p.r. constant.)

10.4. Definition. Let $\mathcal{F}$ be a λ-expander for vocabulary $(\tau; \sigma)$ and let $\mathfrak{M}$ be a τ-structure. An *expansion game* $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ is played as follows: for $i < \lambda$

- $\forall$ chooses $x_{2i} \in M$, and $\alpha_{2i} \in I$, and
- $\exists$ chooses $x_{2i+1} \in M$, $\alpha_{2i+1} \in I$, and $\psi_i \in \{\mathbf{\Phi}_{\mathcal{F}}(i), \neg\mathbf{\Phi}_{\mathcal{F}}(i)\}$.

Player $\exists$ wins if there exists an $\mathcal{F}$-expansion $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, \ldots)$ over $\langle \alpha_i \rangle_{i<\lambda}$ which satisfies ψ_i for $i < \lambda$ and sentences $\exists u \psi(u) \to \psi(c_{2i})$ whenever ψ and $i < \lambda$ are such that $\mathbf{\Phi}_{\mathcal{F}}(i) = \exists u \psi(u)$.

A *co-expansion game* $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$ is played similarly, except that the players switch roles, i.e. $\forall$ chooses the sentences ψ_i, the witnessing constant for $\mathbf{\Phi}_{\mathcal{F}}(i)$ is c_{2i+1} instead of c_{2i}, and $\forall$ wins the play if an $\mathcal{F}$-expansion exists.

Now we can clarify the principal construction method used in this work: having a sentence ϕ in an absolute logic, we build expanders $\mathcal{F}$ and $\mathcal{F}'$ such that if $\mathfrak{M} \models \phi$, then $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$, and otherwise $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$. Then we either transfer the expansion games to game sentences or combine the games and produce a sentence in $M_{\infty\infty}$.

We continue by presenting the tools with which we can show that a player in an expansion game has a winning strategy. Recall that a game is λ-closed if, whenever $\forall$ wins a play, he knows it before λ moves have been played.

10.5. Lemma. *Let $\mathcal{F}$ be a λ-expander and suppose $\mathfrak{M}$ is a structure. The game* $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ *is λ-closed and* $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$ *is λ-open.*

Proof. A play of $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ gives rise to a sequence $x_0, x_1, \ldots$ of elements of $\mathfrak{M}$ and to theories $\mathrm{Th}_{\mathcal{F}}(a)$ and

$$\Psi = \{\, \psi_i \,:\, i < \lambda \,\} \cup \{\, \exists u \psi(u) \to \psi(c_{2i}) \,:\, i < \lambda \text{ and } \Phi_{\mathcal{F}}(i) = \exists u \psi(u) \,\}.$$

Moreover, $\exists$ wins the play if and only if the theory $\mathrm{Th}_{\mathcal{F}}(a) \cup \Psi$ has a model of form $(\mathfrak{M}, x_0, \ldots; \mathfrak{A})$, where each element of $\mathfrak{A}$ is an interpretation of a constant. But since Ψ, when consistent, is a complete Henkin theory, it has a term model $\mathfrak{A}_\Psi$. Thus $\forall$ wins the play if and only if either Ψ is inconsistent or $(\mathfrak{M}, x_0, y_0, \ldots; \mathfrak{A}_\Psi)$ is not a model of some $\mathcal{F}(a{\restriction}i)$. Since $F(w)$ is obtained from sentences in $L_{basic}(\tau \cup \mathrm{ran}(x))$ and $L_{\omega\omega}(\sigma \cup \mathrm{ran}(c))$ with propositional operations, in order to decide whether $(\mathfrak{M}, \ldots; \mathfrak{A}_\Psi)$ is a model of $F(w)$ we need to know only a finite fragment of Ψ and a finite number of constants $x_0, x_1, \ldots$. So, if $\forall$ wins the play, we know it before playing all the λ steps. Thus $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ is λ-closed.

Similarly we see that $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$ is λ-open. □

As we stated in Lemma 4.5, λ-open and λ-closed games are immune to λ-closed forcing in a sense. This is also true for the expansion games.

10.6. Lemma. *Let $\mathcal{F}$ be a λ-expander and let $\mathbb{P}$ be a notion of λ-closed forcing. Suppose $\mathfrak{M} \in \mathrm{Str}(\tau)$.*

(i) If $\mathbb{P}$ forces $\exists$ to win $\mathrm{EG}(\check{\mathfrak{M}}, \check{\mathcal{F}})$*, then $\exists$ wins* $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$.

(ii) If $\mathbb{P}$ forces $\forall$ to win $\mathrm{EG}^*(\check{\mathfrak{M}}, \check{\mathcal{F}})$*, then $\forall$ wins* $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$.

Proof. It is enough to show (i), since (ii) is similar. Note that since $\mathbb{P}$ adds no new sequences of length less than λ, $\mathcal{F}$ is an expander in the generic extension and thus we can construct expansion games there. The result immediately follows by Lemma 4.5, since, if $G = \mathrm{EG}(\mathfrak{M}, \mathcal{F})$, $\mathbb{P}$ forces $\mathrm{EG}(\check{\mathfrak{M}}, \check{\mathcal{F}}) = \tilde{G}_c$. □

The usual way of showing that $\exists$ wins an expansion game $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ is to take a suitable expansion $(\mathfrak{M}; \mathfrak{A})$ and during the expansion game pick such elements from $\mathfrak{A}$ that the theory constructed in the game will be true in $(\mathfrak{M}; \mathfrak{A})$. The following validity game indicates, which kinds of expansions are suitable for this purpose.

10.7. Definition. Let $\mathcal{F}$ be a λ-expander in vocabulary $(\tau;\sigma)$, $\mathfrak{M} \in \mathrm{Str}(\tau)$, and $\mathfrak{A} \in \mathrm{Str}(\sigma)$. A *validity game* $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ is played as follows: for $i < \lambda$

- $\forall$ chooses $x_{2i} \in M$, $c_{2i} \in A$, and $\alpha_{2i} \in I$, and
- $\exists$ chooses $x_{2i+1} \in M$, $c_{2i+1} \in A$, and $\alpha_{2i+1} \in I$.

Denoting $a = \langle \alpha_i \rangle_{i<\lambda}$, player $\exists$ wins the play if $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0, \ldots)$ is a model of $\mathrm{Th}_{\mathcal{F}}(a)$. If $\exists$ wins the game $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$, we say that $\mathcal{F}$ is *valid* in $(\mathfrak{M}; \mathfrak{A})$.

One defines a *co-validity game* $\mathrm{VG}^*(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ in a same way as the validity game $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$, except that $\forall$ wins a play in $\mathrm{VG}^*(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ if $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0, \ldots)$ is a model of $\mathrm{Th}_{\mathcal{F}}(a)$. The expander $\mathcal{F}$ is *co-valid* in $(\mathfrak{M}; \mathfrak{A})$ if $\forall$ wins $\mathrm{VG}^*(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$.

10.8. Lemma. *Let $\mathcal{F}$ be a λ-expander and let $\mathfrak{M}$ be a structure. If $\mathcal{F}$ is valid in $(\mathfrak{M}; \mathfrak{A})$ for some $\mathfrak{A}$, then $\exists$ wins* $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$. *Similarly, if $\mathcal{F}$ is co-valid in $(\mathfrak{M}; \mathfrak{A})$ for some $\mathfrak{A}$, then $\forall$ wins* $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$.

Proof. We show only the expansion game case, since co-expansions are dealt with similarly. Suppose $\exists$ wins $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$. She wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ by playing $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ in the background as follows: at his i^{th} move, $\forall$ picks $x_{2i} \in M$ and $\alpha_{2i} \in I$. First player $\exists$ chooses such an element $c_{2i} \in A$ that $\mathfrak{A} \models \exists u\psi(u) \to \psi(c_{2i})$ if $\Phi_{\mathcal{F}}(i) = \exists u\psi(u)$. Then she lets $\forall$ move $x_{2i}, c_{2i}, \alpha_{2i}$ in the validity game, and gets an answer $x_{2i+1}, c_{2i+1}, \alpha_{2i+1}$. In the expansion game she moves $x_{2i+1} \in M$ and $\alpha_{2i+1} \in I$, and chooses ψ_i in such a way that it is true in $\mathfrak{A}$.

Playing in this way she finally has sequences $\langle x_i \rangle_{i<\lambda}$ and $\langle c_i \rangle_{i<\lambda}$ of elements of $\mathfrak{M}$ and $\mathfrak{A}$, respectively, and an $\mathcal{F}$-branch $a = \langle \alpha_i \rangle_{i<\lambda}$. Moreover, $(\mathfrak{M}, \ldots; \mathfrak{A}, \ldots)$ is a model of $\mathrm{Th}_{\mathcal{F}}(a)$, of each ψ_i $(i < \lambda)$, and of the Henkin sentences. Let $(\mathfrak{B}, c_0, \ldots)$ be the structure $(\mathfrak{A}, c_0, \ldots)$ restricted to those elements which are interpretations of some constant. Since the Henkin sentences $\exists\psi(u) \to \psi(c_{2i})$ hold in $\mathfrak{A}$, $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$, and thus $(\mathfrak{M}; \mathfrak{B})$ is an elementary submodel of $(\mathfrak{M}; \mathfrak{A})$ (relative to language $\mathcal{L}_{exp}(\tau, \sigma, \lambda)$), which implies the claim. □

As an application we show that the basic expanders of Example 10.2 are valid in very many structures.

10.9. Lemma. *Let τ be a vocabulary, $\mathfrak{M} \in \mathrm{Str}(\tau)$ and let ϕ be a set. If $\mathfrak{A}$ is an end extension of $(\mathrm{TC}(\{\mathfrak{M}, \tau, \phi\}), \in, \tau, \phi, \mathfrak{M})$, then $\mathcal{F}(\tau, \phi, \lambda)$ is valid in $(\mathfrak{M}; \mathfrak{A})$, and $\mathcal{F}'(\tau, \phi, \lambda)$ is co-valid in $(\mathfrak{M}; \mathfrak{A})$.*

Proof. We may assume $\mathrm{TC}(\mathfrak{M}, \tau, \phi) \subseteq \mathfrak{A}$. Player $\exists$ wins the validity game $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ by obeying the following rules.

- If t_k is an element of $\mathfrak{M}$, pick $\beta_{2k+1} = 1$ and $x_{2k+1} = t_k$. Otherwise let $\beta_{2k+1} = 0$ and $x_{2k+1} = x_0$.
- If t_k is an element of $\mathrm{TC}(\tau, \phi)$, pick $\alpha_{2k+1} = 1$ and $z_{2k+1} = t_k$. Otherwise let $\alpha_{2k+1} = 0$ and $z_{2k+1} = z_0$.
- Choose $c_{4k+1} = a_k = z_k$ and $c_{4k+3} = b_k = x_k$.

Player $\forall$ wins the co-validity game $\mathrm{VG}^*(\mathfrak{M}, \mathfrak{A}, \mathcal{F}')$ by obeying the following rules:

- If t_k is an element of $\mathfrak{M}$, pick $\beta_{2k} = 1$ and $x_{2k} = t_k$. Otherwise let $\beta_{2k} = 0$ and $x_{2k} = x_0$, where x_0 is any element of $\mathfrak{M}$.
- If t_k is an element of $\mathrm{TC}(\tau, \phi)$, pick $\alpha_{2k} = 1$ and $z_{2k} = t_k$. Otherwise let $\alpha_{2k} = 0$ and $z_{2k} = \emptyset$.
- Choose $c_{4k} = a_k = z_k$ and $c_{4k+2} = b_k = x_k$.

□

11. More expanders

The basic expanders $\mathcal{F}(\tau, \phi, \lambda)$ and $\mathcal{F}'(\tau, \phi, \lambda)$ of example 10.2 as such are not very useful: the last two lemmas in the end of the previous section show that $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}(\tau, \phi, \lambda))$ for every structure $\mathfrak{M}$. However, they become useful if we add more features to them. This is done with regulators.

Recall from the previous section that a λ-expander describes an expansion through a theory in the expansion language $\mathcal{L}_{exp}(\tau, \sigma, \lambda)$. A regulator extends this theory by sentences in the language $L_{\omega\omega}(\sigma_\lambda)$.

11.1. Definition. Let σ be a vocabulary and suppose λ is a cardinal as in the previous section. A *λ-regulator* $\mathcal{R}$ is a function such that $\mathrm{dom}(\mathcal{R}) = I^{<\lambda}$ for some set I and $\mathcal{R}(w)$ is a sentence in $L_{\omega\omega}(\sigma_{\mathrm{len}(w)})$ for each w.

Call a sequence $a\colon \lambda \to I$ a *$\mathcal{R}$-branch.* It defines a theory $\mathrm{Th}_{\mathcal{R}}(a) = \bigcup_{i<\lambda} G(a \restriction i)$. A structure $\mathfrak{A} \in \mathrm{Str}(\sigma_\lambda)$ is *$\mathcal{R}$-regular over a* if $\mathfrak{A} \models \mathrm{Th}_{\mathcal{R}}(a)$ and every element of $\mathfrak{A}$ is an interpretation of a constant symbol.

The *branching cardinal* $\kappa_{\mathcal{R}}$ of a regulator $\mathcal{R}$ is the maximum of $|I|^+$ and λ.

We shall see later that if $\mathcal{F}$ is a λ-expander and $\mathcal{R}$ is a λ-regulator, we can construct such an expander $\mathcal{F}(\mathcal{R})$ that every $\mathcal{F}(\mathcal{R})$-expansion is a $\mathcal{R}$-regular $\mathcal{F}$-expansion.

11.2. Definition. Let $\mathcal{R}$ be a λ-regulator, and suppose $\mathfrak{A} \in \mathrm{Str}(\sigma)$. The validity game $\mathrm{VG}(\mathfrak{A}, \mathcal{R})$ is played as follows: for $i < \lambda$

- $\forall$ chooses $c_{2i} \in A$, and $\alpha_{2i} \in I$, and
- $\exists$ chooses $c_{2i+1} \in A$, and $\alpha_{2i+1} \in I$.

Denoting $a = \langle \alpha_i \rangle_{i<\lambda}$, player $\exists$ wins the play if $(\mathfrak{A}, c_0, \ldots)$ is a model of $\mathrm{Th}_{\mathcal{R}}(a)$. If $\exists$ wins the game $\mathrm{VG}(\mathfrak{A}, \mathcal{R})$, we say that $\mathcal{R}$ is *valid* in $\mathfrak{A}$.

The co-validity game $\mathrm{VG}^*(\mathfrak{A}, \mathcal{R})$ is defined similarly.

11.3. Example. As shown in the previous section, each $\mathcal{F}(\tau, \sigma, \lambda)$-expansion was set-theoretical and had certain properties. Next I present a regulator, by which we can make the expansions well-founded.

Let γ be an ordinal, $\sigma = \{\in, t, p, m\}$, and let λ be an infinite cardinal. Similarly to Example 10.2, rename all the constants in σ_λ: let $t_0 = t$, $t_1 = p$, $t_2 = m$, and $t_{3+i} = c_i$

for each i. Construct $\mathcal{R}\colon \gamma^{<\lambda} \to L_{\omega\omega}(\sigma_\lambda)$ in such a way that for a $\mathcal{R}$-branch $a = \langle \gamma_i \rangle_{i<\lambda}$

$$\mathrm{Th}_{\mathcal{R}}(a) = \{\, t_i \notin t_j \;:\; i, j < \lambda \text{ and } \gamma_{2j+1} \leq \gamma_{2i+1} \,\}.$$

This regulator, which we denote by $\mathcal{R}(\gamma)$, is valid in every $A \subseteq V_\gamma$: the winning strategy of $\exists$ is to choose $\gamma_{2i+1} = \mathrm{rank}(t_i)$. Moreover, every $\mathcal{R}(\gamma)$-regular structure $\mathfrak{A}$ is well-founded: for each i the set-theoretical rank of $t_i^{\mathfrak{A}}$ in $\mathfrak{A}$ is less than or equal to γ_{2i+1}.

The obvious counterpart of the above regulator, $\mathcal{R}'(\gamma)$, is co-valid in every $A \subseteq V_\gamma$, and every $\mathcal{R}'(\gamma)$-regular structure is well-founded.

If $a_k\colon \lambda \to I_k$ for $k = 0, \ldots, n$ are functions, denote by $a_0 \cdot a_1 \cdots a_n$ the function

$$\lambda \to I_0 \times I_1 \times \ldots I_n,\; i \mapsto (a_0(i), a_1(i), \ldots, a_n(i)).$$

11.4. Lemma. *Suppose $\mathcal{F}$ is a λ-expander and $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are λ-regulators. There exists a λ-expander $\mathcal{F}^* = \mathcal{F}(\mathcal{R}_1, \ldots, \mathcal{R}_n)$ such that the following hold.*

(i) If $\mathrm{dom}(\mathcal{F}) = I^{<\lambda}$ *and* $\mathrm{dom}(\mathcal{R}_k) = I_k^{<\lambda}$ *for each k, then* $\mathrm{dom}(\mathcal{F}^*) = I^{*<\lambda}$, *where* $I^* = I \times I_1 \times \cdots \times I_n$.

(ii) If $\mathcal{F}$ is valid in $(\mathfrak{M}, \mathfrak{A})$ and each $\mathcal{R}_k$ is valid in $\mathfrak{A}$, then $\mathcal{F}^*$ is valid in $(\mathfrak{M}, \mathfrak{A})$.

(iii) Let $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0, \ldots)$ be an $\mathcal{F}^*$-expansion over $a_0 \cdot a_1 \cdots a_n$. There exist permutations $\langle c_i^0 \rangle_{i<\lambda}, \ldots, \langle c_i^n \rangle_{i<\lambda}$ of $\langle c_i \rangle_{i<\lambda}$ such that $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}, c_0^0, c_1^0, \ldots)$ is an $\mathcal{F}$-expansion over a_0 and $(\mathfrak{A}, c_0^k, c_1^k, \ldots)$ is $\mathcal{R}_k$-regular over a_k for every $k = 1, \ldots, n$.

(iv) $\kappa_{\mathcal{F}^*} = \max\{\kappa_{\mathcal{F}}, \kappa_{\mathcal{R}_1}, \ldots, \kappa_{\mathcal{R}_n}\}$.

