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**VESA ALA-MATTILA**

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SUOMALAINEN TIEDEAKATEMIA

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Helsinki, October 2011

Vesa Ala-Mattila

## Contents

# 1. Introduction

The main results of this work show that we can always use geometric covering and packing constructions to characterize a Patterson-Sullivan measure $\mu$ of a non-elementary geometrically finite Kleinian group $G$. We mean by this that if the standard covering and packing constructions are modified in a suitable way, we can use either one of them to construct a measure $\nu$ such that $\mu = c\nu$, where $c > 0$ is a constant. The modified constructions are in general defined without reference to Kleinian groups, so they or their variants may prove useful in some other contexts in addition to that of Kleinian groups.

Our results generalize and modify results of D. Sullivan, [Sullivan1984], which show that a measure $\nu$ such as above can sometimes be constructed using the standard covering construction and sometimes the standard packing construction. Sullivan shows also that neither or both of the standard constructions can be used to construct $\nu$ in some situations.

Our modifications of the standard constructions are based on geometric properties of limit sets of Kleinian groups studied first by P. Tukia in [Tukia1985b]. Some estimation results for general conformal measures of Kleinian groups play a crucial role in the proofs of our main results. These estimation results are generalizations and modifications of similar results discussed, for instance, in [SV1995], [Tukia1994b] and [Tukia1994c].

Let us take a closer look at the main results. We begin by introducing some fundamental notions from the theory of Kleinian groups. See Chapter 2 for a more extensive discussion on these topics.

Let $\mathbb{X}^{n+1}$ be either the unit ball $\mathbb{B}^{n+1}$ or the upper half-space $\mathbb{H}^{n+1}$ of the compactified $(n+1)$-dimensional euclidean space $\bar{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$, where $n \in \{1, 2, \ldots\}$. Endow $\mathbb{X}^{n+1}$ with the hyperbolic metric $d$. A subgroup $G$ of the group of all Möbius transformations of $\bar{\mathbb{R}}^{n+1}$ is a *Kleinian group* acting on $\mathbb{X}^{n+1}$ if the elements of $G$ map $\mathbb{X}^{n+1}$ onto itself and $G$ is discrete in the natural topology of Möbius transformations of $\bar{\mathbb{R}}^{n+1}$.

The *limit set* $L(G)$ of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$ can be defined as the set of accumulation points of the $G$-orbit of any point in $\mathbb{X}^{n+1}$. It is well known that $L(G)$ is a subset of $\partial\mathbb{X}^{n+1}$, the topological boundary of $\mathbb{X}^{n+1}$ in $\bar{\mathbb{R}}^{n+1}$, and that $L(G)$ is empty, contains exactly one or two points, or is an uncountable perfect set. We use the standard terminology and say that $G$ is *elementary* if $L(G)$ contains at most two points and *non-elementary* otherwise.

We are particularly interested in *geometrically finite* Kleinian groups. The definition of this notion is rather complicated and we omit the details from this introduction. The notion will be discussed in detail in Chapter 6.

We will next define conformal measures of Kleinian groups. It is easy to give a natural definition for conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$. Let $G$ be a Kleinian group acting on $\mathbb{B}^{n+1}$. Let $s \geq 0$. A measure $\mu$ is an *s-conformal measure* of $G$ if $\mu$ satisfies the following conditions. The $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. The measure $\mu$ is non-trivial and finite and supported by $L(G)$. The measure $\mu$ satisfies the transformation rule

$$\mu(gA) = \int_A |g'|^s d\mu \tag{1.1}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$, where $|g'|$ is the operator norm of the derivative of $g$. It is somewhat more complicated to define conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$. We choose to give the definition using conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$ in the following way. Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Let $s \geq 0$. A measure $\mu$ is an *s-conformal measure* of $G$ if there is a Möbius transformation $h$ of $\bar{\mathbb{R}}^{n+1}$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and an $s$-conformal measure $\nu$ of the Kleinian group $h^{-1}Gh = \{h^{-1} \circ g \circ h : g \in G\}$ acting on $\mathbb{B}^{n+1}$ such that

$$\mu(A) = \int_{h^{-1}A} |h'|^s d\nu \tag{1.2}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. The measures defined by (1.2) are non-trivial measures supported by $L(G)$ whose $\sigma$-algebra of measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. These measures satisfy a transformation rule of the form (1.1) in general, although some minor problems are present in some situations. We will discuss these topics and our motivation for the above definitions in detail in Chapter 2.

If $G$ is a Kleinian group acting on $\mathbb{X}^{n+1}$ such that $L(G) = \partial\mathbb{X}^{n+1}$, then the restriction to the Borel sets of $\bar{\mathbb{R}}^{n+1}$ of the $n$-dimensional Lebesgue measure of $\partial\mathbb{X}^{n+1}$ is an $n$-conformal measure of $G$. This measure is often a useful tool in the study of $G$. The fundamental purpose of conformal measures defined as above is to generalize the situation for Kleinian groups $G$ acting on $\mathbb{X}^{n+1}$ such that $L(G) \neq \partial\mathbb{X}^{n+1}$. The basics of the theory of conformal measures of Kleinian groups are discussed, for example, in [Nicholls1989], [Patterson1987] and [Sullivan1979].

S. J. Patterson discovered, [Patterson1976b], that if $G$ is a *Fuchsian group*, i.e. a Kleinian group acting on $\mathbb{X}^2$ and containing only orientation preserving Möbius transformations, such that $L(G)$ is non-empty, then there are conformal measures of $G$. All of these measures are $\delta$-conformal, where $\delta$ is the *exponent of convergence* of $G$ defined, as usual, by (recall that $d$ denotes the hyperbolic metric of $\mathbb{X}^{n+1}$)

$$\delta = \inf\left\{ s \geq 0 : \sum_{g\in G} e^{-sd(x,g(y))} < \infty \text{ for some } x, y \in \mathbb{X}^{n+1} \right\}. \tag{1.3}$$

D. Sullivan observed, [Sullivan1979], that Patterson's construction works also in the general situation, so any Kleinian group $G$ with a non-empty limit set has $\delta$-conformal measures. Today the conformal measures given by Patterson's construction are known as *Patterson measures* or *Patterson-Sullivan measures*. They are canonical examples of conformal measures of Kleinian groups, but other conformal measures are known to exist, see, for instance, [AFT2007] and [FT2006].

It is well known that the standard measure constructions employing countable coverings or packings of closed balls $\bar{B}^{n+1}(x,t)$ of $\mathbb{R}^{n+1}$, where $x \in \mathbb{R}^{n+1}$ and $t > 0$, and the gauge function $t \mapsto t^s$, where $s \geq 0$ is fixed, construct measures satisfying transformation rules of the form (1.1). (See page 102 for the definitions of these standard constructions.) It is natural, therefore, to ask what can be said about the relation between such measures and a given $s$-conformal measure $\mu$ of a Kleinian group $G$. D. Sullivan studied this question in [Sullivan1984] in the situation where $G$ is non-elementary and geometrically finite and $\mu$ is a Patterson-Sullivan measure of $G$. We give the following definition in order to discuss Sullivan's results.

If $\nu_1$ and $\nu_2$ are measures of $\bar{\mathbb{R}}^{n+1}$ with the same measurable sets such that $\nu_1 = c\nu_2$, where $c > 0$ is a constant, we say that $\nu_1$ and $\nu_2$ are *equivalent* and write $\nu_1 \sim \nu_2$. If $\nu_1$ and $\nu_2$ are not equivalent, we write $\nu_1 \nsim \nu_2$. Note that this is a non-standard definition for the equivalence of measures. This is not a problem, however, since we do not use the standard definitions in this work.

The setting of [Sullivan1984] is the following. Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$. Denote by $\delta$ the exponent of convergence of $G$. Let $\mu$ be a Patterson-Sullivan measure of $G$. Let $m_\delta$ be the measure constructed by the standard covering construction employing the gauge function $t \mapsto t^\delta$. Define the measure $m_\delta^{L(G)}$ by $m_\delta^{L(G)}(A) = m_\delta(A \cap L(G))$ for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. Let $p_\delta$ be the measure constructed by the standard packing construction employing the gauge function $t \mapsto t^\delta$. Define the measure $p_\delta^{L(G)}$ by $p_\delta^{L(G)}(A) = p_\delta(A \cap L(G))$ for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. If $G$ contains parabolic elements, then denote by $k_{\max}$ and $k_{\min}$ the maximum and minimum over the ranks of the parabolic fixed points of $G$. (If $g \in G$ is parabolic, then $g$ has exactly one fixed point, and such points are the *parabolic fixed points* of $G$. If $x$ is a parabolic fixed point of $G$, then the stabilizer of $x$ with respect to $G$ contains a free commutative subgroup of finite index isomorphic to $\mathbb{Z}^k$ for some $k \in \{1, 2, \ldots, n\}$. The *rank* of $x$ is $k$ by definition. See Theorem 2.7 for more details.)

The main results of [Sullivan1984] pertaining to the present discussion are the following. If $G$ contains parabolic elements and $\delta \geq k_{\max}$, then $\mu \sim m_\delta^{L(G)}$. If $G$ contains parabolic elements and $\delta \leq k_{\min}$, then $\mu \sim p_\delta^{L(G)}$. If $G$ contains no parabolic elements, then $\mu \sim m_\delta^{L(G)} \sim p_\delta^{L(G)}$. If $L(G)$ is an $l$-sphere of $\partial\mathbb{X}^{n+1}$ for some $l \in \{1, 2, \ldots, n\}$, then

$\mu \sim m_\delta^{L(G)} \sim p_\delta^{L(G)}$ regardless of whether $G$ contains parabolic elements or not. (We employ the standard convention and say that a euclidean $l$-plane of $\partial\mathbb{H}^{n+1}$ containing the point $\infty$ is an $l$-sphere of $\partial\mathbb{H}^{n+1}$.) On the other hand, if $G$ contains parabolic elements and $k_{\min} < \delta < k_{\max}$, then $\mu \nsim m_\delta^{L(G)}$ and $\mu \nsim p_\delta^{L(G)}$. We will discuss these results of [Sullivan1984] in greater detail in Chapter 7.

We remark that Sullivan actually uses a non-standard definition for the packing construction in [Sullivan1984], but the formulations of his main theorems are the same regardless of which definition is used. Moreover, Sullivan considers only the case $n = 2$ explicitly, but his results generalize easily to the case of any $n \in \{1, 2, \ldots\}$.

We conclude from the above that if $G$ is a non-elementary geometrically finite Kleinian group and $\mu$ a Patterson-Sullivan measure of $G$, then $\mu$ is equivalent to $m_\delta^{L(G)}$, $p_\delta^{L(G)}$, both or neither of these measures depending on additional assumptions. The main contribution of this work is to show that if the covering and packing constructions are modified in a suitable way, one can always use either one of them to construct a measure $\nu$ such that $\mu$ and $\nu$ are equivalent. We emphasize that the modified constructions are defined without reference to Kleinian groups in the general situation, so it is possible that these constructions or their variants turn out to be useful in some contexts other than that of Kleinian groups. The full details of the basic formulations of the modified constructions will be given in Chapter 5 and some variants of these basic formulations will be discussed in Chapter 7.

Let us take a look at the main features of the modifications. In this introduction, we consider explicitly only the covering construction, since the modifications to the packing construction are analogous. Furthermore, we do not consider the construction in the general situation but in the context of a given non-elementary geometrically finite Kleinian group $G$ satisfying some useful assumptions. The assumptions do not restrict the generality of the results in a significant way, but they do guarantee that certain potential minor complications are not present. The full discussion on applying the general modified constructions to geometrically finite Kleinian groups will be given in Chapters 6 and 7.

The setting of the present discussion is the following. Let $G$ be a non-elementary geometrically finite Kleinian group. We assume that $G$ acts on the upper half-space $\mathbb{H}^{n+1} = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$. We assume additionally that $G$ contains parabolic elements and that $L(G)$ is not an $l$-sphere of $\partial\mathbb{H}^{n+1}$ for any $l \in \{1, 2, \ldots, n\}$ because otherwise the situation is covered by the results in [Sullivan1984]. We assume also that $\infty \notin L(G)$. Thus, regarding $\partial\mathbb{H}^{n+1}$ as $\bar{\mathbb{R}}^n$, we have that $L(G)$ is bounded in $\mathbb{R}^n$. We fix two constants $t_0 > 0$ and $v_0 \in ]0, 1[$ before proceeding to further details.

The main ingredient in the modification of the constructions is that close attention is paid to certain geometric properties of the limit set $L(G)$. These geometric properties of limit sets of Kleinian groups were studied first by P. Tukia in [Tukia1985b]. Our treatment of the topic is more explicit in quantitative terms than Tukia's and our context more general. We will formulate and prove our theorems pertaining to this topic in Chapter 4.

The relevant geometric properties of $L(G)$ can be described explicitly in the following way. If $x \in \mathbb{R}^n$ and $t > 0$, we write $\bar{B}^n(x,t) = \bar{B}^{n+1}(x,t) \cap \mathbb{R}^n$, where $\bar{B}^{n+1}(x,t)$ is the standard closed $(n+1)$-dimensional euclidean ball of $\mathbb{R}^{n+1}$ with center $x$ and radius $t$. Given $l \in \{1,2,\ldots,n\}$, we define the *$l$-dimensional flatness function* $\gamma_l$ of $G$ as follows. Let $x \in \mathbb{R}^n$ and $t \in ]0,t_0[$ be such that there is $x' \in L(G)$ with $|x-x'|/t \leq v_0$. Now the ball $\bar{B}^n(x,t)$ has limit points of $G$ relatively close to its center. We define that

$$\gamma_l(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V) \tag{1.4}$$

for these $x$ and $t$, where $\mathcal{F}_l(x,t)$ is the collection of all $l$-dimensional spheres of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x,t)$ and $\rho$ is the Hausdorff metric defined with respect to the euclidean metric in the collection of non-empty and compact sets of $\mathbb{R}^n$, i.e.

$$\rho(A,B) = \sup\{d_{\mathrm{euc}}(y,B), d_{\mathrm{euc}}(z,A) : y \in A, z \in B\} \tag{1.5}$$

for all non-empty and compact sets $A, B \subset \mathbb{R}^n$. (We remind the reader that a euclidean $l$-plane of $\bar{\mathbb{R}}^n$ containing the point $\infty$ is said to be an $l$-sphere of $\bar{\mathbb{R}}^n$.)

The interpretation is that $\gamma_l(x,t)$ measures on a normalized scale how much $L(G)$ resembles an $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$. Our assumption that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ implies that $\bar{B}^n(x,t) \cap L(G)$ is never identical with a set of the form $\bar{B}^n(x,t) \cap V$, where $V \in \mathcal{F}_l(x,t)$, but $\gamma_l(x,t)$ is small in certain important situations to be discussed presently. According to the interpretation, $L(G)$ is close to an $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$ in these situations. The euclidean diameter (possibly $\infty$) of the particular $l$-sphere of $\bar{\mathbb{R}}^n$ is often very large compared to $t$ in these situations, so $L(G)$ is actually close to an $l$-plane of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$. This motivates the term flatness function.

We describe next the behaviour of the functions $\gamma_l$, $l \in \{1,2,\ldots,n\}$, in detail. We assume that $x \in \mathbb{R}^n$ and $t > 0$ are as in (1.4) for the time being. That is, we assume that $t \in ]0,t_0[$ and that there is $x' \in L(G)$ with $|x-x'|/t \leq v_0$, where $t_0 > 0$ and $v_0 \in ]0,1[$ are fixed constants.

Recall that a point fixed by a parabolic element of $G$ is called a parabolic fixed point of $G$. The definition of geometric finiteness of Kleinian groups implies the existence of a finite set $P$ of parabolic fixed points of $G$ such that if $y$ is a parabolic fixed point of $G$, there is exactly one $p \in P$ such that $y = g(p)$ for some $g \in G$. This means that the set of all parabolic fixed points of $G$ can be written as $GP = \{g(p) : g \in G, p \in P\}$. The definition of geometric finiteness implies also the existence of a certain collection $\{H_p : p \in GP\}$ of horoballs of $\mathbb{H}^{n+1}$ called a *complete collection of horoballs* of $G$. (By the standard definition, a *horoball* $B$ of $\mathbb{H}^{n+1}$ based at $y \in \bar{\mathbb{R}}^n$ is an open $(n+1)$-dimensional euclidean ball contained in $\mathbb{H}^{n+1}$ and tangential to $\bar{\mathbb{R}}^n$ at $y$ if $y \neq \infty$; if $y = \infty$, then $B$ is an open half-space of $\mathbb{R}^{n+1}$ contained in $\mathbb{H}^{n+1}$.) The horoball $H_p$, $p \in GP$, is based at $p$, and the horoballs in the complete collection have pairwise disjoint closures. We let $(x,t)$ denote the point in $\mathbb{H}^{n+1}$ whose first $n$-coordinates are given by $x$ and whose $(n+1)$-coordinate is $t$. The intuition is that if $(x,t) \in H_p$ for some $p \in GP$, then $(x,t)$ is in a natural neighbourhood of $p$ so that if $d((x,t),\partial H_p)$ is large, then $(x,t)$ is close to $p$. (Recall that $d$ is the hyperbolic metric of $\mathbb{H}^{n+1}$. We use the convention that $d(y,z) = \infty$ if $y \in \mathbb{H}^{n+1}$ and

$z \in \bar{\mathbb{R}}^n$.) See Chapter 6 for more details on the definition of geometrically finite Kleinian groups.

The major geometric property of $L(G)$ is that if $(x, t) \in H_p$ for some $p \in GP$ and the rank of $p$ is $k \in \{1, 2, \dots, n\}$, then

$$c_1^{-1} e^{-d((x,t),\partial H_p)} \leq \gamma_k(x, t) \leq c_1 e^{-d((x,t),\partial H_p)}, \tag{1.6}$$

where $c_1 > 0$ is a constant. (Recall that $x$ and $t$ are assumed to be as in (1.4) for the time being. Recall the definition of the rank of a parabolic fixed point of $G$ from page 7.) This means that as $d((x, t), \partial H_p)$ increases, i.e. $(x, t)$ approaches $p$ inside $H_p$, $L(G)$ resembles more and more a $k$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x, t)$. In the situation of (1.6), we have additionally that

$$c_2^{-1} \leq \gamma_l(x, t) \leq c_2 \tag{1.7}$$

for all $l \in \{1, 2, \dots, n\} \setminus \{k\}$, where $c_2 > 0$ is a constant, which means that $L(G)$ is uniformly bounded away from any $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x, t)$ for all $l \in \{1, 2, \dots, n\} \setminus \{k\}$. On the other hand, if $(x, t) \notin H_p$ for all $p \in GP$, then

$$c_3^{-1} \leq \gamma_l(x, t) \leq c_3 \tag{1.8}$$

for all $l \in \{1, 2, \dots, n\}$, where $c_3 > 0$ is another constant. The geometric interpretation for (1.8) is the same as for (1.7).

We conclude that we can succinctly describe the geometry of the limit set of a non-elementary geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$ as follows. The limit set resembles a $k$-sphere of $\bar{\mathbb{R}}^n$, $k \in \{1, 2, \dots, n\}$, close to a parabolic fixed point of the group of rank $k$ and no $l$-sphere of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \dots, n\}$ otherwise.

We will also need the following property of $L(G)$ in our modified constructions. Denote the euclidean diameter of a non-empty $A \subset \bar{\mathbb{R}}^n$ by $d_{\text{euc}}(A)$. Suppose that $x \in \mathbb{R}^n$ and $t > 0$ are as in (1.4). Define

$$\beta(x, t) = \frac{1}{t} d_{\text{euc}}(\bar{B}^n(x, t) \cap L(G)). \tag{1.9}$$

The quantity $\beta(x, t)$ measures the diameter of $\bar{B}^n(x, t) \cap L(G)$ on a normalized scale. The main result regarding $\beta(x, t)$ is that there is a constant $c_4 > 0$ such that

$$c_4^{-1} \leq \beta(x, t) \leq c_4, \tag{1.10}$$

where $x \in \mathbb{R}^n$ and $t > 0$ are as in (1.4). We will prove results implying (1.6), (1.7), (1.8) and (1.10) in Chapter 4.

Now that we have discussed the relevant geometric properties of $L(G)$, we turn to the modified covering construction. We start by defining the gauge function of the modified construction. The main idea is that the gauge function takes into account the above quantitative expressions for the relevant geometric properties of $L(G)$. Accordingly, we set the gauge function $\psi$ to be

$$\psi(x, t) = d_{\text{euc}}(\bar{B}^n(x, t) \cap L(G))^\delta \prod_{l=1}^{n} \gamma_l(x, t)^{\delta - l}, \tag{1.11}$$

where $\delta$ is the exponent of convergence of $G$ and $x \in \mathbb{R}^n$ and $t > 0$ are as in (1.4).

The expression on the right hand side of (1.11) consists of two parts. In view of (1.10), we see that the part $d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^\delta$ is comparable to the quantity $t^\delta$ given by the standard gauge function $t \mapsto t^\delta$, so these quantities correspond to one another in a natural way. It is, in fact, the case that if we replace $d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^\delta$ by $t^\delta$ (or $(2t)^\delta$), our main results regarding the modified constructions remain true. We use $d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^\delta$ instead of $t^\delta$ because we want to use a quantity that is connected to the geometry of $L(G)$. The second part of the right hand side of (1.11), the quantity

$$\omega(x,t) = \prod_{l=1}^{n} \gamma_l(x,t)^{\delta - l}, \tag{1.12}$$

quantifies the geometric properties of $\bar{B}^n(x,t) \cap L(G)$. By (1.6), (1.7), (1.8) and the geometric interpretations associated with these formulae, we see that at most one of the quantities $\gamma_l(x,t)$, $l \in \{1,2,\ldots,n\}$, can be small for any given $(x,t)$, and that if $\gamma_k(x,t)$ is small for some $k \in \{1,2,\ldots,n\}$, then $(x,t)$ is close to a parabolic fixed point $p$ of $G$ of rank $k$ (i.e. $(x,t)$ is in the horoball $H_p$ and $d((x,t),\partial H_p)$ is large), $L(G)$ resembles a $k$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$, and $\omega(x,t)$ is small or large if $\delta - k$ is positive or negative. The exponents in (1.12) are of the form $\delta - l$ because of the formula (1.15) to be discussed soon.

Let us give the remaining details of the modified covering construction. We will denote the covering (outer) measure given by the construction by $m$. Let $A \subset L(G)$. Let $\varepsilon \in ]0,t_0[$ and $\nu \in ]0,\nu_0[$. (Recall that $t_0 > 0$ and $\nu_0 \in ]0,1[$ are constants we fixed earlier.) We say that a countable collection $\mathcal{T}$ of closed balls $\bar{B}^n(x,t)$ of $\mathbb{R}^n$ is an *$(\varepsilon,\nu)$-covering* of $A$ if $x \in \mathbb{R}^n$, $t \in ]0,\varepsilon]$, there is $x' \in L(G)$ with $|x - x'|/t \le \nu$, and $A \subset \bigcup \mathcal{T}$. Observe that there are $(\varepsilon,\nu)$-coverings of $A$ since $L(G)$ is a compact set of $\mathbb{R}^n$. We define a preliminary quantity

$$m_\varepsilon^\nu(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \psi(x,t), \tag{1.13}$$

where $\mathcal{T}$ varies in the collection of all $(\varepsilon,\nu)$-coverings of $A$. If $\varepsilon' \in ]0,\varepsilon]$ and $\nu' \in ]0,\nu]$, then the collection of $(\varepsilon',\nu')$-coverings of $A$ is contained in the collection of $(\varepsilon,\nu)$-coverings of $A$, which implies that $m_{\varepsilon'}^{\nu'}(A) \ge m_\varepsilon^\nu(A)$. It is, therefore, natural to define the *$m$-measure* of $A$ to be

$$m(A) = \sup_{\varepsilon\in]0,t_0[,\nu\in]0,\nu_0[} m_\varepsilon^\nu(A). \tag{1.14}$$

It is obvious that this construction is a straightforward modification of the standard covering construction (see page 102 for the definition of the standard construction). Indeed, the arguments needed to show that (1.14) defines an outer measure of $L(G)$ whose $\sigma$-algebra of measurable sets contains all Borel sets of $L(G)$ are essentially the same as those used in the case of the standard construction. See Chapter 5 for a discussion on the differences between the standard construction and the modified construction.

As we mentioned earlier, the modifications to the standard packing construction are analogous: the gauge function $t \mapsto t^\delta$ is again replaced by $\psi$ and the standard packings are replaced by $(\varepsilon, v)$-packings in a similar way as $(\varepsilon, v)$-coverings replace the standard coverings in the above construction. We omit the details of the modified packing construction from this introduction.

The full details of the basic formulations of the modified constructions in a general situation, i.e. a situation where no reference is made to Kleinian groups, will be given in Chapter 5. We will discuss variants of the basic formulations in Chapter 7, both in the general context and in the context of Kleinian groups.

We will next point out the connection between a Patterson-Sullivan measure $\mu$ of the non-elementary geometrically finite Kleinian group $G$ we have been considering and the measures constructed by the modified constructions in the context of $G$. The connection is based on a general estimation theorem which was proved by D. Sullivan in [Sullivan1984] and studied in detail by B. Stratmann and S. L. Velani in [SV1995] in the context of geometrically finite Kleinian groups acting on $\mathbb{B}^{n+1}$.

The formulation of this theorem in the present situation is the following. There is a constant $c_5 > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t > 0$ be as in (1.4), i.e. it is the case that $t \in ]0, t_0[$ and that $|x - x'|/t \leq v_0$ for some $x' \in L(G)$, where $t_0 > 0$ and $v_0 \in ]0, 1[$ are the constants fixed earlier. Then it is true that

$$c_5^{-1}\phi(x,t) \leq \mu(\bar{B}^n(x,t) \cap L(G)) \leq c_5\phi(x,t), \tag{1.15}$$

where

$$\phi(x,t) = t^\delta \prod_{l=1}^{n} \exp((l-\delta)\max\{d_{H_p}((x,t), \partial H_p) : p \in GP, r(p) = l\}), \tag{1.16}$$

where $r(p)$ denotes the rank of $p$ and $d_{H_p}((x,t), \partial H_p)$ equals $d((x,t), \partial H_p)$ if $(x,t) \in H_p$ and 0 otherwise; if $G$ has no parabolic fixed points of rank $l \in \{1, 2, \dots, n\}$, we set that the term corresponding to $l$ in the product in (1.16) equals 1. Observe that $\phi(x,t) = t^\delta e^{d((x,t),\partial H_p)(k-\delta)}$ if $(x,t) \in H_p$ for some $p \in GP$ of rank $k \in \{1, 2, \dots, n\}$ and $\phi(x,t) = t^\delta$ otherwise, since the horoballs in the collection $\{H_p : p \in GP\}$ have pairwise disjoint closures.

We will provide a new proof for the formula (1.15) in Chapter 3. Our proof uses extensions of arguments used by P. Tukia in his papers [Tukia1994b] and [Tukia1994c]. We have found some ideas of [SV1995] useful as well.

Recall the results (1.6), (1.7),(1.8) and (1.10) and the definition (1.11). We see that there is a constant $c_6 > 0$ such that

$$c_6^{-1}\phi(x,t) \leq \psi(x,t) \leq c_6\phi(x,t) \tag{1.17}$$

for all $x \in \mathbb{R}^n$ and $t > 0$ as in (1.15). It follows that there is a constant $c_7 > 0$ such that

$$c_7^{-1}\psi(x,t) \leq \mu(\bar{B}^n(x,t) \cap L(G)) \leq c_7\psi(x,t) \tag{1.18}$$

for these $x \in \mathbb{R}^n$ and $t > 0$. This relation between the gauge function $\psi$ and the Patterson-Sullivan measure $\mu$ establishes the essential connection between $\mu$ and the measures constructed by the modified constructions. Once (1.18) has been proved, it will not be very difficult to use arguments similar to those in [Sullivan1984] to prove the main results of this work. Recall that these results state that we can use the modified covering construction or the modified packing construction to construct a measure $\nu$ such that $\mu$ and $\nu$ are equivalent, i.e. that $\mu = c\nu$, where $c > 0$ is a constant. (In the case of the modified covering construction discussed in this introduction, the measure $\nu$ is defined by $\nu(A) = m(A \cap L(G))$ for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$, where $m$ is defined by (1.14).) This reasoning will be done in Chapter 6.

We end this introduction with an overview of Chapters 2-7.

Chapter 2 considers the background theory of this work. In this chapter, we will introduce the required background notions and results and prove numerous auxiliary results pertaining to them.

In Chapter 3, we will prove some estimation results for conformal measures of Kleinian groups. The formula (1.15) is a direct consequence of these results. The context of Chapter 3 is more general than that of (1.15) in that we do not assume the Kleinian groups considered to be geometrically finite and we consider also other conformal measures besides Patterson-Sullivan measures.

The topic of Chapter 4 is the geometry of the limit set of a non-elementary Kleinian group. Like in Chapter 3, we will not assume that the Kleinian groups considered are geometrically finite. We will prove a number of results, and these results have (1.6), (1.7),(1.8) and (1.10) as immediate consequences.

Chapter 5 contains a discussion on the basic formulations of the modified measure constructions. The context of Chapter 5 is that of general geometric measure theory, so no reference is made to Kleinian groups.

In Chapter 6, we will adapt the main results of Chapters 3, 4 and 5 to the context of geometrically finite Kleinian groups. We will formulate and prove in this chapter the basic version of our main equivalence theorem concerning Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups.

Chapter 7 is the last chapter of this work. It contains a discussion on some variants of the basic versions of the modified constructions given in Chapter 5. Most of the variants considered satisfy a similar equivalence theorem as the constructions introduced in Chapter 5. We will consider also variants which are simpler than the constructions introduced in Chapter 5 but which construct measures with weaker properties: If $G$ is a non-elementary geometrically finite Kleinian group and $\mu$ a Patterson-Sullivan measure of $G$, then any of these variants can be used to construct a measure $\nu$ such that $c^{-1}\nu \leq \mu \leq c\nu$, where $c > 0$ is a constant. (The supervisor of this work, P. Tukia, conjectured the preliminary hypothesis that one of these variants would satisfy the same stronger equivalence result as the constructions introduced in Chapter 5. This hypothesis was of paramount importance - it was the starting point of this work - but it seems now that if the hypothesis is indeed true, one probably needs to use considerably more complicated methods than those used in this work to prove it.) Also, we will discuss the results in [Sullivan1984] relevant to this work in greater detail than we did in this introduction.

## 2. Background theory and auxiliary results

We begin this chapter by discussing the background theory to the extent required by the later chapters. The first part of this discussion considers Möbius transformations and Kleinian groups and the second part considers conformal measures of Kleinian groups. We assume that the reader is familiar with Möbius transformations and Kleinian groups and rely heavily on literature in the first part of the discussion. We do not assume that the reader is familiar with conformal measures of Kleinian groups, so we will give a more detailed account on the topic. After discussing the background theory, we will proceed to prove quite a few auxiliary results pertaining to the topics considered earlier. These results will be needed predominantly in Chapters 3 and 4. Our aim is to formulate the auxiliary results so that they are immediately applicable in the situations encountered in the later chapters.

2.1. **Background theory.** The aim of this section is to establish notation, fix the definitions of basic notions, and present a number of fundamental results. The first part of this section (subsection 2.1.1.) considers Möbius transformations and Kleinian groups. The material in the first part is generally well-known and hence we will omit nearly all proofs, although we will provide specific references on many occasions. The books [Ahlfors1981], [Apanasov2000], [MT1998] and [Nicholls1989], for example, contain material giving a good overview of the topics of the first part. Detailed introductory chapters can be found in the books [Beardon1983] and [Maskit1988], and a long and detailed treatment is given in [Ratcliffe2006]. Furthermore, the paper [Tukia1994a] contains convenient proofs for basic facts about limit sets of Kleinian groups in a general context. The second part of this section (subsection 2.1.2.) considers conformal measures of Kleinian groups. Since this topic is less well-known, we will provide a more detailed account. The basics of the theory of conformal measures of Kleinian groups are discussed, for example, in [Nicholls1989], [Patterson1987] and [Sullivan1979].

2.1.1. *Möbius transformations and Kleinian groups.* Our base space is the compactified $(n+1)$-dimensional euclidean space $\bar{\mathbb{R}}^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$, $n \in \{1, 2, \dots\}$, endowed with the chordal metric $q$ defined as follows. If $x, y \in \mathbb{R}^{n+1}$, then

$$q(x,y) = \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \qquad \text{and} \qquad q(x,\infty) = \frac{2}{\sqrt{1+|x|^2}}. \tag{2.1}$$

We normally use the euclidean metric when considering points and subsets of $\mathbb{R}^{n+1}$. A point $x \in \mathbb{R}^{n+1}$ is often written in coordinate form as $x = (x_1, x_2, \dots, x_{n+1})$. The standard basis of $\mathbb{R}^{n+1}$ is formed by $e_1, e_2, \dots, e_{n+1}$, where $e_1 = (1, 0, \dots, 0)$ etc. The space $\mathbb{R}^k$, $k \in \{1, 2, \dots, n\}$, is usually taken to be the subspace of $\mathbb{R}^{n+1}$ spanned by the first $k$ vectors of the standard basis. Given $X \subset \bar{\mathbb{R}}^{n+1}$, we denote by $\partial X$ the topological boundary of $X$ in $\bar{\mathbb{R}}^{n+1}$ and by $\bar{X}$ the topological closure of $X$ in $\bar{\mathbb{R}}^{n+1}$.

We apply the standard convention and call both the euclidean spheres of $\bar{\mathbb{R}}^{n+1}$ and the euclidean planes of $\bar{\mathbb{R}}^{n+1}$ the spheres of $\bar{\mathbb{R}}^{n+1}$. (Note that a euclidean plane of $\bar{\mathbb{R}}^{n+1}$ contains the point $\infty$ and a euclidean plane of $\mathbb{R}^{n+1}$ does not.) The open euclidean ball of

$\bar{\mathbb{R}}^{n+1}$ with center $x \in \mathbb{R}^{n+1}$ and euclidean radius $t > 0$ is denoted by $B^{n+1}(x,t)$. The corresponding closed ball and sphere are $\bar{B}^{n+1}(x,t)$ and $S^n(x,t)$. Given $k \in \{1,2,\dots,n\}$, we define the open euclidean ball of $\bar{\mathbb{R}}^k$ with center $x \in \mathbb{R}^k$ and euclidean radius $t > 0$ by $B^k(x,t) = B^{n+1}(x,t) \cap \mathbb{R}^k$. The symbols $\bar{B}^k(x,t)$ and $S^{k-1}(x,t)$ have similar meanings. Moreover, we write $\mathbb{B}^k = B^k(0,1)$ and $\mathbb{S}^{k-1} = S^{k-1}(0,1)$ for $k \in \{1,2,\dots,n+1\}$.

We use the standard definition and say that a *Möbius transformation* of $\bar{\mathbb{R}}^{n+1}$ is a finite combination of geometric inversions in $n$-dimensional spheres of $\bar{\mathbb{R}}^{n+1}$. We denote the set of all Möbius transformations of $\bar{\mathbb{R}}^{n+1}$ by Möb$(n+1)$ and assume that any Möbius transformation considered is in Möb$(n+1)$ unless stated otherwise.

Möb$(n+1)$ is a group with respect to the combination of mappings. Since $\bar{\mathbb{R}}^{n+1}$ is compact, we can use the supremum norm of the chordal metric $q$ to define a natural metric and hence a topology for Möb$(n+1)$. Möb$(n+1)$ is, in fact, a topological group ([Beardon1983] Theorem 3.7.1, [Ratcliffe2006] Theorem 5.2.7). It is now possible to speak about discrete subgroups of Möb$(n+1)$.

We take $q$ as the standard metric when considering Möbius transformations. So if $(g_i)_i$ is a sequence in Möb$(n+1)$ and $g \in$ Möb$(n+1)$, then we write $g_i \to g$ *uniformly* to denote that $(g_i)_i$ converges uniformly to $g$ in the metric $q$. Since Möb$(n+1)$ is a topological group, it follows that if $(g_i)_i$ and $(f_i)_i$ are sequences in Möb$(n+1)$ and $g, f \in$ Möb$(n+1)$ are such that $g_i \to g$ uniformly and $f_i \to f$ uniformly, then $g_i^{-1} \to g^{-1}$ uniformly and $g_i \circ f_i \to g \circ f$ uniformly.

We will make some use of the *convergence property* of Möbius transformations stated in the following Theorem 2.2. The convergence property was introduced for groups of quasiconformal mappings of $\bar{\mathbb{R}}^{n+1}$ in [GM1987]. The paper [Tukia1994d] contains an argument that can be used as a proof for Theorem 2.2 in the context of Möbius transformations.

**Theorem 2.2.** *Let* $(g_i)_i$ *be a sequence in* Möb$(n+1)$. *Then* $(g_i)_i$ *has a subsequence* $(g_{i_k})_k$ *such that either* $g_{i_k} \to g$ *uniformly for some mapping* $g \in$ Möb$(n+1)$ *or there are points* $a, b \in \bar{\mathbb{R}}^{n+1}$ *such that* $g_{i_k} \to a$ *uniformly in compact sets of* $\bar{\mathbb{R}}^{n+1} \setminus \{b\}$.

*Proof.* See [Tukia1994d] pages 453-455. □

Let us introduce the models of the $(n+1)$-dimensional hyperbolic space used in this work. We use two common models: the unit ball

$$\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| < 1\} \tag{2.3}$$

and the upper half-space

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}. \tag{2.4}$$

We write $\partial\mathbb{B}^{n+1} = \mathbb{S}^n$ and $\partial\mathbb{H}^{n+1} = \bar{\mathbb{R}}^n$. The symbol $d$ denotes the hyperbolic metric for both of these spaces. We define $d$ using the elements of length

$$\frac{2|dx|}{1-|x|^2} \quad \text{and} \quad \frac{|dx|}{x_{n+1}} \tag{2.5}$$

for $\mathbb{B}^{n+1}$ and $\mathbb{H}^{n+1}$, respectively. When we wish to talk about the $(n+1)$-dimensional hyperbolic space without specifying the model, we will denote the space by the symbol $\mathbb{X}^{n+1}$. We employ the convention that $d(x, y) = \infty$ if either $x \in \mathbb{X}^{n+1}$ and $y \in \partial\mathbb{X}^{n+1}$ or $x, y \in \partial\mathbb{X}^{n+1}$ and $x \neq y$. A point in $\mathbb{H}^{n+1}$ is often written in the form $(x, t)$ where $x \in \mathbb{R}^n$ and $t > 0$.

We define next Kleinian groups as discrete groups of hyperbolic isometries of $\mathbb{X}^{n+1}$. We denote by $\text{Möb}(\mathbb{X}^{n+1})$ the set of all Möbius transformations of $\bar{\mathbb{R}}^{n+1}$ mapping $\mathbb{X}^{n+1}$ onto itself. $\text{Möb}(\mathbb{X}^{n+1})$ is a subgroup of $\text{Möb}(n+1)$, and it is well known that $\text{Möb}(\mathbb{X}^{n+1})$ is the set of hyperbolic isometries of $\mathbb{X}^{n+1}$ ([Apanasov2000] Section 1.3, [Maskit1988] Theorem IV.B.7, [Ratcliffe2006] Theorems 5.2.10 and 5.2.11). (To be exact, the restriction of $g \in \text{Möb}(\mathbb{X}^{n+1})$ to $\mathbb{X}^{n+1}$ is a hyperbolic isometry of $\mathbb{X}^{n+1}$, but it is standard practice to say that $g$ itself is a hyperbolic isometry of $\mathbb{X}^{n+1}$.) We define a *Kleinian group* acting on $\mathbb{X}^{n+1}$ to be a subgroup of $\text{Möb}(\mathbb{X}^{n+1})$ which is discrete in the natural topology of Möbius transformations of $\bar{\mathbb{R}}^{n+1}$. Note that we do not make the often made assumption that Kleinian groups contain only orientation preserving elements.

Kleinian groups acting on $\mathbb{B}^{n+1}$ correspond naturally to Kleinian groups acting on $\mathbb{H}^{n+1}$. The correspondence is realised by any conjugation mapping of the form $G \mapsto fGf^{-1}$, where $G$ is a Kleinian group acting on $\mathbb{B}^{n+1}$, $f$ is a fixed Möbius transformation of $\bar{\mathbb{R}}^{n+1}$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$, and $fGf^{-1} = \{f \circ g \circ f^{-1} : g \in G\}$ is a Kleinian group acting on $\mathbb{H}^{n+1}$. It is common practice in the theory of Kleinian groups to try and formulate theorems so that they remain essentially unchanged under conjugations by Möbius transformations. It follows that, when proving a theorem, it is often convenient to normalize the situation by a conjugation in order to simplify the technical details of the proof. We assume the reader to be well-acquainted with this practice, so we will use it often without detailed explanation. We remark that every conjugating mapping used in this work will be a Möbius transformation.

We classify the elements of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$ in the standard way (see, for example, [Apanasov2000] Section 1.4, [Maskit1988] Section IV.C or [Ratcliffe2006] Section 4.7). Let $g \in G$. If $g$ has a fixed point in $\mathbb{X}^{n+1}$, then $g$ is *elliptic*. If $g$ has exactly one fixed point and this point is in $\partial\mathbb{X}^{n+1}$, then $g$ is *parabolic*. If $g$ has exactly two fixed points and these points are in $\partial\mathbb{X}^{n+1}$, then $g$ is *loxodromic*.

This classification has the following alternative characterization. If $g$ is elliptic, then $g$ can be conjugated into an orthogonal mapping of $\bar{\mathbb{R}}^{n+1}$. If $g$ is parabolic, then $g$ can be conjugated into a mapping of the form $x \mapsto \alpha(x) + x_0$, where $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $\alpha$ is an orthogonal mapping of $\bar{\mathbb{R}}^n$ extended to $\bar{\mathbb{R}}^{n+1}$ such that $\alpha(x_0) = x_0$. If $g$ is loxodromic, then $g$ can be conjugated into a mapping of the form $x \mapsto \lambda\alpha(x)$, where $\lambda > 1$ and $\alpha$ is an orthogonal mapping of $\bar{\mathbb{R}}^n$ extended to $\bar{\mathbb{R}}^{n+1}$.

The above characterization implies the following facts. If $g$ is parabolic and $x$ is the fixed point of $g$, then $g^i \to x$ and $g^{-i} \to x$ uniformly in compact sets contained in $\bar{\mathbb{R}}^{n+1} \setminus \{x\}$. If $g$ is loxodromic with fixed points $x$ and $y$, then one of the fixed points, say $x$, is the *attracting fixed point* of $g$ and the other the *repelling fixed point* of $g$. This means that

$g^i \to x$ uniformly in compact sets contained in $\bar{\mathbb{R}}^{n+1} \setminus \{y\}$. Observe that $y$ is the attracting fixed point of $g^{-1}$ in this situation.

We continue to consider a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$. If $x \in \bar{\mathbb{R}}^{n+1}$, we define that

$$G_x = \{g \in G : g(x) = x\}. \tag{2.6}$$

The set $G_x$ is a subgroup of $G$ and $G_x$ is known as the *stabilizer* of $x$ with respect to $G$. If $x \in \bar{\mathbb{R}}^{n+1}$ is such that $G_x$ contains a parabolic element of $G$, $x$ is said to be a *parabolic fixed point* of $G$. The basic properties of the stabilizer of a parabolic fixed point of $G$ are presented by the following theorem in a normalized situation.

**Theorem 2.7.** *Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\infty$ be a parabolic fixed point of $G$. Then the following claims are true.*

*(i) There is a $G_\infty$-invariant $k$-plane $V \subset \mathbb{R}^n$ for some $k \in \{1, 2, \dots, n\}$ such that $V = G_\infty C = \{g(x) : g \in G_\infty, x \in C\}$ for some compact $C \subset \mathbb{R}^n$.*

*(ii) If $V' \subset \mathbb{R}^n$ is a $G_\infty$-invariant $k'$-plane for some $k' \in \{1, 2, \dots, n\}$ and $V$ and $k$ are as in (i), then $k' \geq k$, $V$ and $V'$ are parallel, and $V' = G_\infty C'$ for some compact $C' \subset \mathbb{R}^n$ if and only if $k' = k$. The number $k$ associated to $G_\infty$ by (i) is thus unique but the $k$-plane $V$ need not be.*

*(iii) If $V = \mathbb{R}^k$ in (i), then $g(x, y, t) = (h(x), \alpha(y), t)$ for all $g \in G_\infty$, $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k}$ and $t \in \mathbb{R}$, where $h$ is a non-loxodromic Möbius transformation of $\bar{\mathbb{R}}^k$ fixing $\infty$ and $\alpha$ is an orthogonal mapping of $\bar{\mathbb{R}}^{n-k}$. In particular, every $g \in G_\infty$ is a euclidean isometry.*

*(iv) There is a free commutative subgroup $G^*_\infty$ of $G_\infty$ of finite index isomorphic to $\mathbb{Z}^k$ such that $G^*_\infty$ acts as a group of translations on $V$, where $k$ and $V$ are as in (i). More specifically, the following is true. Assume that $V = \mathbb{R}^k$ and that $g_1, g_2, \dots, g_k \in G^*_\infty$ are generators for $G^*_\infty$ such that the restriction of $g_j$ to $\mathbb{R}^k$ is the translation $x \mapsto x + x_j$, where $j \in \{1, 2, \dots, k\}$ and $x_j \in \mathbb{R}^k$. Then $x_1, x_2, \dots, x_k$ span $\mathbb{R}^k$ and for every $k$-tuple $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of integers there is exactly one $g \in G^*_\infty$ such that the restriction of $g$ to $\mathbb{R}^k$ is the translation $x \mapsto x + \sum_{j=1}^k \lambda_j x_j$.*

*Proof.* The claims of the theorem are generally well known in the field of Kleinian groups. Explicit proofs for most of the claims are rather complicated and it is impossible to fit them into this work. See Section 2 of [Tukia1985a] for a brief but not self-contained discussion. Expositions that are more detailed but also more scattered in nature can be found in [Apanasov2000] Chapters 2, 3 and 4, [Bowditch1993] Sections 2 and 3, and [Ratcliffe2006] Chapter 5. □

Theorem 2.7 motivates the definition of the rank of a parabolic fixed point of a Kleinian group. Observe that the existence of the unique number $k \in \{1, 2, \dots, n\}$ described in the claims (i) and (ii) of Theorem 2.7 is invariant under conjugations. This means that if $x \in \partial\mathbb{X}^{n+1}$ is a parabolic fixed point of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$, the claims (i) and (ii) of Theorem 2.7 associate a unique number $k \in \{1, 2, \dots, n\}$ to $G_x$. This number is defined to be the *rank* of $x$.

We turn to the dynamics of the action of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$. The discreteness of $G$ in the natural topology of Möb$(n+1)$ is equivalent to the *discontinuity* of the action of $G$ on $\mathbb{X}^{n+1}$ ([Maskit1988] Section IV.E.3, [Ratcliffe2006] Theorem 5.3.5). This discontinuity means that, given $x \in \mathbb{X}^{n+1}$, there is a neighbourhood $U \subset \mathbb{X}^{n+1}$ of $x$ such that $U \cap gU \neq \emptyset$ for only finitely many $g \in G$. It follows that the orbit $Gx$ can accumulate only at $\partial\mathbb{X}^{n+1}$. Since the elements of $G$ are hyperbolic isometries of $\mathbb{X}^{n+1}$, the set of accumulation points of $Gx$ is independent of $x$. This set is known as the *limit set* of $G$ and we denote it by $L(G)$. Note that the discontinuity of the action of $G$ on $\mathbb{X}^{n+1}$ implies that $G$ is countable. Note also that $L(G)$ is closed and $G$-invariant. We remark that the action of $G$ is discontinuous also on $\partial\mathbb{X}^{n+1} \setminus L(G)$ ([Ratcliffe2006] Theorem 12.2.8, [Tukia1994a] Theorem 2L).

It is a well-known but non-trivial fact that $L(G)$ is empty, contains exactly one or two points, or is an uncountable perfect set ([Apanasov2000] Theorem 2.3, [GM1987] Theorem 4.5, [Ratcliffe2006] Theorems 12.2.1 and 12.2.5, [Tukia1994a] Theorem 2S). We employ the normal terminology and say that $G$ is *elementary* if $L(G)$ contains at most two points and *non-elementary* otherwise.

Two kinds of limit points of $G$ are of particular importance for us. These are the conical limit points and the bounded parabolic fixed points of $G$.

A point $x \in \partial\mathbb{X}^{n+1}$ is a *conical limit point* of $G$ if, given any $y \in \mathbb{X}^{n+1}$ and any hyperbolic line $L$ of $\mathbb{X}^{n+1}$ with $x$ as one of its endpoints, there are $g_1, g_2, \ldots \in G$ and $t \geq 0$ such that $g_i(y) \to x$ and $d(g_i(y), L) \leq t$ for every $i \in \{1, 2, \ldots\}$. We let $L_c(G)$ denote the set of conical limit points of $G$. It is trivial that $L_c(G) \subset L(G)$.

On the other hand, any parabolic fixed point $x$ of $G$ is a limit point of $G$, and we say that $x$ is a *bounded parabolic fixed point* of $G$ if there is a compact set $C \subset L(G) \setminus \{x\}$ such that $G_x C = L(G) \setminus \{x\}$, where $G_x$ is the stabilizer of $x$ with respect to $G$.

The notions of the hyperbolic convex hull of $G$ and a horoball of $\mathbb{X}^{n+1}$ are closely connected to the conical limit points and the bounded parabolic fixed points of $G$ in the context of this work.

We denote the *hyperbolic convex hull* of $G$ by $H(G)$. If $L(G)$ contains exactly one point, we set that $H(G) = \emptyset$. If $L(G)$ does not contain exactly one point, we define $H(G)$ to be the smallest closed hyperbolically convex subset of $\mathbb{X}^{n+1}$ whose euclidean closure contains $L(G)$. The definition implies immediately that if $x$ and $y$ are two limit points of $G$ and $L$ is the hyperbolic line of $\mathbb{X}^{n+1}$ with endpoints $x$ and $y$, then $L \subset H(G)$. Note also that $H(G)$ is $G$-invariant.

On the other hand, if $x \in \partial\mathbb{X}^{n+1} \setminus \{\infty\}$, a *horoball* of $\mathbb{X}^{n+1}$ based at $x$ is an open $(n+1)$-dimensional euclidean ball contained in $\mathbb{X}^{n+1}$ and tangential to $\partial\mathbb{X}^{n+1}$ at $x$; the horoballs of $\mathbb{H}^{n+1}$ based at $\infty$ are the open half-spaces of $\mathbb{R}^{n+1}$ contained in $\mathbb{H}^{n+1}$.

The notion that binds together conical limit points, bounded parabolic fixed points, hyperbolic convex hulls and horoballs is that of a geometrically finite Kleinian group. The definition of geometrically finite Kleinian groups is somewhat complicated and we give this definition in Chapter 6 where we need the notion for the first time (see page 89).

The exponent of convergence of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$ is another important fundamental notion in the theory of Kleinian groups. A series of the form

$$P_s(x, y) = \sum_{g \in G} e^{-sd(x,g(y))}, \tag{2.8}$$

$s \geq 0$ and $x, y \in \mathbb{X}^{n+1}$, is called a *Poincaré series* of $G$. The triangle inequality of $d$ and the fact that the elements in $G$ are $d$-isometries imply that the convergence or divergence of $P_s(x, y)$ for a fixed $s \geq 0$ is independent of the points $x$ and $y$. We define the *exponent of convergence* $\delta$ of $G$ by

$$\delta = \inf\{s \geq 0 : P_s(x, y) < \infty \text{ for some } x, y \in \mathbb{X}^{n+1}\}. \tag{2.9}$$

It is always the case that $\delta \leq n$. If $G$ is non-elementary, then $\delta > 0$. See Theorem 1.6.1 and Corollary 3.4.5 of [Nicholls1989] for the proofs of these claims.

The operator norm of the derivative of a Möbius transformation plays a crucial role in the definition of conformal measures of Kleinian groups. Because of this, we take a detailed look at its basic properties.

Let $g \in \text{Möb}(n+1)$. Since $g$ is an orientation preserving or reversing conformal homeomorphism of $\bar{\mathbb{R}}^{n+1}$ onto itself ([Beardon1983] Theorem 3.1.6, [Maskit1988] Section IV.A, [Ratcliffe2006] Theorem 4.1.5), we have that $g'(x) = \tau(x)\alpha(x)$ for all $x \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$, where $\tau(x) \in ]0, \infty[$ and $\alpha(x)$ is an orthogonal transformation of $\bar{\mathbb{R}}^{n+1}$. The quantity $\tau(x)$ is the *operator norm of the derivative* of $g$ at $x$. We write $|g'(x)| = \tau(x)$. The general form of $|g'|$ in $\mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$ and its extension to $\bar{\mathbb{R}}^{n+1}$ depend on whether $g$ fixes $\infty$ or not.

Assume first that $g$ fixes $\infty$. It is well known that $g$ is now a euclidean similarity of $\bar{\mathbb{R}}^{n+1}$ and that, conversely, every euclidean similarity of $\bar{\mathbb{R}}^{n+1}$ is a Möbius transformation of $\bar{\mathbb{R}}^{n+1}$ fixing $\infty$ ([Beardon1983] Theorems 3.1.3 and 3.5.1, [Ratcliffe2006] Theorem 4.3.2). We conclude that $|g'|$ is a finite and positive constant in $\mathbb{R}^{n+1}$ in this case, say $c \in ]0, \infty[$, and it is natural to set that $|g'(\infty)| = c$.

Suppose that $g$ does not fix $\infty$. If $g$ is the inversion in the sphere $S^n(y, v)$, where $y \in \mathbb{R}^{n+1}$ and $v > 0$, then $g(x) = y + v^2(x-y)/|x-y|^2$ for every $x \in \mathbb{R}^{n+1} \setminus \{y\}$, and it is straightforward to calculate that

$$|g'(x)| = \frac{v^2}{|x-y|^2} \tag{2.10}$$

for every $x \in \mathbb{R}^{n+1} \setminus \{y\}$. In any case, there exists a euclidean $n$-sphere $S_g$ of $\bar{\mathbb{R}}^{n+1}$ called the *isometric sphere* of $g$. The following facts pertaining to $S_g$ can be found, for instance, in [Apanasov2000] Proposition 1.7, [Beardon1983] Theorem 3.5.1 (note the comment after the proof of this theorem), [Maskit1988] Section IV.G.3, or [Ratcliffe2006] Theorem 4.3.3. The center of $S_g$ is $g^{-1}(\infty)$. Let us denote the euclidean radius of $S_g$ by $r_g$. The defining property of $S_g$ is that $S_g$ is the unique euclidean $n$-sphere of $\bar{\mathbb{R}}^{n+1}$ mapped by $g$ onto a euclidean $n$-sphere of $\bar{\mathbb{R}}^{n+1}$ of the same euclidean radius. (If $g$ is as in (2.10), then $S_g = S^n(y, v)$.) It is clear that $gS_g = S_{g^{-1}}$. Furthermore, $g$ can be written in the form

$$g = \alpha \circ \sigma, \tag{2.11}$$

where $\alpha$ is a euclidean isometry of $\bar{\mathbb{R}}^{n+1}$ and $\sigma$ is the inversion in $S_g$. We can use (2.10) and the chain rule (2.13) to be discussed presently to conclude that

$$|g'(x)| = |\sigma'(x)| = \frac{r_g^2}{|x - g^{-1}(\infty)|^2} \tag{2.12}$$

for all $x \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$. It is natural to extend $|g'|$ to $\bar{\mathbb{R}}^{n+1}$ by setting $|g'(\infty)| = 0$ and $|g'(g^{-1}(\infty))| = \infty$.

We introduce the convention that $|g'|^0 = 1$ in $\bar{\mathbb{R}}^{n+1}$ for every $g \in \text{Möb}(n+1)$.

The chain rule determines the operator norm of the derivative of a combination of Möbius transformations: if $g, h \in \text{Möb}(n+1)$, then

$$|(g \circ h)'(x)| = |g'(h(x))||h'(x)| \tag{2.13}$$

for every $x \in \mathbb{R}^{n+1} \setminus \{h^{-1}(\infty), (g \circ h)^{-1}(\infty)\}$. The formula (2.13) is valid in other situations as well. In fact, (2.13) is not valid if and only if either $x = \infty$, $g(\infty) \neq \infty$, $h(\infty) \neq \infty$ and $(g \circ h)(\infty) = \infty$, or $x = h^{-1}(\infty)$, $g(\infty) \neq \infty$ and $h(\infty) \neq \infty$.