(v) The mapping $(\mathcal{F}, \mathcal{R}_1, \ldots, \mathcal{R}_n) \mapsto \mathcal{F}^*$ is p.r.

Proof. We show the case $n = 2$ only, the general case being similar. For notational convenience suppose the expander $\mathcal{F}$ produces sentences in vocabulary $(\tau_\lambda; \sigma_\lambda) = (\tau \cup \{x_0, y_0, \ldots\}; \sigma \cup \{c_0, d_0, \ldots\})$ and the regulators $\mathcal{F}'$ and $\mathcal{F}''$ produce sentences in vocabularies $\sigma'_\lambda = \sigma \cup \{c'_0, d'_0, \ldots\}$ and $\sigma''_\lambda = \sigma \cup \{c''_0, d''_0, \ldots\}$, respectively. Let the vocabulary of the expansion language of $\mathcal{F}^*$ be $(\tau_\lambda; \sigma^*_\lambda)$, where $\sigma^*_\lambda = \sigma \cup \{c^*_0, d^*_0, \ldots\}$.

Let functions ζ, ζ', and ζ'' rename constants in such a way that for each i

$$\begin{array}{llll}
\zeta(c_{3i}) = c^*_i, & \zeta(c_{3i+1}) = d^*_{3i+1}, & \zeta(c_{3i+2}) = d^*_{3i+2}, & \zeta(d_i) = d^*_{3i}, \\
\zeta'(c'_{3i}) = c^*_i, & \zeta'(c'_{3i+1}) = d^*_{3i+2}, & \zeta'(c'_{3i+2}) = d^*_{3i}, & \zeta'(d'_i) = d^*_{3i+1}, \\
\zeta''(c''_{3i}) = c^*_i, & \zeta''(c''_{3i+1}) = d^*_{3i}, & \zeta''(c''_{3i+2}) = d^*_{3i+1}, & \zeta''(d''_i) = d^*_{3i+2}.
\end{array}$$

These are the permutations mentioned in (iii).

Suppose the domains of $\mathcal{F}$, $\mathcal{F}'$, and $\mathcal{F}''$ are $I^{<\lambda}$, $I'^{<\lambda}$, and $I''^{<\lambda}$, respectively, and let $I^* = I \times I' \times I''$. For an element $w^* = \langle (a_j, a'_j, a''_j) \rangle_{j<i}$ of $I^{*\leq\lambda}$ let $w = \langle a_j \rangle_{j<i}$ and define similarly w' and w''. Let

$$\mathcal{F}^*(w^*) = \begin{cases} F(w{\restriction}j)[\zeta] & \text{if } \mathrm{len}(w^*) = 3j, \\ F'(w'{\restriction}j)[\zeta'] & \text{if } \mathrm{len}(w^*) = 3j+1, \\ F''(w''{\restriction}j)[\zeta''] & \text{if } \mathrm{len}(w^*) = 3j+2. \end{cases}$$

This makes $\mathcal{F}^*$ an expander. The claims (i), (iv), and (v) are trivial. For (iii) note that if a^* is an $\mathcal{F}^*$-branch, then

$$\mathrm{Th}_{\mathcal{F}^*}(a^*) = \mathrm{Th}_{\mathcal{F}}(a)[\zeta] \cup \mathrm{Th}_{\mathcal{F}'}(a')[\zeta'] \cup \mathrm{Th}_{\mathcal{F}''}(a'')[\zeta''].$$

Thus an $\mathcal{F}^*$-expansion is a model of e.g. $\mathrm{Th}_{\mathcal{F}}(a)[\zeta]$, and hence (iii) holds. So we need to show (Ii).

Claim: If $\exists$ wins the games $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$, $\mathrm{VG}(\mathfrak{A}, \mathcal{F}')$, and $\mathrm{VG}(\mathfrak{A}, \mathcal{F}'')$, she wins also $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$.

Player $\exists$ wins $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$ by playing the other three games in the background. In the validity game $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$ elements are picked from $\mathfrak{M}$, $\mathfrak{A}$, and the parameter set I^*. Elements from $\mathfrak{M}$ are picked only in the games $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$ and $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$, so their reduction is trivial, as is the reduction of the parameters. Thus the whole problem lies in reducing the elements picked from $\mathfrak{A}$. The following strategy is applied.

First $\forall$ picks $c_0^* \in A$ in $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$. Player $\exists$ lets $\forall$ pick c_0^* in every phantom game (i.e. $c_0 = c_0' = c_0'' = c_0^*$), and gets three elements d_0, d_0', d_0'' in return, each from a different phantom game. She reserves her next three moves in $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$ for these elements and continues likewise. This sets the reducing function $f_\exists$. Meanwhile she continues the phantom games as follows. The first move of $\forall$ in $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ was c_0^*, and then $\exists$ answered d_0. She lets $\forall$ pick d_0' and d_0'' at his next two moves, then c_1^*, and so on. The other phantom games are played in a similar manner. See Diagram 2 for the whole construction.

Suppose that $\exists$ wins each play on the right. Now, if $a^* = \langle (a_i, a_i', a_i''), (b_i, b_i', b_i'') \rangle_{i<\lambda}$ and a, a', and a'' are as above, then for example

$$(\mathfrak{M}, x_0, y_0, \ldots; \mathfrak{A}, c_0^*, d_0, d_0', d_0'', \ldots) \text{ is a model of } \mathrm{Th}_{\mathcal{F}}(a),$$

which by our variable renamings implies

$$(\mathfrak{M}, x_0, y_0, \ldots; \mathfrak{A}, c_0^*, d_0, c_1^*, d_0', c_2^*, d_0'', \ldots) \text{ is a model of } \mathrm{Th}_{\mathcal{F}}(a)[\zeta].$$

Thus $\exists$ wins the play on the left. $\square$

As an application of the lemma consider the expander $\mathcal{F} = \mathcal{F}(\tau, \phi)$ of Example 10.2 and the regulator $\mathcal{R} = \mathcal{R}(\gamma)$ of Example 11.3. Now $\mathcal{F}(\mathcal{R})$ is an expander. Moreover, if $\mathfrak{M}$ is a structure in vocabulary τ, and γ is so large that $\mathfrak{M}$, τ, and ϕ are elements of V_γ, then $\mathcal{F}(\mathcal{R})$ is valid in $(\mathfrak{M}; V_\gamma, \in, \tau, \phi, \mathfrak{M})$. On the other hand, if $(\mathfrak{M}, \ldots; A, E, t, p, m, \ldots)$ is a $\mathcal{F}(\mathcal{R})$-expansion, then (A, E) is well-founded and $t^{\mathfrak{A}}$, $p^{\mathfrak{A}}$, and $m^{\mathfrak{A}}$ collapse into τ^S, ϕ^S, and into a model isomorphic with $\mathfrak{M}^{XS}$ for certain sets X and S.

12. Expanders and logics

The final stage in our sketch is to turn an expansion game into a sentence of a logic. That is, having an expander $\mathcal{F}$ we construct a sentence ϕ such that $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ if and only if ϕ is true in $\mathfrak{M}$. The expansion game is so designed that the transformation into a game sentence is particularly easy.

$\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$	$\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$	$\mathrm{VG}(\mathfrak{A}, \mathcal{R}_1)$	$\mathrm{VG}(\mathfrak{A}, \mathcal{R}_2)$
$\forall : x_0, c_0^*, (a_0, a_0', a_0'')$			
	$\forall : x_0, c_0^*, a_0$	$\forall : c_0^*, a_0'$	$\forall : c_0^*, a_0''$
	$\exists : y_0, d_0, b_0$	$\exists : d_0', b_0'$	$\exists : d_0'', b_0''$
$\exists : y_0, d_0, (b_0, b_0', b_0'')$			
$\forall : x_1, c_1^*, (a_1, a_1', a_1'')$			
	$\forall : x_1, d_0', a_1$	$\forall : d_0'', a_1'$	$\forall : d_0, a_1''$
	$\exists : y_1, d_1, b_1$	$\exists : d_1', b_1'$	$\exists : d_1'', b_1''$
$\exists : y_1, d_0', (b_1, b_1', b_1'')$			
$\forall : x_2, c_2^*, (a_2, a_2', a_2'')$			
	$\forall : x_2, d_0'', a_2$	$\forall : d_0, a_2'$	$\forall : d_0', a_2''$
	$\exists : y_2, d_2, b_2$	$\exists : d_2', b_2'$	$\exists : d_2'', b_2''$
$\exists : y_2, d_0'', (b_2, b_2', b_2'')$			
$\forall : x_3, c_3^*, (a_3, a_3', a_3'')$			
	$\forall : x_3, c_1^*, a_3$	$\forall : c_1^*, a_3'$	$\forall : c_1^*, a_3''$
	$\exists : y_3, d_3, b_3$	$\exists : d_3', b_3'$	$\exists : d_3'', b_3''$
$\exists : y_2, d_1, (b_3, b_3', b_3'')$			
$\forall : x_4, c_4^*, (a_4, a_4', a_4'')$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Diagram 2: The phantoms of $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F}^*)$.

12.1. Lemma. *Suppose $\mathcal{F}$ is a λ-expander. There exist game sentences*

$$\phi_{\mathcal{F}} = \big(\forall x_{2i} \bigwedge_{a_{2i}\in A} \exists x_{2i+1} \bigvee_{a_{2i+1}\in A}\big)_{i<\lambda} \bigwedge_{i<\lambda} \psi_{a_0\ldots a_{2i+1}} \quad \textit{and}$$

$$\phi^*_{\mathcal{F}} = \big(\exists x_{2i} \bigvee_{a_{2i}\in A} \forall x_{2i+1} \bigwedge_{a_{2i+1}\in A}\big)_{i<\lambda} \bigwedge_{i<\lambda} \psi^*_{a_0\ldots a_{2i+1}}$$

in $V_{\kappa\lambda}$, where each ψ_w and ψ^*_w are in $L_{\omega\omega}(\tau_\lambda)$, and the following claims hold.

(i) $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ if and only if $\mathfrak{M} \models \phi_{\mathcal{F}}$.

(ii) $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$ if and only if $\mathfrak{M} \models \phi^*_{\mathcal{F}}$.

(iii) $\kappa = \kappa_{\mathcal{F}}$.

(iv) The mappings $\mathcal{F} \mapsto \phi_{\mathcal{F}}$ and $\mathcal{F} \mapsto \phi^*_{\mathcal{F}}$ are p.r.

Proof. The game sentence $\phi_{\mathcal{F}}$ directly codes the expansion game of the expander $\mathcal{F}$. The problem of the construction lies in eliminating the extra expanding sort from the sentences given by the expander: if the expansion language had no sentences of the expanding sort, one could directly set ψ_w as $\mathcal{F}(w)$. Now recall that during the expansion game a complete theory about the expanding sort is constructed using the enumeration $\Phi_{\mathcal{F}}$ of the language $L_{\omega\omega}(\sigma_\lambda)$. This complete theory can be used to eliminate the extra sort. The elimination rule is simple: replace each initial sentence of the expansion language occurring in the complete theory with "$x_0 = x_0$".

To start with the technical details, suppose that the domain of $\mathcal{F}$ is $I^{<\lambda}$. Let $A = I \times 2$. For $w = \langle (\alpha_j, \gamma_j) \rangle_{j<i} \in A^{<\lambda}$ let $F(w)$ be the sentence $\mathcal{F}(\langle \alpha_j \rangle_{j<i})$, and let

$$G(w) = \begin{cases} \exists u \psi(u) \to \psi(c_{2k}) & \text{if } \operatorname{len}(w) = 4k \text{ and } \Phi_{\mathcal{F}}(k) = \exists u \psi(u), \\ \Phi_{\mathcal{F}}(k) & \text{if } \operatorname{len}(w) = 4k+2 \text{ and } \gamma_{2k+1} = 0, \\ \neg \Phi_{\mathcal{F}}(k) & \text{if } \operatorname{len}(w) = 4k+2 \text{ and } \gamma_{2k+1} = 1, \text{ and} \\ x_0 = x_0 & \text{otherwise.} \end{cases}$$

Thus F gives the same sentences as $\mathcal{F}$, and G enumerates the complete theory on the expanding sort. For $w \in A^{\leq\lambda}$ denote the cumulated theory by $\bar{F}(w) = \{ F(w \restriction i) : i \leq \operatorname{len}(w) \text{ and } i < \lambda \}$, and let $\bar{G}(w)$ be obtained similarly from G.

The sentences of the expansion language were obtained from *initial sentences* in $L_{basic}(\tau_\lambda)$ and $L_{\omega\omega}(\sigma_\lambda)$ by propositional connectives. Let $H(v, w)$ be obtained from the sentence $F(v)$ by

- replacing each initial subsentence of $F(v)$ which exists in $\bar{G}(w)$ with $x_0 = x_0$, and
- replacing each initial subsentence of $F(v)$ the negation of which exists in $\bar{G}(w)$ with $x_0 \neq x_0$.

However, if $F(v)$ has initial subsentences which, or the negations of which, do not exist in $\bar{G}(w)$, let $H(v, w)$ be undefined. This function H does the sort elimination, but we cannot simply choose ψ_w as $H(w, w)$, since the initial subsentences of sort 1 occurring in $F(w)$ might not yet be present in $\bar{G}(w)$. Moreover, we must set ψ_w identically false if it turns out that $\bar{G}(w)$ is inconsistent.

Define a function $f \colon A^{<\lambda} \to \mathrm{Ord}$ in such a manner that for each w and $i < f(w)$ the sentence $H(w \restriction i, w)$ is defined: let $f(\langle\rangle) = 0$,

$$f(w^\wedge \langle a \rangle) = \begin{cases} f(w) + 1 & \text{if } H(w \restriction f(w), w^\wedge \langle a \rangle) \text{ is defined,} \\ f(w) & \text{otherwise,} \end{cases}$$

and let $f(w) = \sup_{i < \operatorname{len}(w)} f(w \restriction i)$ if $\operatorname{len}(w)$ is a limit. Define

$$F^*(w) = \begin{cases} H(w \restriction f(w), w) & \text{if defined, and} \\ x_0 = x_0 & \text{otherwise,} \end{cases}$$

and let $\bar{F}^*$ be obtained from F^* in the same way as $\bar{F}$ is obtained from F. Now the definitions imply: if $a \colon \lambda \to A$, then $\bar{F}^*(a) = \{ H(a \restriction i, a) : i < \lambda \}$. Finally let for

$w \in A^{<\lambda}$ of even successor length

$$\psi_w = \begin{cases} F^*(w \restriction k) & \text{if } \mathrm{len}(w) = 4k+2, \\ x_0 \neq x_0 & \text{if } \mathrm{len}(w) = 4k+4 \text{ and there exists a sentence } \eta \in \bar{G}(w) \\ & \text{such that } \neg\eta \in \bar{G}(w), \text{ and} \\ x_0 = x_0 & \text{otherwise.} \end{cases}$$

This ends the construction of the game sentence $\phi_{\mathcal{F}}$. The other game sentence $\phi^*_{\mathcal{F}}$ is constructed likewise. The claims (i) and (ii) are shown similarly, and claim (iii) is trivial, so we only show (i) and (iv).

Claim A: $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ if and only if $\mathfrak{M} \models \phi_{\mathcal{F}}$.

"$\Longrightarrow$" Suppose $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$. Player $\exists$ wins the semantic game of $\mathfrak{M} \models \phi_{\mathcal{F}}$ by playing $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ in the background as follows. Suppose $\forall$ moves $x_{2i} \in \mathfrak{M}$ and $(\alpha_{2i}, \gamma_{2i}) \in A$. Player $\exists$ lets $\forall$ move x_{2i}, α_{2i} in the expansion game and gets an answer $x_{2i+1}, \alpha_{2i+1}, \psi_i$. Let her move $x_{2i+1} \in \mathfrak{M}$ and $(\alpha_{2i+1}, \gamma_{2i+1}) \in B$ in the semantic game where $\gamma_{2i+1} = 0$, if $\psi_i = \Phi_{\mathcal{F}}(i)$, and $\gamma_{2i+1} = 1$, otherwise.

Suppose then that $\exists$ wins the play in the expansion game. Then we have a sequence $a = \langle(\alpha_i, \gamma_i)\rangle_{i<\lambda}$ and elements $x_0, x_1, \ldots$ such that for some structure $\mathfrak{A}$

$$(\mathfrak{M}, x_0, x_1, \ldots; \mathfrak{A}) \models \bar{F}(a) \cup \bar{G}(a).$$

Clearly $(\mathfrak{M}, x_0, \ldots) \models H(a \restriction i, a)$ for every $i < \lambda$ (H replaces true subformulas with true subformulas and false ones with false ones); thus each ψ_w is satisfied in $(\mathfrak{M}, x_0, \ldots)$, and $\exists$ has won the semantic game.

"$\Longleftarrow$" Suppose $\exists$ wins the semantic game of $\mathfrak{M} \models \phi_{\mathcal{F}}$. Player $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ by playing $\mathrm{S}(\mathfrak{M}, \phi_{\mathcal{F}})$ in the background, mapping the moves similarly to the previous case. Suppose $\exists$ to have won the play in the semantic game. Denote $a = \langle(\alpha_i, \gamma_i)\rangle_{i<\lambda}$. Now

$$(\mathfrak{M}, x_0, \ldots) \models \psi_{a \restriction i} \text{ for every even successor ordinal } i < \lambda.$$

This implies that $\bar{G}(a)$ does not contain any sentence with its negation, and is thus a consistent and complete Henkin theory. Let $\mathfrak{A}$ be the term model of $\bar{G}(a)$. Effecting the replacement $F(w) \mapsto H(w, a)$ backwards we see that $(\mathfrak{M}, \ldots; \mathfrak{A})$ is a model of $\bar{F}(a)$. Thus $\exists$ wins the expansion game.

Claim B: The mapping $\mathcal{F} \mapsto \phi_{\mathcal{F}}$ is p.r.

Mapping $\mathcal{F} \mapsto F$ is trivially p.r. Since the enumeration $\Phi_{\mathcal{F}}$ is p.r. relative to λ, the mappings $\mathcal{F} \mapsto G$, $\mathcal{F} \mapsto \bar{F}$, and $\mathcal{F} \mapsto \bar{G}$ are p.r. Thus $\mathcal{F} \mapsto H$ is p.r., and since f was defined by primitive recursion, $\mathcal{F} \mapsto F^*$ is p.r. Thus $(\mathcal{F}, w) \mapsto \psi_w$ is p.r., which implies that $\mathcal{F} \mapsto \phi_{\mathcal{F}}$ is p.r. □

Recall again our principal construction method: given a sentence, say ϕ, in a logic we try to construct a pair of λ-expanders $(\mathcal{F}, \mathcal{F}')$ such that if ϕ is true in a structure, say $\mathfrak{M}$, then $\exists$ wins EG($\mathfrak{M}, \mathcal{F}$), and otherwise $\forall$ wins EG*($\mathfrak{M}, \mathcal{F}$). Plays of the expansion games give rise to set-theoretical expansions of the structure $\mathfrak{M}$, which corroborate certain facts concerning Kueker-approximations $\mathfrak{M}^{XS}$ and ϕ^S of the structure $\mathfrak{M}$ and the sentence ϕ; these approximations are determined by the moves made during the plays. For example, an $\mathcal{F}$-expansion may verify that ϕ^S is true in $\mathfrak{M}^{XS}$. Suppose then that we are able to play both the expansion game and the co-expansion game simultaneously, and that at the end two plays define the same Kueker-approximations. Now, in favourable conditions, the expansions corroborate controversial facts: in our canonical example the $\mathcal{F}$-expansion indicates that ϕ^S is true in $\mathfrak{M}^{XS}$, and $\mathcal{F}'$-expansion indicates it to be false. This implies that if we play the games simultaneously, either $\exists$ loses the play in the expansion game or $\forall$ loses the play in the co-expansion game. Since the former game is closed and the latter is open, the player who will lose (his/her) game will know it before the end of the game. This fact can be used to truncate the game and to transfer the pair of games into a sentence in $M_{\infty\lambda}$.

What was required from the pair of expanders above was that one could play them simultaneously tying certain moves together, and that this kind of simultaneous playing resulted in either $\exists$ losing the expansion game or $\forall$ losing the co-expansion game. Next we define a couple of concepts to express these requirements: the former is expressed by saying that the $\mathcal{F}$-branch and the $\mathcal{F}'$-branch meet on a common ground, and the latter by saying that the expanders are disjoint.

12.2. Definition. Let H be a set. λ-expanders $\mathcal{F}$ and $\mathcal{F}'$ with domains $I^{<\lambda}$ and $I'^{<\lambda}$, respectively, lie on a *common ground* H if $I = H \times \cdots$ and $I' = H \times \cdots$. Moreover, if $a = \langle (a_i, \ldots) \rangle_{i<\lambda}$ is an $\mathcal{F}$-branch and $a' = \langle (a'_i, \ldots) \rangle_{i<\lambda}$ is an $\mathcal{F}'$-branch such that $a_i = a'_i$ for each i, we say that the branches a and a' *meet* on H.

Expanders $\mathcal{F}$ and $\mathcal{F}'$ are *disjoint* if for every $\mathcal{F}$-branch a and for every $\mathcal{F}'$-branch a' either the branches do not meet on a common ground, or there exists no structure $(\mathfrak{M}; \mathfrak{A}; \mathfrak{B})$ such that $(\mathfrak{M}; \mathfrak{A}) \models \mathrm{Th}_{\mathcal{F}}(a)$ and $(\mathfrak{M}; \mathfrak{B}) \models \mathrm{Th}_{\mathcal{F}'}(a')$.

12.3. Example. The basic expanders $\mathcal{F}(\tau, \phi, \lambda)$ and $\mathcal{F}'(\tau, \phi, \lambda)$ lie on a common ground TC(τ, ϕ). Moreover, if $(\mathfrak{M}, x_0, \ldots; \mathfrak{A})$ is an $\mathcal{F}$-expansion over a and $(\mathfrak{M}, x_0, \ldots; \mathfrak{A}')$ is an $\mathcal{F}'$-expansion over a' such that a and a' meet on TC(τ, ϕ), the structural interpretations of m in $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic, and t in $\mathfrak{A}$ collapses into the same set as t in $\mathfrak{A}'$, as does p in $\mathfrak{A}$ and $\mathfrak{A}'$.

12.4. Lemma. *If $\mathcal{F}$ and $\mathcal{F}'$ are disjoint λ-expanders and $\mathfrak{M}$ is a structure, either $\exists$ does not win* EG($\mathfrak{M}, \mathcal{F}$) *or $\forall$ does not win* EG*($\mathfrak{M}, \mathcal{F}'$).

Proof. Let dom($\mathcal{F}$) $= (H \times I)^{<\lambda}$ and dom($\mathcal{F}'$) $= (H \times I')^{<\lambda}$, i.e. suppose that H is a common ground for $\mathcal{F}$ and $\mathcal{F}'$. Suppose, for contradiction, that $\exists$ wins EG($\mathfrak{M}, \mathcal{F}$) and $\forall$ wins EG*($\mathfrak{M}, \mathcal{F}'$). Play the two games simultanously as follows: for $i < \lambda$

- pick $x_{2i} \in M$, $a_{2i} \in H$, $b'_{2i} \in I'$ and ψ'_i by the winning strategy of $\forall$ in EG($\mathfrak{M}, \mathcal{F}'$), and let $\forall$ play $x_{2i}, (a_{2i}, b_{2i})$ in EG($\mathfrak{M}, \mathcal{F}$), where b_{2i} is arbitrary, and

- pick $x_{2i+1} \in M$, $a_{2i+1} \in H$, $b_{2i+1} \in I$, and ψ_i by the winning strategy of $\exists$ in $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$, and let $\exists$ play $x_{2i+1}, (a_{2i+1}, b'_{2i+1})$ in $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$ with an arbitrary b'_{2i+1}.

Let $a = \langle (a_i, b_i) \rangle_{i<\lambda}$ and $a' = \langle (a_i, b'_i) \rangle_{i<\lambda}$. The structure $(\mathfrak{M}, x_0, \ldots)$ has both an $\mathcal{F}$-expansion over a and an $\mathcal{F}'$-expansion over a', and the $\mathcal{F}$-branch a and the $\mathcal{F}'$-branch a' meet, which contradicts the disjointness of the expanders. $\square$

12.5. Lemma. *Suppose $\mathcal{F}$ and $\mathcal{F}'$ are disjoint λ-expanders. There exists a $M_{\kappa\lambda}$ sentence ϕ such that the following conditions hold.*

(i) If $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$*, then $\mathfrak{M} \models \phi$.*
(ii) If $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$*, then $\mathfrak{M} \not\models \phi$.*
(iii) $\kappa = \max\{\kappa_{\mathcal{F}}, \kappa_{\mathcal{F}'}\}$.
(iv) The mapping $(\mathcal{F}, \mathcal{F}') \mapsto \phi$ is p.r.

Proof. Let

$$\phi_{\mathcal{F}} = \Big(\forall x_i \bigwedge_{a_i \in A} \exists y_i \bigvee_{b_i \in A}\Big)_{i<\lambda} \bigwedge_{i<\lambda} \psi_{a_0 b_0 \ldots a_i b_i} \quad \text{and}$$
$$\phi^*_{\mathcal{F}'} = \Big(\exists x_i \bigvee_{a_i \in A'} \forall y_i \bigwedge_{b_i \in A'}\Big)_{i<\lambda} \bigwedge_{i<\lambda} \psi^*_{a_0 b_0 \ldots a_i b_i}$$

be the game sentences given by Lemma 12.1 such that

$\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$ if and only if $\mathfrak{M} \models \phi_{\mathcal{F}}$, and
$\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$ if and only if $\mathfrak{M} \models \phi^*_{\mathcal{F}'}$.

For notational convenience assume that $A = H \times I$ and $A' = H \times I'$, with H the common ground of $\mathcal{F}$ and $\mathcal{F}'$ where the games are disjoint.

First consider sequences $\bar{u} = \langle u_i \rangle_i$ such that, for each i, u_{4i} is the variable symbol x_i, $u_{4i+1} = (s_i, a_i, a'_i)$ is in $H \times I \times I'$, u_{4i+2} is the variable symbol y_i, and $u_{4i+3} = (t_i, b_i, b'_i)$ is in $H \times I \times I'$. Given such $\bar{u}$, let $w_{\bar{u}} = \langle (s_i, a_i), (t_i, b_i) \rangle_{4i+3<\mathrm{len}(\bar{u})}$ and denote

$$\Psi_{\bar{u}} = \{\, \psi_w \ : \ w = w_{\bar{u}} \restriction i \text{ for some } i \,\}.$$

Similarly, denote $w'_{\bar{u}} = \langle (s_i, a'_i), (t_i, b'_i) \rangle_{4i+3<\mathrm{len}(\bar{u})}$, and likewise obtain $\Psi'_{\bar{u}}$ from $\phi^*_{\mathcal{F}'}$. Let the syntax tree T of ϕ consist of those sequences $\bar{u}$ of the above form for which the theory $\Psi_{\bar{u}\restriction i} \cup \Psi'_{\bar{u}\restriction i}$ is consistent for every $i < \mathrm{len}(\bar{u})$.

Claim A: T is λ-leaftree.

Clearly T is a $\lambda + 1$-tree. Suppose, for contradiction, that $\bar{u}$ is a leaf of T having height λ. By first-order compactness $\Psi_{\bar{u}} \cup \Psi'_{\bar{u}}$ is consistent. Now the length of both $w_{\bar{u}}$ and $w'_{\bar{u}}$ is λ, and they give rise to an $\mathcal{F}$-branch a and an $\mathcal{F}'$-branch a' which meet. Let $(\mathfrak{M}, x_0, \ldots)$ be a model for $\Psi_{\bar{u}} \cup \Psi'_{\bar{u}}$. As shown in the proof of Lemma 12.1, there exist structures $\mathfrak{A}$ and $\mathfrak{A}'$ such that $(\mathfrak{M}, \ldots; \mathfrak{A})$ is a model of $\mathrm{Th}_{\mathcal{F}}(a)$ and $(\mathfrak{M}, \ldots; \mathfrak{A}')$ is a model of $\mathrm{Th}_{\mathcal{F}}(a')$. This contradicts the disjointness of $\mathcal{F}$ and $\mathcal{F}'$.

Now, if $\bar{u}$ is a leaf of T, then $\mathrm{len}(\bar{u}) = 4(i+1)$ for some i, and the theory $\Psi_{\bar{u}} \cup \Psi'_{\bar{u}}$ is inconsistent. Define labelling L as follows: let $L(\bar{u}) = \bigwedge \Psi_{\bar{u}}$ if $\bar{u}$ is a leaf, and otherwise, when $\mathrm{len}(\bar{u}) = 4i$, let

$$\begin{array}{lcl} L(\bar{u}) = \forall x_i, & & L(\bar{u}^\wedge\langle x_i\rangle) = \bigwedge, \\ L(\bar{u}^\wedge\langle x_i, \bar{a}_i\rangle) = \exists x_i, & \text{and} & L(\bar{u}^\wedge\langle x_i, \bar{a}_i, y_i\rangle) = \bigvee. \end{array}$$

This makes up a sentence $\phi = (T, L)$ in $M_{\kappa\lambda}$, where κ need not be larger than the maximum of $|H \times I \times I'|^+$ and λ.

Claim B: If $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$, then $\mathfrak{M} \models \phi$.

Suppose $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$. Then $\mathfrak{M} \models \phi_{\mathcal{F}}$. Player $\exists$ wins $\mathrm{S}(\mathfrak{M}, \phi)$ by playing $\mathrm{S}(\mathfrak{M}, \phi_{\mathcal{F}})$ in the background as follows: when $\forall$ picks $x_i \in M$ and $(s_i, a_i, a'_i) \in H \times I \times I'$ in $\mathrm{S}(\mathfrak{M}, \phi)$, $\exists$ lets $\forall$ pick x_i and $(s_i, a_i) \in A$ in $\mathrm{S}(\mathfrak{M}, \phi_{\mathcal{F}})$, and gets an answer $y_i \in M$, $(t_i, b_i) \in A$. She moves y_i and (t_i, b_i, b'_i) for an arbitrary b'_i in $\mathrm{S}(\mathfrak{M}, \phi)$. When the semantic game $\mathrm{S}(\mathfrak{M}, \phi)$ ends, she continues $\mathrm{S}(\mathfrak{M}, \phi_{\mathcal{F}})$ in an arbitrary way. If $\exists$ wins the play of $\mathrm{S}(\mathfrak{M}, \phi_{\mathcal{F}})$, then trivially $\exists$ wins the corresponding play of $\mathrm{S}(\mathfrak{M}, \phi)$.

Claim C: If $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$, then $\mathfrak{M} \models \phi$.

Suppose $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$. Then $\mathfrak{M} \models \phi^*_{\mathcal{F}'}$. Player $\forall$ wins $\mathrm{S}(\mathfrak{M}, \phi)$ by playing $\mathrm{S}(\mathfrak{M}, \phi^*_{\mathcal{F}'})$ in the background in a similar way to the claim B. Suppose then that we have played up to a leaf $\bar{u}$ in $\mathrm{S}(\mathfrak{M}, \phi)$, $\forall$ playing in $\mathrm{S}(\mathfrak{M}, \phi^*_{\mathcal{F}'})$ with a winning strategy. Now we know that $(\mathfrak{M}, \ldots) \models \Psi'_{\bar{u}}$, and since $\Psi_{\bar{u}} \cup \Psi'_{\bar{u}}$ is inconsistent, $(\mathfrak{M}, \ldots)$ is not a model of $\Psi_{\bar{u}}$. Thus $\forall$ won the play of $\mathrm{S}(\mathfrak{M}, \phi)$.

Claim D: The mapping $(\mathcal{F}, \mathcal{F}') \mapsto \phi$ is p.r.

Lemma 12.1 implies that the mappings $(\mathcal{F}, \bar{u}) \mapsto \Psi_{\bar{u}}$ and $(\mathcal{F}', \bar{u}) \mapsto \Psi'_{\bar{u}}$ are p.r. The testing of whether $\Psi_{\bar{u}} \cup \Psi'_{\bar{u}}$ is consistent is p.r., since the theories are quantifier-free (see the proof of 12.1). Thus $(\mathcal{F}, \mathcal{F}') \mapsto T$ is p.r. The labelling is trivially p.r. □

IV Logics absolute relative to various set theories

In this final part of this work we show certain characterizations and limits for the expressive power of logics absolute relative to various set theories. We start from weak set theories and proceed towards stronger ones. In the three first sections we present new proofs for certain known facts; for example, we will see that $L_{\infty\omega}$ is a maximal logic absolute relative to KP + Inf, and that every logic, absolute relative to ZFC, can be approximated with $L_{\infty\omega}$. In the two last sections we investigate to what extent this special position of $L_{\infty\omega}$ among logics applies to M-languages.

13. Logics absolute relative to KPU

We begin applying the concepts introduced in the previous part by presenting a result of Väänänen in [V2]: if the syntax of a logic, first-order relative to KPU, is finite, then the logic is a sublogic of $L_{\omega\omega}$. Since we are dealing with a logic, first-order relative to a set theory with urelements, we assume here that the set-theoretical universe contains urelements, that vocabularies and sentences are pure sets, and that the elements of the structures are urelements.

13.1. Lemma. *Let $\mathcal{L}$ be first-order relative to* KPU*, let τ be a hereditarily finite vocabulary, and let ϕ be a hereditarily finite sentence in $\mathcal{L}(\tau)$. There exists a pair of disjoint ω-expanders $\mathcal{F}$, $\mathcal{F}'$ such that the following conditions hold.*

(i) If $\mathfrak{M} \models_{\mathcal{L}} \phi$, then $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$.

(ii) If $\mathfrak{M} \not\models_{\mathcal{L}} \phi$, then $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$.

(iii) The branching cardinal of $\mathcal{F}$ and $\mathcal{F}'$ is ω.

(iv) The mapping $(\tau, \phi) \mapsto \Phi^+$ is p.r.

Proof. Let $\sigma = (\in, U, t, p, m, s_R)_{R \in \tau}$, where $\in$ is a binary relation, U is an unary relation, and all the other symbols are constants. We construct the expander $\mathcal{F}$ by modifying the basic expander of Example 10.2 as follows. Let τ_ω and σ_ω be the vocabularies τ and σ, augmented with ω new constant symbols $x_0, x_1, \ldots$ and $a_0, b_0, a_1, b_1 \ldots$, respectively. For each hereditarily finite set x let $\mu_x(v)$ be a first-order sentence such that whenever $(A, E) \models \mu_x[a]$, then a is in the well-founded part of (A, E) and collapses into x. Let the domain of $\mathcal{F}$ be $I^{<\omega}$, where $I = \{0, 1\}$. Let $t_0, t_1, \ldots$ enumerate all the constants in σ_ω. Define $\mathcal{F}$ in such a way that for any $\mathcal{F}$-branch $a = \langle \alpha_i \rangle_{i<\omega}$ the theory $\mathrm{Th}_{\mathcal{F}}(a)$ contains the following sentences.

(1) Finite subset of KPU.

(2) $\mu_\phi(p) \wedge \mu_\tau(t) \wedge \bigwedge_{R \in \tau} \mu_R(s_R)$.

(3) $m \in \mathrm{Str}(t) \wedge p \in \mathcal{L}(t) \wedge m \models_{\mathcal{L}(t)} p$.

(4) $\mathrm{dom}(m) \subseteq U$.

(5) $b_i \in \mathrm{dom}(m)$ for every $i \in \omega$.

(6) $b_i = b_j \leftrightarrow x_i = x_j$ for every $i, j \in \omega$.

(7) $\left.\begin{array}{ll} t_i \notin \mathrm{dom}(m) & \text{if } \alpha_{2i+1} = 0 \\ t_i = b_{2i+1} & \text{if } \alpha_{2i+1} = 1 \end{array}\right\}$ for every $i \in \omega$.