2.1.2. *Conformal measures.* The purpose of this subsection is to introduce conformal measures of Kleinian groups. It is simple to give a natural definition for conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$, but the situation is somewhat more complicated in the case of Kleinian groups acting on $\mathbb{H}^{n+1}$. We will, therefore, start by considering Kleinian groups acting on $\mathbb{B}^{n+1}$. The general theory of conformal measures of Kleinian groups is discussed, for instance, in [Nicholls1989], [Patterson1987] and [Sullivan1979].

Let $G$ be a Kleinian group acting on $\mathbb{B}^{n+1}$. Let $s \geq 0$. A measure $\mu$ of $\bar{\mathbb{R}}^{n+1}$ is an *s-conformal measure* of $G$ if the following conditions are satisfied. The $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. The measure $\mu$ is non-trivial and finite and supported by $L(G) \subset \mathbb{S}^n = \partial\mathbb{B}^{n+1}$. It is the case that

$$\mu(gA) = \int_A |g'|^s d\mu \tag{2.14}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$.

If $G$ is a Kleinian group acting on $\mathbb{B}^{n+1}$ such that $L(G) = \mathbb{S}^n$, then the restriction to the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$ of the $n$-dimensional Lebesgue measure of $\mathbb{S}^n$ is an $n$-conformal measure of $G$, and this measure is a useful tool in the study of $G$. The basic purpose of the above definition is to generalize this for $G$ such that $L(G) \neq \mathbb{S}^n$.

The definition (2.14) is natural in the sense that the euclidean metric is a natural metric of $\mathbb{S}^n$ and $|g'|$ is the operator norm of the derivative of $g \in \text{Möb}(\mathbb{B}^{n+1})$ with respect to the euclidean metric. It is true, in fact, that one can use other natural metrics of $\mathbb{S}^n$ to define conformal measures for Kleinian groups acting on $\mathbb{B}^{n+1}$. This can be done as follows.

Let $y \in \mathbb{B}^{n+1}$ and let $D_y$ be the hyperbolic visual angle metric of $\mathbb{S}^n$ based at $y$. That is, if $x_1, x_2 \in \mathbb{S}^n$, then $D_y(x_1, x_2)$ is the angle between the hyperbolic rays from $y$ to $x_1$ and $x_2$. Given $f \in \text{Möb}(\mathbb{B}^{n+1})$ and $x_0 \in \mathbb{S}^n$, we define that

$$|\partial_{D_y} f(x_0)| = \lim_{x \in \mathbb{S}^n, x \to x_0} \frac{D_y(f(x), f(x_0))}{D_y(x, x_0)}. \tag{2.15}$$

The function $|\partial_{D_y} f|$ is the operator norm of the derivative of $f$ corresponding to $D_y$.

Let $G$ be a Kleinian group acting on $\mathbb{B}^{n+1}$. Let $s \geq 0$. It is natural to say that a non-trivial and finite measure $\mu$ supported by $L(G)$ is an $s$-conformal measure of $G$ defined with respect to the metric $D_y$, $y \in \mathbb{B}^{n+1}$, if the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$ and

$$\mu(gA) = \int_A |\partial_{D_y} g|^s d\mu \tag{2.16}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$.

If $y_1, y_2 \in \mathbb{B}^{n+1}$, there is a natural way to transform from the metric $D_{y_1}$ to the metric $D_{y_2}$. It follows that if $\mu_{y_1}$ is an $s$-conformal measure of $G$ defined with respect to $D_{y_1}$, there is an $s$-conformal measure $\mu_{y_2}$ of $G$ defined with respect to $D_{y_2}$ that is obtained from $\mu_{y_1}$ using a formula that employs the natural correspondence between $D_{y_1}$ and $D_{y_2}$. (We omit the details of the formulae mentioned in this discussion since we have no explicit need for them. A reader interested in the explicit forms of the formulae can consult, for example, Chapters 3 and 4 of [Nicholls1989].) One can think, therefore, that $\mu_{y_1}$ and $\mu_{y_2}$ are representations of a single object. The connection between the definitions (2.14) and (2.16) is that if $y = 0$ in (2.16), then $|\partial_{D_y} g| = |g'|$, i.e. the two definitions coincide in this case. We conclude that it is natural to let (2.14) be the definition for a conformal measure of a Kleinian group acting on $\mathbb{B}^{n+1}$. We will use this definition for the rest of this work.

It is a remarkable fact that conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$ exist in every non-trivial situation: if $G$ is a Kleinian group acting on $\mathbb{B}^{n+1}$ and $L(G)$ is non-empty, then $G$ has conformal measures and there is an explicit method to construct such measures. More specifically, the construction constructs families of measures satisfying (2.16). This classical construction method was discovered by S. J. Patterson for all $G$ which are *Fuchsian groups*, Kleinian groups acting on $\mathbb{B}^2$ and containing only orientation preserving elements, see [Patterson1976b]. D. Sullivan observed, [Sullivan1979], that the method of Patterson generalizes immediately to the case of an arbitrary Kleinian group $G$ acting on $\mathbb{B}^{n+1}$. A detailed account on Patterson's construction in the general situation can be found, for instance, in Chapter 3 of [Nicholls1989]. Briefer expositions on the construction are given, for example, in [Patterson1987] and [Sullivan1979]. Today the measures constructed by Patterson's method are called *Patterson measures* or *Patterson-Sullivan measures*. We will use the latter term.

A fundamental property of a Patterson-Sullivan measure $\mu$ of a Kleinian group $G$ acting on $\mathbb{B}^{n+1}$ is that $\mu$ is $\delta$-conformal, where $\delta$ is the exponent of convergence of $G$ (see (2.9)). A related important result states that if $G$ is non-elementary and $\nu$ is an $s$-conformal measure of $G$ for some $s \geq 0$, then $s \geq \delta$ ([Nicholls1989] Corollary 4.5.3, [Patterson1987] Section 3 Theorem 2, [Sullivan1979] Corollary 4). Recall that we quoted, following definition (2.9), a result stating that if $G$ is non-elementary, then $\delta > 0$. It is, therefore, correct to assume that $s > 0$ when discussing $s$-conformal measures of non-elementary Kleinian groups.

Patterson-Sullivan measures are canonical examples of conformal measures of Kleinian groups, but other conformal measures are known to exist. The papers [AFT2007] and

[FT2006], for example, consider an alternative method for constructing conformal measures of Kleinian groups.

We will next discuss conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$. Our discussion resembles in part the discussion given in [Tukia1994b], although our discussion is more detailed.

Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Observe that the hyperbolic visual angle metrics of $\bar{\mathbb{R}}^n$ are just as natural as the hyperbolic visual angle metrics of $\mathbb{S}^n$ and that the hyperbolic visual angle metrics of the two models correspond exactly to each other. It would be perfectly natural, therefore, to define conformal measures of $G$ using the definition (2.16) with $y \in \mathbb{H}^{n+1}$. The book [Nicholls1989], for example, uses this definition. When Patterson's construction is applied to Kleinian groups acting on $\mathbb{H}^{n+1}$, the construction gives families of measures satisfying (2.16). (So if we used (2.16) as the definition in the case of Kleinian groups acting on $\mathbb{H}^{n+1}$, these measures would be the Patterson-Sullivan measures.)

The main problem with the definition (2.16) in the case of Kleinian groups acting on $\mathbb{H}^{n+1}$ is that the transformation formulae resulting from the definition are rather unwieldy. Our aim is to define conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$ so that the measures satisfy simpler transformation rules of the form (2.14). We will employ the notion of a conformal image of a measure in this task. This notion is defined as follows.

Let $\mu$ be a measure of $\bar{\mathbb{R}}^{n+1}$ such that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. Suppose that $\mu(\infty) = 0$. Let $s \geq 0$ and $h \in \text{Möb}(n+1)$. We define formally a set function $h_*^s\mu$ by

$$h_*^s\mu(A) = \int_{h^{-1}A} |h'|^s d\mu \tag{2.17}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. We say that $h_*^s\mu$ is a *conformal image* of $\mu$. It is evident that if $h_*^s\mu$ is well-defined, then $h_*^s\mu$ is a measure of $\bar{\mathbb{R}}^{n+1}$ whose $\sigma$-algebra of measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$.

**Lemma 2.18.** *The set function $h_*^s\mu$ defined by (2.17) is well-defined.*

*Proof.* By the convention introduced earlier, $|h'|^s = 1$ in $\bar{\mathbb{R}}^{n+1}$ if $s = 0$, so $h_*^s\mu$ is well-defined in this case. The set function $h_*^s\mu$ is well-defined also if $h(\infty) = \infty$, since then $|h'|^s$ is a finite and positive constant in $\bar{\mathbb{R}}^{n+1}$.

Let us assume that $s > 0$ and that $h(\infty) \neq \infty$. The definition (2.17) is non-problematic if $|h'|^s \in ]0, \infty[$ in $h^{-1}A$, i.e. if $A \subset \mathbb{R}^{n+1} \setminus \{h(\infty)\}$ (recall (2.12)). It is natural to set that

$$h_*^s\mu(\infty) = |h'(h^{-1}(\infty))|^s\mu(h^{-1}(\infty)) = \infty \cdot \mu(h^{-1}(\infty)) = 0 \text{ or } \infty$$

according to whether $\mu(h^{-1}(\infty)) = 0$ or not. On the other hand, since by definition $\mu(\infty) = 0$, we have that $h_*^s\mu(h(\infty)) = |h'(\infty)|^s\mu(\infty) = 0$. We conclude that $h_*^s\mu$ is well-defined in all cases. □

We continue by proving the following two lemmas that consider essential properties of conformal images of measures.

**Lemma 2.19.** *Let $\mu$ be a measure of $\bar{\mathbb{R}}^{n+1}$ such that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$ and that $\mu(\infty) = 0$. Let $h \in$ Möb$(n+1)$ and $s \geq 0$ and write $\hat{\mu} = h^s_* \mu$. Let $f \in$ Möb$(n+1)$. Then*

$$\int_A |f'|^s d\hat{\mu} = \int_{h^{-1}A} (|f'|^s \circ h)|h'|^s d\mu \tag{2.20}$$

*for all Borel sets $A$ of $\mathbb{R}^{n+1}$. If $\mu(h^{-1}(\infty)) = 0$, $s = 0$ or $f(\infty) = \infty$, the formula (2.20) is valid for all Borel sets $A$ of $\bar{\mathbb{R}}^{n+1}$.*

*Proof.* If $s = 0$ or $f(\infty) = \infty$, then $|f'|^s$ is a finite and positive constant in $\bar{\mathbb{R}}^{n+1}$ (recall that $|f'|^0 = 1$ in $\bar{\mathbb{R}}^{n+1}$ by convention), so (2.20) is clearly true for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ in this case. We assume that $s > 0$ and $f(\infty) \neq \infty$.

Suppose that $A \subset \mathbb{R}^{n+1} \setminus \{f^{-1}(\infty)\}$ is a non-empty Borel set. Since now $|f'|^s \in ]0,\infty[$ in $A$ (recall (2.12)), the left hand side of (2.20) is well-defined. It is true that $(|f'|^s \circ h)|h'|^s \in ]0,\infty[$ in $h^{-1}A \cap \mathbb{R}^n$ and that

$$\int_{\{\infty\}} (|f'|^s \circ h)|h'|^s d\mu = 0, \tag{2.21}$$

since $\mu(\infty) = 0$, so the right hand side of (2.20) is well-defined as well. Since $|f'|^s \in ]0,\infty[$ in $A$ and the integrals on both sides of (2.20) are countably additive, we can assume that $M^{-1} \leq |f'|^s \leq M$ in $A$, where $M > 0$ is a constant. It is evident that if $\hat{\mu}(A) = \infty$, then both sides of (2.20) are equal to $\infty$, so suppose that $\hat{\mu}(A) < \infty$. Let $\varepsilon > 0$. Because of (2.12), we can divide $A$ into non-empty and pairwise disjoint Borel sets $A_k$, $k \in \{1, 2, \ldots, k_\varepsilon\}$, such that $\sup_{A_k} |f'|^s - \inf_{A_k} |f'|^s \leq \varepsilon$ for every $k \in \{1, 2, \ldots, k_\varepsilon\}$. Now

$$\begin{aligned}
\int_A |f'|^s d\hat{\mu} &\leq \sum_{k=1}^{k_\varepsilon} \sup_{A_k} |f'|^s \hat{\mu}(A_k) = \sum_{k=1}^{k_\varepsilon} \int_{h^{-1}A_k} \left( \sup_{h^{-1}A_k} (|f'|^s \circ h) \right) |h'|^s d\mu \\
&\leq \sum_{k=1}^{k_\varepsilon} \int_{h^{-1}A_k} ((|f'|^s \circ h) + \varepsilon)|h'|^s d\mu = \int_{h^{-1}A} (|f'|^s \circ h)|h'|^s d\mu + \varepsilon\hat{\mu}(A).
\end{aligned}$$

We obtain similarly that

$$\int_A |f'|^s d\hat{\mu} \geq \int_{h^{-1}A} (|f'|^s \circ h)|h'|^s d\mu - \varepsilon\hat{\mu}(A).$$

Since $\varepsilon > 0$ was arbitrary and $\hat{\mu}(A) < \infty$, we see that (2.20) is true in the considered case.

Suppose next that $A = \{f^{-1}(\infty)\}$. If $f^{-1}(\infty) = h(\infty)$, then the right hand side of (2.20) equals 0 because of (2.21), and the left hand side of (2.20) equals 0 since $\mu(\infty) = 0$ implies that $\hat{\mu}(h(\infty)) = |h'(\infty)|^s \mu(\infty) = 0$. Suppose that $f^{-1}(\infty) \neq h(\infty)$. Since $(f \circ h)^{-1}(\infty) \neq \infty$, we have that $|h'((f \circ h)^{-1}(\infty))|^s > 0$ (recall (2.12)). Now

$$\begin{aligned}
\int_A |f'|^s d\hat{\mu} &= |f'(f^{-1}(\infty))|^s \hat{\mu}(f^{-1}(\infty)) \\
&= \infty \cdot |h'((f \circ h)^{-1}(\infty))|^s \mu((f \circ h)^{-1}(\infty)) \\
&= \int_{h^{-1}A} (|f'|^s \circ h)|h'|^s d\mu,
\end{aligned}$$

so both sides of (2.20) are equal to 0 or $\infty$ according to whether $\mu((f \circ h)^{-1}(\infty)) = 0$ or not. We have established that (2.20) is true for all Borel sets $A$ of $\mathbb{R}^{n+1}$.

Suppose finally that $\mu(h^{-1}(\infty)) = 0$. Now $\hat{\mu}(\infty) = 0$. It follows that (2.20) is true with both sides equal to 0 in case $A = \{\infty\}$. We conclude that if $\mu(h^{-1}(\infty)) = 0$, then (2.20) is valid for all Borel sets $A$ of $\bar{\mathbb{R}}^{n+1}$. We have proved every claim of the lemma. □

**Lemma 2.22.** *Let $\mu$ be a measure of $\bar{\mathbb{R}}^{n+1}$ such that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$ and that $\mu(\infty) = 0$. Let $h \in$ Möb$(n+1)$ and $s \geq 0$ and write $\hat{\mu} = h^s_*\mu$. Let $f \in$ Möb$(n+1)$ be such that*

$$\mu(fA) = \int_A |f'|^s d\mu \tag{2.23}$$

*for every Borel set A of $\bar{\mathbb{R}}^{n+1}$. Then*

$$\hat{\mu}((h \circ f \circ h^{-1})A) = \int_A |(h \circ f \circ h^{-1})'|^s d\hat{\mu} \tag{2.24}$$

*for every Borel set A of $\mathbb{R}^{n+1}$. If $\mu(h^{-1}(\infty)) = 0$, $s = 0$ or $(h \circ f \circ h^{-1})(\infty) = \infty$, then (2.24) is valid for every Borel set A of $\bar{\mathbb{R}}^{n+1}$.*

*Proof.* Observe that (2.23) implies that $f^s_*\mu = \mu$. Note that

$$0 = \mu(\infty) = \mu(f(f^{-1}(\infty))) = |f'(f^{-1}(\infty))|^s \mu(f^{-1}(\infty))$$

and that $|f'(f^{-1}(\infty))|^s \neq 0$, so $\mu(f^{-1}(\infty)) = 0$. Let $A$ be a Borel set of $\mathbb{R}^{n+1}$. Recall the pathological cases connected to the chain rule (2.13). To prove (2.24), we perform the following calculation (we use the fact that $f^s_*\mu = \mu$ in the first step; the second and sixth step follow from Lemma 2.19):

$$\begin{aligned}
\hat{\mu}((h \circ f \circ h^{-1})A) &= \int_{fh^{-1}A} |h'|^s df^s_*\mu = \int_{h^{-1}A} (|h'|^s \circ f)|f'|^s d\mu \\
&= \int_{h^{-1}A} |(h \circ f)'|^s d\mu = \int_{h^{-1}A} |((h \circ f \circ h^{-1}) \circ h)'|^s d\mu \\
&= \int_{h^{-1}A} (|(h \circ f \circ h^{-1})'|^s \circ h)|h'|^s d\mu = \int_A |(h \circ f \circ h^{-1})'|^s d\hat{\mu}.
\end{aligned}$$

Recall the convention that $|g'|^0 = 1$ in $\bar{\mathbb{R}}^{n+1}$ for any $g \in$ Möb$(n+1)$. It is easy to see that the above calculation is valid for any Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ if $\mu(h^{-1}(\infty)) = 0$, $s = 0$ or $(h \circ f \circ h^{-1})(\infty) = \infty$. □

To obtain a preliminary motivation for our definition of conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$, we show that there is a natural connection between conformal images of measures and conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$. The connection is that if a Kleinian group acting on $\mathbb{B}^{n+1}$ is conjugated to another such group, then the conformal measures of the resulting group are the conformal images of the conformal measures of the original group with respect to the conjugating mapping.

**Theorem 2.25.** *Let $G$ be a Kleinian group acting on $\mathbb{B}^{n+1}$. Let $s \geq 0$. Let $h \in$ Möb$(\mathbb{B}^{n+1})$. Then $\nu$ is an $s$-conformal measure of $hGh^{-1}$ if and only if $\nu = h^s_*\mu$ for some $s$-conformal measure $\mu$ of $G$.*

*Proof.* Let $\mu$ be an $s$-conformal measure of $G$. Recall that either $|h'|$ is a positive and finite constant in $\bar{\mathbb{R}}^{n+1}$ or $|h'|$ satisfies (2.12). Since $L(G)$ and $L(hGh^{-1}) = hL(G)$ are subsets of $\mathbb{S}^n$, it is clear that $h^s_*\mu$ is a non-trivial and finite measure supported by $L(hGh^{-1})$ whose $\sigma$-algebra of measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. Since $\mu(h^{-1}(\infty)) = 0$, Lemma 2.22 implies that $h^s_*\mu$ satisfies a transformation formula of the form (2.14) with respect to $hGh^{-1}$. We have shown that $h^s_*\mu$ is an $s$-conformal measure of $hGh^{-1}$.

Let $\nu$ be an $s$-conformal measure of $hGh^{-1}$. The above argument shows that $(h^{-1})^s_*\nu$ is an $s$-conformal measure of $G$. Let $A$ be a Borel set of $\bar{\mathbb{R}}^{n+1}$. Recall the pathological cases connected to the chain rule (2.13). Since $\nu(\infty) = 0 = \nu(h(\infty))$, we can calculate that (we use Lemma 2.19 in the second step)

$$\begin{aligned} h^s_*((h^{-1})^s_*\nu)(A) &= \int_{h^{-1}A} |h'|^s d(h^{-1})^s_*\nu = \int_A (|h'|^s \circ h^{-1})|(h^{-1})'|^s d\nu \\ &= \int_A |(h \circ h^{-1})'|^s d\nu = \nu(A). \end{aligned}$$

So $\nu = h^s_*((h^{-1})^s_*\nu)$, which completes the proof. □

Theorem 2.25 motivates us to define conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$ as conformal images of conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$.

Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Let $s \geq 0$. A set function $\mu$ is an *$s$-conformal measure* of $G$ if there is $h \in$ Möb$(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and an $s$-conformal measure $\nu$ of the Kleinian group $h^{-1}Gh$ acting on $\mathbb{B}^{n+1}$ such that

$$\mu = h^s_*\nu. \tag{2.26}$$

To discuss this definition further, we prove the following two theorems.

**Theorem 2.27.** *Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Let $s \geq 0$. Let $\mu$ be an $s$-conformal measure of $G$ as defined by (2.26). Then $\mu$ is a non-trivial measure of $\bar{\mathbb{R}}^{n+1}$ supported by $L(G)$, the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$, and the $\mu$-measure of every bounded Borel set of $\mathbb{R}^{n+1}$ is finite. Furthermore, if $s > 0$, then the following claims are true. It is the case that either $\mu(\infty) = 0$ or $\mu(\infty) = \infty$. If $\mu(\infty) = 0$, then*

$$\mu(gA) = \int_A |g'|^s d\mu \tag{2.28}$$

*for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$. If $\mu(\infty) = \infty$, then (2.28) is valid for every Borel set $A$ of $\mathbb{R}^{n+1}$ and every $g \in G$. On the other hand, if $s = 0$, then $\mu(\infty) = \nu(h^{-1}(\infty))$ and (2.28) is valid for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$.*

*Proof.* The definitions (2.14) and (2.17) and the formula (2.12) imply immediately that $\mu$ is a non-trivial measure of $\bar{\mathbb{R}}^{n+1}$ supported by $L(G)$, that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$, and that the $\mu$-measure of every bounded Borel set of $\mathbb{R}^{n+1}$ is finite. Suppose that $s > 0$. It is the case that

$$\mu(\infty) = |h'(h^{-1}(\infty))|^s \nu(h^{-1}(\infty)) = \infty \cdot \nu(h^{-1}(\infty)) = 0 \text{ or } \infty$$

according to whether $\nu(h^{-1}(\infty)) = 0$ or not. Note that if $g \in G$, then

$$\nu((h^{-1} \circ g \circ h)A) = \int_A |(h^{-1} \circ g \circ h)'|^s d\nu$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. The remaining claims follow easily from Lemma 2.22 and the convention that $|f'|^0 = 1$ in $\bar{\mathbb{R}}^{n+1}$ for every $f \in \text{Möb}(n+1)$. $\square$

**Theorem 2.29.** *Let $G$ be a Kleinian group acting on $\mathbb{H}^{n+1}$. Suppose that $L(G) \neq \emptyset$. Suppose that if $\infty \in L(G)$, then there is $f \in G$ such that $f(\infty) \neq \infty$. Let $\mu$ be a measure of $\bar{\mathbb{R}}^{n+1}$ which satisfies the following conditions. The $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. The measure $\mu$ is supported by $L(G)$. The measure $\mu$ is non-trivial and the $\mu$-measure of every bounded Borel set of $\mathbb{R}^{n+1}$ is finite. It is the case that $\mu(\infty) = 0$. There is $s \geq 0$ such that*

$$\mu(gA) = \int_A |g'|^s d\mu \tag{2.30}$$

*for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$. Let $h \in$ Möb$(n+1)$ map $\mathbb{H}^{n+1}$ onto $\mathbb{B}^{n+1}$. Write $\hat{\mu} = h^s_* \mu$. It is true in this situation that $\hat{\mu}$ is an $s$-conformal measure of $hGh^{-1}$ and that $\mu = (h^{-1})^s_* \hat{\mu}$, so $\mu$ is an $s$-conformal measure of $G$.*

*Proof.* The measure $\hat{\mu}$ is well-defined since $\mu(\infty) = 0$. It is clear that the $\sigma$-algebra of $\hat{\mu}$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$ and that $\hat{\mu}$ is supported by $L(hGh^{-1})$. Since $\mu$ is non-trivial and $\mu(\infty) = 0$, there is $r > 0$ such that $\mu(E) > 0$ for $E = L(G) \cap B^{n+1}(0, r)$. Now (recall (2.12))

$$\hat{\mu}(hE) = \int_E |h'|^s d\mu \geq \inf_E |h'|^s \mu(E) > 0.$$

We conclude that $\hat{\mu}$ is non-trivial. We show next that $\hat{\mu}$ is finite. Let $U$ be an open neighbourhood of $\infty$. It is true that

$$\hat{\mu}(\bar{\mathbb{R}}^{n+1}) = \hat{\mu}(hU) + \hat{\mu}(h(\bar{\mathbb{R}}^{n+1} \setminus U)).$$

Since $\mu$ is supported by $L(G) \subset \bar{\mathbb{R}}^n$, $h^{-1}(\infty) \notin \bar{\mathbb{R}}^n$, and the $\mu$-measure of every bounded Borel set of $\mathbb{R}^{n+1}$ is finite, it is the case that (we use (2.12) again)

$$\hat{\mu}(h(\bar{\mathbb{R}}^{n+1} \setminus U)) = \int_{\bar{\mathbb{R}}^n \setminus U} |h'|^s d\mu \leq \sup_{\bar{\mathbb{R}}^n \setminus U} |h'|^s \mu(\bar{\mathbb{R}}^n \setminus U) < \infty.$$

If $\infty \notin L(G)$, we assume that $U$ was chosen so that $L(G) \subset \bar{\mathbb{R}}^{n+1} \setminus U$. We conclude that if $\infty \notin L(G)$, then $\hat{\mu}$ is finite. Let us assume for the moment that $\infty \in L(G)$. Recall that now there is $f \in G$ such that $f(\infty) \neq \infty$. Let us assume that $U$ was chosen so that $fU$ is bounded in $\mathbb{R}^{n+1}$, so $\mu(fU) < \infty$. Recall the pathological cases connected to the chain rule (2.13). Note that $(h \circ f^{-1})(\infty) \neq \infty$ and that $(h \circ f^{-1})^{-1}(\infty) \notin \bar{\mathbb{R}}^n$, so $|(h \circ f^{-1})'|$ is bounded in $\bar{\mathbb{R}}^n$ by (2.12). We can now calculate that (the first step uses the fact that $\mu = (f^{-1})^s_* \mu$ by

(2.30); the observation that $\mu(f(\infty)) = |f'(\infty)|^s\mu(\infty) = 0$ shows that Lemma 2.19 implies the second step)

$$\begin{aligned}\hat{\mu}(hU) &= \int_U |h'|^s d(f^{-1})^s_*\mu = \int_{fU} (|h'|^s \circ f^{-1})|(f^{-1})'|^s d\mu = \int_{fU} |(h \circ f^{-1})'|^s d\mu \\ &= \int_{fU\cap\bar{\mathbb{R}}^n} |(h \circ f^{-1})'|^s d\mu \le \sup_{fU\cap\bar{\mathbb{R}}^n} |(h \circ f^{-1})'|^s \mu(fU \cap \bar{\mathbb{R}}^n) < \infty.\end{aligned}$$

We conclude that $\hat{\mu}$ is finite regardless of whether $L(G)$ contains $\infty$ or not. Lemma 2.22 implies that $\hat{\mu}$ satisfies a formula of the form (2.14) with respect to $hGh^{-1}$. We have shown that $\hat{\mu}$ is an $s$-conformal measure of $hGh^{-1}$.

Since $\hat{\mu}(\infty) = 0$, the measure $(h^{-1})^s_*\hat{\mu}$ is well-defined. The argument to show that $\mu = (h^{-1})^s_*\hat{\mu}$ is exactly the same we used to show that $\nu = h^s_*((h^{-1})^s_*\nu)$ in the last paragraph of the proof of Theorem 2.25. □

Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $s > 0$. We say that a measure $\mu$ of $\bar{\mathbb{R}}^{n+1}$ satisfies Condition (C) with respect to $G$ if $\mu$ satisfies the conditions listed in Theorem 2.27 without the explicit assumption that $\mu$ is an $s$-conformal measure of $G$. That is, if $\mu$ satisfies Condition (C) with respect to $G$, then $\mu$ is a non-trivial measure supported by $L(G)$ with the Borel sets of $\bar{\mathbb{R}}^{n+1}$ as measurable sets, $\mu(A) < \infty$ for every bounded Borel set $A$ of $\mathbb{R}^{n+1}$, $\mu(\infty) = 0$ or $\infty$, and $\mu$ satisfies (2.28).