(8) ${s_R}^m(b_{j_1}, \ldots, b_{j_n}) \iff R(x_{j_1}, \ldots, x_{j_n})$ for every $j_1, \ldots, j_n \in \omega$ and for any n-ary relation symbol $R \in \tau$, and similarly for the other symbols in τ.

Let $\mathcal{F}'$ be defined similarly, except that constants a_i are used instead of b_i, and in the sentences (7) α_{2i} is used instead of α_{2i+1}.

The claims (iii) and (iv) hold trivially, and the expanders clearly lie on a common ground ($\emptyset$). The claims (i) and (ii) are shown similarly; we show only (i) and that the expanders are disjoint.

Suppose $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$. Let A be a transitive model of KPU such that $\mathfrak{M}, \phi, \tau \in A$ and the sentences (1)–(4) hold in $\mathfrak{A} = (A, \in, A{\restriction}U, \tau, \phi, \mathfrak{M}, \ldots)$. By Lemma 10.8 we need to show that $\mathcal{F}$ is valid in $(\mathfrak{M}; \mathfrak{A})$. Player $\exists$ wins the validity game $\mathrm{VG}(\mathfrak{M}, \mathfrak{A}, \mathcal{F})$ as follows. Suppose $\forall$ moves $x_{2i} \in M$, and $a_i \in A$. We may assume about the enumeration t_i $(i < \omega)$ that t_i has already been chosen during the play. Let $\exists$ pick $x_{2i+1} = t_i$ and $\alpha_i = 0$ if $t_i \in M$, and $x_{2i+1} = x_0$ and $\alpha_i = 1$ otherwise; and let her pick $b_i = x_i$. Clearly now (1)–(9) hold in $(\mathfrak{M}; \mathfrak{A})$. Thus (i) holds.

Consider first an $\mathcal{F}$-expansion $(\mathfrak{M}, x_0, \ldots; A, E, U, t, p, m, \ldots)$ over $\langle \alpha_i \rangle_{i<\omega}$. Denote $\mathfrak{A} = (A, E, U)$ and $X = \{x_0, x_1, \ldots\}$. Since $\mathfrak{A}$ is a model of (1), the relation E is extensional. Because $\mathfrak{A}$ satisfies (4)–(6), there exists a collapsing function $c\colon \mathrm{Wf}(\mathfrak{A}) \to (V, \in, U)$ such that $c(b_i) = x_i$. The sentence (2) indicates that each s_R like t and p are in $\mathrm{Wf}(\mathfrak{A})$, $c(s_R) = R$, $c(t) = \tau$, and $c(p) = \phi$. Sentences (4)–(7) imply that the universe of m collapses into X, the universe of $\mathfrak{M}{\restriction}X$. Now the vocabulary of $c(m)$ is $c(t) = \tau$, and thus (7)–(9) imply $c(m) = \mathfrak{M}{\restriction}X$. Since both $\mathfrak{A}$ and its well-founded part $\mathrm{Wf}(\mathfrak{A})$ are models of (a sufficiently large part of) KPU, $\mathrm{Wf}(\mathfrak{A})$ models (3) by the absoluteness of $\mathcal{L}$. Using the absoluteness of $\mathcal{L}$ between $c''\mathrm{Wf}(\mathfrak{A})$ and the real universe we finally see that $\mathfrak{M}{\restriction}X \models_{\mathcal{L}} \phi$.

Similarly, if an $\mathcal{F}'$-expansion $(\mathfrak{M}, x_0, \ldots; A', \ldots)$ exists, then $\mathfrak{M}{\restriction}X \not\models_{\mathcal{L}} \phi$. Thus the $\mathcal{F}'$-expansion cannot exist together with an $\mathcal{F}$-expansion $(\mathfrak{M}, x_0, \ldots; A, \ldots)$, and we have seen that the expanders are disjoint. □

13.2. Theorem. *Let $\mathcal{L}$ be first-order relative to* KPU. *There exists a primitive recursive translation $t\colon (\mathcal{L} \times \mathrm{Voc}){\restriction}HF \to L_{\omega\omega}$ such that*

$$\mathfrak{M} \models \phi \iff \mathfrak{M} \models t(\phi).$$

Proof. If τ and $\phi \in \mathcal{L}(\tau)$ are hereditarily finite, the previous lemma gives us a disjoint pair $(\mathcal{F}, \mathcal{F}')$ of expanders on a common ground. With Lemma 12.5 this pair turns into a sentence in $M_{\omega\omega}$, and the final turn into a sentence of $L_{\omega\omega}$ is made with Lemma 5.5. □

Since $L_{\omega\omega}$ is first-order relative to KPU (Theorem 7.16), we have the following result, which appears already in [V2] Corollary 3.1.5:

13.3. Corollary ([V2]). *$L_{\omega\omega}$ is the strongest finite logic which is first-order relative to KPU.* □

There is not much room for extending this result. As shown in Theorem 7.16, the logic $L_{\omega\omega}(Q_0)$ is first-order relative to KP and relative to KPU+Inf, but as it is well known, the quantifier Q_0 is not definable in $L_{\omega\omega}$. We cannot even extend the result to logics which are absolute relative to KPU: consider the following (somewhat artificial) logic $\mathcal{L}$, for which

$$\phi \in \mathcal{L}(\tau) \iff \phi \in L_{\omega\omega}(Q_0)(\tau) \text{ and the axiom of infinity is true, and}$$
$$\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi \iff \mathfrak{M} \models_{L_{\omega\omega}(Q_0)(\tau)} \phi.$$

This logic is absolute relative to KPU and semantically equivalent to $L_{\omega\omega}(Q_0)$.

14. Logics absolute relative to KP+Inf

The logic $L_{\infty\omega}$ has a remarkable position among absolute logics. Barwise has namely shown that $L_{\infty\omega}$ is the maximal logic, absolute relative to KP + Inf (see [B2]). We begin this section by showing this result in its full power: each logic, absolute relative to KP + Inf, has such a translation into $L_{\infty\omega}$ that admissible sets containing ω are closed under it.

14.1. Definition. If $\mathcal{L}$ is a logic and A is a transitive set, the logic $\mathcal{L}_A$ is defined by

$$\phi \in \mathcal{L}_A(\tau) \iff \tau, \phi \in A \quad \text{and} \quad [\phi \in \mathcal{L}(\tau)]^A$$
$$\mathfrak{M} \models_{\mathcal{L}_A} \phi \iff \mathfrak{M} \models_{\mathcal{L}} \phi.$$

We denote $L_A = (L_{\infty\omega})_A$. If $\mathcal{L}$ is absolute relative to KP+Inf, for every vocabulary τ and for every sentence ϕ there exists an admissible set A (e.g. H_κ for $\kappa = |\mathrm{TC}(\tau,\phi)|^+$) such that $\phi \in \mathcal{L}_A(\tau)$. Moreover, if $\mathcal{L}$ is first-order relative to KP, then $\mathcal{L}_A = \mathcal{L} \cap A$ for admissible sets A. Otherwise the inclusion $\mathcal{L}_A \subseteq \mathcal{L} \cap A$ may be strict.

14.2. Lemma. *If $\mathcal{L}$ is absolute relative to* KP+Inf, *then for each ordinal γ, for each vocabulary τ, and for each sentence $\phi \in \mathcal{L}(\tau)$ there exists a disjoint pair $(\mathcal{F}_\gamma, \mathcal{F}'_\gamma)$ of ω-expanders so that the following conditions hold.*

(i) *Suppose A is admissible and $\phi \in \mathcal{L}_A$. There exists $\gamma \in A$ such that*

$$\mathfrak{M} \models_{\mathcal{L}} \phi \Rightarrow \exists \text{ } \textit{wins} \text{ } \mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma), \textit{ and}$$
$$\mathfrak{M} \not\models_{\mathcal{L}} \phi \Rightarrow \forall \text{ } \textit{wins} \text{ } \mathrm{EG}^*(\mathfrak{M}, \mathcal{F}'_\gamma).$$

(ii) $\kappa_{\mathcal{F}} = \kappa_{\mathcal{F}'} = \max\{|\mathrm{TC}(\tau,\phi)|^+, \gamma^+\}$.

(iii) *The mapping $(\tau,\phi,\gamma) \mapsto (\mathcal{F}_\gamma, \mathcal{F}'_\gamma)$ is p.r.*

Moreover, if the logic $\mathcal{L}$ is first-order relative to KP+Inf, *we may assume $\mathcal{F}_\gamma = \mathcal{F}_0$ and $\mathcal{F}'_\gamma = \mathcal{F}'_0$ for every γ.*

Proof. Let $\sigma = (\in, t, p, m)$ and let $\mathcal{F}_b = \mathcal{F}(\tau, \phi, \omega)$ be the basic expander of Example 10.2. Let τ_ω and σ_ω be τ and σ, augmented with constants $x_0, x_1, \ldots$ and $c_0, c_1, \ldots$, and let $t_0, t_1, \ldots$ enumerate the constants of σ_ω. Denote $a = c_1$, $g = c_3$, $n_\omega = c_5$, and $n_i = c_{6+2i+1}$ for $i < \omega$. Define a regulator $\mathcal{R}_\gamma$ as follows: let its domain be $I^{<\omega}$, where $I = \omega \times (\gamma + 1)$, and let its values be such that for $b = \langle(\alpha_i, \gamma_i)\rangle_{i<\omega}$ the theory $\mathrm{Th}_{\mathcal{R}}(b)$ contains the following sentences:

(1) A finite subset of KP + Inf.

(2) $\theta(t, p, a)$, where θ is the Δ_0-sentence for which

$$\mathrm{KP} + \mathrm{Inf} \vdash \exists x \theta(t, p, x) \leftrightarrow p \in \mathcal{L}(t).$$

(3) $m \in \text{Str}(t) \wedge m \models_{\mathcal{L}(t)} p$.

(4) $\mu_i(n_i) \wedge n_i \in n_\omega$ for every $i < \omega$.

(5) $\left.\begin{array}{ll} t_i \notin n_\omega & \text{if } \alpha_{2i+1} = 0, \\ t_i = n_j & \text{if } \alpha_{2i+1} = 1 + j, \end{array}\right\}$ for every $i \in \omega$.

(6) g is a one-to-one function from $\text{dom}(m)$ to n_ω.

(7) $t_i \neq t_i$ if $\gamma_{2i+1} = \gamma$, for $i < 3$.

(8) $t_i \notin t_j$ if $\gamma \neq \gamma_{2j+1} \leq \gamma_{2i+1}$, for every $i, j < \omega$.

Let $\mathcal{F}_\gamma = \mathcal{F}_b(\mathcal{R}_\gamma)$. Construct $\mathcal{F}'_\gamma$ similarly by starting from the basic expander $\mathcal{F}'(\tau, \phi, \omega)$, and adding a regulator $\mathcal{R}'_\gamma$, similar to $\mathcal{R}_\gamma$, except that the constant names a, g, and n_i for $i \leq \omega$ refer to constants c_j with even indices, α_{2i} is used instead of α_{2i+1} in (5), γ_{2i} is used instead of γ_{2i+1} in (7) and (8), and (3) is replaced by

(3′) $m \in \text{Str}(t) \wedge m \not\models_{\mathcal{L}(t)} p$.

The claims (ii) and (iii) are trivial, so we need to show (i) and that $(\mathcal{F}, \mathcal{F}')$ is a disjoint pair of expanders.

To see (i), suppose $\phi \in \mathcal{L}_A(t)$ and $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$. Choose such $r \in A$ that $\theta(\tau, \phi, r)$ holds in $(A, \in)$. Let $\gamma = \max\{\text{rank}(\tau), \text{rank}(\phi), \text{rank}(r)\} + 1 \in A$. Let $\mathbb{P}$ be a forcing which forces the model $\mathfrak{M}$ to be countable. Because of Corollary 10.6 it is enough to show that $\exists$ wins $\text{EG}(\mathfrak{M}, \mathcal{F})$ in the generic extension of the universe relative to $\mathbb{P}$. Thus, let us next work in the generic extension. Let B be an admissible set such that $A \subseteq B$, $\mathfrak{M} \in B$, and the generic enumeration $f: M \to \omega$ is in B. Let $\mathfrak{B} = (B, \in, \tau, \phi, \mathfrak{M})$. The expander $\mathcal{F}_b$ is valid in $(\mathfrak{M}; \mathfrak{B})$, since $\text{TC}(\tau, \phi, \mathfrak{M}) \subseteq B$. We need to show that $\mathcal{R}_\gamma$ is valid in $\mathfrak{B}$: Lemma 11.4 implies that then $\mathcal{F}_\gamma$ is valid in $(\mathfrak{M}; \mathfrak{B})$, so thus $\exists$ wins $\text{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$.

Player $\exists$ wins $\text{VG}(\mathfrak{B}, \mathcal{R}_\gamma)$ by playing as follows. Suppose $\forall$ picks $c_{2i} \in B$. We may assume about the enumeration t_i $(i < \omega)$ that an interpretation of t_i has already been chosen. Let $\exists$ choose $\alpha_{2i+1} = t_i + 1$ if t_i is an element of ω, and $\alpha_{2i+1} = 0$ otherwise, and let her choose $\gamma_{2i+1} = \text{rank}(t_i)$ if $\text{rank}(t_i) < \gamma$, and γ otherwise. Let her pick $a = r$, $g = f$, $n_\omega = \omega$, and $n_k = k$ for $k < \omega$.

The expanders $\mathcal{F}(\tau, \phi, \omega)$ and $\mathcal{F}'(\tau, \phi, \omega)$ have a common ground $\text{TC}(\tau, \phi)$, and so the expanders $\mathcal{F}_\gamma$ and $\mathcal{F}'_\gamma$ have a common ground $\text{TC}(\tau, \phi)$. We need to show that they are disjoint. Suppose $(\mathfrak{M}, x_0, \ldots; A, E, t, p, m, \ldots)$ is an $\mathcal{F}_\gamma$-expansion over an $\mathcal{F}_\gamma$-branch b. Then the structure $(\mathfrak{M}, \ldots; A, E, t, p, m, \ldots)$ is an $\mathcal{F}_b$-expansion and $\mathfrak{A} = (A, E, t, p, m)$ is $\mathcal{R}_\gamma$-regular. By Lemma 10.3 we know that t and p are in $\text{Wf}(A, E)$, and they collapse into τ^S and ϕ^S, respectively, where S is determined by b. Moreover, the structural interpretation $\mathfrak{N}$ of m in (A, E) is isomorphic to $\mathfrak{M}^{XS}$, where $X = \{x_0, y_0, \ldots\}$.

Since $\mathfrak{A} \models (1)$, both (A, E) and $\text{Wf}(A, E)$ are models of (a finite fragment of) KP. The sentences (4)–(5) imply that n_i is in $\text{Wf}(A, E)$ and collapses into i for $i \leq \omega$. Thus both (A, E) and $\text{Wf}(A, E)$ are models of the axiom of infinity. The sentences (7)–(8) imply: $t_k \in \text{Wf}(A, E)$ whenever $\gamma_{2k+1} < \gamma$, so (7) indicates $a \in \text{Wf}(A, E)$. Thus the Δ_0-sentence (2) is true in $\text{Wf}(A, E)$. The sentence (16) indicates that $g: \text{dom}(m) \to n_\omega$ is a one-to-one mapping in (A, E). Thus $g: \text{dom}(m) \to ran(g)$ is a bijection, there is

$m' \in \mathrm{Wf}(A, E)$ for which $g\colon m \cong m'$, and m' collapses into a structure isomorphic to $\mathfrak{M}^{XS}$. Since $\mathcal{L}$ is absolute relative to KP+Inf and $\mathrm{Wf}(A, E) \models (2)$, we have $m' \in Str(t)$, $p \in \mathcal{L}(t)$, and $m' \models p$ in $\mathrm{Wf}(A, E)$. This implies:

$$\mathfrak{M}^{XS} \in \mathrm{Str}(\tau^S), \quad \phi^S \in \mathcal{L}(\tau^S), \quad \text{and} \quad \mathfrak{M}^{XS} \models_{\mathcal{L}(\tau^S)} \phi^S.$$

Similarly one sees that if $(\mathfrak{M}, x_0, \ldots; \mathfrak{B})$ is an $\mathcal{F}'_\gamma$-expansion over an $\mathcal{F}'_\gamma$-branch b', then

$$\mathfrak{M}^{XS'} \in \mathrm{Str}(\tau^{S'}), \quad \phi^{S'} \in \mathcal{L}(\tau^{S'}), \quad \text{and} \quad \mathfrak{M}^{XS'} \not\models_{\mathcal{L}(\tau^{S'})} \phi^{S'}.$$

But if the branches b and b' meet on their common ground $\mathrm{TC}(\tau, \phi)$, the sets S and S' are the same set. Thus $\mathcal{F}_\gamma$ and $\mathcal{F}'_\gamma$ are disjoint.

Finally consider a logic $\mathcal{L}$ which is first-order relative to KP+Inf. Leave out the parameter γ from the regulator $\mathcal{R}_\gamma$: let $I = \omega$, remove the sentences (7)–(8) from the theories, and replace (2) with

(2′) $p \in \mathcal{L}(t)$.

The proof of the (i) is similar to above. In the proof of the disjointness of the expanders the implication

$$(A, E) \models p \in \mathcal{L}(t) \implies \mathrm{Wf}(A, E) \models p \in \mathcal{L}(t)$$

is now easy, since as a $\Delta_1^{\mathrm{KP+Inf}}$-sentence $p \in \mathcal{L}(t)$ is absolute relative to these two models of KP+Inf. □

14.3. Theorem ([B2]). *(i) Let $\mathcal{L}$ be absolute relative to* KP+Inf. *Then $\mathcal{L}_A \leq L_A$ for any admissible set $A \ni \omega$.*

(ii) Let $\mathcal{L}$ be first-order relative to KP+Inf. *There exists a translation $t\colon \mathcal{L} \to L_{\infty\omega}$, primitive recursive in a mapping which maps a well-founded tree to its ordinal.*

Proof. (i) Let $\phi \in \mathcal{L}_A$. By Lemma 14.2 there exists a pair of disjoint expanders $(\mathcal{F}, \mathcal{F}')$ in A such that if $\mathfrak{M} \models \phi$, then $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$, and otherwise $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F})$. By Lemma 12.5 this turns into a $M_{\infty\omega}$-sentence in A and finally, by Lemma 5.3, into a sentence of L_A.

(ii) is shown similarly. □

The original proof of Barwise is based on three ideas: first, by an easy Löwenheim–Skolem argument, one obviously needs to show only the result for countable admissible sets. Secondly, the expression "$\mathfrak{M} \models_{\mathcal{L}} \phi$" is translated into a $\Delta(L_{\infty\omega})$-expression by introducing a new set-theoretical sort. Finally, this new sort is eliminated by interpolation in a countable admissible fragment.

In the above approach, we could as well have used the same Löwenheim–Skolem argument to avoid forcing when showing the claim (i) in the proof of 14.2. However, the chosen approach produced slightly more refined result: in the case of logics which are first-order relative to KP+Inf, we have an estimate for the complexity of the transformation into $L_{\infty\omega}$. The Löwenheim–Skolem-argument would have given the estimate only for countable admissible fragments.

In the above result we have replaced the use of interpolation of $L_{\infty\omega}$ on countable admissible fragments with a new construction. A natural question is, whether we can show the interpolation result with this new method. The answer is affirmative, and the construction is sketched below. Instead of just giving the Δ-interpolation result, we show a stronger separation result, the original result appearing in [L] and [B1]. The result is given in a single-sorted form; the many-sorted form can be proven in a similar way with obvious modifications.

Let $\mathcal{L}$ and $\mathcal{L}'$ be logics. We say that $\mathcal{L}'$ *allows separation for* $\mathcal{L}$ if the following condition is satisfied. For every pairwise disjoint vocabulary τ, ρ, and ρ' and for every sentence $\phi \in \mathcal{L}(\tau \cup \rho)$ and $\phi' \in \mathcal{L}(\tau \cup \rho')$ such that $\phi \wedge \phi'$ has no model, there exists a sentence $\psi \in \mathcal{L}'(\tau)$ satisfying

$$(\mathfrak{M}, \vec{R}) \models \phi \implies \mathfrak{M} \models \psi \quad \text{and}$$
$$(\mathfrak{M}, \vec{R}') \models \phi' \implies \mathfrak{M} \not\models \psi.$$

14.4. Theorem. *Let $\mathcal{L}$ be absolute relative to* KP + Inf *and let A be a countable admissible set containing ω. Then L_A allows separation for $\mathcal{L}_A$.*

Proof. We modify the proof of 14.2. For simplicity, assume that $\mathcal{L}$ is first-order relative to KP + Inf: to obtain the result for the absolute case one needs a similar modification to the translation result above. The first task is to construct ω-expanders $\mathcal{F}$ and $\mathcal{F}'$ such that

$$\exists \vec{R}\ (\mathfrak{M}, \vec{R}) \models \phi \implies \exists \text{ wins } \mathrm{EG}(\mathfrak{M}, \mathcal{F}), \quad \text{and}$$
$$\exists \vec{R}'\ (\mathfrak{M}, \vec{R}') \models \phi' \implies \forall \text{ wins } \mathrm{EG}^*(\mathfrak{M}, \mathcal{F}').$$

Moreover, if $(\mathfrak{M}, x_0, \ldots; B, E, \ldots)$ is an $\mathcal{F}$-expansion over q, then ϕ^S should hold in some expansion of $\mathfrak{M}^{XS}$, where $X = \{x_0, \ldots\}$ and S is given by the $\mathcal{F}$-branch q; and similar should hold for the expander $\mathcal{F}'$. These expanders can be constructed similarly to the expanders in the proof of 14.2. The problem in the construction is that the expanders $\mathcal{F}$ and $\mathcal{F}'$ are not disjoint on their common ground $Z = \mathrm{TC}(\tau, \rho, \rho', \phi, \phi')$. This is because, even if the sentence $\phi \wedge \phi'$ had no models, its approximation may have them.

We solve the problem as follows. Recall that Z is an element of A, a countable admissible fragment. Let $h\colon \omega \to Z$ enumerate Z. It is not hard to modify $\mathcal{F}'$ in such a way that if $(\mathfrak{M}, \vec{R}') \models \phi'$ for some $\vec{R}'$, then $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$ by a strategy where his every second move on the common ground Z is picked with the enumeration h. (This requires some rearrangements among the constant names.) Let us combine the expanders $\mathcal{F}$ and $\mathcal{F}'$ (or actually the game sentences given by Lemma 12.1) just as presented in Lemma 12.5, except that we make $\forall$, in the semantic game of the M-sentence, to pick his every second choice in Z with the enumeration h. The result is a sentence θ in $M_{\infty\omega}$: if the resulting sentence had a long branch, it would give rise to a structure $\mathfrak{M}$ and expansions which would indicate $(\mathfrak{M}^{XS}, \ldots) \models \phi^S$ and $(\mathfrak{M}^{XS}, \ldots) \models \phi'^S$. This is a contradiction: now $S = Z$, $\phi^S = \phi$, and $\phi'^S = \phi'$, since h enumerates the common ground Z.

However, the sentence θ above does not necessarily exist in A, since it is constructed using the enumeration h. Let $B \supseteq A$ be the least admissible set containing h. The sets A and B have the same ordinals, and so the ordinal α of the syntax tree of θ exists in A. Construct a sentence $\psi \in M_{\infty\omega}\lceil A$ by restricting the game sentence attached to $\mathcal{F}$ by Lemma 12.1 with the tree B_α: add conjunctions of form $\bigwedge_{\alpha_{i+1}<\alpha_i}$ to its game prefix and cut the branch when $\alpha_k = 0$. Now it is clear that $\mathfrak{M} \models \psi$ if $(\mathfrak{M}, \mathfrak{A}) \models \phi$ for some $\mathfrak{A}$. On the other hand, if $(\mathfrak{M}, \mathfrak{A}') \models \phi'$ for some $\mathfrak{A}'$, then $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$ with a strategy using the enumeration h. Thus he wins the semantic game $\mathrm{S}(\mathfrak{M}, \theta)$. When playing the semantic game $\mathrm{S}(\mathfrak{M}, \psi)$, player $\forall$ can choose the α_i's by operating $\mathrm{S}(\mathfrak{M}, \theta)$ in the background and thus win it, too. □

15. Logics absolute relative to a standard set theory

In the previous two sections we saw that there exist maximal logics first-order relative to KPU and absolute relative to KP + Inf. Now, if we strengthen the theory beyond KP + Σ_1-sep, these kinds of maximal logics no longer exist. The following diagonal argument is a standard way of proving this result. We call set theory a *standard* theory if its vocabulary is $\{\in\}$.

15.1. Theorem. *Let T be a standard set theory containing* KP + Σ_1*-sep. There exists no maximal logic $\mathcal{L}$, absolute relative to T.*

Proof. Let $\mathcal{L}$ be absolute relative to T. It is enough to define a model class $\mathcal{K}$ which is Δ_1-definable in T but not definable in $\mathcal{L}$: we may then add this model class as a single sentence to $\mathcal{L}$ producing a strictly stronger logic, absolute relative to T. The model class $\mathcal{K}$ consists of those structures (A, E, p, a) where p codes a sentence of $\mathcal{L}$, false in (A, E, p, a).

To be exact, let P be a Δ_0-predicate such that

$$T \vdash \phi \in \mathcal{L}(\tau) \leftrightarrow \exists x\, P(\tau, \phi, x).$$

Let $\tau = \{\in, c, d\}$ be a hereditarily finite vocabulary. Let $\mathcal{K}$ be the class of those structures $(A, E, p, a) \in \mathrm{Str}(\tau)$ where (A, E) is extensional and well-founded, and there exists a Mostowski collapsing function c of (A, E) such that $P(\tau, cp, ca)$ and $(A, E, p, a) \not\models_{\mathcal{L}(\tau)} cp$. It is clear that $\mathcal{K}$ is Δ_1-definable in T: the existential quantification over Mostowski collapsing functions can be changed to universal. To see that $\mathcal{K}$ is not definable in $\mathcal{L}$, suppose for contradiction that $\phi \in \mathcal{L}(\tau)$ defines $\mathcal{K}$. Choose such x that $P(\tau, \phi, x)$ and let A be a transitive set containing both x and ϕ. Now

$$(A, \in, \phi, x) \models_{\mathcal{L}} \phi \iff (A, \in, \phi, x) \in \mathcal{K} \iff (A, \in, \phi, x) \not\models_{\mathcal{L}} \phi,$$

which is a contradiction. □

We next give a new proof for the result of Burgess in [Bu]: if $\mathcal{L}$ is absolute relative to a standard set theory (e.g. ZFC or some of its extensions), there exists a p.r. approximation mapping $\mathcal{L} \times \mathrm{Ord} \to L_{\infty\omega}$.

15.2. Lemma. *For each vocabulary τ, for each formula $\phi \in \mathcal{L}(\tau)$, and for each ordinal γ there exists an expander $\mathcal{F}_\gamma$ such that the following conditions hold.*

(i) $\mathfrak{M} \models_{\mathcal{L}} \phi \iff \exists$ *wins* $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$ *for some* γ
$\iff \exists$ *wins* $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$ *for some* $\gamma < \max\{|\mathfrak{M}|, |\mathrm{TC}(\tau,\phi)|\}^+$.

(ii) If $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$, she wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\delta)$ for every $\delta > \gamma$.

(iii) The branching cardinal of $\mathcal{F}_\gamma$ is $\max\{|\mathrm{TC}(\tau,\phi)|^+, |\gamma|^+\}$.

(iv) The mapping $(\tau, \phi, \gamma) \mapsto \mathcal{F}_\gamma$ is p.r.

Proof. Let $\mathcal{F}_b = \mathcal{F}(\tau, \phi, \omega)$ be the basic expander of Example 10.2. Let $\mathcal{R}_\gamma$ be the regulator of Example 11.3. Let $\mathcal{R}$ be a trivial regulator, where $\mathrm{dom}(\mathcal{R}) = \{0\}^{<\omega}$ and, for an $\mathcal{R}$-branch q, the theory $\mathrm{Th}_{\mathcal{R}}(q)$ contains a single sentence,

(1) $m \in \mathrm{Str}(t) \wedge p \in \mathcal{L}(t) \wedge m \models_{\mathcal{L}(t)} p$.

Let $\mathcal{F}_\gamma = \mathcal{F}_b(\mathcal{R}_\gamma, \mathcal{R})$. The claims (ii)–(iv) are easy, so it is enough to show (i).

Claim A: If $\mathfrak{M} \in \mathrm{Str}(\tau)$, $\phi \in \mathcal{L}$, and $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$, there exists γ less than $\max\{|\mathrm{TC}(\tau,\phi)|^+, |\mathfrak{M}|^+\}$ such that $\exists$ wins the expansion game.

Let $\kappa = \max\{|\mathrm{TC}(\tau,\phi)|^+, |\mathfrak{M}|^+\}$. We may assume $\mathfrak{M} \in H_\kappa$. Since Σ_1-formulas reflect onto H_κ, the sentence (1) is true in H_κ, and in fact (1) is true in some V_γ where $\gamma < \kappa$. But now $\mathcal{F}_b$ is valid in $(\mathfrak{M}; V_\gamma, \in, \tau, \phi, \mathfrak{M})$, and both $\mathcal{R}_\gamma$ and $\mathcal{R}$ are valid in $(V_\gamma, \in, \tau, \phi, \mathfrak{M})$. Thus $\mathcal{F}_\gamma$ is valid in an expansion of $\mathfrak{M}$, and $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$.

Claim B: If $(\mathfrak{M}, x_0, \ldots; A, E, t, p, m, \ldots)$ is an $\mathcal{F}_\gamma$-expansion over q, then

$$\mathfrak{M}^{XS} \in \mathrm{Str}(\tau^S), \quad \phi^S \in \mathcal{L}(\tau^S), \quad \text{and} \quad \mathfrak{M}^{XS} \models_{\mathcal{L}(\tau^S)} \phi^S,$$

where $X = \{x_0, x_1, x_2, \ldots\}$ and S is determined by q.

Since an $\mathcal{F}_\gamma$-expansion is a $\mathcal{R}_\gamma$-regular $\mathcal{F}_b$-expansion, the structure (A, E) is well-founded and the elements t, p, and m collapse into τ^S, ϕ^S, and $\mathfrak{M}^{XS}$, respectively. The claim holds, since (A, E) is $\mathcal{R}$-regular and the sentence (1) is a Σ_1-sentence.

Claim C: $\mathfrak{M} \models_{\mathcal{L}} \phi \implies \exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$ for some $\gamma < \max\{|\mathfrak{M}|, |\mathrm{TC}(\tau,\phi)|\}^+$
$\implies \exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F}_\gamma)$ for some γ
$\implies \mathfrak{M} \models_{\mathcal{L}} \phi$.

The first implication is the claim A, the second one is trivial, so we need to show the third one. Suppose $\mathfrak{M} \not\models_{\mathcal{L}} \phi$. Construct expanders $\mathcal{F}'_\gamma$ from the basic expander $\mathcal{F}'(\tau, \phi, \omega)$, from the regulators $\mathcal{R}'_\gamma$, and from a regulator similar to $\mathcal{R}$, except that instead of (1) we have

(1′) $m \in \mathrm{Str}(t) \wedge p \in \mathcal{L}(t) \wedge m \not\models_{\mathcal{L}(t)} p$.

As above, we can show that the claims A and B hold for these expanders, too. The claim B implies that the expanders $\mathcal{F}_\gamma$ and $\mathcal{F}'_\gamma$ are disjoint for each γ. By the claim A and (ii) there is γ such that $\forall$ wins $\mathrm{EG}^*(\mathcal{F}'_\delta, \mathfrak{M})$ whenever $\delta \geq \gamma$. Thus $\exists$ cannot win $\mathrm{EG}(\mathcal{F}_\delta, \mathfrak{M})$ for any δ. □

By Lemma 12.1 this lemma directly implies the following approximation result.

15.3. Corollary. *Let $\mathcal{L}$ be absolute relative to a standard set theory. There exists a primitive recursive mapping $a\colon \mathrm{Voc} \times \mathcal{L} \times \mathrm{Ord} \to V_{\infty\omega}$ such that the following conditions hold.*

(i) If $\phi \in \mathcal{L}(\tau)$ and $\mathfrak{M} \in \mathrm{Str}(\tau)$, then

$$\begin{aligned}\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi &\iff \mathfrak{M} \models a(\tau, \phi, \gamma) \text{ for some } \gamma \\ &\iff \mathfrak{M} \models a(\tau, \phi, \gamma) \text{ for some } \gamma < \max\{|\mathfrak{M}|, |\mathrm{TC}(\tau, \phi)|\}^+.\end{aligned}$$

(ii) If $\gamma \leq \delta$, then $a(\tau, \phi, \gamma) \Rightarrow a(\tau, \phi, \delta)$.

(iii) $a(\tau, \phi, \gamma) \in V_{\kappa\omega}$, where $\kappa = \max\{|\mathrm{TC}(\tau, \phi)|^+, \gamma^+\}$.

□

The proof of the following proposition is given for example in [Bu].

15.4. Proposition. *There exists a primitive recursive mapping $b\colon \mathrm{Voc} \times V_{\infty\omega} \times \mathrm{Ord} \to L_{\infty\omega}$ such that the following conditions hold.*

(i) If $\phi \in V_{\infty\omega}(\tau)$ and $\mathfrak{M} \in \mathrm{Str}(\tau)$, then

$$\begin{aligned}\mathfrak{M} \models \phi &\iff \mathfrak{M} \models b(\tau, \phi, \gamma) \text{ for some } \gamma \\ &\iff \mathfrak{M} \models b(\tau, \phi, \gamma) \text{ for some } \gamma < \max\{|\mathfrak{M}|, |\mathrm{TC}(\tau, \phi)|\}^+.\end{aligned}$$

(ii) If $\gamma \leq \delta$, then $b(\tau, \phi, \gamma) \Rightarrow b(\tau, \phi, \delta)$.

(iii) $b(\tau, \phi, \gamma) \in L_{\kappa\omega}$, where $\kappa = \max\{|\mathrm{TC}(\tau, \phi)|^+, \gamma^+\}$.

□

15.5. Corollary ([Bu]). *Let $\mathcal{L}$ be absolute relative to a standard set theory. There exists a primitive recursive mapping $A\colon \mathrm{Voc} \times \mathcal{L} \times \mathrm{Ord} \to L_{\infty\omega}$ such that the following conditions hold.*

(i) If $\phi \in \mathcal{L}(\tau)$ and $\mathfrak{M} \in \mathrm{Str}(\tau)$, then

$$\begin{aligned}\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi &\iff \mathfrak{M} \models A(\tau, \phi, \gamma) \text{ for some } \gamma \\ &\iff \mathfrak{M} \models A(\tau, \phi, \gamma) \text{ for some } \gamma < \max\{|\mathfrak{M}|, |\mathrm{TC}(\tau, \phi)|\}^+.\end{aligned}$$

(ii) If $\gamma < \delta$, then $A(\tau, \phi, \gamma) \Rightarrow A(\tau, \phi, \delta)$.

(iii) $A(\tau, \phi, \gamma) \in L_{\kappa\omega}$, where $\kappa = \max\{|\mathrm{TC}(\tau, \phi)|^+, \gamma^+\}$.

Proof. If a and b are the approximation mappings of the previous corollary and proposition, choose

$$A(\tau, \phi, \gamma) = \bigvee_{\delta<\gamma} b(\tau, a(\tau, \phi, \delta), \gamma).$$

□

This result is actually somewhat stronger than the original result of Burgess. His argument is valid only when the vocabulary is finite and when the logic is absolute relative to a forcing which forces a set to be countable. Our argument works for all vocabularies and for all logics absolute relative to a set theory in a standard vocabulary.