Suppose that $\mu$ is a measure of $\bar{\mathbb{R}}^{n+1}$ such that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. Suppose that $\mu(\infty) = 0$. Theorems 2.27 and 2.29 show that $\mu$ is an $s$-conformal measure of $G$ if and only if $\mu$ satisfies Condition (C) with respect to $G$. (It is well known that if $\Gamma$ is a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$, then no point in $\bar{\mathbb{R}}^{n+1}$ is fixed by every element of $\Gamma$, see, for example, Theorem 2T of [Tukia1994a].) Moreover, if $\mu$ satisfies Condition (C) with respect to $G$ and $h \in$ Möb$(n+1)$ maps $\mathbb{H}^{n+1}$ onto $\mathbb{B}^{n+1}$, then $h^s_*\mu$ is an $s$-conformal measure of the Kleinian group $hGh^{-1}$ acting on $\mathbb{B}^{n+1}$.

In fact, the above is true also if $\mu(\infty) = \infty$, although the arguments we have given so far prove only the case $\mu(\infty) = 0$. Furthermore, the current definition (2.17) of a conformal image of a measure does not even cover the case $\mu(\infty) = \infty$: if $\mu(\infty) = \infty$ and $h \in$ Möb$(n+1)$ maps $\mathbb{H}^{n+1}$ onto $\mathbb{B}^{n+1}$, then (recall (2.12))

$$h^s_*\mu(h(\infty)) = |h'(\infty)|^s\mu(\infty) = 0 \cdot \infty,$$

which is problematic since $h^s_*\mu(h(\infty))$ ought to be a finite atom in the present situation. To solve the problem, we redefine the set function $f^s_*\mu$, $f \in$ Möb$(n+1)$, as follows if $\mu(\infty) = \infty$. If $A \subset \bar{\mathbb{R}}^{n+1} \setminus \{f(\infty)\}$ is a Borel set, then $f^s_*\mu(A)$ is defined as in (2.17), and $f^s_*\mu(f(\infty))$ is defined by

$$f^s_*\mu(f(\infty)) = |(f \circ g^{-1})'(g(\infty))|^s\mu(g(\infty)),$$

where $g$ is any element of $G$ not fixing $\infty$. (It is not difficult to see that this definition is independent of the particular $g$.)

With the new definition for conformal images of measures, Theorem 2.29 can be extended to the case $\mu(\infty) = \infty$ (note that (2.30) is assumed to hold only for all Borel sets $A$

of $\mathbb{R}^{n+1}$ if $\mu(\infty) = \infty$), and thus the definition (2.26) and Condition (C) are actually equivalent for non-elementary groups. This means that, when defining conformal measures of non-elementary Kleinian groups acting on $\mathbb{H}^{n+1}$, we could take Condition (C) as the defining condition and prove the definition (2.26) as a theorem. This method would have the appropriate property that conformal measures of non-elementary Kleinian groups acting on $\mathbb{H}^{n+1}$ would be defined without reference to Kleinian groups acting on $\mathbb{B}^{n+1}$ or their conformal measures.

We will not, however, prove the extension of Theorem 2.29 to the case $\mu(\infty) = \infty$. Our reason is basically practical: the proof of the extension is rather tedious and the conclusion rather irrelevant from the point of view of our main results, since Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups are atomless (this will be proved in Chapter 6 in Theorem 6.21). On the other hand, it is vital for our later results (results of Chapter 3 and their applications) that a conformal measure of a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$ can be expressed as a conformal image of a conformal measure of a Kleinian group acting on $\mathbb{B}^{n+1}$. Therefore, we simplify the situation and take (2.26) as the definition and prove Theorem 2.29 only in the case $\mu(\infty) = 0$, which is sufficient for our later needs. Since we define conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$ using conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$, it is perfectly natural to think that the latter are the primary objects of study of this work and that the former are auxiliary objects used in this study.

We end our introductory discussion on conformal measures by giving the exact definition of Patterson-Sullivan measures. As mentioned earlier, the Patterson-Sullivan measures of Kleinian groups acting on $\mathbb{B}^{n+1}$ are obtained from Patterson's construction. If $G$ is a Kleinian group acting on $\mathbb{H}^{n+1}$ with the exponent of convergence $\delta$, then $\mu$ is a Patterson-Sullivan measure of $G$ if $\mu$ satisfies (2.26) with $s = \delta$ so that $\nu$ is a Patterson-Sullivan measure of $h^{-1}Gh$.

2.2. **Auxiliary results.** This section contains auxiliary results of technical nature pertaining to the background material discussed in the previous section. These results will be needed predominantly in Chapters 3 and 4. We will not give a detailed account on the motivations of the results, since such an account would be unnecessarily long. However, we will provide some motivation for the more complex results. To compensate for the lack of detailed motivations, we will formulate the results so that they will be easily applicable in the situations encountered in the later chapters.

2.2.1. *Results on hyperbolic spaces and Kleinian groups.* We start off with seven results concerning hyperbolic spaces and Kleinian groups. The first three results are elementary observations regarding horoballs and hyperbolic convex hulls of non-elementary Kleinian groups (see page 18 for the definitions). The fourth result is a somewhat more technical result that we will need in the proofs of Theorems 4.10 and 4.46. The fifth result considers well-known basic properties of a parabolic fixed point of a non-elementary Kleinian group (the facts stated by the result are often mentioned in the literature without proof). The sixth result is a fundamental compactness result on bounded parabolic fixed points of non-elementary Kleinian groups (recall the definition of a bounded parabolic fixed point from page 18). Many later results concerning compactness will refer to this result directly or indirectly. The result and its proof are straightforward generalizations of Lemma 2A of [Tukia1994c] and its proof. The seventh result considers the distribution of points of a set of the form $G_\infty z$, where $G_\infty$ is the stabilizer of $\infty$ with respect to a non-elementary Kleinian group $G$ acting on $\mathbb{H}^{n+1}$ and $z \neq \infty$ is a limit point of $G$, under the assumption that $\infty$ is a (bounded) parabolic fixed point of $G$. The result will be important in the proofs of Theorems 2.49, 3.3 and 3.6, where we will deduce estimates for conformal measures. Results similar to the seventh result are used in [Nicholls1989], [SV1995] and [Tukia1994c], for example, although no detailed proofs are given.

Recall that we denote the hyperbolic convex hull of a Kleinian group $G$ by $H(G)$, the limit set of $G$ by $L(G)$, and the stabilizer of $x \in \bar{\mathbb{R}}^{n+1}$ with respect to $G$ by $G_x$. Recall also that a point in $\mathbb{H}^{n+1}$ is often written as $(x, t)$, where $x \in \mathbb{R}^n$ and $t > 0$, and that the hyperbolic metric of $\mathbb{X}^{n+1}$ is denoted by $d$.

**Lemma 2.31.** *Let $x_0 \in \partial\mathbb{X}^{n+1}$ and let $H$ be a horoball of $\mathbb{X}^{n+1}$ based at $x_0$. Let $u > 0$. Then*

$$\{x \in H : d(x, \partial H) > u\} = H' \quad \textit{and} \quad \{x \in H : d(x, \partial H) = u\} = \partial H' \cap \mathbb{X}^{n+1},$$

*where $H' \subset H$ is a horoball of $\mathbb{X}^{n+1}$ based at $x_0$.*

*Proof.* The claim is clearly conjugation invariant so we can assume that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $x_0 = \infty$. Now $H$ is an open half-space of $\mathbb{R}^{n+1}$ contained in $\mathbb{H}^{n+1}$. The claim is obvious in this case. □

**Lemma 2.32.** *Let $H$ be a horoball of $\mathbb{H}^{n+1}$ based at* 0*. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $(x, t) \in H$. Then*

$$e^{d((x,t),\partial H)} = \frac{d_{\mathrm{euc}}(H)t}{|x|^2 + t^2}. \tag{2.33}$$

*Proof.* Denote by $\sigma$ the inversion $y \mapsto y/|y|^2$ in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Observe that $\sigma H = \{y \in \mathbb{H}^{n+1} : y_{n+1} > 1/d_{\mathrm{euc}}(H)\}$. Note that $\sigma$ is a hyperbolic isometry of $\mathbb{H}^{n+1}$ and that $d((y, u_1), (y, u_2)) = |\log(u_1/u_2)|$ for every $y \in \mathbb{R}^n$ and $u_1, u_2 > 0$. We calculate that

$$\begin{aligned} d((x,t),\partial H) &= d(\sigma(x,t),\partial\sigma H) \\ &= d\left(\left(\frac{x}{|x|^2+t^2},\frac{t}{|x|^2+t^2}\right),\left(\frac{x}{|x|^2+t^2},\frac{1}{d_{\mathrm{euc}}(H)}\right)\right) \\ &= \log\frac{d_{\mathrm{euc}}(H)t}{|x|^2+t^2}. \end{aligned}$$

Equation (2.33) follows. □

**Lemma 2.34.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $t_0 > 0$ and $v_0 \in ]0, 1[$. Then there is a constant $c > 0$ satisfying the following. If $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ are such that $|x - x'|/t \leq v_0$ for some $x' \in L(G) \cap \mathbb{R}^n$, then $d((x, t), H(G)) \leq c$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. If $\infty \in L(G)$, the existence of $c$ is clear since the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x'$ and $\infty$ is contained in $H(G)$ in this case. So suppose that $\infty \notin L(G)$. Since $L(G)$ is closed in $\bar{\mathbb{R}}^{n+1}$, $L(G)$ is now bounded in $\mathbb{R}^n$. Let $t_1 \in ]0, t_0[$ be small compared to the euclidean diameter of $L(G)$. If $t \in [t_1, t_0[$, then $d((x,t), y_0)$ is bounded by a constant for any fixed $y_0 \in H(G)$. Thus, the constant $c$ exists in this case. If $t \in ]0, t_1[$, there is $y \in L(G) \setminus \bar{B}^n(x,t)$ such that $d_{\mathrm{euc}}(y, S^{n-1}(x,t)) \geq c_0 t$ for some constant $c_0 > 0$. The distance $d((x,t), z)$ is bounded by a constant in this case, where $z$ is the intersection point of $S^n(x,t)$ and the hyperbolic line $L(y, x')$ of $\mathbb{H}^{n+1}$ with endpoints $y$ and $x'$. The existence of $c$ follows since $L(y, x') \subset H(G)$. □

**Lemma 2.35.** *Let $H$ be a horoball of $\mathbb{H}^{n+1}$ based at $0$. Denote by $\sigma$ the inversion $z \mapsto z/|z|^2$ in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Let $w \in ]0,1[$ and $M > 0$. Then there is a constant $u > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $(x,t) \in H$ with $d((x,t), \partial H) > u$. Let $y \in \mathbb{R}^n$ be such that $\bar{B}^n(y, wt) \subset B^n(x,t)$. Then*

$$d_{\mathrm{euc}}(\sigma(y), \sigma S^{n-1}(y, wt)) \geq M. \tag{2.36}$$

*Proof.* Let us choose $u > 0$. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $(x,t) \in H$ with $d((x,t), \partial H) > u$. Let $y \in \mathbb{R}^n$ be such that $\bar{B}^n(y, wt) \subset B^n(x,t)$. We show that if $u$ is chosen large enough, then (2.36) is true. We note that (2.36) is trivial if $\bar{B}^n(0, wt) \subset B^n(x,t)$ and $y = 0$. We can hence assume that $y \neq 0$. Now

$$d_{\mathrm{euc}}(\sigma(y), \sigma S^{n-1}(y, wt)) = \frac{1}{|y|} - \frac{1}{|y| + wt} = \frac{wt}{|y|(|y| + wt)} \geq \frac{wt}{(|x| + (1-w)t)(|x| + t)}.$$

We have that $\{z \in H : d(z, \partial H) > u\} = H_u$ by Lemma 2.31, where $H_u \subset H$ is a horoball of $\mathbb{H}^{n+1}$ based at 0. Let $r_u$ be the euclidean radius of $H_u$. It is the case that $r_u \to 0$ if $u \to \infty$. Suppose first that $|x| \leq t$. Then

$$\frac{wt}{(|x| + (1-w)t)(|x| + t)} \geq \frac{w}{2(2-w)t} \geq \frac{w}{4(2-w)r_u}.$$

We see that if $u$ is chosen large enough, then (2.36) is valid in this case. Suppose next that $|x| > t$. Note that if $z \in \mathbb{R}^n$ and $s > 0$, then

$$(z, s) \in \partial H_u \qquad \text{if and only if} \qquad s = r_u \pm \sqrt{r_u^2 - |z|^2}.$$

We conclude that

$$\begin{aligned} \frac{wt}{(|x| + (1-w)t)(|x| + t)} &\geq \frac{w\left(r_u - \sqrt{r_u^2 - |x|^2}\right)}{(|x| + (1-w)t)(|x| + t)} \\ &= \frac{w|x|^2}{(|x| + (1-w)t)(|x| + t)\left(r_u + \sqrt{r_u^2 - |x|^2}\right)} \\ &\geq \frac{w}{4(2-w)r_u}. \end{aligned}$$

So if $u$ is chosen large enough, (2.36) is valid in this case as well. □

**Lemma 2.37.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\infty$ be a parabolic fixed point of $G$ of rank $k \in \{1, 2, \ldots, n\}$. Let $V \subset \mathbb{R}^n$ be a $G_\infty$-invariant $k$-plane as described in the claim (i) of Theorem 2.7. Then the following claims are true. (i) The* *distance $d_{\mathrm{euc}}(x, L(G))$ is bounded by a constant for every $x \in V$. (ii) The point $\infty$ is a bounded parabolic fixed point of $G$ if and only if $d_{\mathrm{euc}}(x, V)$ is bounded by a constant for every $x \in L(G) \cap \mathbb{R}^n$.*

*Proof.* According to the claim (i) of Theorem 2.7, there is a compact set $K \subset \mathbb{R}^n$ such that $G_\infty K = V$.

We prove (i) first. We begin by observing that the following three claims are true. Since $K$ is compact and $G$ is non-elementary, there is a constant $a_0 > 0$ such that $d_{\mathrm{euc}}(z, L(G)) \leq a_0$ for every $z \in K$. The elements in $G_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. The set $L(G)$ is $G_\infty$-invariant. Let $x \in V$ be arbitrary. There is $g \in G_\infty$ and $y \in K$ such that $g(y) = x$. The three claims mentioned above imply that $d_{\mathrm{euc}}(x, L(G)) = d_{\mathrm{euc}}(y, L(G)) \leq a_0$. We have proved (i).

We consider the claim (ii). Suppose first that $\infty$ is a bounded parabolic fixed point of $G$. This means that there is a compact set $C \subset L(G) \cap \mathbb{R}^n$ such that $G_\infty C = L(G) \cap \mathbb{R}^n$. Let $x \in L(G) \cap \mathbb{R}^n$. We need to show that $d_{\mathrm{euc}}(x, V)$ is bounded by a constant. There is $g \in G_\infty$ such that $x = g(y)$ for some $y \in C$, and there is a constant $c_0 > 0$ such that $d_{\mathrm{euc}}(y, V) \leq c_0$ since $C$ is compact. It follows that $d_{\mathrm{euc}}(x, V) \leq c_0$ since $V$ is $G_\infty$-invariant and the elements of $G_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7.

Assume next that there is a constant $c_1 > 0$ such that $d_{\mathrm{euc}}(x, V) \leq c_1$ for all $x \in L(G) \cap \mathbb{R}^n$. We need to show that $\infty$ is a bounded parabolic fixed point of $G$. Recall that $K$ is a compact set such that $G_\infty K = V$. Define $K' = \{y \in \mathbb{R}^n : d_{\mathrm{euc}}(y, K) \leq c_1\}$. Let $x \in L(G) \cap \mathbb{R}^n$. Let $x^* \in V$ be such that $|x - x^*| = d_{\mathrm{euc}}(x, V)$. Let $g \in G_\infty$ be such that $x^* = g(y^*)$ for some $y^* \in K$. Now $g^{-1}(x) \in K'$ since $g$ is a euclidean isometry. We see that, since $L(G)$ is $G_\infty$-invariant, it is true that $L(G) \cap \mathbb{R}^n = G_\infty K''$, where the set $K'' = K' \cap L(G)$ is compact. It is true, therefore, that $\infty$ is a bounded parabolic fixed point of $G$. The proof of (ii) is finished. □

**Lemma 2.38.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $p \in \partial\mathbb{X}^{n+1}$ be a bounded parabolic fixed point of $G$ of rank $k \in \{1, 2, \ldots, n\}$. Let $u \geq 0$. Let $H_1 \subset H_2$ be horoballs of $\mathbb{X}^{n+1}$ based at $p$. Then there is a compact set $C \subset \mathbb{X}^{n+1}$ such that the following holds. Let $\alpha \in$ Möb$(n+1)$ map $\mathbb{X}^{n+1}$ onto $\mathbb{B}^{n+1}$ or $\mathbb{H}^{n+1}$. Write $G^\alpha = \alpha G \alpha^{-1}$, $\bar{N}(H(G^\alpha), u) = \{x \in \alpha\mathbb{X}^{n+1} : d(x, H(G^\alpha)) \leq u\}$ and $H_j^\alpha = \alpha H_j$ for $j = 1, 2$. It is now the case that*

$$G^\alpha_{\alpha(p)} \alpha C = (\bar{H}_2^\alpha \setminus H_1^\alpha) \cap \bar{N}(H(G^\alpha), u). \tag{2.39}$$

*Proof.* It is evident that the claim is conjugation invariant. This means that we can assume that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$, $p = \infty$ and $\alpha = \mathrm{id}$. We can assume also that $\mathbb{R}^k$ is $G_\infty$-invariant so that $\mathbb{R}^k = G_\infty K$ for some compact $K \subset \mathbb{R}^k$ (see the claim (i) of Theorem 2.7). We assume that $u = 0$ for the time being. This means that $\bar{N}(H(G), u) = H(G)$.

The fact that $\infty$ is a bounded parabolic fixed point of $G$ means that $d_{\mathrm{euc}}(y, \mathbb{R}^k)$ is bounded by a constant for every $y \in L(G) \cap \mathbb{R}^n$ by the claim (ii) of Lemma 2.37. It follows that we can choose $r > 0$ such that

$$L(G) \cap \mathbb{R}^n \subset \mathbb{R}^k \times \bar{B}^{n-k}(0, r) = A, \tag{2.40}$$

where $\bar{B}^{n-k}(0,r) = \{y \in \mathbb{R}^{n-k} : |y| \le r\}$. (We use notation that assumes implicitly that $k < n$. It is obvious how the notation is to be altered in case $k = n$.) Write $D = A \times ]0,\infty[$ and note that $(\bar{H}_2 \setminus H_1) \setminus \{\infty\} = \mathbb{R}^n \times [v_1, v_2]$ for some $0 < v_1 \le v_2$. Recall the details of the action of $G_\infty$ on $\mathbb{R}^{n+1}$ from the claim (iii) of Theorem 2.7. We conclude that

$$D \cap (\bar{H}_2 \setminus H_1) = G_\infty(K \times \bar{B}^{n-k}(0,r) \times [v_1, v_2]). \tag{2.41}$$

Observe that $D$ closed in $\mathbb{H}^{n+1}$ and hyperbolically convex and that $\bar{D}$ contains $L(G)$ by (2.40). We obtain that $H(G) \subset D$ by the definition of $H(G)$. Recall that $H(G)$ is $G_\infty$-invariant. We see now that

$$(\bar{H}_2 \setminus H_1) \cap H(G) = G_\infty C$$

by (2.41), where

$$C = (K \times \bar{B}^{n-k}(0,r) \times [v_1, v_2]) \cap H(G).$$

We have proved (2.39) in case $u = 0$.

Let $u > 0$. Suppose that $x \in (\bar{H}_2 \setminus H_1) \cap \bar{N}(H(G), u)$. Let $y \in H(G)$ be such that $d(x,y) \le u$. We can fix horoballs $H_1' \subset H_2'$ of $\mathbb{H}^{n+1}$ based at $\infty$ and determined by $u$ such that $y \in \bar{H}_2' \setminus H_1'$. The first part of this proof implies the existence of a compact set $C' \subset \mathbb{H}^{n+1}$ such that $G_\infty C' = (\bar{H}_2' \setminus H_1') \cap H(G)$. It is the case that $y \in gC'$ for some $g \in G_\infty$, and so $x \in g\bar{N}(C', u)$, where $\bar{N}(C', u) = \{z \in \mathbb{H}^{n+1} : d(z, C') \le u\}$. Note that $H_1$, $\bar{H}_2$ and $\bar{N}(H(G), u)$ are $G_\infty$-invariant. We obtain that

$$(\bar{H}_2 \setminus H_1) \cap \bar{N}(H(G), u) = G_\infty C'',$$

where

$$C'' = (\bar{H}_2 \setminus H_1) \cap \bar{N}(H(G), u) \cap \bar{N}(C', u),$$

which finishes the proof. □

**Lemma 2.42.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Assume that $\infty$ is a parabolic fixed point of $G$ of rank $k \in \{1, 2, \ldots, n\}$. Let $z_0 \in L(G) \cap \mathbb{R}^n$. Then the following claims are true.*

*(i) There are constants $c_0 > 0$, $c_1 > 0$, $r_0 > 0$ and $R_0 > 0$ satisfying the following. Write $A_i = \{z \in \mathbb{R}^n : c_0 i \le |z| < c_0(i+1)\}$ for $i \in \{1, 2, \ldots\}$. Now $A_i$ contains at least $c_1^{-1} i^{k-1}$ pairwise disjoint balls of the form $gB^n(z_0, r_0)$, $g \in G_\infty$, for every $i \in \{1, 2, \ldots\}$. Furthermore, if $\infty$ is a bounded parabolic fixed point of $G$, the set $A_i \cap L(G)$ has a covering of balls of the form $gB^n(z_0, R_0)$, $g \in G_\infty$, for every $i \in \{1, 2, \ldots\}$ such that the covering contains at most $c_1 i^{k-1}$ balls.*

*(ii) Let $v \in [0, 1[$. Assume that $\infty$ is a bounded parabolic fixed point of $G$. Then there are constants $c_2 > 0$, $c_3 > 0$, $r_1 > 0$ and $R_1 > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $|x - x'|/t \le v$ for some $x' \in L(G) \cap \mathbb{R}^n$ and that $t \ge c_2$. Then $\bar{B}^n(x,t)$ contains at least $c_3^{-1} t^k$ pairwise disjoint balls of the form $gB^n(z_0, r_1)$, $g \in G_\infty$. Moreover, the set $\bar{B}^n(x,t) \cap L(G)$ has a covering of balls of the form $gB^n(z_0, R_1)$, $g \in G_\infty$, such that the covering contains at most $c_3 t^k$ balls.*

*Proof.* Let $V \subset \mathbb{R}^n$ be a $G_\infty$-invariant $k$-plane as described in the claim (i) of Theorem 2.7. We denote by $\mathrm{Vol}(X)$ the volume of $X \subset V$ in $V$. Let $G_\infty^*$ be a free commutative subgroup of $G_\infty$ of finite index isomorphic to $\mathbb{Z}^k$ as described in the claim (iv) of Theorem 2.7.

We consider the claim (i). Let us choose a number $c_0 > 0$ as follows. Suppose that we momentarily conjugate the situation to such as described in the claim (iv) of Theorem 2.7 using a euclidean isometry. We choose $c_0$ so that $c_0$ is large compared to $|x_1|, |x_2|, \ldots, |x_k|$ in the conjugated situation, where $x_1, x_2, \ldots, x_k \in \mathbb{R}^k$ are the vectors appearing in the claim (iv) of Theorem 2.7. Let $z_1$ be the orthogonal projection of $z_0$ to $V$. We write

$$A'_i = \{z \in \mathbb{R}^n : c_0 i \leq |z - z_1| < c_0(i+1)\}$$

for $i \in \{1, 2, \ldots\}$. Recall that the elements in $G_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. Due to the nature of the action of $G^*_\infty$ on $V$ as described in the claim (iv) of Theorem 2.7, we can choose a small number $s_0 > 0$ such that the $k$-balls $B(g) = B^n(g(z_1), s_0) \cap V$, $g \in G^*_\infty$, are pairwise disjoint. On the other hand, again due to the nature of the action of $G^*_\infty$ on $V$, we can choose $S_0 > 0$ such that the $k$-balls $B'(g) = B^n(g(z_1), S_0) \cap V$, $g \in G^*_\infty$, cover $V$. Write $b_0 = \mathrm{Vol}(B^n(z_1, 1) \cap V)$. Write also $N_i = \{g \in G^*_\infty : g(z_1) \in A'_i\}$ for $i \in \{1, 2, \ldots\}$.

We fix $i \in \{1, 2, \ldots\}$. Our aim is to show that $c_1^{-1} i^{k-1} \leq |N_i| \leq c_1 i^{k-1}$ for some constant $c_1 > 0$, where $|N_i|$ denotes the number of elements in $N_i$. We can adjust $c_0$ and $s_0$ if necessary so that if $g \in N_i$, then

$$B(g) \subset \{z \in V : c_0 i - s_0 \leq |z - z_1| \leq c_0(i+1) + s_0\} = D_i.$$

Since the $k$-balls $B(g)$, $g \in G^*_\infty$, are pairwise disjoint,

$$\mathrm{Vol}(D_i) \geq \mathrm{Vol}\left(\bigcup_{g \in N_i} B(g)\right) = |N_i| \mathrm{Vol}(B(\mathrm{id})) = b_0 s_0^k |N_i|.$$

It is clear that we can write that

$$\mathrm{Vol}(D_i) = b_0(c_0(i+1) + s_0)^k - b_0(c_0 i - s_0)^k = i^{k-1}\left(a_0 + \frac{a_1}{i} + \ldots + \frac{a_{k-1}}{i^{k-1}}\right),$$

where $a_0, a_1, \ldots, a_{k-1}$ are constants and $a_0 > 0$. We obtain that $|N_i| \leq c_1 i^{k-1}$ for a suitable constant $c_1 > 0$. Suppose next that $c_0$ is large compared to $S_0$ and define

$$D'_i = \{z \in V : c_0 i + S_0 \leq |z - z_1| \leq c_0(i+1) - S_0\}$$

and

$$M_i = \{g \in G^*_\infty : B'(g) \cap D'_i \neq \emptyset\}.$$

Now the $k$-balls $B'(g)$, $g \in M_i$, cover $D'_i$ and $M_i \subset N_i$. It follows that

$$\mathrm{Vol}(D'_i) \leq \mathrm{Vol}\left(\bigcup_{g \in M_i} B'(g)\right) \leq |M_i| \mathrm{Vol}(B'(\mathrm{id})) \leq b_0 S_0^k |N_i|.$$

It is evident that we can use similar reasoning as above to show that if $c_1$ is adjusted accordingly, then $|N_i| \geq c_1^{-1} i^{k-1}$.

We continue to consider a fixed $i \in \{1, 2, \ldots\}$. It is clear, after the above reasoning, that if $r_0 > 0$ is chosen small enough and $c_0$ and $c_1$ are adjusted if needed, then $A'_i$ contains at least $c_1^{-1} i^{k-1}$ pairwise disjoint balls of the form $gB^n(z_1, r_0)$, $g \in G^*_\infty$. Recall the details of the action of the elements in $G^*_\infty$ on $\mathbb{R}^n$ from the claim (iii) of Theorem 2.7. It is not difficult to see that if $c_0$, $c_1$ and $r_0$ are adjusted if needed, then $A'_i$ contains at least $c_1^{-1} i^{k-1}$

pairwise disjoint balls of the form $gB^n(z_0, r_0)$, $g \in G^*_\infty$. We see finally that if $c_0$, $c_1$ and $r_0$ are adjusted once more, then we can replace $A'_i$ by $A_i$ in the previous claim ($A_i$ appears in the claim (i) of the present lemma). We have proved the first part of (i).