Burgess's argument is as follows: suppose first that the model $\mathfrak{M}$ is countable and the sentence $\phi \in \mathcal{L}(\tau)$ is hereditarily countable. Now an element m in a certain metric space $\omega^{\omega^n} \times \ldots \times 2^{\omega^n} \times \ldots \times \omega \times \ldots$ depending on the vocabulary τ codes the model $\mathfrak{M}$, and an element $p \subseteq \omega^2$ codes the formula ϕ through isomorphism $(\mathrm{TC}(\{\phi\}, \in) \cong (\omega, p)$. Such a Σ^1_2-relation R is known to exist that

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff R(m, p).$$

Using certain normal forms and absoluteness results in descriptive set theory we are able to write

$$\begin{aligned} R(m,p) &\iff \exists \alpha < \omega_1\, \exists z \in L_\alpha(m,p)\big(F(m,p,z) \text{ is a well-ordering of length } < \alpha\big) \\ &\iff \exists \alpha < \omega_1\, \exists z \subseteq \omega^2\big((\omega, z) \cong (\alpha, \in) \wedge P(m,p,z)\big), \end{aligned}$$

where F is a recursive functional, P a Σ^1_1-set, and $L_\alpha(m,p)$ the α^{th} level of the sets constructible from m and p.

Given a model $\mathfrak{M}$ and a sentence ϕ, let $\mathbb{P}$ be a forcing which forces $\mathfrak{M}$ and ϕ to be countable. Now there are canonical $\mathbb{P}$-names m and p for the codes of $\mathfrak{M}$ and ϕ in the extended universe. Using the absoluteness of the logic we obtain

$$\begin{aligned} \mathfrak{M} \models_{\mathcal{L}} \phi &\iff \mathbb{P} \Vdash \check{\mathfrak{M}} \models \check{\phi} \\ &\iff \mathbb{P} \Vdash R(m,p) \\ &\iff \mathbb{P} \Vdash \exists \alpha < \omega_1 \exists z \subseteq \omega^2\big((\omega, z) \cong (\alpha, \in) \wedge P(m,p,z)\big). \end{aligned}$$

Let then $\mathbb{Q}_\alpha$ be a forcing which makes an ordinal α countable. Denote by a a canonical $\mathbb{Q}_\alpha$-name for a set z such that $(\alpha, \in) \cong (\omega, z)$ in the extended universe. We proceed by showing

$$\mathbb{P} \Vdash \exists \alpha < \omega_1 \exists z \subseteq \omega^2\big((\omega, z) \cong (\alpha, \in) \wedge P(m,p,z)\big) \iff \exists \alpha\, \mathbb{P} \times \mathbb{Q}_\alpha \Vdash P(m,p,a).$$

The final step in the construction is to turn the expression $\mathbb{P} \times \mathbb{Q}_\alpha \Vdash P(m,p,a)$ into a game formula ψ_α in such a way that

$$\mathfrak{M} \models_{\mathcal{L}} \phi \iff \exists \alpha\, \mathfrak{M} \models \psi_\alpha.$$

What do the approximating game formulas then express? First, since the truth definition of the logic is equivalent to a Σ_1-expression, $\mathfrak{M} \models_{\mathcal{L}} \phi$ if and only if there exists a *witness*, an element which binds the model and sentence together. Now, in Corollary 15.3 the formula $a(\tau, \phi, \alpha)$ indicates "there is a witness in V_α to $\mathfrak{M}^{XS} \models_{\mathcal{L}} \phi^S$ for almost every countable X and S". Instead the approximating formula ψ_α of Burgess indicates: "there is a witness to $\mathfrak{M} \models_{\mathcal{L}} \phi$ which is potentially constructible from $\mathfrak{M}$ and ϕ at level α", where the expression "potentially constructible" means that the witness in question is constructible in a certain generic extension of the universe.

16. Logics absolute relative to ZFC($\mathcal{P}_\kappa$)

The results in the last three sections show that the restrictive nature of absoluteness relative to standard set theories is quite well understood. In particular we have seen that the logic $L_{\infty\omega}$ has a special position in this respect: every logic, absolute relative to KP+Inf, is a sublogic of $L_{\infty\omega}$, and every logic, absolute relative to a standard theory, can be approximated with $L_{\infty\omega}$. However, as regards uncountability, it turns out that $L_{\infty\omega}$ lacks expressive power. For example, a sentence of $L_{\infty\omega}$ cannot express that an equivalence relation has an uncountable number of equivalence classes, except by introducing new symbols in the vocabulary. Similarly, the standard set theories are not at their best for dealing with uncountable cardinals: uncountability is not absolute relative to them.

New and improved logics have been introduced for describing uncountability: among these are for example $L_{\omega\omega}(Q_1)$, $L_{\infty\omega}(Q_1)$, $L_{\infty\omega_1}$, and the M-languages. The logics $L_{\omega\omega}(Q_1)$ and $L_{\infty\omega}(Q_1)$ are interesting; for example, they have a complete proof system but not very strong expressive power. The logic $L_{\infty\omega_1}$ does not inherit the favorable position of $L_{\infty\omega}$, and thus much research has been done lately on M-languages. It has turned out that many properties of $L_{\infty\omega}$ and $L_{\omega_1\omega}$ are shared by $M_{\infty\infty}$ and by $M_{\kappa^+\kappa}$ for regular cardinals κ. However, the properties are usually not preserved as such: where $L_{\omega_1\omega}$ is nice and straight, $M_{\kappa^+\kappa}$ is complicated and full of dependencies on strong set-theoretical axioms.

The set theory ZFC($\mathcal{P}_{\omega_1}$) makes absolute many predicates which distinguish between countability and uncountability. In fact, if $(A, E, P, k) \subseteq_{\text{end}} (A', E', P', k')$ are two models of ZFC($\mathcal{P}_{\omega_1}$), they have exactly the same countable sets (of elements of A, naturally). Moreover, for example the logics $L_{\omega\omega}(Q_1)$, $V_{\infty\omega}[Q_1]$, L_{pos}, and $L_{\infty\omega_1}$ are first-order relative to ZFC($\mathcal{P}_{\omega_1}$). Thus is natural to investigate which kinds of logics are absolute relative to ZFC($\mathcal{P}_{\omega_1}$) or ZFC($\mathcal{P}_\kappa$) for a regular cardinal κ. It turns out that every such logic can be translated into $M^{\text{det}}_{\infty\lambda}$, where $\lambda = \kappa^{<\kappa}$.

16.1. Lemma. *Let κ be an uncountable cardinal, and let $\lambda = \kappa^{<\kappa}$. Let a logic $\mathcal{L}$ be absolute relative to $T \supseteq$ ZFC($\mathcal{P}_\kappa$). For every vocabulary τ and for every sentence $\phi \in \mathcal{L}(\tau)$ there exists a pair $(\mathcal{F}, \mathcal{F}')$ of disjoint λ-expanders such that the following conditions hold.*

(i) *If $\mathfrak{M} \in$ Str(τ) and $\mathfrak{M} \models_{\mathcal{L}} \phi$, then $\exists$ wins* EG$(\mathfrak{M}, \mathcal{F})$.
(ii) *If $\mathfrak{M} \in$ Str(τ) and $\mathfrak{M} \not\models_{\mathcal{L}} \phi$, then $\forall$ wins* EG$^*(\mathfrak{M}, \mathcal{F}')$.

(iii) $\kappa_{\mathcal{F}} = \kappa_{\mathcal{F}'} = \max\{|\mathrm{TC}(\tau,\phi)|^+, \lambda^+\}$.
(iv) The mapping $(\tau,\phi) \mapsto (\mathcal{F},\mathcal{F}')$ *is p.r. in* π_κ, *a surjective function* $\lambda \to \lambda^{<\kappa}$.

Proof. We need some coding mappings. Note first that $\lambda^{<\kappa} = \lambda$. Let $\pi_\kappa: \lambda \to \lambda^{<\kappa}$ be surjective (here $\lambda^{<\kappa}$ is the set of functions), and let $\pi: \lambda \times \lambda \to \lambda$ be the canonical p.r. bijection. Let

$$\begin{array}{ll} \pi_1: \lambda \to \mathcal{P}_\kappa(\lambda), & \pi_1(i) = \mathrm{ran}(\pi_\kappa(i)), \\ \pi_2: \lambda \to \lambda^\omega, & \pi_2(i)(n) = \begin{cases} \min\{\pi_\kappa(i)(n), i\} & \text{if } n \in \mathrm{dom}(\pi_\kappa(i)), \text{ and} \\ i & \text{otherwise.} \end{cases} \end{array}$$

Both these mappings are onto, p.r. relative to π_κ, and $\mathrm{ran}(\pi_2(i)) \subseteq (i+1)$ for every i.

Let $\mathcal{F}_b = \mathcal{F}(\tau,\phi,\lambda)$ be the λ-expander of Example 10.2. We may assume that the expanding vocabulary $\sigma = (\in, P, n_\lambda, t, p, m)$, where P is a binary predicate and n_λ is a constant. We next define a λ-regulator $\mathcal{R}$ such that if a structure $\mathfrak{A} = (A, E, P, n_\lambda, t, p, m, \ldots)$ is $\mathcal{R}$-regular, it is well-founded and extensional, n_λ collapses into λ, P is the relation "$\mathcal{P}_\kappa(x) = y$", and $m \models_{\mathcal{L}} p$. Recall that the vocabulary σ_λ is σ augmented with λ constant symbols $c_0, c_1, \ldots$. Give alias names to constants c_i with odd indices as follows: for $i, j < \lambda$ let

$$\begin{array}{lll} n_i = c_{12i+1}, & e_i = c_{12i+3}, & p_i = c_{12i+5}, \\ q_i = c_{12i+7}, & f_i = c_{12i+9}, & r_{i,j} = c_{12\pi(i,j)+11}, \end{array}$$

and let $t_0, t_1, \ldots$ enumerate all the constants of σ_λ. Given a constant symbol x in σ_λ, let $\ulcorner x \urcorner \in \lambda$ be such that x is the same constant as $t_{\ulcorner x \urcorner}$. Let the domain of $\mathcal{R}$ be $I^{<\lambda}$, where $I = (\lambda+1) \times (\kappa+1) \times \omega$, and define $\mathcal{R}$ in such a way that if $a = \langle(\alpha_i, \beta_i, \gamma_i)\rangle_{i<\lambda}$ is an $\mathcal{R}$-branch, $\mathrm{Th}_{\mathcal{R}}(a)$ contains the following sentences.

(1) A finite subset of T.

(2) The Σ_1-form of $m \in \mathrm{Str}(t) \wedge p \in \mathcal{L}(t)$.

(3) The Σ_1-form of $m \models_{\mathcal{L}(t)} p$.

(4) $\left.\begin{array}{ll} n_i \in n_j & \text{if } i < j \\ n_i \notin n_j & \text{if } i \geq j \end{array}\right\}$ for every $i, j \leq \lambda$.

(5) $\left.\begin{array}{ll} t_i \notin n_j & \text{if } \alpha_{2i+1} = 0 \\ t_i = n_j & \text{if } \alpha_{2i+1} = 1 + j \end{array}\right\}$ for every $i < \lambda, j \leq \lambda$.

(6) $p_i \subseteq n_\lambda$ for every $i < \lambda$.

(7) $\left.\begin{array}{ll} n_j \in p_i & \text{if } j \in \pi_1(i) \\ n_j \notin p_i & \text{if } j \notin \pi_1(i) \end{array}\right\}$ for every $i, j < \lambda$.

(8) $P(t_i, t_j) \to t_j = e_i$ for every $i, j < \lambda$.

(9) $P(t_i, e_i)$ for every $i < \lambda$.

(10) $\forall u \in e_i (u \subseteq t_i)$ for every $i < \lambda$.

(11) $t_j \in q_i$ for every $j < i < \lambda$.

(12) $f_i\colon q_i \to n_i$ one-to-one for every $i < \lambda$.

(13) $\forall u(u \subseteq q_j \wedge u \subseteq t_i \to u \in e_i)$ for every $i, j < \lambda$.

(14) $\left.\begin{array}{ll} t_i \subseteq q_j & \text{if } \beta_{2i+1} < \kappa \text{ and } j = \ulcorner r_{i,\beta_{2i+1}} \urcorner, \\ t_i \notin e_j & \text{if } \beta_{2i+1} = \kappa \end{array}\right\}$ for every $i, j < \lambda$.

(15) $t_i \notin t_j \vee t_i = t_j$ if $i = \pi_2(k)(\gamma_{2k+1} + 1)$ and $j = \pi_2(k)(\gamma_{2k+1})$ for some $k < \lambda$.

Claim A: Suppose $(A, E, P, n_\lambda, t, p, m)$ is $\mathcal{R}$-regular. Then (A, E) is well-founded and extensional, $cn_\lambda = \lambda$, $P(x) = y$ iff $\mathcal{P}_\kappa(cx) = cy$, $cm \in \mathrm{Str}(ct)$, $cp \in \mathcal{L}(ct)$, and $cm \models_{\mathcal{L}(ct)} cp$, where c is the Mostowski collapsing function of (A, E).

The sentences (15) imply that (A, E) is well-founded: let t_{i_k} for $k < \omega$ be arbitrary. Choose $i < \lambda$ such that $\pi_2(i)(k) = i_k$ for every $k < \omega$. Letting $k = \pi_2(i)(\gamma_{2i+1})$ we have $\neg(t_{i_{k+1}} E t_{i_k})$ or $t_{i_{k+1}} = t_{i_k}$. The sentence (1) makes (A, E) extensional. Thus the Mostowski collapsing function c exists. Now (4)–(5) imply $cn_i = i$ for $i \leq \lambda$, and (6)–(7) imply $cp_i = \pi_1(i)$, so we have $\mathcal{P}_\kappa(\lambda) = \{\, cp_i : i < \lambda \,\}$.

Moreover, the sentences (10)–(14) imply $ce_i = \mathcal{P}_\kappa(ct_i)$ for every $i < \lambda$: suppose first $x \in ce_i$, i.e. $x = ct_k$ and $(A, E) \models t_k \in e_i$. By (10) and (14), $(A, E) \models t_k \subseteq t_i$ and $(A, E) \models t_k \subseteq q_j$ for some $j < \lambda$. Using (12) and some axioms of ZF we get a one-to-one function $g \in A$ from t_k to n_j. Since $cn_j = j$, $x = ct_k \in \mathcal{P}_\kappa(t_i)$. On the other hand, suppose $x \in \mathcal{P}_\kappa(ct_i)$. By König's lemma the cofinality of λ is at least κ, so $x \subseteq \{\, ct_k : k < j \,\}$ for some $j < \lambda$. Thus $x \subseteq cq_j$ by (11). The sentence (12) indicates that $cf_j\colon cq_j \to j$ is one-to-one. Now $\{\, cf_j(y) : y \in x \,\} = \pi_1(k) \in \mathcal{P}_\kappa(\lambda)$ for some $k < \lambda$. Using some axioms of ZF we get $u \in A$ such that $(A, E) \models f_j''u = p_k$. Now $cu = x$, and (13) implies $x \in ce_i$.

Since (A, E, P) satisfies the sentences (8)–(9), this implies

$$(A, E, P) \models P(x, y) \iff cx = \mathcal{P}_\kappa(cy)$$

for every $x, y \in A$. Since (2) and (3) are Σ_1-sentences, we finally have $cm \in \mathrm{Str}(ct)$, $cp \in \mathcal{L}(ct)$, and $cm \models_{\mathcal{L}(ct)} cp$.

Let $\mathcal{F} = \mathcal{F}_b(\mathcal{R})$. Since an $\mathcal{F}$-expansion is a $\mathcal{R}$-regular $\mathcal{F}_b$-expansion, the claim A immediately implies that if $(\mathfrak{M}, x_0, \ldots; A, E, P, n_\lambda, t, p, m, \ldots)$ is an $\mathcal{F}$-expansion over $q \cdot r$, then

$$\phi^S \in \mathcal{L}(\tau^S), \quad \mathfrak{M}^{XS} \in \mathrm{Str}(\tau^S), \quad \text{and} \quad \mathfrak{M}^{XS} \models_{\mathcal{L}(\tau^S)} \phi^S,$$

where $X = \{\, x_\xi : \xi < \lambda \,\}$ and S is determined by the $\mathcal{F}_b$-branch q.

Claim B: If $\mathfrak{M} \in \mathrm{Str}(\tau)$, $\phi \in \mathcal{L}(\tau)$, and $\mathfrak{M} \models_{\mathcal{L}(\tau)} \phi$, then $\exists$ wins $\mathrm{EG}(\mathfrak{M}, \mathcal{F})$.

Let $\mu > \lambda$ be such that $\mathfrak{M}$, τ, and ϕ are elements of H_μ, $\forall \nu < \mu\ (\nu^{<\lambda} < \mu)$, and that H_μ is a model of (1). Let $P(x, y) \iff y = \mathcal{P}_\kappa(x)$. Now Extended Levy theorem 3.2 implies that (2) and (3) hold in $\mathfrak{A} = (H_\mu, \in, P, \lambda, \tau, \phi, \mathfrak{M})$. Since $\mathcal{F}_b$ is valid in $(\mathfrak{M}; \mathfrak{A})$, it is enough to see that $\mathcal{R}$ is valid in $\mathfrak{A}$: then $\mathcal{F}$ is valid in $(\mathfrak{M};\mathfrak{A})$ and $\exists$ wins the expansion game.

Suppose $\forall$ picks an element $c_{2i} \in H_\mu$. The enumeration t_i $(i < \lambda)$ shows that $t_0, \dots, t_i$ have already been chosen. Let $\exists$ move as follows:

- $\alpha_{2i+1} = 1 + t_i$ if $t_i \in \{0, \dots, \lambda\}$, and $\alpha_{2i+1} = 0$ otherwise.
- $\beta_{2i+1} = \min\{|t_i|, \kappa\}$.
- γ_{2i+1} is the least $n \in \omega$ such that either $t_{h(n+1)} \notin t_{h(n)}$ or $t_{h(n+1)} = t_{h(n)}$, where $h = \pi_2(i)\colon \omega \to i$.
- c_{2i+1} is such that for every i: $n_i = i$, $e_i = \mathcal{P}_\kappa(t_i)$, $p_i = \pi_1(i)$, $q_i = \{\, t_j : j < i \,\}$, $f_i(t_j) = j$ for each $j < i$, and $r_{i,j}$ for $j < |t_i|$ enumerates t_i if $|t_i| < \kappa$, and otherwise $r_{i,j} = \emptyset$.

These make the sentences (1)–(15) true. For example, to see (14) suppose $i, j \leq \lambda$. If $\beta_{2i+1} = \kappa$, we have $|t_i| \geq \kappa$, so $t_i \notin \mathcal{P}_\kappa(t_j) = e_j$. On the other hand, suppose $\beta_{2i+1} < \kappa$. Now $|t_i| = \beta_{2i+1} < \kappa$, and if $j = \ulcorner r_{i,\beta_{2i+1}} \urcorner$, then

$$t_i = \{\, r_{i,k} : k < \beta_{2i+1} \,\} \subseteq \{\, t_k : k < j \,\} = q_j,$$

since the mapping $k \mapsto \ulcorner r_{i,k} \urcorner$ is strictly increasing. Thus $\exists$ wins the game.

Construct $\mathcal{F}'$ similarly to $\mathcal{F}$: start from the basic expander $\mathcal{F}'(\tau, \phi, \lambda)$ and add a regulator $\mathcal{R}'$ similar to $\mathcal{R}$, except that in the sentences (1)–(15) the constants $n_i, e_i, \dots$ are the constants c_i with even indices, parameters $\alpha_{2i}, \dots$ are used instead of $\alpha_{2i+1}, \dots$, and the sentence (3) is replaced with

(3′) The Σ_1-form of $m \not\models_{\mathcal{L}(t)} p$.

Similary to the claim B one sees that $\forall$ wins $\mathrm{EG}^*(\mathfrak{M}, \mathcal{F}')$ if $\mathfrak{M} \in \mathrm{Str}(\tau)$ and $\mathfrak{M} \not\models_{\mathcal{L}(\tau)} \phi$. Thus (i) and (ii) hold. The claims (iii) and (iv) are easy, so we need to show that $\mathcal{F}$ and $\mathcal{F}'$ are disjoint on the common basis $\mathrm{TC}(\tau, \phi)$ of $\mathcal{F}(\tau, \phi, \lambda)$ and $\mathcal{F}'(\tau, \phi, \lambda)$. But, as above, one can see that if $(\mathfrak{M}, x_0, \dots; \mathfrak{B}, \dots)$ is an $\mathcal{F}'$-expansion over $q' \cdot r'$, then $\mathfrak{M}^{XS'} \not\models_{\mathcal{L}(\tau^{S'})} \phi^{S'}$, where $X = \{x_0, y_0, \dots\}$ and S' is determined by q'. Moreover, if an $\mathcal{F}$-branch $q \cdot r$ and an $\mathcal{F}'$-branch $q' \cdot r'$ meet, then q and q' also meet, which implies $S = S'$. Thus the expanders are disjoint. □

16.2. Theorem. *Suppose $\kappa > \omega$ and $\lambda = \kappa^{<\kappa}$. If $\mathcal{L}$ is absolute relative to $T \supseteq \mathrm{ZFC}(\mathcal{P}_\kappa)$, there exists a translation $t\colon \mathrm{Voc} \times \mathcal{L} \to M^{\det}_{\infty\lambda}$, primitive recursive in π_κ, such that $t(\tau, \phi) \in M_{\mu\lambda}$ when $\mu \geq \max\{|\mathrm{TC}(\tau, \phi)|^+, \lambda^+\}$.*

Proof. The disjoint pair $(\mathcal{F}, \mathcal{F}')$ of expanders given by the previous lemma can be turned into a sentence in $M^{\det}_{\infty\lambda}$ by Lemma 12.5. □

This result is analogous to Theorem 14.3, which states the existence of a translation $\mathcal{L} \to L_{\infty\omega}$ for every logic, absolute relative to KP + Inf. Since the former translation is primitive recursive relative to π_κ, admissible fragments containing π_κ are closed under it.

For other results in this area, see [V1], [V2] Corollary 3.4.5, and [O], which show that a class of structures is Δ_1-definable in ZFC($\mathcal{P}_{\omega_1}$) with parameters from $H(\omega_2)$ exactly when it is definable in $\Delta_{\mathrm{RPC}}(L_{\omega_2\omega_1})$. Heikkilä further extends this result by applying the separation theorem to the $\Delta_{\mathrm{RPC}}(L_{\omega_2\omega_1})$-definition and getting a class, definable in $M_{\omega_2\omega}$; see [He] Theorem 9.9.

Note that the construction in Lemma 16.1 can be used almost as such to show that every class of structures, Δ_1-definable in some theory containing ZFC($\mathcal{P}_\kappa$) with parameters from $H(\lambda^+)$, is definable in $M^{\det}_{\lambda^+\lambda}$, where $\lambda = \kappa^{<\kappa}$. Suppose the class in question is $\mathcal{K}_a$, where a is the parameter. We essentially need an expander $\mathcal{F}$ (and the respective co-expander) such that

- if $\mathfrak{M} \in \mathcal{K}_a$, then $\exists$ wins EG($\mathfrak{M}, \mathcal{F}$), and
- if $(\mathfrak{M}, \ldots; A, E, x, \ldots)$ is an $\mathcal{F}$-expansion, then x collapses into the parameter a and $\mathfrak{M}^{XS}$ is in $\mathcal{K}_a$.

This is achieved by adding a constant for each element of TC(a) in the expansion language, an by enumerating those sentences in the expander which imply that x collapses into a.

In Section 14 we showed a separation theorem (Theorem 14.4), which implies that L_A allows separation for itself when $A \ni \omega$ is countable and admissible. What can we say about the analogous results in this case? Tuuri has shown various separation theorems in [T], proving for example that $M_{\lambda^+\lambda}$ allows separation for $L_{\kappa^+\kappa}$ when $\lambda = \kappa^{<\kappa}$ and κ is regular. In [He] Theorem 8.9 Heikkilä refines this result and shows that the separation holds in admissible sets A which are *λ-closed* (i.e. $\lambda \in A$ and for every $x \in A$ the set $\mathcal{P}_\lambda(x)$ is in A) and *locally λ-enumerable* (i.e. for every $x \in A$ of cardinality less than or equal to λ there exists a surjective enumeration $\lambda \to x$ in A). We next sketch a new proof of this fact.

16.3. Theorem. *Let κ be an uncountable cardinal, $\lambda = \kappa^{<\kappa}$, and suppose A is a κ-closed, locally λ-enumerable admissible set. Let $\mathcal{L}$ be absolute relative to a theory containing* ZFC($\mathcal{P}_\kappa$). *Then $M_{\lambda^+\lambda} \cap A$ allows separation for $\mathcal{L}_{\lambda^+} \cap A$.*

Proof. Let τ, ρ, ρ' be pairwise disjoint vocabularies, and suppose $\phi \in \mathcal{L}(\tau \cup \rho)$ and $\phi' \in \mathcal{L}(\tau \cup \rho')$ such that $\phi \wedge \phi'$ has no model. Suppose, moreover, that the vocabularies and the sentences are elements of a κ-closed, locally λ-enumerable admissible set A, and that they are of hereditarily cardinality less than or equal to λ. Let $Z = \mathrm{TC}(\tau, \rho, \rho', \phi, \phi')$. Since $|Z| \leq \lambda$ and $\lambda^{<\kappa} = \lambda$, the set A contains functions $h\colon \lambda \to Z$ and $\pi_\kappa\colon \lambda \to \lambda^{<\kappa}$.

First construct λ-expanders $\mathcal{F}$ and $\mathcal{F}'$ such that for every structure $\mathfrak{M} \in \mathrm{Str}(\tau)$

$$\begin{array}{lcl} (\mathfrak{M}, \vec{R}) \models \phi \text{ for some } \vec{R} & \Longrightarrow & \exists \text{ wins EG}(\mathfrak{M}, \mathcal{F}), \text{ and} \\ (\mathfrak{M}, \vec{R}') \models \phi' \text{ for some } \vec{R}' & \Longrightarrow & \forall \text{ wins EG}^*(\mathfrak{M}, \mathcal{F}'). \end{array}$$

Moreover, construct the expanders in such a way that, if $(\mathfrak{M}, x_0, \ldots; \mathfrak{A})$ is an $\mathcal{F}$-expansion, $(\mathfrak{M}^X, \vec{R}) \models \phi$ for some $\vec{R}$, where $X = \{x_0, \ldots\}$, and similarly for $\mathcal{F}'$. There is nothing difficult in this; the expanders of Lemma 16.1 serve their purpose well after some modifications: we use the function h to enumerate those sentences which imply that the constant p collapses into ϕ (or ϕ'). Since the construction of the expanders is p.r. relative to h and π_κ, the expanders are in the admissible set A. Moreover, they are disjoint, and thus, by Lemma 12.5, the required sentence ψ in $M_{\lambda^+\lambda}$ exists and is an element of A. □

16.4. Corollary ([T],[He]). *Let $\kappa > \omega$, $\lambda = \kappa^{<\kappa}$, and suppose A is a κ-closed, locally λ-enumerable admissible set. Then $M_{\lambda^+\lambda} \cap A$ allows separation for $L_{\lambda^+\kappa} \cap A$.* □

We could now question whether the translation 16.2 is the best possible: for example, is there a strict subclass of $M^{\mathrm{det}}_{\infty\lambda}$, which serves as a destination of the translation? Note first that it is consistent to assume $\kappa^{<\kappa} = \kappa$ (i.e. $\lambda = \kappa$). Next we show that — under this assumption — a large part of $M^{\mathrm{det}}_{\infty\kappa}$ is already covered.

16.5. Lemma. *Suppose $\kappa^{<\kappa} = \kappa$. Logic $\mathcal{L}$ is absolute relative to $T \supseteq \mathrm{ZFC}(\mathcal{P}_\kappa)$ if and only if there exists $K \subseteq M^{\mathrm{det}}_{\infty\kappa}$ such that K is Σ_1-definable in T, $T \vdash K \subseteq M^{\mathrm{det}}_{\infty\kappa}$, and $\mathcal{L} = M^{\mathrm{det}}_{\infty\kappa}\restriction K$.*

Proof. "$\Longleftarrow$" We need to show that the semantics of $M^{\mathrm{det}}_{\infty\kappa}\restriction K$ is absolute relative to T. But since every $\phi \in K$ is a determined $M_{\infty\kappa}$-sentence in every model of T, Lemma 8.3 implies the absoluteness of $\mathcal{L}$.

"$\Longrightarrow$" Let $\mathcal{L}$ be absolute relative T. Let $t\colon \mathcal{L} \to M^{\mathrm{det}}_{\infty\kappa}$ be the translation given by Theorem 16.2. Since t is p.r. relative to π_κ, it is defined by a Σ_1-formula P such that for every ϕ and ψ

$$t(\phi) = \psi \iff P(\pi_\kappa, \phi, \psi).$$

Let

$$\psi \in K \iff \exists\pi\exists\phi(\phi \in \mathcal{L} \wedge \pi\colon \kappa \to \kappa^{<\kappa} \text{ is bijective} \wedge P(\pi, \phi, \psi)).$$

□

We have already seen in Section 8 that $M^{\mathrm{det}}_{\infty\kappa}$ is not absolute relative to $\mathrm{ZFC}(\mathcal{P}_\kappa)$, since its syntax is not upwards persistent. The reason is trivial: the predicate "T is a κ-leaftree" is not absolute relative to $\mathrm{ZFC}(\mathcal{P}_\kappa)$. Now we can show a stronger result: by a construction similar to 15.1 we see that there exists no maximal logic absolute relative to a theory $T \supseteq \mathrm{ZFC}(\mathcal{P}_\kappa)$. Thus Translation theorem 16.2 implies:

16.6. Corollary. *Suppose $\kappa = \kappa^{<\kappa}$. There exists no logic, absolute relative to an extension of* $\mathrm{ZFC}(\mathcal{P}_\kappa)$*, with the same expressive power as $M^{\mathrm{det}}_{\infty\kappa}$.* □

The following questions are still left open.

16.7. Open Questions. (1) If $\phi \in M^{\mathrm{det}}_{\infty\kappa}$, is there a $\Sigma_1(\mathcal{P}_\kappa)$-definable subclass K such that $\phi \in K$?

(2) Since $L_{\infty\kappa}$ is absolute relative to $\mathrm{ZFC}(\mathcal{P}_\kappa)$, we know: if $\kappa = \kappa^{<\kappa}$,

$$\begin{aligned}\mathfrak{A} \equiv \mathfrak{B}(M^{\mathrm{det}}_{\infty\kappa}) &\implies \mathfrak{A} \equiv \mathfrak{B}(\mathcal{L}) \text{ for every } \mathcal{L}\text{, absolute relative to } \mathrm{ZFC}(\mathcal{P}_\kappa)\\ &\implies \mathfrak{A} \equiv \mathfrak{B}(L_{\infty\kappa}).\end{aligned}$$

Which of the converse implications hold?

17. Logics absolute relative to ω_1-closed forcing

In the previous section we saw that the logics absolute relative to ZFC($\mathcal{P}_\kappa$) and $M^{\mathrm{det}}_{\infty\lambda}$ for $\lambda = \kappa^{<\kappa}$ are related in the same way as are the logics absolute relative to KP + Inf and $L_{\infty\omega}$. The results in Section 15 illustrate the special position of $L_{\infty\omega}$ among logics absolute relative to ZFC as well. In this last section we investigate whether the M-languages have analogous properties in relation to those logics which are absolute relative to κ-closed forcing. We shall see that the analogy partly holds, partly fails.

17.1. Lemma (Shelah). *Let κ be a regular cardinal. Player $\exists$ wins* $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$ *if and only if there is a κ-closed notion of forcing which makes $\mathfrak{A}$ and $\mathfrak{B}$ isomorphic.*

Proof."$\Longrightarrow$" Let S be a winning strategy of $\exists$ in $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$. Each non-maximal position $u \in S$ is a sequence $(u_i)_{i<\xi}$ where $\xi < \kappa$. Moreover, if we denote

$$\begin{cases} a_i = u_{3i+1} \text{ and } b_i = u_{3i+2} & \text{if } u_{3i+1} \text{ is in } \mathfrak{A} \text{ and } u_{3i+2} \text{ is in } \mathfrak{B}, \text{ and} \\ a_i = u_{3i+2} \text{ and } b_i = u_{3i+1} & \text{otherwise,} \end{cases}$$

the partial isomorphism $p(u) : a_i \mapsto b_i$ is of cardinality $< \kappa$. Let

$$\mathbb{P} = \{\, p(u) \,:\, u \in S \text{ is not maximal}\,\}.$$

Now $\mathbb{P}$ forces $\mathfrak{A} \cong \mathfrak{B}$, and $\mathbb{P}$ is κ-closed, since κ is regular.

"$\Longleftarrow$" Since $\mathbb{P}$ forces the models to be isomorphic, there exists a $\mathbb{P}$-name f and a condition $p_0 \in \mathbb{P}$ which forces f to be an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Player $\exists$ wins $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$ by playing as follows. Suppose we have played ξ turns, i.e. elements $a_i \in A$ and $b_i \in B$ for $i < \xi$ have been so picked that $a_i \mapsto b_i$ is a partial isomorphism. Suppose, moreover, that p_ξ forces $f(a_i) = b_i$ for every $i < \xi$. Let $\forall$ move, say, $a_\xi \in A$. Let $\exists$ choose $b_\xi \in B$ and a condition $p_{\xi+1} \leq p_\xi$ such that $p_{\xi+1}$ forces $f(a_\xi) = b_\xi$. Since $p_{\xi+1}$ forces f to be an isomorphism, $a_i \mapsto b_i$ $(i \leq \xi)$ is a partial isomorphism. Finally, if ξ is a limit and, for each $i < \xi$, p_i forces $f(a_j) = b_j$ $(j < i)$, let p_ξ be a lower bound for $\{\, p_i \,:\, i < \xi\,\}$. □

17.2. Proposition ([Kt]). $\mathfrak{A} \equiv \mathfrak{B}\ (V_{\infty\kappa}) \iff \exists$ *wins* $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$. □

17.3. Proposition ([Hy]). *There exists an approximation mapping from $V_{\infty\kappa}$ to $M_{\infty\kappa}$.* □

17.4. Corollary. *Let κ be regular. The following claims are equivalent:*

(i) $\mathfrak{A} \equiv \mathfrak{B}(M_{\infty\kappa})$.

(ii) $\exists$ *wins* $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$.

(iii) There is a κ-closed forcing $\mathbb{P}$ which makes $\mathfrak{A}$ and $\mathfrak{B}$ isomorphic.

Proof. The equivalence (ii) $\iff$ (iii) is Lemma 17.1. Proposition 17.3 implies $\mathfrak{A} \equiv \mathfrak{B}(M_{\infty\kappa})$ iff $\mathfrak{A} \equiv \mathfrak{B}(V_{\infty\kappa})$, so the equivalence (i) $\iff$ (ii) follows from 17.2. □

17.5. Corollary. *Let κ be regular, and suppose logic $\mathcal{L}$ is absolute relative to κ-closed forcing. Then*

$$\mathfrak{A} \equiv \mathfrak{B}\ (M_{\infty\kappa}) \implies \mathfrak{A} \equiv \mathfrak{B}\ (\mathcal{L}).$$

□

In this respect absoluteness relative to κ-closed forcing and the logic $M_{\infty\kappa}$ behave similarly to absoluteness and the logic $L_{\infty\omega}$: as shown e.g. in [B2], every logic $\mathcal{L}$, absolute relative to a standard set theory, has so called *Karp-property*

$$\mathfrak{A} \equiv \mathfrak{B}\ (L_{\infty\omega}) \implies \mathfrak{A} \equiv \mathfrak{B}\ (\mathcal{L}).$$

Proposition 17.3 is analogous to the existence of an approximation from $V_{\infty\omega}$ to $L_{\infty\omega}$ (Proposition 15.4). Thus it is natural to ask: do we have an approximation mapping analogous to Burgess's approximation 15.5; i.e. if $\mathcal{L}$ is absolute relative to κ-closed forcing, is there an approximation mapping from $\mathcal{L}$ to $M_{\infty\kappa}$? Next we show that the analogy fails when $\kappa = \omega_1$, the main reason being that $M_{\infty\omega_1}$, unlike $L_{\infty\omega}$, does not have Scott sentences, i.e. there are structures $\mathfrak{A}$ such that no sentence ϕ of $M_{\infty\omega_1}$ satisfies

$$\mathfrak{B} \models_{\mathcal{L}} \phi \implies \mathfrak{A} \equiv \mathfrak{B}(M_{\infty\omega_1})$$

for every structure $\mathfrak{B}$.