We assume until the end of the proof of the claim (i) that $\infty$ is a bounded parabolic fixed point of $G$. The distances $d_{\text{euc}}(x, V)$, $x \in L(G) \cap \mathbb{R}^n$, are bounded by a constant by the claim (ii) of Lemma 2.37. Recalling the details of our argument hitherto, it is not difficult to see that if $R_0 > 0$ is chosen large enough, the set $A'_i \cap L(G)$ has a covering by balls of the form $gB^n(z_1, R_0)$, $g \in G^*_\infty$, such that the covering contains at most $c_1 i^{k-1}$ balls. We see like in the proof of the first part of (i) that if the constants $c_0$, $c_1$ and $R_0$ are adjusted, then we can claim that the set $A_i \cap L(G)$ has a covering by balls of the form $gB^n(z_0, R_0)$, $g \in G^*_\infty$, such that the covering contains at most $c_1 i^{k-1}$ balls. We have proved the second part of (i).

We consider the claim (ii). Let $v \in [0, 1[$ and assume that $\infty$ is a bounded parabolic fixed point of $G$. Let $c_2 > 0$ be a number that is large in the same sense as $c_0$ was at the beginning of the proof of (i). We again denote the orthogonal projection of $z_0$ to $V$ by $z_1$. Let the numbers $s_0 > 0$ and $S_0 > 0$ and the balls $B(g)$ and $B'(g)$, $g \in G^*_\infty$, be as in the proof of (i). Write again $b_0 = \text{Vol}(B^n(z_1, 1) \cap V)$.

We fix $x \in \mathbb{R}^n$, $t \geq c_2$ and $x' \in L(G) \cap \mathbb{R}^n$ as in (ii). Recall that the distances $d_{\text{euc}}(y, V)$, $y \in L(G) \cap \mathbb{R}^n$, are bounded by a constant by the claim (ii) of Lemma 2.37. We can thus guarantee, by increasing $c_2$ if necessary, that $\bar{B}^n(x, t)$ contains points of the form $g(z_1)$, $g \in G^*_\infty$. Write $N(x, t) = \{g \in G^*_\infty : g(z_1) \in \bar{B}^n(x, t)\}$. Let us show that there is a constant $c_3 > 0$ such that $c_3^{-1} t^k \leq |N(x, t)| \leq c_3 t^k$.

Observe that if $c_2$ is chosen large enough, we can always choose $\hat{x} \in V$ such that $|x' - \hat{x}|/t \leq w_0$, where $w_0 > 0$ is a constant we can choose as small as we want. It follows that if $w_0$ is chosen small enough, then there are constants $u_0 > 0$ and $U_0 > 0$ such that $B^n(\hat{x}, u_0 t) \subset \bar{B}^n(x, t) \subset B^n(\hat{x}, U_0 t)$. Define

$$N(\hat{x}, U_0 t) = \{g \in G^*_\infty : g(z_1) \in B^n(\hat{x}, U_0 t)\}.$$

If $g \in N(\hat{x}, U_0 t)$, then $B(g) \subset B^n(\hat{x}, U_0 t + s_0) \cap V = \hat{B}$. The balls $B(g)$, $g \in G^*_\infty$, are pairwise disjoint, so

$$\text{Vol}(\hat{B}) \geq |N(\hat{x}, U_0 t)| \text{Vol}(B(\text{id})) \geq b_0 s_0^k |N(x, t)|.$$

Since $\text{Vol}(\hat{B}) = b_0 (U_0 t + s_0)^k$ and $t \geq c_2$, there is a constant $c_3 > 0$ such that $|N(x, t)| \leq c_3 t^k$. Assume next that $c_2$ is so large that $u_0 t$ is large compared to $S_0$. Let us write $\hat{B}' = B^n(\hat{x}, u_0 t - S_0) \cap V$ and define

$$M(\hat{x}, u_0 t) = \{g \in G^*_\infty : B'(g) \cap \hat{B}' \neq \emptyset\}.$$

It is true that $M(\hat{x}, u_0 t) \subset N(x, t)$. Since the $k$-balls $B'(g)$, $g \in M(\hat{x}, u_0 t)$, cover $\hat{B}'$, it is the case that

$$\text{Vol}(\hat{B}') \leq |M(\hat{x}, u_0 t)| \text{Vol}(B'(\text{id})) \leq b_0 S_0^k |N(x, t)|.$$

Since $\text{Vol}(\hat{B}') = b_0 (u_0 t - S_0)^k$, we see that if $c_3$ is adjusted, it is true that $|N(x, t)| \geq c_3^{-1} t^k$.

The rest of the proof of (ii) is analogous to the corresponding part of the proof of (i). We can claim first that there are constants $r_1 > 0$ and $R_1 > 0$ such that if $c_2$ and $c_3$ are adjusted, then the following two claims are true. The ball $\bar{B}^n(x, t)$ contains at least $c_3^{-1} t^k$

pairwise disjoint balls of the form $gB^n(z_1, r_1)$, $g \in G^*_\infty$. The set $\bar{B}^n(x,t) \cap L(G)$ has a covering of balls of the form $gB^n(z_1, R_1)$, $g \in G^*_\infty$, such that the covering contains at most $c_3 t^k$ balls. We can claim, moreover, that if $r_1$, $R_1$, $c_2$ and $c_3$ are adjusted, we can restate the previous two claims with the balls of the form $gB^n(z_1, r_1)$ and $gB^n(z_1, R_1)$, $g \in G^*_\infty$, replaced by balls of the form $gB^n(z_0, r_1)$ and $gB^n(z_0, R_1)$, $g \in G^*_\infty$. We have proved (ii). □

2.2.2. *Results featuring conformal measures.* We prove next a series of five results featuring conformal measures of Kleinian groups. The first result is a useful technical lemma. The next two results are elementary observations regarding conformal measures. The fourth result is an important theorem about the exponent of convergence of a non-elementary Kleinian group and the fifth result a corollary of the theorem. This theorem, Theorem 2.49, and its proof are essentially contained in [Tukia1994c] (see the proof of Theorem 2B) but the result is not stated as an explicit theorem in this paper. The corollary of Theorem 2.49, Corollary 2.50, is a well-known result in the field of Kleinian groups. The standard expositions on the result, see, for example, [Beardon1968] and [Patterson1976a], consider explicitly only classical Kleinian groups, i.e. groups acting on $\mathbb{X}^2$ or $\mathbb{X}^3$ and containing only orientation preserving elements, and the given proofs do not involve conformal measures.

Recall that we mentioned on page 21 that if a non-elementary Kleinian group $G$ has an $s$-conformal measure and the exponent of convergence of $G$ is $\delta$, then $s \geq \delta$. Recall also that $\delta > 0$ according to a result mentioned after the definition (2.9) of the exponent of convergence. We see thus that it is appropriate to assume that $s > 0$ when considering an $s$-conformal measure of a non-elementary Kleinian group. We remind the reader that a conformal measure of a Kleinian group $G$ is supported by the limit set $L(G)$ by definition.

**Lemma 2.43.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Suppose that $h \in$ Möb$(n+1)$ is a Möbius transformation mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and that $\nu$ is an $s$-conformal measure of $h^{-1}Gh$ such that $\mu = h^s_* \nu$. Let $f \in$ Möb$(\mathbb{H}^{n+1})$. Write $\mu^f = (f \circ h)^s_* \nu$, so $\mu^f$ is an $s$-conformal measure of $fGf^{-1}$. In this situation, the following claims are true. It is true that*

$$\int_A |(f^{-1})'|^s d\mu^f = \mu(f^{-1}A) \tag{2.44}$$

*for every Borel set $A$ of $\mathbb{R}^{n+1}$. Moreover,*

$$\mu^{f \circ g} = \mu^f \tag{2.45}$$

*for every $g \in G$, so in particular*

$$\mu^g = \mu \tag{2.46}$$

*for every $g \in G$.*

*Proof.* We prove first (2.44). Let $A$ be a Borel set of $\mathbb{R}^{n+1}$. Recall the pathological cases connected to the chain rule (2.13). Now (we use Lemma 2.19 in the first step)

$$\begin{aligned}\int_A |(f^{-1})'|^s d\mu^f &= \int_{(f\circ h)^{-1}A} (|(f^{-1})'|^s \circ (f\circ h))|(f\circ h)'|^s d\nu \\ &= \int_{(f\circ h)^{-1}A} |(f^{-1}\circ (f\circ h))'|^s d\nu = \int_{h^{-1}f^{-1}A} |h'|^s d\nu = \mu(f^{-1}A).\end{aligned}$$

We prove next (2.45) and (2.46). Let $g \in G$. The $\sigma$-algebra of measurable sets for both $\mu^{f\circ g}$ and $\mu^f$ is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. Let $A$ be a Borel set of $\bar{\mathbb{R}}^{n+1}$. We can assume that $A \subset fL(G)$ since $\mu^{f\circ g}$ and $\mu^f$ are supported by $fL(G)$. Note that $\nu = (h^{-1}\circ g^{-1}\circ h)^s_*\nu$ by (2.14) and (2.17). Now (we use Lemma 2.19 in the third step)

$$\begin{aligned}\mu^{f\circ g}(A) &= \int_{(f\circ g\circ h)^{-1}A} |(f\circ g\circ h)'|^s d\nu = \int_{(h^{-1}\circ g^{-1}\circ h)((f\circ h)^{-1}A)} |(f\circ g\circ h)'|^s d(h^{-1}\circ g^{-1}\circ h)^s_*\nu \\ &= \int_{(f\circ h)^{-1}A} (|(f\circ g\circ h)'|^s \circ (h^{-1}\circ g^{-1}\circ h))|(h^{-1}\circ g^{-1}\circ h)'|^s d\nu \\ &= \int_{(f\circ h)^{-1}A} |((f\circ g\circ h)\circ (h^{-1}\circ g^{-1}\circ h))'|^s d\nu = \int_{(f\circ h)^{-1}A} |(f\circ h)'|^s d\nu = \mu^f(A).\end{aligned}$$

We have proved (2.45) and (2.46). □

**Lemma 2.47.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Let $U$ be an open set of $\bar{\mathbb{R}}^{n+1}$ intersecting $L(G)$. Then $\mu(U) > 0$.*

*Proof.* We start by showing that $L(G) \subset GU$. We quote the following result from the literature, see, for instance, [Apanasov2000] Theorem 2.14, [GM1987] Theorem 6.17 or [Tukia1994a] Theorem 2R. It is true that if $V$ and $W$ are disjoint open sets of $\bar{\mathbb{R}}^{n+1}$ intersecting $L(G)$, then there is a loxodromic $g \in G$ such that one fixed point of $g$ is in $V$ and the other in $W$.

Let $x \in L(G)$. Choose two disjoint open sets $V_1$ and $V_2$ of $\bar{\mathbb{R}}^{n+1}$ intersecting $L(G)$ such that $V_1 \subset U$ and $x \notin V_2$. According to the result quoted above, there is a loxodromic $g \in G$ with fixed points $z_1, z_2$ such that $z_1 \in V_1$ and $z_2 \in V_2$. We can assume that $z_2$ is the attracting fixed point of $g$. Since $z_2$ is the attracting fixed point of $g$, there is $i \in \{1, 2, \ldots\}$ such that $g^i(\bar{\mathbb{R}}^{n+1} \setminus V_1) \subset V_2$. Hence $x \in \bar{\mathbb{R}}^{n+1} \setminus V_2 \subset g^i V_1 \subset GU$ and so $L(G) \subset GU$.

Let us prove that $\mu(U) > 0$. We assume that $\mu(U) = 0$. If $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$, we replace $U$ by a smaller set if needed and assume that $\infty \notin U$. Now

$$\mu(L(G)) \le \mu(GU) \le \sum_{g\in G} \mu(gU) = \sum_{g\in G} \int_U |g'|^s d\mu = \sum_{g\in G} 0 = 0,$$

which is a contradiction since $\mu$ is non-trivial. Hence $\mu(U) > 0$. □

**Lemma 2.48.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Assume that $\infty$ is a parabolic fixed point of $G$. Let $y_0 \in L(G) \cap \mathbb{R}^n$ and $r > 0$. Then the set $gB^n(y_0, r)$ is a euclidean $n$-ball and $\mu(gB^n(y_0, r)) = \mu(B^n(y_0, r)) > 0$ for every $g \in G_\infty$.*

*Proof.* The elements in $G_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. Hence $gB^n(y_0, r)$ is a euclidean $n$-ball and $|g'| = 1$ in $\bar{\mathbb{R}}^{n+1}$ for every $g \in G_\infty$. We deduce that

$$\mu(gB^n(y_0, r)) = \int_{B^n(y_0,r)} |g'|^s d\mu = \mu(B^n(y_0, r))$$

for every $g \in G_\infty$. It is the case that $\mu(B^n(y_0, r)) > 0$ since $y_0 \in L(G)$ and $G$ is non-elementary, see Lemma 2.47. □

**Theorem 2.49.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $p \in \partial\mathbb{X}^{n+1}$ be a parabolic fixed point of $G$ of rank $k \in \{1, 2, \ldots, n\}$. Suppose that $\mu$ is an $s$-conformal measure of $G$ for some $s > 0$. Then it is true that $s > k/2$.*

*Proof.* There is a non-elementary Kleinian group $\Gamma$ acting on $\mathbb{B}^{n+1}$, an $s$-conformal measure $\nu$ of $\Gamma$ and $h \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{X}^{n+1}$ such that $G = h\Gamma h^{-1}$ and $\mu = h_*^s\nu$. This is trivial if $\mathbb{X}^{n+1} = \mathbb{B}^{n+1}$ – we choose $\Gamma = G$, $\nu = \mu$ and $h = \text{id}$ in this case – and if $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$, the existence of $\Gamma$, $\nu$ and $h$ follows from the definition of $\mu$. The point $q = h^{-1}(p)$ is a parabolic fixed point of $\Gamma$ of rank $k$. Let $f \in \text{Möb}(n+1)$ map $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$. Suppose that $f(q) = 0$ and that $f^{-1}(\infty) \in L(\Gamma)$. Now $f\Gamma f^{-1}$ is a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$ with a limit point $\infty$, a parabolic fixed point 0 of rank $k$, and an $s$-conformal measure $f_*^s\nu$. We conclude that we can assume about the given $\mathbb{X}^{n+1}$, $G$ and $p$ that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$, $p = 0$ and $\infty \in L(G)$.

Let $\Gamma$, $\nu$ and $h$ be as above. Let $\sigma$ be the inversion $x \mapsto x/|x|^2$ in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Write $\hat{G} = \sigma G\sigma$ and $\hat{\mu} = \mu^\sigma = (\sigma \circ h)_*^s\nu$. Now $\hat{\mu}$ is an $s$-conformal measure of $\hat{G}$.

The point $\infty$ is a parabolic fixed point of $\hat{G}$ of rank $k$. Recall that the elements in $\hat{G}_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. According to the claim (i) of Lemma 2.42, there are constants $c_0 > 0$, $c_1 > 0$ and $r_0 > 0$ which satisfy the following. The set $A_i = B^n(0, c_0(i+1)) \setminus B^n(0, c_0 i)$ contains at least $c_1 i^{k-1}$ pairwise disjoint balls of the form $gB^n(0, r_0)$, $g \in \hat{G}_\infty$, for every $i \in \{1, 2, \ldots\}$. Since $0 \in L(\hat{G})$, we obtain from Lemma 2.48 that $\hat{\mu}(gB^n(0, r_0)) = \hat{\mu}(B^n(0, r_0)) > 0$ for every $g \in \hat{G}_\infty$.

Let us assume that $s \leq k/2$. We need to derive a contradiction to finish our proof. Write $s = k/2 - t$, where $t \geq 0$. Observe that $|\sigma'(x)| = 1/|x|^2$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$ by (2.10). We can now estimate for $i \in \{1, 2, \ldots\}$ that (we use (2.44) in the first step)

$$\mu(\sigma A_i) = \int_{A_i} |\sigma'|^s d\hat{\mu} \geq \frac{\hat{\mu}(A_i)}{(c_0(i+1))^{2s}} \geq \frac{c_1\hat{\mu}(B^n(0, r_0))}{4^s c_0^{2s}} i^{-2s+k-1} = c_2 i^{-1+2t},$$

where $c_2 = 4^{-s}c_0^{-2s}c_1\hat{\mu}(B^n(0, r_0)) > 0$. This implies that

$$\mu(\bar{B}^n(0, c_0^{-1}) \setminus \{0\}) = \mu(\sigma(\mathbb{R}^n \setminus B^n(0, c_0))) = \mu\left(\sigma \bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(\sigma A_i) \geq c_2 \sum_{i=1}^{\infty} i^{-1+2t} = \infty.$$

This result is a contradiction since the $\mu$-measure of every bounded Borel set of $\mathbb{R}^{n+1}$ is finite by Theorem 2.27. We conclude that $s > k/2$. □

We remind the reader that the result stated by the following corollary has been considered in [Beardon1968] and [Patterson1976a].

**Corollary 2.50.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$ with the exponent of convergence $\delta$. Suppose that there is a parabolic fixed point of $G$ of rank $k \in \{1, 2, \ldots, n\}$. Then $\delta > k/2$.*

*Proof.* The claim follows from Theorem 2.49 since there are Patterson-Sullivan measures of $G$ which all are $\delta$-conformal measures of $G$, see page 21. □

2.2.3. *Results on Möbius transformations.* We continue by proving three simple results involving operator norms of the derivatives of Möbius transformations. The first result is mentioned explicitly, for instance, in [Ahlfors1981] and [Nicholls1989], but we give a detailed proof for the result since the proof is easy and the result will be used frequently in the later chapters.

**Lemma 2.51.** *Let $g \in$ Möb$(n+1)$. Then*

$$|g(x) - g(y)| = |g'(x)|^{1/2}|g'(y)|^{1/2}|x - y| \tag{2.52}$$

*for all $x, y \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$.*

*Proof.* If $g(\infty) = \infty$, then $g$ is a euclidean similarity of $\bar{\mathbb{R}}^{n+1}$ and (2.52) is trivial. Suppose that $g(\infty) \neq \infty$. Recall from (2.11) that now $g = \alpha \circ \sigma$, where $\alpha$ is a euclidean isometry of $\bar{\mathbb{R}}^{n+1}$ and $\sigma$ is the inversion in the isometric sphere $S_g = S^n(g^{-1}(\infty), r_g)$ of $g$. Recall also that

$$\sigma(x) = g^{-1}(\infty) + r_g^2 \frac{x - g^{-1}(\infty)}{|x - g^{-1}(\infty)|^2} \qquad \text{and} \qquad |\sigma'(x)| = \frac{r_g^2}{|x - g^{-1}(\infty)|^2}$$

for every $x \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$, see (2.10). We calculate that (we write $g^{-1}(\infty) = z$ in the following calculation)

$$\begin{aligned}
|\sigma(x) - \sigma(y)|^2 &= r_g^4 \left| \frac{x - z}{|x - z|^2} - \frac{y - z}{|y - z|^2} \right|^2 = r_g^4 \left( \frac{1}{|x - z|^2} - \frac{2(x - z) \cdot (y - z)}{|x - z|^2 |y - z|^2} + \frac{1}{|y - z|^2} \right) \\
&= \frac{r_g^4(|y - z|^2 - 2(x - z) \cdot (y - z) + |x - z|^2)}{|x - z|^2 |y - z|^2} = \frac{r_g^4 |x - y|^2}{|x - z|^2 |y - z|^2} \\
&= |\sigma'(x)||\sigma'(y)||x - y|^2
\end{aligned}$$

for all $x, y \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$, so the claim is valid for $\sigma$. We obtain, therefore, that

$$\begin{aligned}
|g(x) - g(y)| &= |\alpha(\sigma(x)) - \alpha(\sigma(y))| = |\sigma(x) - \sigma(y)| \\
&= |\sigma'(x)|^{1/2}|\sigma'(y)|^{1/2}|x - y| = |g'(x)|^{1/2}|g'(y)|^{1/2}|x - y|
\end{aligned}$$

for all $x, y \in \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$ by (2.12). □

**Lemma 2.53.** *Let $g, g_1, g_2, \ldots \in$ Möb$(n+1)$ be such that $g_i \to g$ uniformly. Then $|g_i'| \to |g'|$ uniformly in compact subsets of $\mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$.*

*Proof.* Let $C \subset \mathbb{R}^{n+1} \setminus \{g^{-1}(\infty)\}$ be compact. We may assume that $C \subset \mathbb{R}^{n+1} \setminus \{g_i^{-1}(\infty)\}$ for all $i$ since $g_i \to g$ uniformly. Let $y$ and $z$ be two fixed points in $\mathbb{R}^{n+1} \setminus C$ such that $g_i(y) \neq \infty \neq g_i(z)$ for all $i$ and $g(y) \neq \infty \neq g(z)$. Given $x \in C$ and $i \in \{1, 2, \ldots\}$, set

$$T_{xy}^i = \frac{|g_i'(x)|^{1/2}|g_i'(y)|^{1/2}}{|g'(x)|^{1/2}|g'(y)|^{1/2}} = \frac{|g_i(x) - g_i(y)|}{|g(x) - g(y)|},$$

where we have used (2.52); define $T^i_{xz}$ and $T^i_{yz}$ similarly. The uniform convergence $g_i \to g$ implies that $T^i_{xy}$ and $T^i_{xz}$ are uniformly close to 1 in $C$ for all large enough $i$. Additionally, $T^i_{yz}$ is close to 1 for all large enough $i$. It is obvious that the quantity

$$\frac{|g_i'(x)|}{|g'(x)|} = \frac{T^i_{xy}}{T^i_{yz}} T^i_{xz}$$

is uniformly close to 1 in $C$ for all large enough $i$, which proves our claim. □

**Lemma 2.54.** *Let $C \subset \mathbb{R}^{n+1}$ be compact. Let $A \subset$ Möb$(n+1)$ be compact. Suppose that $\infty \notin g^{-1}C$ for all $g \in A$. Then there exists a constant $c > 0$ such that $c^{-1} \leq |g'(x)| \leq c$ for every $g \in A$ and every $x \in g^{-1}C$.*

*Proof.* Assume that there are $g_1, g_2, \ldots \in A$ and $x_1 \in g_1^{-1}C, x_2 \in g_2^{-1}C, \ldots$ such that $|g_i'(x_i)|$ either decreases to 0 or increases to $\infty$. We may assume that $g_i \to g \in A$ uniformly and that $g_i(x_i) \to y \in C$. Now $g_i^{-1} \to g^{-1}$ uniformly and $g(\infty) \notin C$. Lemma 2.53 implies that $|(g_i^{-1})'| \to |(g^{-1})'|$ uniformly in $C$. Since id $= g_i^{-1} \circ g_i$, we can use the chain rule (2.13) to deduce that

$$|(g^{-1})'(y)| = \lim_{i\to\infty} |(g_i^{-1})'(g_i(x_i))| = \lim_{i\to\infty} |g_i'(x_i)|^{-1} \in \{0, \infty\}.$$

Recall that if $f \in$ Möb$(n+1)$, then $|f'(z)| \in \{0, \infty\}$ if and only if $f(\infty) \neq \infty$ and $z \in \{\infty, f^{-1}(\infty)\}$. Hence $y \in \{\infty, g(\infty)\}$, which is a contradiction. Our claim follows. □

2.2.4. *Compactness results for non-elementary Kleinian groups.* The next four results are technical compactness results for non-elementary Kleinian groups. We will use these results in Chapters 3 and 4. Although the topics of these two chapters are very different from one another, there are strong correspondences between the main results of the chapters, which leads to a number of similarities in the proofs, including the application of the results proved in this subsection. For example, Theorems 3.1 and 4.5 correspond to one another and they both use Lemma 2.55 in a similar way. Similarly, Theorems 3.14 and 4.37 correspond to one another and Lemma 2.59 is used in essentially the same way in their proofs. There are also other such correspondences.

It is not really possible to discuss these correspondences in detail at this point in a sensible way. The core idea is that the results of this subsection encapsulate the technical aspects which the proofs of corresponding results of Chapters 3 and 4 have in common. The aim is to take care of this host of technicalities so that it will be easier to concentrate on the actual topics in Chapters 3 and 4.

We remind the reader that a point in $\mathbb{H}^{n+1}$ is often written in the form $(x, t)$, where $x \in \mathbb{R}^n$ and $t > 0$, that $d$ denotes the hyperbolic metric of $\mathbb{H}^{n+1}$, and that bounded parabolic fixed points, hyperbolic convex hulls and horoballs were defined on page 18.

**Lemma 2.55.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $v \in [0, 1[$. Let $C \subset \mathbb{H}^{n+1}$ be compact. Given $x \in \mathbb{R}^n$ and $t > 0$ such that $(x, t) \in C$, write*

$$M_{(x,t)} = \{\alpha \in \text{Möb}(\mathbb{H}^{n+1}) : \alpha(x,t) = (x,t), \bar{B}^n(x, vt) \cap \alpha L(G) \neq \emptyset\}.$$

*Write also $M_C = \bigcup_{(x,t)\in C} M_{(x,t)}$. Then the following claim is true. If $(\alpha_i)_i$ is a sequence in $M_C$ such that $\alpha_i \in M_{(x_i,t_i)}$ for some $(x_i,t_i) \in C$, then there is $(z,w) \in C$ and $\beta \in M_{(z,w)}$ such that $\alpha_{i_k} \to \beta$ uniformly for some subsequence $(\alpha_{i_k})_k$. In particular, $M_C$ is compact.*

*Proof.* Let the sequences $(\alpha_i)_i$ and $((x_i,t_i))_i$ be as in the claim. We can assume that $(x_i,t_i) \to (z,w) \in C$ because $C$ is compact. Recall the convergence property of Möbius transformations described by Theorem 2.2. Suppose that we can assume, using Theorem 2.2, that there are points $a, b \in \bar{\mathbb{R}}^{n+1}$ such that $\alpha_i \to a$ uniformly in compact subsets of $\bar{\mathbb{R}}^{n+1} \setminus \{b\}$. Our first aim is to show that this situation contains a contradiction.

It is the case that $\alpha_i(y) \to a$ for every $y \in \bar{\mathbb{R}}^{n+1} \setminus \{a,b\}$. This fact and the fact that $\alpha_i\mathbb{H}^{n+1} = \mathbb{H}^{n+1}$ for every $i$ imply that $a \in \bar{\mathbb{R}}^n$. One sees immediately that, since $\alpha_i \to a$ uniformly in compact subsets of $\bar{\mathbb{R}}^{n+1} \setminus \{b\}$, it is true that $\alpha_i^{-1} \to b$ uniformly in compact subsets of $\bar{\mathbb{R}}^{n+1} \setminus \{a\}$. We can repeat the above argument for $(\alpha_i^{-1})_i$ and conclude that $b \in \bar{\mathbb{R}}^n$. Now $a \neq (z,w) \neq b$ and

$$\begin{aligned} d(\alpha_i(z,w),(z,w)) &\leq d(\alpha_i(z,w),\alpha_i(x_i,t_i)) + d(\alpha_i(x_i,t_i),(z,w)) \\ &= d((z,w),(x_i,t_i)) + d((x_i,t_i),(z,w)) \to 0, \end{aligned}$$

which contradicts the fact that $\alpha_i(z,w) \to a \in \bar{\mathbb{R}}^n$. We conclude that no points $a$ and $b$ such as described above exist.

Using Theorem 2.2, we suppose next that there is $\beta \in \text{Möb}(n+1)$ such that $\alpha_i \to \beta$ uniformly. Our proof will be complete once we show that $\beta \in M_{(z,w)}$.

Since $\bar{\mathbb{H}}^{n+1}$ is compact and $\alpha_i\bar{\mathbb{H}}^{n+1} = \bar{\mathbb{H}}^{n+1}$ for every $i$, it follows that $\beta\bar{\mathbb{H}}^{n+1} \subset \bar{\mathbb{H}}^{n+1}$. And since $\alpha_i^{-1} \to \beta^{-1}$ uniformly, we see that $\beta^{-1}\bar{\mathbb{H}}^{n+1} \subset \bar{\mathbb{H}}^{n+1}$. It follows that $\beta \in \text{Möb}(\mathbb{H}^{n+1})$. On the other hand, $\alpha_i(x_i,t_i) = (x_i,t_i) \to (z,w)$ and $\alpha_i(x_i,t_i) \to \beta(z,w)$, so $\beta(z,w) = (z,w)$. Let $y_i \in L(G)$ be such that $\alpha_i(y_i) \in \bar{B}^n(x_i, vt_i)$. Since $L(G)$ is compact, we can suppose that $y_i \to \zeta \in L(G)$. It is obvious that $\beta(\zeta) = \lim_{i\to\infty} \alpha_i(y_i) \neq \infty$. Given $\varepsilon > 0$, we can find $i_0 \in \{1,2,\ldots\}$ such that

$$|\beta(\zeta) - z| \leq |\beta(\zeta) - \alpha_{i_0}(y_{i_0})| + |\alpha_{i_0}(y_{i_0}) - x_{i_0}| + |x_{i_0} - z| \leq \varepsilon + vt_{i_0} + \varepsilon \leq vw + 3\varepsilon.$$

Hence $\beta(\zeta) \in \bar{B}^n(z, vw)$ and so $\bar{B}^n(z,vw) \cap \beta L(G) \neq \emptyset$. We have proved that $\beta \in M_{(z,w)}$, which concludes the proof. □

Recall that if $G$ is a Kleinian group acting on $\mathbb{X}^{n+1}$ and $x \in \bar{\mathbb{R}}^{n+1}$, then the stabilizer of $x$ with respect to $G$ is denoted by $G_x$.