Given a tree T, let $R(T)$ be the tree of finite sequences $\langle s_0, \dots, s_n\rangle$ of elements of T ordered by the relation

$$\langle s_0, \dots, s_m\rangle \le \langle t_0, \dots, t_m\rangle \iff m \le n \wedge \forall i < m\,(s_i = t_i) \wedge s_m \le_T t_m.$$

For $s = \langle s_i\rangle_{i\le m} \in R(T)$, denote by $\mathrm{ls}(s) = s_m$ the *last element* in s and by $\mathrm{ph}(s) = m$ the *phase* of s.

Let T_0 be the many-rooted tree of sequences $t\colon \alpha \to \omega_1$, where $0 < \alpha < \omega_1$, ordered by end extension.

17.6. Lemma. *For every ω_1-tree $T \neq \emptyset$ there is a many-rooted ω_1-tree T_1 such that $\exists$ wins* $\mathrm{EF}_T(R(T_0), R(T_1))$ *and $R(T_0) \not\equiv R(T_1)(M_{\infty\omega_1})$.*

Proof. ([HT]) We may assume T is a leaftree. Let

$$T_2 = \Big(\bigoplus_{\alpha<\omega_1} (\alpha+1)\Big)\cdot T \quad\text{and}\quad T_1 = T_2 \otimes T_0.$$

The tree T_2 has a single root, while the other trees T_0, T_1, $R(T_0)$, and $R(T_1)$ have $\aleph_1$ roots. All limits in every tree are unique, and the trees T_1 and T_2 are leaftrees.

Claim A: $R(T_0) \not\equiv R(T_1)(M_{\infty\omega_1})$.

The tree $R(T_0)$ clearly has branches of length ω_1, since the tree T_0 has. However, since T_1 has no branches of length ω_1 and ω_1 is regular, $R(T_1)$ is a ω_1-tree. Thus there is no ω_1-closed forcing which forces the models isomorphic (see Lemma 8.1), and the claim follows from Corollary 17.4.

Denote by EF* an EF-game between trees where $\forall$ is not allowed to choose a node if its all predecessors have not yet been chosen.

Claim B: If $\exists$ wins $\mathrm{EF}^*_{T_2}(R(T_0), R(T_1))$, then $\exists$ wins $\mathrm{EF}_T(R(T_0), R(T_1))$.

Player $\exists$ wins the game $\mathrm{EF}_T(R(T_0), R(T_1))$ by playing $\mathrm{EF}^*_{T_2}(R(T_0), R(T_1))$ in the background as follows. Suppose we have already played i moves, and suppose $\forall$ picks $t_i \in T$ and, say, $a_i \in R(T_0)$ (the other case being similar). Let α_i be the height of a_i in $R(T_0)$, and suppose $\langle a_i^\beta \rangle_{\beta \leq \alpha_i}$ is the path in $R(T_0)$ for which $a_i^{\alpha_i} = a_i$. The nodes of T_2 are (essentially) tuples $(g, (\beta, \alpha), s)$, where $\beta \leq \alpha$, $s \in T$, and $g\colon \mathrm{pred}_T(s) \to \kappa$. Let $g_i\colon t_j \mapsto \alpha_j$ for $j < i$, and let $t_i^\beta = (g_i, (\beta, \alpha_i), t_i)$ for each $\beta \leq \alpha_i$. Player $\exists$ plays $\alpha_i + 1$ moves in the background game $\mathrm{EF}^*_{T_2}(R(T_0), R(T_1)))$: she lets $\forall$ move $t_i^\beta \in T_2$ and a_i^β for $\beta \leq \alpha_i$, and gets the elements $b_i^\beta \in R(T_1)$ $(\beta \leq \alpha_i)$ in return. Let her finally move $b_i = b_i^{\alpha_i}$ in $\mathrm{EF}_T(R(T_0), R(T_1))$.

If $\exists$ wins the resulting play of $\mathrm{EF}^*_{T_2}(R(T_0), R(T_1)))$, the mapping $a_i^\beta \mapsto b_i^\beta$ is a partial isomorphism. Thus its restriction $a_i \mapsto b_i$ is a partial isomorphism, and $\exists$ wins the corresponding play of $\mathrm{EF}_T(R(T_0), R(T_1))$.

Claim C: $\exists$ wins $\mathrm{EF}^*_{T_2}(R(T_0), R(T_1))$.

Let $f_0\colon T_1 = T_2 \otimes T_0 \to T_2$ be the canonical projection, and $f = f_0 \circ ls\colon R(T_1) \to T_2$. On each turn i of the EF*-game $\forall$ picks an element t_i of T_2, and the players pick elements $a_i \in R(T_0)$ and $b_i \in R(T_1)$. Since $\forall$ is allowed to pick an element only after all its predecessors have already been picked, $\exists$ wins a play, if for each i one of the following conditions holds:

- Both a_i and b_i are roots.
- There exists $j < i$ such that $a_i = a_j$ and $b_i = b_j$.
- Neither a_i nor b_i has been chosen before the turn i, and there exists $j < i$ such that $a_i \in \mathrm{succ}(a_j)$ and $b_i \in \mathrm{succ}(b_j)$.
- There exists i_j $(j < \omega)$ such that a_i is the limit of a_{i_j} $(j < \omega)$ and b_i is the limit of b_{i_j} $(j < \omega)$.

Player $\exists$ wins the game by always so picking her element that

$$(*) \qquad \mathrm{ph}(a_i) \leq \mathrm{ph}(b_i) \leq \mathrm{ph}(a_i) + 1, \quad \text{and} \quad \mathrm{ph}(b_i) = \mathrm{ph}(a_i) + 1 \implies f(b_i) \leq t_i.$$

We need to show that $\exists$ can follow this strategy.

Suppose $\forall$ chooses $t_k \in T_2$ and $a_k \in R(T_0)$. If $a_k = a_i$ for some $i < k$, let $\exists$ choose $b_k = b_i$. Thus we may assume $a_k \neq a_i$ $(i < k)$ and need to find an element $b_k \in R(T_1)$ such that both the winning conditions and $(*)$ are satisfied. There are several cases:

1° a_k is a root of $R(T_0)$. The tree T_1 has $\aleph_1$ roots: the nodes (t, s) where t is the root of T_2 and $s\colon \{0\} \to \omega_1$ is a root of T_0. Since only countably many elements have been chosen, $\exists$ is able to choose a root b_k of $R(T_1)$ such that

$$\forall i < k\,(b_i \neq b_k), \quad f(b_k) \leq t_k \text{ is the root of } T_2, \quad \text{and} \quad ph(a_k) = ph(b_k) = 0.$$

2° $a_k \in \mathrm{succ}(a_i)$.

(a) $ph(a_k) = ph(a_i)$ and $ph(b_i) = ph(a_i) + 1$. Now $f(b_i) \leq t_i < t_k$, i.e. $ls(b_i) = (t, s)$, where $t \leq t_i$ and $s \in T_0$. The node t has a successor $t' \leq t_k$, and the node s has $\aleph_1$ successors $s^\wedge\langle\alpha\rangle$ $(\alpha < \omega_1)$. Thus b_i has $\aleph_1$ successors b such that

$$ph(b) = ph(b_i) \quad \text{and} \quad f(b) = t' \leq t_k.$$

Let b_k be one of those which have not yet been chosen.

(b) Otherwise. Every element $b_i{}^\wedge\langle(t, s)\rangle$, where t is the root of T_2 and s is a root of T_0, is in $\mathrm{succ}(b_i)$. Let b_k be one of those which have not yet been chosen. Now $ph(b_k) = ph(b_i) + 1$ and $f(b_k) = t \leq t_k$, so in any case $(*)$ is satisfied.

3° a_k is a limit node in $R(T_0)$. Let $X = \{\, i < k \,:\, a_i \leq a_k \,\}$,

$$p = \sup\{\, ph(a_i) \,:\, i \in X \,\}, \quad \text{and} \quad q = \sup\{\, ph(b_i) \,:\, i \in X \,\}.$$

Since all the predecessors of a_k have been chosen and the phase of the nodes of $R(T_0)$ does not change on limits, $ph(a_k) = p$. Now $(*)$ implies $p \leq q \leq p + 1$. Let

$$Y = \{\, i < k \,:\, ph(a_i) = p,\ ph(b_i) = q,\ a_i < a_k \,\}.$$

The set $\{\, f(b_i) \,:\, i \in Y \,\}$ is a chain in T_2. Since all the branches of T_2 are of successor length, there is a unique

$$(s, t) = \sup\{\, ls(b_i) \,:\, i \in Y \,\},$$

and thus the chain $\{\, b_i \,:\, i < k \wedge a_i \leq a_k \,\}$ has a unique supremum b_k for which $ph(b_k) = q$. This b_k has not yet been chosen: if $b_j = b_k$ for some $j < k$, by uniqueness of the limits of $R(T_0)$ it would be $a_k \leq a_j$, which contradicts the requirement that a $\forall$ is not allowed to pick elements before all its predecessors have been picked. Finally, if $q = p + 1$, $f(b_i) \leq t_i < t_k$ when $i \in Y$. Thus $f(b_k) = t \leq t_k$.

Assume then that $\forall$ picks $b_k \in R(T_1)$. If $b_k = b_i$ for some $i < k$, we again let $\exists$ choose $a_k = a_i$. Suppose $b_k \neq b_i$ $(i < k)$ and consider the following cases:

1° b_k is a root. Since the tree $R(T_0)$ has $\aleph_1$ roots, $\exists$ is able to choose a root a_k which has not yet been chosen. Now $ph(b_k) = ph(a_k) = 0$.

2° $b_k \in \mathrm{succ}(b_i)$.

(a) $ph(a_i) < ph(b_i) < ph(b_k)$. Now $f(b_k)$ is the root of T_2 below t_k, and since a_i has $\aleph_1$ successors of phase $ph(a_i) + 1$, $\exists$ is able to choose $a_k \in \mathrm{succ}(a_i)$ having the same phase as b_i.

(b) Otherwise $\exists$ is able to choose a_k such that $ph(a_k) = ph(b_k)$.

3° b_k is a limit node. Let $X = \{ i < k : b_i < b_k \}$, and denote

$$q = ph(b_k) = \sup\{ ph(b_i) : i \in X \} \quad \text{and} \quad p = \sup\{ ph(a_i) : i \in X \}.$$

As in the previous case, $p \leq q \leq p + 1$. Let $\exists$ choose $a_k = \sup\{ a_i : i \in X \}$. If $q = p + 1$, $f(b_i) \leq t_i < t_k$ whenever $i < k$, $b_i < b_k$, $ph(b_i) = q$, and $ph(a_i) = p$. Thus $f(b_k) \leq t_k$.

□

17.7. Lemma. *For a regular cardinal κ, the predicate $\mathfrak{A} \equiv \mathfrak{B}\ (M_{\infty\kappa})$ is absolute relative to κ-closed forcing.*

Proof. By Lemma 17.4

$$\mathfrak{A} \equiv \mathfrak{B}\ (M_{\infty\kappa}) \iff \exists \text{ wins } \mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B}).$$

The game $\mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$ is κ-closed, and if $G = \mathrm{EF}_\kappa(\mathfrak{A}, \mathfrak{B})$, a κ-closed forcing forces $\tilde{G}_c = \mathrm{EF}_\kappa(\check{\mathfrak{A}}, \check{\mathfrak{B}})$. Thus by Lemma 4.5 the equivalence is absolute relative to κ-closed forcing. □

17.8. Theorem. *There exists a logic $\mathcal{L}$, absolute relative to ω_1-closed forcing, but having no conjunctive nor disjunctive approximation to $M_{\infty\omega_1}$.*

Proof. Take a logic $\mathcal{L}$ with a sentence ϕ such that

$$\mathfrak{A} \models_{\mathcal{L}} \phi \iff \mathfrak{A} \equiv R(T_0)\ (M_{\infty\omega_1}),$$

and close $\mathcal{L}$ under negation. The structure $R(T_0)$ is absolute relative to ω_1-closed forcing, since such a forcing preserves countable sequences. By Lemma 17.7 the logic $\mathcal{L}$ is absolute relative to ω_1-closed forcing.

Claim A: There is no class $X \subseteq M_{\infty\omega_1}$ such that

$$\mathfrak{A} \models_{\mathcal{L}} \phi \iff \exists \psi \in X\ (\mathfrak{A} \models \psi).$$

For contradiction, suppose X is such a class. Since $R(T_0) \models \phi$, there is $\psi \in X$ such that $R(T_0) \models \psi$. By 5.1 and 17.6 there exists a tree T_1 such that $R(T_1) \models \psi$ but $R(T_1) \not\models \phi$, which is a contradiction. Similarly we show:

Claim B: There is no class $X \subseteq M_{\infty\omega_1}$ such that

$$\mathfrak{A} \models_{\mathcal{L}} \neg\phi \iff \forall \psi \in X\ (\mathfrak{A} \models \psi).$$

□

One could now ask whether the above counterexample only shows that absoluteness relative to ω_1-closed forcing is not the "right" degree of absoluteness for an approximation to $M_{\infty\omega_1}$. However, the counterexample is stronger than it seems. Namely, the structure $R(T_0)$ is actually absolute relative to ZFC($\mathcal{P}_{\omega_1}$). Moreover, if $\mathcal{L}$ is a "strong" logic absolute relative to the "right degree", $\mathcal{L}$ should be an extension of $M_{\infty\omega_1}$. This in turn implies that $\mathcal{L}$ is capable of distinguishing whether a structure is equivalent with $R(T_0)$ relative to $M_{\infty\omega_1}$. The above counterexample shows, that there is no logic which can make this distinction with a single sentence and has an approximation to $M_{\infty\omega_1}$.

References

[B1] BARWISE, K.J.: Infinitary logic and admissible sets. - Doctoral dissertation. Stanford University, 1967.

[B2] BARWISE, K.J.: Absolute logics and $L_{\infty\omega}$. - Ann. Math. Logic 4, 1972, 309–340.

[B3] BARWISE, K.J.: Mostowski's collapsing function and the closed unbounded filter. - Fund. Math. 82, 1974, 95–103.

[B4] BARWISE, K.J.: Admissible sets and structures: an approach to definability theory. - Perspect. Math. Logic. Springer-Verlag, 1975.

[BHK] BAUMGARTNER, J.E., L.A. HARRINGTON, and E.M. KLEINBERG: Adding a closed unbounded set. - J. Symbolic Logic 41/2, 1976, 481–482.

[Bu] BURGESS, J.P.: Descriptive set theory and infinitary languages. - In: Set theory. Foundations of mathematics. Zb. Rad. Mat. Inst. Beograd (N.S.) 2/10, 1977, 9–30.

[D] DEVLIN, K.J.: Aspects of constructibility. - Lecture Notes in Math. 354. Springer-Verlag, 1973.

[E] EBBINGHAUS, H.-D.: Extended logics: the general framework. - In: Model-Theoretic Logics, edited by J. Barwise and S. Feferman. Perspect. Math. Logic. Springer-Verlag, 1985.

[F] FEFERMAN, S.: Persistent and invariant formulas for outer extensions. - Compositio Math. 20, 1968, 29–52.

[FK] FEFERMAN, S., and G. KREISEL: Persistent and invariant formulas relative to theories of higher order. - Bull. Amer. Math. Soc. 72, 1966, 480–485.

[He] HEIKKILÄ, H.: Definability and infinitely deep languages. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 89, 1993.

[Hy] HYTTINEN, T.: Games and infinitary languages. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 64, 1987.

[HT] HYTTINEN, T., and H. TUURI: Constructing strongly equivalent nonisomorphic models for unstable theories. - Ann. Pure Appl. Logic 52, 1991, 203–248.

[J] JECH, T.: Set theory. - Academic Press, 1978.

[JK] JENSEN, R.B., and C. KARP: Primitive recursive set functions. - In: Axiomatic Set Theory, Part I, edited by D.S. Scott. Proc. Sympos. Pure Math. 13, 1971, 143–176.

[Kp] KARP, C.: Finite-quantifier equivalence. - In: The Theory of Models, edited by J.W. Addison, L.A. Henkin, and A. Tarski. North-Holland, 1965, 407–412.

[Kt] KARTTUNEN, M.: Model theory for infinitely deep languages. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 50, 1984.

[Kn] KUNEN, K.: Set theory. - North-Holland, 1980.

[K1] KUEKER, D.W.: Löwenheim–Skolem and interpolation theorems. - Bull. Amer. Math. Soc. 78/2, 1972, 211–215.

[K2] KUEKER, D.W.: Countable approximations and Löwenheim–Skolem theorems. - Ann. Math. Logic 11, 1977, 57–103.

[L] LÓPEZ-ESCOBAR, E.G.K.: An interpolation theorem for denumerably long formulas. - Fund. Math. 57, 1965, 253–272.

[M] MALITZ, J.J.: Infinitary analogs of theorems from first order model theory. - J. Symbolic Logic 36, 1971, 216–228.

[O] OIKKONEN, J.: Second order definability, game quantifiers and related expressions. - Soc. Sci. Fenn. Comment. Phys.-Mat. 48/1, 1978, 39–101.

[T] TUURI, H.: Relative separation theorems for $L_{\kappa^+\kappa}$. - Notre Dame J. Formal Logic 33/3, 1992, 383–401.

[Va] VAUGHT, R.L.: Descriptive set theory in $L_{\omega_1\omega}$. - In: Cambridge summer school in mathematical logic, ed. by A.R.D. Mathias and H. Rogers. Lecture Notes in Math. 337. Springer-Verlag, 1973, 574–598.

[V1] VÄÄNÄNEN, J.: Applications of set theory to generalized quantifiers. - PhD thesis, University of Manchester, Oct. 1977.

[V2] VÄÄNÄNEN, J.: Set-theoretic definability of logics. - In: Model-Theoretic Logics, edited by J. Barwise and S. Feferman. Perspect. Math. Logic. Springer-Verlag, 1985.