**Lemma 2.56.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $p \in \mathbb{R}^n$ be a bounded parabolic fixed point of $G$. Let $H_p$ be a horoball of $\mathbb{H}^{n+1}$ based at $p$. Let $v \in [0,1[$. Let $u > 0$. Let $A \subset \text{Möb}(\mathbb{H}^{n+1})$ be a non-empty and compact set of Möbius transformations mapping $H_p$ onto itself. Then there is a compact set $C \subset \mathbb{H}^{n+1}$ satisfying the following. Let $\alpha \in A$ and let $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ be a euclidean similarity. Write $f = \lambda \circ \alpha$. Now if $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in fH_p$ with $d((x,t),\partial fH_p) \leq u$ and there is $x' \in fL(G) \cap \mathbb{R}^n$ with $|x - x'|/t \leq v$, then $(x,t) \in (f \circ g)C$ for some $g \in G_p$.*

*Proof.* Let $\beta \in A$. Lemma 2.34 implies that there is a number $c_\beta > 0$ such that the following holds. If $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in H_p$ (so $t < d_{\text{euc}}(H_p)$) and that $|x - x'|/t \leq v$ for some $x' \in \beta L(G) \cap \mathbb{R}^n$, then $d((x,t),\beta H(G)) \leq c_\beta$, where $H(G)$ denotes the hyperbolic convex hull of $G$. We claim that the numbers $c_\beta, \beta \in A$, are bounded by a constant.

Suppose that such a constant does not exist. It follows that there are sequences $(\beta_i)_i$, $(x_i)_i$, $(t_i)_i$ and $(x'_i)_i$ in $A$, $\mathbb{R}^n$, $]0, d_{\mathrm{euc}}(H_p)[$ and $\beta_i L(G) \cap \mathbb{R}^n$, respectively, satisfying the following. It is the case that $(x_i, t_i) \in H_p$ and $|x_i - x'_i|/t_i \leq v$ for every $i \in \{1, 2, \ldots\}$. Furthermore, $d((x_i, t_i), \beta_i H(G)) \to \infty$. Choose $y_0 \in L(G) \setminus \{p\}$ and let $y_1$ be the intersection point of $\partial H_p$ and the hyperbolic line $L$ of $\mathbb{H}^{n+1}$ with endpoints $p$ and $y_0$. Since $A$ is compact and the elements in $A$ map $H_p$ onto itself, we can assume that $\beta_i(y_0) \to y'_0 \in \bar{\mathbb{R}}^n \setminus \{p\}$ and $\beta_i(y_1) \to y'_1 \in \partial H_p \setminus \{p\}$. We can assume also that $(x_i, t_i) \to z \in \bar{H}_p$.

Suppose that $z \neq p$. Then $d((x_i, t_i), \beta_i(y_1))$ is bounded by a constant for every $i \in \{1, 2, \ldots\}$. This is a contradiction since $\beta_i(y_1) \in \beta_i L \subset \beta_i H(G)$ for every $i \in \{1, 2, \ldots\}$. Suppose that $z = p$. We can assume that $\beta_i(y_0) \notin \bar{B}^n(x_i, t_i)$ for every $i \in \{1, 2, \ldots\}$. Denote by $L_i$ the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x'_i$ and $\beta_i(y_0)$ and by $z_i$ the intersection point of $S^n(x_i, t_i)$ and $L_i$ for $i \in \{1, 2, \ldots\}$. Now $d((x_i, t_i), z_i)$ is bounded by a constant for every $i \in \{1, 2, \ldots\}$, which is contradictory since $z_i \in L_i \subset \beta_i H(G)$. We conclude that the numbers $c_\beta, \beta \in A$, are bounded by a constant, say $w > 0$.

Let $f = \lambda \circ \alpha$ be as in the claim. It is clear that we can assume that $\lambda = \mathrm{id}$, i.e. that $f \in A$. Let $x$, $t$ and $x'$ be as in the claim. We obtain from above that $(x, t) \in \bar{N}(fH(G), w)$, where

$$\bar{N}(fH(G), w) = \{z \in \mathbb{H}^{n+1} : d(z, fH(G)) \leq w\}.$$

We use Lemma 2.31 to deduce that there is a horoball $H'_p \subset H_p$ of $\mathbb{H}^{n+1}$ based at $p$ such that $H'_p = \{z \in H_p : d(z, \partial H_p) > u\}$. The mapping $f$ maps $H_p$ onto itself and is a hyperbolic isometry of $\mathbb{H}^{n+1}$, so $f$ maps $H'_p$ onto itself. We conclude that $(x, t) \in f\bar{H}_p \setminus fH'_p = \bar{H}_p \setminus H'_p$. Lemma 2.38 implies that there is a compact set $C \subset \mathbb{H}^{n+1}$ such that

$$(f\bar{H}_p \setminus fH'_p) \cap \bar{N}(fH(G), w) = (fG_pf^{-1})(fC).$$

Our claim follows. □

**Lemma 2.57.** *Let $G$, $p$, $H_p$, $v$ and $A$ be as in Lemma 2.56. Then there are constants $u > 0$, $c > 0$, $w \in [0, 1[$ and $s > 0$ which satisfy the following. Let $\alpha \in A$. Suppose that $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x, t) \in H_p$ with $d((x, t), \partial H_p) > u$ and that $|x - x'|/t \leq v$ for some $x' \in \alpha L(G) \cap \mathbb{R}^n$. Then the following claims are true. The set $\alpha^{-1}\bar{B}^n(x, t)$ is a euclidean $n$-ball of $\bar{\mathbb{R}}^n$, say $\bar{B}^n(\hat{x}, \hat{t})$. It is true that $c^{-1} \leq |\alpha'| \leq c$ in $\bar{B}^n(\hat{x}, \hat{t})$. It is the case that $c^{-1}t \leq \hat{t} \leq ct$ and that $|\hat{x} - \alpha^{-1}(x')|/\hat{t} \leq w$. Moreover, it is true that $(\hat{x}, \hat{t}) \in H_p$ so that*

$$d((x, t), \partial H_p) - s \leq d((\hat{x}, \hat{t}), \partial H_p) \leq d((x, t), \partial H_p) + s. \tag{2.58}$$

*Proof.* The compactness of the sets $A$ and $\{\alpha^{-1} : \alpha \in A\}$ and the fact that the elements in $A$ fix $p$ imply that there is a constant $a > 0$ such that $\alpha(\infty), \alpha^{-1}(\infty) \notin \bar{B}^n(p, a)$ for every $\alpha \in A$. The same reasons imply that there is a constant $a' \in ]0, a/2[$ such that $\alpha^{-1}\bar{B}^n(p, a') \subset B^n(p, a/2)$ for every $\alpha \in A$. We use Lemma 2.54 to deduce the existence of a constant $c > 0$ such that $c^{-1} \leq |\alpha'(z)| \leq c$ for every $\alpha \in A$ and $z \in \alpha^{-1}\bar{B}^n(p, a)$. We see that $c^{-1} \leq |(\alpha^{-1})'| \leq c$ in $\bar{B}^n(p, a)$ for every $\alpha \in A$ since $|(\alpha^{-1})'| = |\alpha'|^{-1} \circ \alpha^{-1}$ in $\bar{B}^n(p, a)$ for every $\alpha \in A$.

Let us recall Lemma 2.31 and note that $\{z \in H_p : d(z, \partial H_p) > u\} = H'_p$ for every $u > 0$, where $H'_p \subset H_p$ is a horoball of $\mathbb{H}^{n+1}$ based at $p$, which allows us to fix $u > 0$ so that $\bar{B}^n(x,t) \subset B^n(p,a')$ for every $x \in \mathbb{R}^n$ and $t > 0$ such that $(x,t) \in H_p$ with $d((x,t), \partial H_p) > u$.

Let us fix $\alpha \in A$, $x \in \mathbb{R}^n$ and $t > 0$ such that $(x,t) \in H_p$ with $d((x,t), \partial H_p) > u$ and that $|x - x'|/t \le v$ for some $x' \in \alpha L(G) \cap \mathbb{R}^n$. Our aim is to show that $\alpha$, $x$, $t$ and $x'$ satisfy the claims made in the lemma.

Since $\bar{B}^n(x,t) \subset B^n(p,a') \subset B^n(p,a)$ and $\alpha(\infty) \notin \bar{B}^n(p,a)$, it is true that $\alpha^{-1}\bar{B}^n(x,t) = \bar{B}^n(\hat{x},\hat{t})$ for some $\hat{x} \in \mathbb{R}^n$ and $\hat{t} > 0$. Bear in mind that $\bar{B}^n(\hat{x},\hat{t}) \subset B^n(p,a/2)$. We obtain from above that $c^{-1} \le |\alpha'| \le c$ in $\bar{B}^n(\hat{x},\hat{t})$. We recall (2.52) and calculate that

$$\hat{t} = \frac{1}{2} \sup_{z_1,z_2 \in S^{n-1}(x,t)} |\alpha^{-1}(z_1) - \alpha^{-1}(z_2)| \in [c^{-1}t, ct].$$

Similarly,

$$d_{\mathrm{euc}}(\alpha^{-1}(x'), S^{n-1}(\hat{x},\hat{t})) = \inf_{z \in S^{n-1}(x,t)} |\alpha^{-1}(x') - \alpha^{-1}(z)| \ge c^{-1}(1-v)t \ge c^{-2}(1-v)\hat{t}.$$

We conclude that $|\hat{x} - \alpha^{-1}(x')|/\hat{t} \le w$ for some constant $w \in [0,1[$. Similar reasoning gives that $|\hat{x} - \alpha^{-1}(x)|/\hat{t} \le b$ for some constant $b \in [0,1[$. Now if $L$ is the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x$ and $\infty$, then $L$ intersects $S^n(x,t)$ orthogonally at $(x,t)$ and $\alpha^{-1}L$ is the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $\alpha^{-1}(x)$ and $\alpha^{-1}(\infty)$ intersecting $S^n(\hat{x},\hat{t})$ orthogonally at $\alpha^{-1}(x,t)$. The existence of $b$ and the facts that $\alpha^{-1}(\infty) \notin \bar{B}^n(p,a)$ and $\bar{B}^n(\hat{x},\hat{t}) \subset B^n(p,a/2)$ imply that $d((\hat{x},\hat{t}), \alpha^{-1}(x,t)) \le s$ for some constant $s > 0$. We can adjust $u$ if needed so that always $(\hat{x},\hat{t}) \in H_p$ since $d(\alpha^{-1}(x,t), \partial H_p) = d((x,t), \partial H_p)$. Note finally that

$$d((\hat{x},\hat{t}), \partial H_p) \le d((\hat{x},\hat{t}), \alpha^{-1}(x,t)) + d(\alpha^{-1}(x,t), \partial H_p) \le s + d((x,t), \partial H_p)$$

and

$$d((x,t), \partial H_p) = d(\alpha^{-1}(x,t), \partial H_p) \le d(\alpha^{-1}(x,t), (\hat{x},\hat{t})) + d((\hat{x},\hat{t}), \partial H_p) \le s + d((\hat{x},\hat{t}), \partial H_p).$$

We see that (2.58) is valid. We have proved all the claims of the lemma. □

**Lemma 2.59.** *Let $G$, $p$ and $H_p$ be as in Lemma 2.56. Assume that the collection of horoballs $\{gH_p : g \in G\}$ is pairwise disjoint. Then there is a non-empty and compact set $A \subset$* Möb$(\mathbb{H}^{n+1})$ *satisfying the following. The elements in $A$ map $H_p$ onto itself. If $g \in G$ and $g(p) \in \mathbb{R}^n$, then there is $f \in G_p$ such that $g \circ f = \lambda \circ \alpha$, where $\lambda \in$* Möb$(\mathbb{H}^{n+1})$ *is a euclidean similarity mapping $H_p$ onto $gH_p$ and $\alpha \in A$.*

*Proof.* If $g \in G$ and $g(p) \in \mathbb{R}^n$, write $z_g = (g(p), d_{\mathrm{euc}}(gH_p)) \in \mathbb{H}^{n+1}$. We claim that there is a constant $u \ge 0$ such that $d(z_g, H(G)) \le u$ for all $g \in G$ such that $g(p) \in \mathbb{R}^n$, where $H(G)$ is the hyperbolic convex hull of $G$. The claim is trivial with $u = 0$ if $\infty \in L(G)$, since then the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $g(p)$ and $\infty$ is contained in $H(G)$ for every $g \in G$ such that $g(p) \in \mathbb{R}^n$. Suppose temporarily that $\infty \notin L(G)$. It follows that $L(G)$ is bounded in $\mathbb{R}^n$. The pairwise disjointness of the horoballs in $\{gH_p : g \in G\}$ implies that the euclidean diameters of these horoballs are bounded by a constant. The existence of $u$ follows now from Lemma 2.34.

We use next Lemma 2.38 with $H_1 = H_p = H_2$ to deduce the existence of a compact set $C \subset \mathbb{H}^{n+1}$ such that $G_pC = \partial H_p \cap \bar{N}(H(G), u)$, where $\bar{N}(H(G), u) = \{x \in \mathbb{H}^{n+1} : d(x, H(G)) \leq u\}$. Consider a fixed $g \in G$ such that $g(p) \in \mathbb{R}^n$. Since $z_g \in \partial gH_p \cap \bar{N}(H(G), u)$, it is true that $g^{-1}(z_g) \in \partial H_p \cap \bar{N}(H(G), u)$. This means that there is $f \in G_p$ such that $g^{-1}(z_g) = f(y_g)$ for some $y_g \in C$. Let $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ be a euclidean similarity mapping $H_p$ onto $gH_p$. Then $\lambda(z_{\text{id}}) = z_g$. Write $\alpha = \lambda^{-1} \circ (g \circ f)$. The mapping $\alpha$ maps $H_p$ onto itself and $y_g$ to $z_{\text{id}}$. The mapping $\alpha$ is, therefore, contained in the set

$$A = \{\tau \in \text{Möb}(\mathbb{H}^{n+1}) : \tau H_p = H_p, \tau^{-1}(z_{\text{id}}) \in C\}.$$

We finish our proof by showing that $A$ is compact.

Let $(\tau_i)_i$ be a sequence in $A$. Recall the convergence property of Möbius transformations described by Theorem 2.2. We use Theorem 2.2 and assume first that $\tau_i \to a$ uniformly in compact sets of $\bar{\mathbb{R}}^{n+1} \setminus \{b\}$, where $a, b \in \bar{\mathbb{R}}^{n+1}$ are fixed points. We show that this situation contains a contradiction.

Since $\tau_i(x) \to a$ for every $x \in \bar{\mathbb{R}}^{n+1} \setminus \{a, b\}$ and each $\tau_i$ maps $\mathbb{H}^{n+1}$ onto itself, it is evident that $a \in \bar{\mathbb{R}}^n$. The same reasoning gives that $b \in \bar{\mathbb{R}}^n$ since the assumption that $\tau_i \to a$ uniformly in compact sets of $\bar{\mathbb{R}}^{n+1} \setminus \{b\}$ implies that $\tau_i^{-1} \to b$ uniformly in compact sets of $\bar{\mathbb{R}}^{n+1} \setminus \{a\}$. Now $a \neq z_{\text{id}} \neq b$, so we obtain the contradiction $\tau_i^{-1}(z_{\text{id}}) \to b \in C \subset \mathbb{H}^{n+1}$.

It follows that we can use Theorem 2.2 to justify the assumption that there is $\tau \in \text{Möb}(n+1)$ such that $\tau_i \to \tau$ uniformly. We complete the proof by showing that $\tau \in A$. The compactness of $\bar{\mathbb{H}}^{n+1}$ and the fact that $\tau_i\bar{\mathbb{H}}^{n+1} = \bar{\mathbb{H}}^{n+1}$ for every $i$ imply that $\tau\bar{\mathbb{H}}^{n+1} \subset \bar{\mathbb{H}}^{n+1}$. And since $\tau_i^{-1} \to \tau^{-1}$ uniformly, it is true that $\tau^{-1}\bar{\mathbb{H}}^{n+1} \subset \bar{\mathbb{H}}^{n+1}$. We conclude that $\tau \in \text{Möb}(\mathbb{H}^{n+1})$. Observe next that $\tau^{-1}(z_{\text{id}}) = \lim_{i\to\infty} \tau_i^{-1}(z_{\text{id}}) \in C$ by the compactness of $C$. The set $\bar{H}_p$, like $\bar{\mathbb{H}}^{n+1}$, is compact and mapped onto itself by every $\tau_i$. We can argue as above to show that $\tau\bar{H}_p = \bar{H}_p$. Hence $\tau H_p = H_p$. We have shown that $\tau \in A$, which proves the compactness of $A$. □

Let us remark that it is actually unnecessary to assume in Lemma 2.59 that the collection $\{gH_p : g \in G\}$ of horoballs of $\mathbb{H}^{n+1}$ should be pairwise disjoint: if the horoball $H_p$ is chosen small enough in the situation of Lemma 2.59, then the collection $\{gH_p : g \in G\}$ is automatically pairwise disjoint. This fact is non-trivial and probably cannot be considered well-known, and so we add the disjointness assumption to Lemma 2.59 for the sake of simplicity. This property of bounded parabolic fixed points and related topics are discussed, for instance, in [Apanasov2000] Chapter 3 (particularly Theorem 3.15), [Bowditch1993] Section 4 (particularly Proposition 4.4) and [Ratcliffe2006] Chapter 12 (particularly Theorems 12.6.4 and 12.6.5).

2.2.5. *Geometric results.* We end this section with the following five geometric results which we will need predominantly in Chapter 4. The first result shows that if the limit set $L(G)$ of a non-elementary Kleinian group $G$ acting on $\mathbb{X}^{n+1}$ is not an $l$-sphere of $\partial\mathbb{X}^{n+1}$, $l \in \{1, 2, \ldots, n\}$, then $L(G)$ is not an $l$-sphere of $\partial\mathbb{X}^{n+1}$ in any neighbourhood of any point in $L(G)$. (Recall that we call the euclidean $l$-planes of $\bar{\mathbb{R}}^n$ and the euclidean $l$-spheres of $\bar{\mathbb{R}}^n$ the $l$-spheres of $\bar{\mathbb{R}}^n$ for $l \in \{1, 2, \ldots, n\}$.) The second result discusses a compactness property of $l$-spheres of $\bar{\mathbb{R}}^n$, $l \in \{1, 2, \ldots, n\}$. The third and fourth results are technical results related to this compactness property. The fifth result is a distinct technical lemma needed in the proof of Theorem 4.10.

**Lemma 2.60.** *Suppose that $G$ is a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $l \in \{1, 2, \ldots, n\}$ and suppose that $L(G)$ is not an $l$-sphere of $\partial\mathbb{X}^{n+1}$. Let $U$ be an open set of $\bar{\mathbb{R}}^{n+1}$ intersecting $L(G)$. Then $U \cap L(G) \neq U \cap V$ for every $l$-sphere $V$ of $\partial\mathbb{X}^{n+1}$.*

*Proof.* Let us assume that there is an $l$-sphere $V$ of $\partial\mathbb{X}^{n+1}$ such that $U \cap L(G) = U \cap V$. Our aim is to show that this implies the contradiction $L(G) = V$.

Recall that we quoted the following result from the literature at the beginning of the proof of Lemma 2.47. If $W_1$ and $W_2$ are disjoint open sets of $\bar{\mathbb{R}}^{n+1}$ intersecting $L(G)$, then there is a loxodromic $g \in G$ such that one fixed point of $g$ is in $W_1$ and the other in $W_2$.

We choose $W_1$ and $W_2$ as above such that $W_1 \subset U$. Let $g \in G$ be loxodromic with fixed points $x_1 \in W_1$ and $x_2 \in W_2$. We can assume that $x_2$ is the attracting fixed point of $g$. We can assume also that $x_j \in \mathbb{R}^{n+1}$ and that $W_j = B^{n+1}(x_j, r_j)$ for some small $r_j > 0$ for $j = 1, 2$. Since $x_2$ is the attracting fixed point of $g$, there is $i \in \{1, 2, \ldots\}$ such that $g^i(\bar{\mathbb{R}}^{n+1} \setminus W_1) \subset W_2$ and so $\bar{\mathbb{R}}^{n+1} \setminus W_2 \subset g^i W_1$. We claim that $L(G) \setminus W_2 = g^i V \setminus W_2$.

Note first that $W_1 \cap L(G) = W_1 \cap V$ since $W_1 \subset U$. Let $x \in L(G) \setminus W_2$. Since $x \in \bar{\mathbb{R}}^{n+1} \setminus W_2$, it is true that $x \in g^i W_1$. Now $x \in g^i(W_1 \cap L(G)) = g^i(W_1 \cap V)$ since $L(G)$ is $G$-invariant. Hence $x \in g^i V \setminus W_2$. Next, let $x \in g^i V \setminus W_2$. Again, $x \in g^i W_1$ so $x \in g^i(W_1 \cap V) = g^i(W_1 \cap L(G))$. We conclude that $x \in L(G) \setminus W_2$. Therefore, $L(G) \setminus W_2 = g^i V \setminus W_2$.

We see that for every $r_1$ and $r_2$ as above there is $i(r_1, r_2) \in \{1, 2, \ldots\}$ such that $L(G) \setminus W_2 = g^{i(r_1,r_2)} V \setminus W_2$. It is clear that if $r_1$ and $r_2$ are small enough, we have that $g^{i(r_1,r_2)} V = V$ and hence that $L(G) = V$. We have reached the wanted contradiction and the proof is therefore complete. □

Recall that $q$ denotes the chordal metric of $\bar{\mathbb{R}}^{n+1}$ defined by (2.1). If $(A_i)_i$ is a sequence of non-empty subsets of $\bar{\mathbb{R}}^{n+1}$ and $A$ is a non-empty subset of $\bar{\mathbb{R}}^{n+1}$, we say that $A_i \to A$ with respect to $q$ in case the following is true. Given any $\varepsilon > 0$, there is $i_\varepsilon \in \{1, 2, \ldots\}$ such that if $i \geq i_\varepsilon$, then $q(x, A) \leq \varepsilon$ for every $x \in A_i$ and $q(y, A_i) \leq \varepsilon$ for every $y \in A$.

**Lemma 2.61.** *Let $Z \subset \mathbb{R}^n$ be non-empty and compact. Let $l \in \{1, 2, \ldots, n\}$. Let $(V_i)_i$ be a sequence of $l$-spheres of $\bar{\mathbb{R}}^n$ intersecting $Z$. Then there is a subsequence $(V_{i_k})_k$ and a set $W$ which satisfy the following. Either $W$ contains exactly one point and this point is in $Z$, or $W$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ intersecting $Z$. Furthermore, $V_{i_k} \to W$ with respect to $q$.*

*Proof.* The claim is trivial if $l = n$, so assume that $l \in \{1, 2, \ldots, n-1\}$. Given $i \in \{1, 2, \ldots\}$, let $z_i \in V_i \cap Z$. Since $Z$ is compact, we can assume that $z_i \to z$ for some $z \in Z$.

Fix $i \in \{1, 2, \ldots\}$ for the moment. If $V_i$ is an $l$-plane of $\bar{\mathbb{R}}^n$, let $v_1^i, v_2^i, \ldots, v_l^i \in \mathbb{S}^{n-1} = \partial\mathbb{B}^n$ form an affine orthonormal basis for $V_i$. That is, $v_1^i, v_2^i, \ldots, v_l^i$ form an orthonormal basis for the $l$-plane $V_i - z_i$ of $\bar{\mathbb{R}}^n$ through 0. If $V_i$ is a euclidean $l$-sphere of $\bar{\mathbb{R}}^n$ with center $\zeta_i$, let $v_1^i, v_2^i, \ldots, v_l^i \in \mathbb{S}^{n-1}$ form an affine orthonormal basis for $T_i$, where $T_i$ is an $l$-plane of $\bar{\mathbb{R}}^n$ tangential to $V_i$ at $z_i$ such that $V_i$ and $T_i$ are contained in the $(l+1)$-plane of $\bar{\mathbb{R}}^n$ through $z_i$ spanned affininely by the orthonormal basis formed by $v_1^i, v_2^i, \ldots, v_l^i, (\zeta_i - z_i)/|\zeta_i - z_i|$. We can assume that $v_1^i \to w_1 \in \mathbb{S}^{n-1}$, $v_2^i \to w_2 \in \mathbb{S}^{n-1}$, …, $v_l^i \to w_l \in \mathbb{S}^{n-1}$.

Suppose that there is a subsequence $(V_{i_k})_k$ such that $d_{\mathrm{euc}}(V_{i_k}) \to 0$. Then the following is true. If $\varepsilon > 0$, then there is $k_\varepsilon \in \{1, 2, \ldots\}$ such that $V_{i_k} \subset \bar{B}^n(z, \varepsilon)$ for every $k \geq k_\varepsilon$. The claim of the lemma is clearly true with $W = \{z\}$ in this case.

Suppose next that there is a subsequence $(V_{i_k})_k$ such that $d_{\mathrm{euc}}(V_{i_k}) \to \infty$. Let $T$ be the $l$-plane of $\bar{\mathbb{R}}^n$ through $z$ spanned affinely by the orthonormal basis formed by $w_1, w_2, \ldots, w_l$. It is clear that the claim of the lemma is true with $W = T$ in this case.

Finally, suppose that there is a constant $c_0 > 0$ such that $c_0^{-1} \leq d_{\mathrm{euc}}(V_i) \leq c_0$ for every $i \in \{1, 2, \ldots\}$. We can assume that $d_{\mathrm{euc}}(V_i) \to d_0$ for some $d_0 \in [c_0^{-1}, c_0]$. Recall that the center of $V_i$ is $\zeta_i$ and that $v^i_j \to w_j$ for $j \in \{1, 2, \ldots, l\}$. We can assume that $(\zeta_i - z_i)/|\zeta_i - z_i| \to w \in \mathbb{S}^{n-1}$. Let $T$ be the $l$-plane of $\bar{\mathbb{R}}^n$ through $z$ spanned affinely by the orthonormal basis formed by $w_1, w_2, \ldots, w_l$. Let $U$ be the euclidean $l$-sphere of $\bar{\mathbb{R}}^n$ with euclidean radius $d_0/2$ and center $z + (d_0/2)w$ such that $T$ is tangential to $U$ at $z$ and $U$ and $T$ are contained in the $(l+1)$-plane of $\bar{\mathbb{R}}^n$ through $z$ spanned affinely by the orthonormal basis formed by $w_1, w_2, \ldots, w_l, w$. It is clear that the claim of the lemma is true with $W = U$ in this case. $\square$

Let us denote by $\rho$ the Hausdorff metric defined with respect to the euclidean metric in the space of all non-empty compact subsets of $\mathbb{R}^n$, i.e.

$$\rho(A, B) = \sup\{d_{\mathrm{euc}}(y, B), d_{\mathrm{euc}}(z, A) : y \in A, z \in B\} \tag{2.62}$$

for every non-empty compact $A, B \subset \mathbb{R}^n$.

**Lemma 2.63.** *Suppose that $G$ is a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $l \in \{1, 2, \ldots, n\}$ and suppose that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. Let $f, f_1, f_2, \ldots \in$ Möb$(\mathbb{H}^{n+1})$ be such that $f_i \to f$ uniformly. Let $(x, t), (x_1, t_1), (x_2, t_2), \ldots \in \mathbb{H}^{n+1}$ be such that $(x_i, t_i) \to (x, t)$. Suppose that $B^n(x, t) \cap fL(G) \neq \emptyset$. Let $(V_i)_i$ be a sequence of $l$-spheres of $\bar{\mathbb{R}}^n$ such that $V_i$ intersects $\bar{B}^n(x_i, t_i)$. Suppose that $W$ is a set such that either $W = \{y\}$ for some $y \in \bar{B}^n(x, t)$ or $W$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x, t)$. Suppose that $V_i \to W$ with respect to $q$. In this situation, there is a constant $c > 0$ satisfying the following. For every large enough $i \in \{1, 2, \ldots\}$ there is a point $z_i$ such that $z_i \in B^n(x_i, t_i) \cap f_iL(G)$ and $d_{\mathrm{euc}}(z_i, V_i) \geq c$, or $z_i \in B^n(x_i, t_i) \cap V_i$ and $d_{\mathrm{euc}}(z_i, f_iL(G)) \geq c$. In particular,*

$$\rho(\bar{B}^n(x_i, t_i) \cap f_iL(G), \bar{B}^n(x_i, t_i) \cap V_i) \geq c \tag{2.64}$$

*for all large enough $i \in \{1, 2, \ldots\}$.*

*Proof.* Lemma 2.60 implies that $B^n(x, t) \cap fL(G) \neq B^n(x, t) \cap W$. It follows that there is a point $z$ such that $z \in B^n(x, t) \cap fL(G)$ but $z \notin W$ or $z \in B^n(x, t) \cap W$ but $z \notin fL(G)$.

Suppose that there is a point $z$ such that $z \in B^n(x, t) \cap fL(G)$ but $z \notin W$. Let $c_0 \in ]0, d_{\mathrm{euc}}(z, W)[$ be such that $\bar{B}^n(z, c_0) \subset B^n(x, t)$. It is clear that if $i$ is large enough, then $f_i(f^{-1}(z)) \in B^n(z, c_0/2) \subset B^n(x_i, t_i)$ and $d_{\mathrm{euc}}(f_i(f^{-1}(z)), V_i) \geq c_0/2$. We can choose $z_i = f_i(f^{-1}(z))$ and $c = c_0/2$ in this case.

Suppose next that there is a point $z$ such that $z \in B^n(x, t) \cap W$ but $z \notin fL(G)$. Let $c_1 \in ]0, d_{\mathrm{euc}}(z, fL(G))[$ be such that $\bar{B}^n(z, c_1) \subset B^n(x, t)$. It is clear that if $i$ is large enough, then there is $z_i \in B^n(z, c_1/2) \cap V_i \subset B^n(x_i, t_i)$ such that $d_{\mathrm{euc}}(z_i, f_iL(G)) \geq c_1/2$. It follows that $c = c_1/2$ in this case. $\square$

**Lemma 2.65.** *Let $a > 0$ and $b \in [0, 1[$. Let $l_0, l_1 \in \{1, 2, \ldots, n\}$, $l_0 \neq l_1$. Then there is a constant $c > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t > 0$. Let $V_j$, $j = 0, 1$, be an $l_j$-sphere of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x, t)$ such that $d_{\text{euc}}(V_j) \geq at$ and $d_{\text{euc}}(x, V_j) \leq bt$. Then*

$$\rho(\bar{B}^n(x, t) \cap V_0, \bar{B}^n(x, t) \cap V_1) \geq ct. \tag{2.66}$$

*Proof.* We can clearly assume that $x = 0$ and $t = 1$. Suppose that a constant $c$ as in the claim does not exist. Then there is a sequence $(V_i)_i$ of $l_0$-spheres of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x, t)$ and a sequence $(U_i)_i$ of $l_1$-spheres of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x, t)$ such that $d_{\text{euc}}(V_i), d_{\text{euc}}(U_i) \geq a$ and $d_{\text{euc}}(x, V_i), d_{\text{euc}}(x, U_i) \leq b$ for every $i$ and that $\rho(\bar{B}^n(x, t) \cap V_i, \bar{B}^n(x, t) \cap U_i) \to 0$. We apply Lemma 2.61 to $(V_i)_i$ and $(U_i)_i$. It is clear that we can assume that there is an $l_0$-sphere $V$ of $\bar{\mathbb{R}}^n$ and an $l_1$-sphere $U$ of $\bar{\mathbb{R}}^n$ such that $d_{\text{euc}}(V), d_{\text{euc}}(U) \geq a$, that $d_{\text{euc}}(x, V), d_{\text{euc}}(x, U) \leq b$, and that $V_i \to V$ with respect to $q$ and $U_i \to U$ with respect to $q$. We now have that $B^n(x, t) \cap V \neq B^n(x, t) \cap U$. Therefore, there is a point $z$ such that $z \in B^n(x, t) \cap V$ but $z \notin U$ or $z \in B^n(x, t) \cap U$ but $z \notin V$. Suppose that $z \in B^n(x, t) \cap V$ but $z \notin U$. Let $c_0 \in ]0, d_{\text{euc}}(z, U)[$ be such that $\bar{B}^n(z, c_0) \subset B^n(x, t)$. We obtain that if $i$ is large enough, there is $y_i \in B^n(z, c_0/2) \cap V_i$ such that $d_{\text{euc}}(y_i, \bar{B}^n(x, t) \cap U_i) \geq c_0/2$, and so $\rho(\bar{B}^n(x, t) \cap V_i, \bar{B}^n(x, t) \cap U_i) \geq c_0/2$ in this case. It is clear that if $c_0$ is adjusted, the same conclusion holds in case there is a point $z$ such that $z \in B^n(x, t) \cap U$ but $z \notin V$. We see that the situation is contradictory, which proves the claim of the lemma. □

**Lemma 2.67.** *Let $\alpha_0 \in ] - 1, 1]$. Then there are constants $a \in ]0, 1[$ and $c > 0$ satisfying the following. Let $x \in \mathbb{R}^n \setminus \{0\}$, $t > 0$ and $y \in S^{n-1}(x, (1 - a)t)$ be such that the cosine of the angle between $x$ and $y - x$ is larger than or equal to $\alpha_0$. Then $|y| \geq c(|x| + t)$.*

*Proof.* We start by fixing some $a \in ]0, 1[$. We will see later how small $a$ must be chosen. Let $x \in \mathbb{R}^n \setminus \{0\}$, $t > 0$ and $y \in S^{n-1}(x, (1 - a)t)$ be as in the claim. Given $z \in \mathbb{S}^{n-1} = \partial \mathbb{B}^n$, we denote by $\alpha$ the cosine of the angle between $x$ and $z$. Now

$$\begin{aligned} \frac{|x + (1 - a)tz|^2}{(|x| + t)^2} &= \frac{|x|^2 + 2(1 - a)|x|t\alpha + (1 - a)^2 t^2}{(|x| + t)^2} \\ &= \frac{|x|^2 + t^2}{(|x| + t)^2} + \frac{2|x|t\alpha}{(|x| + t)^2} - \frac{at((2 - a)t + 2|x|\alpha)}{(|x| + t)^2} \end{aligned}$$

for all $z \in \mathbb{S}^{n-1}$. Note that $(|x|^2 + t^2)/(|x| + t)^2 \geq 1/2$ and $2|x|t/(|x| + t)^2 \in [0, 1/2]$. We see that if we take $z = (y - x)/|y - x|$ in the above calculation, we obtain that

$$\frac{|x|^2 + t^2}{(|x| + t)^2} + \frac{2|x|t\alpha_1}{(|x| + t)^2} \geq c_0$$

for some constant $c_0 > 0$ determined by $\alpha_0$, where $\alpha_1$ is the cosine of the angle between $x$ and $y - x$. Since $x + (1 - a)tz = y$ if $z = (y - x)/|y - x|$, it follows that

$$\frac{|y|^2}{(|x| + t)^2} \geq c_0 - \frac{at((2 - a)t + 2|x|\alpha_1)}{(|x| + t)^2}. \tag{2.68}$$

Denote by $m$ the second term on the right hand side of (2.68). Then

$$|m| \leq a\left(\frac{(2 - a)t^2}{(|x| + t)^2} + \frac{2|x|t|\alpha_1|}{(|x| + t)^2}\right) \leq (3 - a)a.$$

We see that if $a$ was fixed so small that $c_0 - (3-a)a > 0$, our claim follows. □

## 3. Estimation results for conformal measures

We will prove in this chapter a number of estimation results regarding conformal measures of Kleinian groups. We will use these results in Chapter 6 in the context of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups. The natural context of the results is, however, far more general: the conformal measures need not be Patterson-Sullivan measures and the Kleinian groups need not be geometrically finite. Accordingly, we will formulate our results in a context whose generality, we think, is natural given the arguments that we will be employing in the proofs. The last result of this chapter, Theorem 3.14, collects the preceding results of the chapter into a result that will be directly applicable in the context of geometrically finite Kleinian groups in Chapter 6.

Technically speaking, we will be dealing with the situation introduced in Lemma 2.43. We consider a non-elementary Kleinian group $G$ acting on $\mathbb{H}^{n+1}$ and an $s$-conformal measure $\mu$ of $G$ for some $s > 0$ such that $\mu = h^s_* \nu$, where $h \in \text{Möb}(n+1)$ maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and $\nu$ is an $s$-conformal measure of $h^{-1}Gh$. Given $f \in \text{Möb}(\mathbb{H}^{n+1})$, we use the notation $\mu^f = (f \circ h)^s_* \nu$, and the most general of the following estimation results will feature measures of this form.

Recall from our discussion on conformal measures in Chapter 2 that the natural point of view of this work is that conformal measures of Kleinian groups acting on $\mathbb{B}^{n+1}$ are the primary objects of study and that conformal measures of Kleinian groups acting on $\mathbb{H}^{n+1}$ are auxiliary objects used in this study. This principle is at work in this chapter: we focus on Kleinian groups acting on $\mathbb{H}^{n+1}$ and their conformal measures because this situation is technically simpler than that of Kleinian groups acting on $\mathbb{B}^{n+1}$.

We begin with a theorem similar to Lemma 2C of [Tukia1994b], Theorem 3.1. The proof we give for the theorem is similar to the proof of Lemma 2C in [Tukia1994b]. We remind the reader that a point in $\mathbb{H}^{n+1}$ is often written in the form $(x,t)$, where $x \in \mathbb{R}^n$ and $t > 0$, and that if $x \in \mathbb{R}^n$ and $t > 0$, then $\bar{B}^n(x,t) = \bar{B}^{n+1}(x,t) \cap \mathbb{R}^n$ and the symbols $B^n(x,t)$ and $S^{n-1}(x,t)$ have similar meanings. We also remind the reader that a conformal measure of a Kleinian group $G$ is supported by the limit set $L(G)$ by definition. Note that both of the very similar symbols $v$ and $\nu$ are used in the claim and proof of Theorem 3.1, but this should not be a problem, since the symbols denote objects which are very different from one another.

**Theorem 3.1.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Let $h \in \text{Möb}(n+1)$ map $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and let $\nu$ be an $s$-conformal measure of $h^{-1}Gh$ such that $\mu = h^s_*\nu$. Let $v \in [0,1[$. Let $C \subset \mathbb{H}^{n+1}$ be compact. Then there is a constant $c > 0$ such that the following is true. If $f \in \text{Möb}(\mathbb{H}^{n+1})$, $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in fC$ and $|x - x'|/t \le v$ for some $x' \in fL(G) \cap \mathbb{R}^n$, then*

$$c^{-1}t^s \le \mu^f(\bar{B}^n(x,t)) \le ct^s, \tag{3.2}$$

*where $\mu^f = (f \circ h)^s_* \nu$.*

*Proof.* Let $f$, $x$, $t$ and $x'$ be as in the claim. Let $(\hat{x}, \hat{t}) \in C$ be such that $f(\hat{x}, \hat{t}) = (x, t)$. Let $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ be a euclidean similarity mapping $(\hat{x}, \hat{t})$ to $(x, t)$. Write $\gamma = \lambda^{-1} \circ f$, so $f = \lambda \circ \gamma$. Define

$$M_{(y,u)} = \{\alpha \in \text{Möb}(\mathbb{H}^{n+1}) : \alpha(y, u) = (y, u), \bar{B}^n(y, vu) \cap \alpha L(G) \neq \emptyset\}$$

for every $(y, u) \in C$. Define also $M_C = \bigcup_{(y,u)\in C} M_{(y,u)}$. It is the case that $\gamma \in M_{(\hat{x},\hat{t})}$. Recall that $\mu^f = (f \circ h)^s_* \nu$. Now

$$\begin{aligned} \mu^f(\bar{B}^n(x, t)) &= \int_{(f\circ h)^{-1}\bar{B}^n(x,t)} |(f \circ h)'|^s d\nu = \int_{(\lambda\circ\gamma\circ h)^{-1}\bar{B}^n(x,t)} |(\lambda \circ \gamma \circ h)'|^s d\nu \\ &= \int_{(\gamma\circ h)^{-1}\bar{B}^n(\hat{x},\hat{t})} (|\lambda'|^s \circ (\gamma \circ h))|(\gamma \circ h)'|^s d\nu = \left(\frac{t}{\hat{t}}\right)^s \int_{(\gamma\circ h)^{-1}\bar{B}^n(\hat{x},\hat{t})} |(\gamma \circ h)'|^s d\nu. \end{aligned}$$

Lemma 2.55 implies that $M_C$ is compact. The set $\{\alpha \circ h : \alpha \in M_C\}$ is compact as well. We can choose a compact set $D \subset \mathbb{R}^n$ such that $\bar{B}^n(\hat{x}, \hat{t}) \subset D$ in every case considered. Since $\infty \notin (\alpha \circ h)^{-1}D \subset \mathbb{S}^n = \partial\mathbb{B}^{n+1}$ for all $\alpha \in M_C$, we can use Lemma 2.54 to conclude that there is a constant $c_0 > 0$ such that $c_0^{-1} \leq |(\alpha \circ h)'(y)| \leq c_0$ for every $\alpha \in M_C$ and every $y \in (\alpha \circ h)^{-1}D$. We obtain, therefore, that

$$c_0^{-s} \left(\frac{t}{\hat{t}}\right)^s \nu((\gamma \circ h)^{-1}\bar{B}^n(\hat{x}, \hat{t})) \leq \mu^f(\bar{B}^n(x, t)) \leq c_0^s \left(\frac{t}{\hat{t}}\right)^s \nu((\gamma \circ h)^{-1}\bar{B}^n(\hat{x}, \hat{t})).$$

The definition of $\nu$ implies that $\nu((\gamma \circ h)^{-1}\bar{B}^n(\hat{x}, \hat{t})) \leq \nu(\mathbb{S}^n) < \infty$ and the compactness of $C$ implies that $\hat{t}$ is bounded from below and above by positive constants. So to prove our claim, it suffices to show that $\nu((\alpha \circ h)^{-1}\bar{B}^n(y, u))$, where $(y, u) \in C$ and $\alpha \in M_{(y,u)}$ are arbitrary, is bounded from below by a positive constant.

Note first that since $\bar{B}^n(y, vu) \cap \alpha L(G) \neq \emptyset$, where $(y, u) \in C$ and $\alpha \in M_{(y,u)}$, it is true that $(\alpha \circ h)^{-1}B^{n+1}(y, u) \cap L(h^{-1}Gh) \neq \emptyset$. Lemma 2.47 implies that $\nu((\alpha \circ h)^{-1}\bar{B}^n(y, u)) > 0$ for every $(y, u) \in C$ and every $\alpha \in M_{(y,u)}$.

Suppose that we can find a sequence $((y_i, u_i))_i$ in $C$ and a sequence $(\alpha_i)_i$ in $M_C$ with $\alpha_i \in M_{(y_i,u_i)}$ such that $\nu((\alpha_i \circ h)^{-1}\bar{B}^n(y_i, u_i)) \to 0$. We need to deduce a contradiction to finish the proof. According to Lemma 2.55, we can assume that $(y_i, u_i) \to (z, w) \in C$ and $\alpha_i \to \beta$ uniformly for some $\beta \in M_{(z,w)}$. Now $\alpha_i \circ h \to \beta \circ h$ uniformly. It is evident that we can choose $\zeta \in (\beta \circ h)^{-1}B^{n+1}(z, w) \cap L(h^{-1}Gh)$ and $\varepsilon > 0$ so that $B^{n+1}(\zeta, \varepsilon) \cap \mathbb{S}^n \subset (\alpha_i \circ h)^{-1}\bar{B}^n(y_i, u_i)$ for all large enough $i$. Applying Lemma 2.47 again, we obtain the contradiction

$$0 < \nu(B^{n+1}(\zeta, \varepsilon) \cap \mathbb{S}^n) \leq \nu((\alpha_i \circ h)^{-1}\bar{B}^n(y_i, u_i)) \to 0,$$

which concludes our proof. □

We prove next three estimation results of increasing generality involving bounded parabolic fixed points and horoballs. The first of these results, Theorem 3.3, gives estimates for quantities of the form $\mu(\bar{B}^n(p, t))$, where $\mu$ is a conformal measure of a non-elementary Kleinian group $G$ acting on $\mathbb{H}^{n+1}$, $p \in \mathbb{R}^n$ is a bounded parabolic fixed point of $G$ such that $\mu(p) = 0$, and $(p, t)$ is contained in a given horoball $H_p$ of $\mathbb{H}^{n+1}$ based at $p$. Theorem 3.3 is essentially a special case of Theorem 2B of [Tukia1994c]. The next theorem, Theorem 3.6, generalizes the estimates of Theorem 3.3 to include quantities of the form $\mu(\bar{B}^n(x, t))$, where $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in H_p$ and $|x - x'|/t \le \nu$ for some $x' \in L(G) \cap \mathbb{R}^n$ and a fixed number $\nu \in [0,1[$. We found some ideas of [SV1995] very useful when developing our proof for Theorem 3.6. The paper [SV1995] considers similar estimation results for Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups acting on $\mathbb{B}^{n+1}$. The last theorem, Theorem 3.12, generalizes Theorem 3.6 so that the quantities estimated are of the form $\mu^f(\bar{B}^n(x,t))$, where $f \in \text{Möb}(\mathbb{H}^{n+1})$ is a Möbius transformation of a particular form and $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in fH_p$ and that $|x - x'|/t \le \nu$ for some $x' \in fL(G) \cap \mathbb{R}^n$. The way Theorem 3.12 generalizes Theorem 3.6 is very similar to the way Theorem 2B of [Tukia1994c] generalizes our Theorem 3.3.

**Theorem 3.3.** *Let $G$, $\mu$, $s$, $h$ and $\nu$ be as in Theorem 3.1. Suppose that $p \in \mathbb{R}^n$ is a bounded parabolic fixed point of $G$ of rank $k \in \{1,2,\ldots,n\}$ such that $\mu(p) = 0$. Let $H_p$ be a horoball of $\mathbb{H}^{n+1}$ based at $p$. Then*

$$c^{-1}t^{2s-k} \le \mu(\bar{B}^n(p,t)) \le ct^{2s-k} \tag{3.4}$$

*for every $t > 0$ such that $(p,t) \in H_p$, where $c > 0$ is a constant.*

*Proof.* We can assume that $p = 0$. Let $\sigma$ be the inversion $x \mapsto x/|x|^2$ in $\mathbb{S}^n$. Denote $\hat{G} = \sigma G \sigma$ and $\hat{\mu} = \mu^\sigma = (\sigma \circ h)^s_* \nu$. Then $\hat{\mu}$ is an $s$-conformal measure of $\hat{G}$.

The point $\infty$ is a bounded parabolic fixed point of $\hat{G}$ of rank $k$. Recall that the elements in $\hat{G}_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. Choose $y_0 \in L(\hat{G}) \cap \mathbb{R}^n$. According to the claim (i) of Lemma 2.42, there are constants $c_0 > 0$, $c_1 > 0$, $r_0 > 0$ and $R_0 > 0$ satisfying the following. Write $A_i = \{z \in \mathbb{R}^n : c_0 i \le |z| < c_0(i+1)\}$ for $i \in \{1,2,\ldots\}$ . It is now the case that $A_i$ contains at least $c_1^{-1} i^{k-1}$ pairwise disjoint balls of the form $gB^n(y_0,r_0)$, $g \in \hat{G}_\infty$, for every $i \in \{1,2,\ldots\}$. Furthermore, the set $A_i \cap L(\hat{G})$, $i \in \{1,2,\ldots\}$, has a covering of open balls of the form $gB^n(y_0,R_0)$, $g \in \hat{G}_\infty$, containing at most $c_1 i^{k-1}$ elements. It is clear that we can assume that $c_0 = 1$ for the sake of notational convenience.

Let $i \in \{1,2,\ldots\}$. We use (2.44) to conclude that

$$\int_{A_i} |\sigma'|^s d\hat{\mu} = \mu(\sigma A_i).$$

Recall that $|\sigma'(z)| = |z|^{-2}$ for $z \in \mathbb{R}^{n+1} \setminus \{0\}$ (see (2.10)) and that we set $c_0 = 1$ above. We see that

$$(i+1)^{-2s}\hat{\mu}(A_i) \le \mu(\sigma A_i) \le i^{-2s}\hat{\mu}(A_i).$$

To estimate $\hat{\mu}(A_i) = \hat{\mu}(A_i \cap L(\hat{G}))$, note that $\hat{\mu}(gB^n(y_0,r_0)) = \hat{\mu}(B^n(y_0,r_0)) > 0$ and $\hat{\mu}(gB^n(y_0,R_0)) = \hat{\mu}(B^n(y_0,R_0)) > 0$ for every $g \in \hat{G}_\infty$ by Lemma 2.48. We can now use the properties of the constants $c_0$, $c_1$, $r_0$ and $R_0$ established above to conclude that

$$c_1^{-1} i^{k-1}\hat{\mu}(B^n(y_0,r_0)) \le \hat{\mu}(A_i) \le c_1 i^{k-1}\hat{\mu}(B^n(y_0,R_0)).$$

We obtain that there is a constant $c_2 > 0$ such that

$$c_2^{-1} i^{-2s+k-1} \le \mu(\sigma A_i) \le c_2 i^{-2s+k-1} \tag{3.5}$$

for every $i \in \{1,2,\ldots\}$.

Let $t > 0$ be such that $(p,t) \in H_p$, i.e. $t \in ]0, d_{\mathrm{euc}}(H_p)[$. We wish to estimate $\mu(\bar{B}^n(p,t))$. Since $\mu(p) = 0$ by assumption, it is the case that

$$\mu(\bar{B}^n(p,t)) = \mu(\bar{B}^n(p,t) \setminus \{0\}) = \mu(\sigma(\mathbb{R}^n \setminus B^n(0,t^{-1}))).$$

Assume that $t^{-1} \in \{2,3,\ldots\}$. Recall that we have assumed that $c_0 = 1$. The set $\mathbb{R}^n \setminus B^n(0,t^{-1})$ is the disjoint union of the sets $A_{t^{-1}}, A_{t^{-1}+1}, \ldots$, and so $\mu(\bar{B}^n(p,t)) = \sum_{i=t^{-1}}^{\infty} \mu(\sigma A_i)$. The estimate (3.5) implies that

$$c_2^{-1} \sum_{i=t^{-1}}^{\infty} i^{-2s+k-1} \le \mu(\bar{B}^n(p,t)) \le c_2 \sum_{i=t^{-1}}^{\infty} i^{-2s+k-1}.$$

According to Theorem 2.49, it is true that $s > k/2$. Hence $2s - k + 1 = u > 1$. A well-known elementary result implies that there is a constant $c_3 > 0$ such that if $i_0 \in \{1,2,\ldots\}$, then $c_3^{-1} i_0^{1-u} \le \sum_{i=i_0}^{\infty} i^{-u} \le c_3 i_0^{1-u}$. Now $1 - u = k - 2s$, so it follows that

$$c_2^{-1} c_3^{-1} t^{2s-k} \le \mu(\bar{B}^n(p,t)) \le c_2 c_3 t^{2s-k}.$$

We have proved our claim in case $t^{-1} \in \{2,3,\ldots\}$.

Suppose next that there are $v_1, v_2 \in ]0,1[$ such that $t^{-1} + v_1, t^{-1} - v_2 \in \{2,3,\ldots\}$. Using the reasoning of the previous case, we see that

$$\mu(\bar{B}^n(p,t)) \ge \mu(\sigma(\mathbb{R}^n \setminus B^n(0,t^{-1}+v_1))) \ge c_2^{-1} c_3^{-1} (t^{-1}+v_1)^{-2s+k} \ge 2^{-2s+k} c_2^{-1} c_3^{-1} t^{2s-k}$$

and

$$\mu(\bar{B}^n(p,t)) \le \mu(\sigma(\mathbb{R}^n \setminus B^n(0,t^{-1}-v_2))) \le c_2 c_3 (t^{-1}-v_2)^{-2s+k} \le 2^{2s-k} c_2 c_3 t^{2s-k},$$

so our claim is valid also in this case.

The final case we need to consider is such that $t \in ]1/2, d_{\mathrm{euc}}(H_p)[$. Lemma 2.47 implies that $\mu(\bar{B}^n(p,w)) > 0$ for every $w > 0$ and Theorem 2.27 implies that $\mu(\bar{B}^n(p,w)) < \infty$ for every $w > 0$. Therefore, there is a constant $c_4 > 0$ such that $c_4^{-1} \le \mu(\bar{B}^n(p,t)) \le c_4$ for every $t$ considered here. We see that the claim of the theorem is trivial for these $t$. We have proved our claim in all possible cases. □

Recall that $d$ denotes the hyperbolic metric of $\mathbb{H}^{n+1}$. We point out again that although we will use both of the very similar symbols $v$ and $\nu$ in the following, this should not be a problem since the symbols denote objects which are very different from one another.

**Theorem 3.6.** *Let $G$, $\mu$, $s$, $h$, $\nu$, $p$, $k$ and $H_p$ be as in Theorem 3.3. Let $v \in [0,1[$. Then there exists a constant $c > 0$ such that the following holds. If $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x,t) \in H_p$ and that $|x - x'|/t \le v$ for some $x' \in L(G) \cap \mathbb{R}^n$, then*

$$c^{-1} t^s e^{d((x,t),\partial H_p)(k-s)} \le \mu(\bar{B}^n(x,t)) \le c t^s e^{d((x,t),\partial H_p)(k-s)}. \tag{3.7}$$

*Proof.* Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. Consider first the situation such that $d((x,t),\partial H_p) \le d_0$, where we have chosen $d_0 > 0$ freely. Lemma 2.56 implies that there is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x,t) \in G_p C$. The formula (2.46) implies that $\mu^g(\bar{B}^n(x,t)) = (g \circ h)_*^s \nu(\bar{B}^n(x,t)) = \mu(\bar{B}^n(x,t))$ for every $g \in G_p$. We can, therefore, apply Theorem 3.1 to deduce that there is a constant $c_0 > 0$ such that

$$c_0^{-1} t^s \le \mu(\bar{B}^n(x,t)) \le c_0 t^s.$$

Our claim is obviously true in the considered case. We can hence use, during the rest of this proof, the additional assumption that $d((x,t),\partial H_p) > d_0$, where we can choose $d_0 > 0$ as large as we want.

Let us assume like in the proof of Theorem 3.3 that $p = 0$ and that $\sigma$ denotes the inversion in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Write $\hat{G} = \sigma G \sigma$ and $\hat{\mu} = \mu^\sigma = (\sigma \circ h)^s_* \nu$. We continue to consider $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ as in the claim. Bear in mind that we have made the additional assumption that $d((x,t),\partial H_p) > d_0$, where we can choose $d_0 > 0$ freely.

According to Theorem 3.3, there is a constant $c_1 > 0$ such that

$$c_1^{-1} u^{2s-k} \leq \mu(\bar{B}^n(p,u)) \leq c_1 u^{2s-k} \tag{3.8}$$

for every $u \in ]0, d_{\mathrm{euc}}(H_p)[$. Note that $d((p,u),\partial H_p) = \log(d_{\mathrm{euc}}(H_p)/u)$ for every $u \in ]0, d_{\mathrm{euc}}(H_p)[$. We see that the claim is true in case $x = p$. So let us assume that $x \neq p$ for the rest of this proof.

Choose a small $\varepsilon > 0$. We will eventually see how small $\varepsilon$ needs to be chosen. If $y \in \mathbb{R}^n \setminus \{0\}$, then $d((y,|y|),(y,w)) = \varepsilon$ if and only if $w = e^{\pm\varepsilon}|y|$. We will consider separately the situations where $t \leq e^{-\varepsilon}|x|$, $t \geq e^{\varepsilon}|x|$ and $e^{-\varepsilon}|x| < t < e^{\varepsilon}|x|$.

Assume that $t \leq e^{-\varepsilon}|x|$. Now

$$\sigma \bar{B}^n(x,t) = \bar{B}^n\left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right) = \bar{B}^n(\hat{x},\hat{t}). \tag{3.9}$$

We wish to apply the claim (ii) of Lemma 2.42 to $\bar{B}^n(\hat{x},\hat{t})$. To do this, we observe that

$$\hat{d} = d_{\mathrm{euc}}(\sigma(x'), S^{n-1}(\hat{x},\hat{t})) \geq \frac{1}{|x| + vt} - \frac{1}{|x| + t} = \frac{(1-v)t}{(|x|+t)(|x|+vt)}.$$

We have thus that

$$\frac{\hat{d}}{\hat{t}} \geq \frac{(1-v)t}{(|x|+t)(|x|+vt)} \cdot \frac{|x|^2 - t^2}{t} \geq \frac{(1-v)(1-e^{-\varepsilon})}{1 + ve^{-\varepsilon}} > 0.$$

We conclude that there is a constant $\hat{v} \in [0,1[$ such that $|\hat{x} - \sigma(x')|/\hat{t} \leq \hat{v}$. On the other hand, $\{z \in H_p : d(z,\partial H_p) > d_0\} = H'_p$ for some horoball $H'_p \subset H_p$ of $\mathbb{H}^{n+1}$ based at $p$ by Lemma 2.31. It is the case that

$$(\hat{x},\hat{t}) \in \sigma H'_p = \{z \in \mathbb{H}^{n+1} : z_{n+1} > 1/d_{\mathrm{euc}}(H'_p)\}.$$

Since $d_{\mathrm{euc}}(H'_p) \to 0$ if $d_0 \to \infty$, we see that if $d_0$ is chosen large enough, we have that $\hat{t}$ is larger than any given positive number. Let us choose $y_0 \in L(\hat{G}) \cap \mathbb{R}^n$. We can now use the claim (ii) of Lemma 2.42 to deduce the existence of constants $c_2 > 0$, $r_0 > 0$ and $R_0 > 0$ satisfying the following two claims. The ball $\bar{B}^n(\hat{x},\hat{t})$ contains at least $c_2^{-1}\hat{t}^k$ pairwise disjoint balls of the form $gB^n(y_0,r_0)$, $g \in \hat{G}_\infty$. The set $\bar{B}^n(\hat{x},\hat{t}) \cap L(\hat{G})$ has a covering of balls of the form $gB^n(y_0,R_0)$, $g \in \hat{G}_\infty$, containing at most $c_2\hat{t}^k$ elements.

To estimate $\hat{\mu}(\bar{B}^n(\hat{x},\hat{t})) = \hat{\mu}(\bar{B}^n(\hat{x},\hat{t}) \cap L(\hat{G}))$, note that $\hat{\mu}(gB^n(y_0,r_0)) = \hat{\mu}(B^n(y_0,r_0)) > 0$ and $\hat{\mu}(gB^n(y_0,R_0)) = \hat{\mu}(B^n(y_0,R_0)) > 0$ for every $g \in \hat{G}_\infty$ by Lemma 2.48. We obtain that

$$c_2^{-1}\hat{t}^k \hat{\mu}(B^n(y_0,r_0)) \leq \hat{\mu}(\bar{B}^n(\hat{x},\hat{t})) \leq c_2 \hat{t}^k \hat{\mu}(B^n(y_0,R_0)). \tag{3.10}$$

We use next (2.44) to conclude that

$$\int_{\bar{B}^n(\hat{x},\hat{t})} |\sigma'|^s d\hat{\mu} = \mu(\bar{B}^n(x,t)).$$

Combining this with the fact that $(|x|-t)^2 \le |\sigma'| \le (|x|+t)^2$ in $\bar{B}^n(\hat{x},\hat{t})$ (see (2.10)), we can use (3.9) and (3.10) to deduce that there is a constant $c_3 > 0$ such that

$$c_3^{-1}(|x|-t)^{2s}\left(\frac{t}{|x|^2-t^2}\right)^k \le \mu(\bar{B}^n(x,t)) \le c_3(|x|+t)^{2s}\left(\frac{t}{|x|^2-t^2}\right)^k.$$

Lemma 2.32 gives that

$$e^{d((x,t),\partial H_p)} = \frac{d_{\mathrm{euc}}(H_p)t}{|x|^2+t^2}. \tag{3.11}$$

It is easy to show that there is a constant $c_4 > 0$ such that

$$(|x|-t)^{2s}\left(\frac{t}{|x|^2-t^2}\right)^k t^{-s} e^{d((x,t),\partial H_p)(s-k)} \ge c_4^{-1}$$

and

$$(|x|+t)^{2s}\left(\frac{t}{|x|^2-t^2}\right)^k t^{-s} e^{d((x,t),\partial H_p)(s-k)} \le c_4.$$

We see that we have proved our claim in case $t \le e^{-\varepsilon}|x|$.

Suppose next that $t \ge e^{\varepsilon}|x|$. We have that

$$\bar{B}^n(0,t-|x|) \subset \bar{B}^n(x,t) \subset \bar{B}^n(0,t+|x|).$$

It is clear that if $c_1$ is adjusted if necessary, we can use (3.8) to deduce that

$$c_1^{-1}(t-|x|)^{2s-k} \le \mu(\bar{B}^n(x,t)) \le c_1(t+|x|)^{2s-k}.$$

One sees easily that (recall (3.11))

$$(t-|x|)^{2s-k} t^{-s} e^{d((x,t),\partial H_p)(s-k)} \ge c_5^{-1}$$

and

$$(t+|x|)^{2s-k} t^{-s} e^{d((x,t),\partial H_p)(s-k)} \le c_5,$$

where $c_5 > 0$ is a constant. We see that the claim of the theorem is valid in case $t \ge e^{\varepsilon}|x|$.

Assume finally that $e^{-\varepsilon}|x| < t < e^{\varepsilon}|x|$. Suppose that $d_0$ was chosen so large that $(x, e^{-\varepsilon}|x|), (x, e^{\varepsilon}|x|) \in H_p$. We can guarantee by assuming that $\varepsilon$ was chosen small enough that $x' \in B^n(x, e^{-\varepsilon}|x|)$ and that $|x-x'|/(e^{-\varepsilon}|x|) \le v'$ for some fixed $v' \in [0,1[$. According to what we have already proved above, there is a constant $c_6 > 0$ such that

$$\mu(\bar{B}^n(x,t)) \le \mu(\bar{B}^n(x,e^{\varepsilon}|x|)) \le c_6(e^{\varepsilon}|x|)^s e^{d((x,e^{\varepsilon}|x|),\partial H_p)(k-s)}.$$

Our assumption implies that $|x| \le e^{\varepsilon}t$. Since $d((x,t),(x,e^{\varepsilon}|x|)) \le 2\varepsilon$, it is easy to see that

$$d((x,t),\partial H_p) - 2\varepsilon \le d((x,e^{\varepsilon}|x|),\partial H_p) \le d((x,t),\partial H_p) + 2\varepsilon.$$

It is evident that the upper estimate of (3.7) follows immediately. And it is clear that we can prove the lower estimate of (3.7) by applying a similar argument to the fact that $\mu(\bar{B}^n(x,t)) \ge \mu(\bar{B}^n(x,e^{-\varepsilon}|x|))$. We have proved our claim in all situations. $\square$

**Theorem 3.12.** *Let $G$, $\mu$, $s$, $h$, $\nu$, $p$, $k$, $H_p$ and $v$ be as in Theorem 3.6. Let $A \subset \text{Möb}(\mathbb{H}^{n+1})$ be a non-empty and compact set of Möbius transformations mapping $H_p$ onto itself. Then there exists a constant $c > 0$ such that the following is true. Let $\alpha \in A$ and let $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ be a euclidean similarity. Write $f = \lambda \circ \alpha$. Now if $x \in \mathbb{R}^n$ and $t > 0$ are such that $(x, t) \in fH_p$ and there is $x' \in fL(G) \cap \mathbb{R}^n$ with $|x - x'|/t \leq v$, then*

$$c^{-1} t^s e^{d((x,t),\partial fH_p)(k-s)} \leq \mu^f(\bar{B}^n(x,t)) \leq ct^s e^{d((x,t),\partial fH_p)(k-s)}, \tag{3.13}$$

*where $\mu^f = (f \circ h)^s_* \nu$.*

*Proof.* Let $f = \lambda \circ \alpha$, $x$, $t$ and $x'$ be as in the claim. Let us first consider the case such that $d((x,t), \partial fH_p) \leq d_0$ for some freely chosen $d_0 > 0$. Lemma 2.56 implies that there is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x,t) \in (f \circ g)C$ for some $g \in G_p$. We can use the formula (2.45) and Theorem 3.1 to conclude that there is a constant $c_0 > 0$ such that

$$c_0^{-1} t^s \leq \mu^{f\circ g}(\bar{B}^n(x,t)) = \mu^f(\bar{B}^n(x,t)) \leq c_0 t^s.$$

Therefore, our claim is valid in the considered case. This means that we can apply, for the rest of this proof, the additional assumption that $d((x,t), \partial fH_p) > d_0$, where $d_0 > 0$ can be chosen freely.

Let us write $y = \lambda^{-1}(x)$, $y' = \lambda^{-1}(x')$ and $u = |\lambda'|^{-1} t$. Then $\bar{B}^n(y,u) = \lambda^{-1}\bar{B}^n(x,t)$, $(y,u) \in H_p$ with $d((y,u), \partial H_p) = d((x,t), \partial fH_p) > d_0$, and $y' \in \alpha L(G) \cap \mathbb{R}^n$ with $|y - y'|/u = |x - x'|/t \leq v$. Observe that Lemma 2.57 can be applied to the present situation. This means that the following four statements are true when we choose $d_0$ large enough. It is true that $\alpha^{-1}\bar{B}^n(y,u) = f^{-1}\bar{B}^n(x,t) = \bar{B}^n(\hat{x},\hat{t})$ for some $\hat{x} \in \mathbb{R}^n$ and $\hat{t} > 0$. We have that $c_1^{-1}|\lambda'|^{-1} t \leq \hat{t} \leq c_1 |\lambda'|^{-1} t$ and $|\hat{x} - f^{-1}(x')|/\hat{t} \leq \hat{v}$ for some constants $c_1 > 0$ and $\hat{v} \in [0, 1[$. It is the case that $c_1^{-1} \leq |\alpha'| \leq c_1$ in $\bar{B}^n(\hat{x},\hat{t})$. It is true that $(\hat{x},\hat{t}) \in H_p$ so that

$$d((x,t), \partial fH_p) - c_2 \leq d((\hat{x},\hat{t}), \partial H_p) \leq d((x,t), \partial fH_p) + c_2,$$

where $c_2 > 0$ is a constant.

We conclude that Theorem 3.6 can be used to estimate $\mu(\bar{B}^n(\hat{x},\hat{t}))$. We see that there is a constant $c_3 > 0$ such that

$$c_3^{-1} \hat{t}^s e^{d((\hat{x},\hat{t}),\partial H_p)(k-s)} \leq \mu(\bar{B}^n(\hat{x},\hat{t})) \leq c_3 \hat{t}^s e^{d((\hat{x},\hat{t}),\partial H_p)(k-s)}.$$

We can use the facts established in the previous paragraph to deduce that

$$c_4^{-1} |\lambda'|^{-s} t^s e^{d((x,t),\partial fH_p)(k-s)} \leq \mu(\bar{B}^n(\hat{x},\hat{t})) \leq c_4 |\lambda'|^{-s} t^s e^{d((x,t),\partial fH_p)(k-s)},$$

where $c_4 > 0$ is a constant. Recall that $\mu^f = (f \circ h)^s_* \nu$ and that $c_1^{-1} \leq |\alpha'| \leq c_1$ in $\bar{B}^n(\hat{x},\hat{t})$. We calculate that

$$\begin{aligned}
\mu^f(\bar{B}^n(x,t)) &= \int_{(f\circ h)^{-1}\bar{B}^n(x,t)} |(f \circ h)'|^s d\nu = \int_{h^{-1}\bar{B}^n(\hat{x},\hat{t})} (|f'|^s \circ h)|h'|^s d\nu \\
&= \int_{h^{-1}\bar{B}^n(\hat{x},\hat{t})} (|\lambda'|^s \circ (\alpha \circ h))(|\alpha'|^s \circ h)|h'|^s d\nu \\
&\leq c_1^s |\lambda'|^s \int_{h^{-1}\bar{B}^n(\hat{x},\hat{t})} |h'|^s d\nu = c_1^s |\lambda'|^s \mu(\bar{B}^n(\hat{x},\hat{t})).
\end{aligned}$$

A similar calculation gives that

$$\mu^f(\bar{B}^n(x,t)) \geq c_1^{-s}|\lambda'|^s \mu(\bar{B}^n(\hat{x},\hat{t})).$$

The above results imply (3.13). □

We end this chapter with the following theorem that combines the earlier estimation results into an estimation result that is immediately applicable in Chapter 6 in the context of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups.

**Theorem 3.14.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Let $v \in [0,1[$. Let $P \subset \mathbb{R}^n$ be a finite, possibly empty, set of bounded parabolic fixed points of $G$ such that if $p_1$ and $p_2$ are two points in $P$, then $Gp_1 \cap Gp_2 = \emptyset$. Assume that $\mu(p) = 0$ for all $p \in P$. Suppose that there is a pairwise disjoint collection $\{H_p : p \in GP\}$ of horoballs of $\mathbb{H}^{n+1}$ such that $H_p$, $p \in GP$, is based at $p$ and that $gH_p = H_{g(p)}$ for every $g \in G$ and every $p \in GP$. Let $C \subset \mathbb{H}^{n+1}$ be a compact set.*

*The following is true in this situation. There is a constant $c > 0$ such that the following holds. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that there is $x' \in L(G) \cap \mathbb{R}^n$ with $|x - x'|/t \leq v$. Then*

$$c^{-1}t^s \leq \mu(\bar{B}^n(x,t)) \leq ct^s \tag{3.15}$$

*if $(x,t) \in GC$, and*

$$c^{-1}t^s e^{d((x,t),\partial H_p)(k(p)-s)} \leq \mu(\bar{B}^n(x,t)) \leq ct^s e^{d((x,t),\partial H_p)(k(p)-s)} \tag{3.16}$$

*if $(x,t) \in H_p$ for some $p \in GP \cap \mathbb{R}^n$, where $k(p)$ is the rank of $p$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. Suppose first that $(x,t) \in GC$. This means that $(x,t) \in gC$ for some $g \in G$ and that $x' \in gL(G) \cap \mathbb{R}^n$ since $L(G)$ is $G$-invariant. The formula (2.46) implies that $\mu^g(\bar{B}^n(x,t)) = \mu(\bar{B}^n(x,t))$. Theorem 3.1 implies now that there is a constant $c_0 > 0$ such that

$$c_0^{-1}t^s \leq \mu(\bar{B}^n(x,t)) \leq c_0 t^s$$

in the considered case. We have proved (3.15).

Let $q \in P$. Let $k \in \{1,2,\dots,n\}$ be the rank of $q$. The collection $\{gH_q : g \in G\}$ of horoballs of $\mathbb{H}^{n+1}$ is pairwise disjoint by assumption. Lemma 2.59 implies that there is a compact set $A \subset \text{Möb}(\mathbb{H}^{n+1})$ satisfying the following two claims. The elements in $A$ map $H_q$ onto itself. If $g \in G$ and $g(q) \in \mathbb{R}^n$, then there is $f \in G_q$ such that $g \circ f = \lambda \circ \alpha$, where $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ is a euclidean similarity mapping $H_q$ onto $H_{g(q)}$ and $\alpha \in A$.

Suppose that $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ are as in the claim and additionally that $(x,t) \in H_{g(q)}$ for some $g \in G$ such that $g(q) \in \mathbb{R}^n$. According to the above, there is $f \in G_q$, $\alpha \in A$ and a euclidean similarity $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ mapping $H_q$ onto $H_{g(q)}$ such that $g \circ f = \lambda \circ \alpha$. Note that $H_{g(q)} = H_{(g\circ f)(q)} = (g \circ f)H_q = (\lambda \circ \alpha)H_q$ and furthermore that $x' \in (g \circ f)L(G) \cap \mathbb{R}^n = (\lambda \circ \alpha)L(G) \cap \mathbb{R}^n$ because $L(G)$ is $G$-invariant. Note also that $\mu^{\lambda\circ\alpha}(\bar{B}^n(x,t)) = \mu^{g\circ f}(\bar{B}^n(x,t)) = \mu(\bar{B}^n(x,t))$ by (2.46). We can now use Theorem 3.12 to establish that there is a constant $c_1 > 0$ such that

$$c_1^{-1}t^s e^{d((x,t),\partial H_{g(q)})(k-s)} \leq \mu(\bar{B}^n(x,t)) \leq c_1 t^s e^{d((x,t),\partial H_{g(q)})(k-s)}.$$

This means that we have proved (3.16) for all $(x,t)$ such that $(x,t) \in H_p$ for some $p \in Gq \cap \mathbb{R}^n$. Since $P$ is finite by assumption, we see that (3.16) is valid for all $(x,t)$ such that $(x,t) \in H_p$ for some $p \in GP \cap \mathbb{R}^n$, which completes the proof. □

## 4. Geometry of limit sets of Kleinian groups

This chapter is devoted to the study of geometry of limit sets of non-elementary Kleinian groups. Like in Chapter 3, we will consider Kleinian groups acting on $\mathbb{H}^{n+1}$ because the technical details of the arguments are simpler for these groups than for Kleinian groups acting on $\mathbb{B}^{n+1}$. Moreover, we will formulate and prove our results in a context which is more general than the one in which we will actually apply the results. The general settings of the main results of this chapter are the same as in Chapter 3: the settings of Theorem 4.5 and Theorem 3.1 correspond to one another, as well as the settings of Theorems 4.10 and 4.28 and Theorems 3.3, 3.6 and 3.12. Like Theorem 3.14 in Chapter 3, Theorem 4.37 will present the main results of this chapter in one result that is immediately applicable in Chapter 6 in the context of non-elementary geometrically finite Kleinian groups. After we have proved the main results of this chapter, we will prove additional results which we will need in Chapter 7 when discussing some variants of the main results of this work.

4.1. **The main results.** Let us define explicitly the geometric properties of limit sets of non-elementary Kleinian groups which are the objects of study in this chapter. These properties were studied first by P. Tukia in [Tukia1985b]. Our exposition on the topic is more explicit in quantitative terms and our context is more general.

Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. We are interested in certain geometric properties of sets of the form $\bar{B}^n(x,t) \cap fL(G)$, where $x \in \mathbb{R}^n$, $t > 0$ and $f \in \text{Möb}(\mathbb{H}^{n+1})$. More specifically, given $f \in \text{Möb}(\mathbb{H}^{n+1})$, $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap fL(G) \neq \emptyset$, we define that

$$\beta^f(x,t) = \frac{1}{t} d_{\text{euc}}(\bar{B}^n(x,t) \cap fL(G)) \tag{4.1}$$

and

$$\gamma_l^f(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap fL(G), \bar{B}^n(x,t) \cap V) \tag{4.2}$$

for every $l \in \{1,2,\ldots,n\}$, where $\mathcal{F}_l(x,t)$ is the collection of $l$-spheres of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(x,t)$ and $\rho$ is the Hausdorff metric defined with respect to the euclidean metric in the space of all non-empty and compact subsets of $\mathbb{R}^n$, i.e.

$$\rho(A,B) = \sup\{d_{\text{euc}}(y,B), d_{\text{euc}}(z,A) : y \in A, z \in B\} \tag{4.3}$$

for all non-empty and compact $A, B \subset \mathbb{R}^n$. (Recall that we call both the euclidean spheres of $\bar{\mathbb{R}}^n$ and the euclidean planes of $\bar{\mathbb{R}}^n$ the spheres of $\bar{\mathbb{R}}^n$.) If $f \in G$, we write simply $\beta^f = \beta$ and $\gamma_l^f = \gamma_l$ since $fL(G) = L(G)$ in this case.

The quantity $\beta^f(x,t)$ is the euclidean diameter of $\bar{B}^n(x,t) \cap fL(G)$ on a normalized scale and $\gamma_l^f(x,t)$ measures, again on a normalized scale, how closely $fL(G)$ resembles an $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$. Due to this geometric interpretation, we name the functions $\gamma_l^f$ *flatness functions*.

We prove the following preliminary result before going into our main results.

**Theorem 4.4.** *Suppose that $G$ is a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $l \in \{1, 2, \dots, n\}$ and suppose that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. Let $f \in \text{Möb}(\mathbb{H}^{n+1})$. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $B^n(x, t) \cap fL(G) \neq \emptyset$. Then $\gamma_l^f(x, t) > 0$.*

*Proof.* We can assume that $f = \text{id}$ since $fL(G)$ is the limit set of the Kleinian group $fGf^{-1}$ acting on $\mathbb{H}^{n+1}$. We follow the above convention and write $\gamma_l^f = \gamma_l$. Suppose that $\gamma_l(x, t) = 0$. Our aim is to derive a contradiction.

The assumption $\gamma_l(x, t) = 0$ implies that there is a sequence $(V_i)_i$ in $\mathcal{F}_l(x, t)$ such that

$$\rho_i = \rho(\bar{B}^n(x, t) \cap L(G), \bar{B}^n(x, t) \cap V_i) \to 0.$$

We use Lemma 2.61 with $Z = \bar{B}^n(x, t)$ and assume that the claim of Lemma 2.61 is true for the sequence $(V_i)_i$ itself. Let $W$ be the set given by Lemma 2.61. We can now use Lemma 2.63 to conclude that there is a constant $c_0 > 0$ such that $\rho_i \geq c_0$ for all large enough $i$. The situation is contradictory, and so we conclude that $\gamma_l(x, t) > 0$. □

The following four theorems are the main results of this chapter. As mentioned above, the settings of Theorem 4.5 and Theorem 3.1 correspond to one another, as well as the settings of Theorems 4.10 and 4.28 and Theorems 3.3, 3.6 and 3.12. Moreover, Theorem 4.37 will express the obtained results in one result that we can directly apply in Chapter 6 in the context of non-elementary geometrically finite Kleinian groups.

**Theorem 4.5.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $C \subset \mathbb{H}^{n+1}$ be compact. Let $v \in [0, 1[$. Then there are constants $b > 0$ and $c > 0$ such that the following holds. Let $f \in \text{Möb}(\mathbb{H}^{n+1})$, $x \in \mathbb{R}^n$ and $t > 0$ be such that $(x, t) \in fC$ and that $|x - x'|/t \leq v$ for some $x' \in fL(G) \cap \mathbb{R}^n$. Then*

$$b^{-1} \leq \beta^f(x, t) \leq b, \tag{4.6}$$

*and if $l \in \{1, 2, \dots, n\}$ and $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, then*

$$c^{-1} \leq \gamma_l^f(x, t) \leq c. \tag{4.7}$$

*Proof.* Observe first that the upper bounds of (4.6) and (4.7) follow immediately from the definitions (4.1) and (4.2). Let $f$, $x$, $t$ and $x'$ be as in the claim. Let $(\hat{x}, \hat{t}) \in C$ be such that $f(\hat{x}, \hat{t}) = (x, t)$. Let $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ be a euclidean similarity mapping $(\hat{x}, \hat{t})$ to $(x, t)$. Write $\alpha = \lambda^{-1} \circ f$ so $f = \lambda \circ \alpha$. Define that

$$M_{(y,u)} = \{\tau \in \text{Möb}(\mathbb{H}^{n+1}) : \tau(y, u) = (y, u), \bar{B}^n(y, vu) \cap \tau L(G) \neq \emptyset\}$$

for every $(y, u) \in C$. We see that $\alpha \in M_{(\hat{x}, \hat{t})}$.

Note that $|y - z| = (t/\hat{t})|\lambda^{-1}(y) - \lambda^{-1}(z)|$ for every $y, z \in \mathbb{R}^{n+1}$. Note also that $V \in \mathcal{F}_l(\hat{x}, \hat{t})$ if and only if $\lambda V \in \mathcal{F}_l(x, t)$ for every $l \in \{1, 2, \dots, n\}$. Write $A = \bar{B}^n(\hat{x}, \hat{t}) \cap \alpha L(G)$ and $D_V = \bar{B}^n(\hat{x}, \hat{t}) \cap V$ for every $l \in \{1, 2, \dots, n\}$ and every $V \in \mathcal{F}_l(\hat{x}, \hat{t})$. It is now the case that

$$\beta^f(x, t) = \frac{1}{t} \sup_{y, z \in \lambda \bar{B}^n(\hat{x}, \hat{t}) \cap (\lambda \circ \alpha) L(G)} |y - z| = \frac{1}{\hat{t}} d_{\text{euc}}(\bar{B}^n(\hat{x}, \hat{t}) \cap \alpha L(G))$$

and

$$\begin{aligned}\gamma_l^f(x,t) &= \frac{1}{t}\inf_{V\in\mathcal{F}_l(\hat{x},\hat{t})}\sup\left\{\inf_{z'\in\lambda D_V}|y-z'|,\ \inf_{y'\in\lambda A}|y'-z| : y\in\lambda A, z\in\lambda D_V\right\}\\ &= \frac{1}{\hat{t}}\inf_{V\in\mathcal{F}_l(\hat{x},\hat{t})}\rho(\bar{B}^n(\hat{x},\hat{t})\cap\alpha L(G),\bar{B}^n(\hat{x},\hat{t})\cap V)\end{aligned}$$

for every $l\in\{1,2,\dots,n\}$. Observe that $\hat{t}$ is bounded between two positive constants due to the compactness of $C$. We see that the lower bound of (4.6) is valid if we show that the quantities

$$\beta((y,u),\tau) = d_{\mathrm{euc}}(\bar{B}^n(y,u)\cap\tau L(G)) \tag{4.8}$$

are greater than some positive constant, where $(y,u)\in C$ and $\tau\in M_{(y,u)}$ are arbitrary. Similarly, we can establish the lower bound of (4.7) by showing that the quantities

$$\gamma_l((y,u),\tau) = \inf_{V\in\mathcal{F}_l(y,u)}\rho(\bar{B}^n(y,u)\cap\tau L(G),\bar{B}^n(y,u)\cap V) \tag{4.9}$$

are greater than some positive constant, where $(y,u)\in C$ and $\tau\in M_{(y,u)}$ are arbitrary and $l\in\{1,2,\dots,n\}$ is such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. The quantities defined by (4.8) and (4.9) are all positive. This is evident for (4.8), and the positivity of the quantities defined by (4.9) follows from Theorem 4.4.

Suppose that the quantities defined by (4.8) are not greater than some positive constant. This means that we can find $(y_i,u_i)\in C$ and $\tau_i\in M_{(y_i,u_i)}$ such that $\beta((y_i,u_i),\tau_i)\to 0$. According to Lemma 2.55, we can assume that there is $(z,w)\in C$ and $\omega\in M_{(z,w)}$ such that $(y_i,u_i)\to(z,w)$ and $\tau_i\to\omega$ uniformly. We can choose $\zeta_1,\zeta_2\in L(G)$ and $\varepsilon>0$ such that $\bar{B}^n(\omega(\zeta_j),2\varepsilon)\subset B^n(z,w)$, $j=1,2$, and that $\bar{B}^n(\omega(\zeta_1),2\varepsilon)\cap\bar{B}^n(\omega(\zeta_2),2\varepsilon)=\emptyset$. Since $\tau_i\to\omega$ uniformly, we can assume that $\tau_i(\zeta_j)\in B^n(\omega(\zeta_j),\varepsilon)\subset B^n(y_i,u_i)$ for $j=1,2$ and every $i$. We conclude that

$$\beta((y_i,u_i),\tau_i)\geq|\tau_i(\zeta_1)-\tau_i(\zeta_2)|\geq 2\varepsilon>0$$

for every $i$, which is a contradiction. It follows that the quantities defined by (4.8) are greater than some positive constant.

Suppose next that the quantities defined by (4.9) are not greater than some positive constant. This means that we can find $(y_i,u_i)\in C$ and $\tau_i\in M_{(y_i,u_i)}$ such that $\gamma_l((y_i,u_i),\tau_i)\to 0$, where $l\in\{1,2,\dots,n\}$ is a fixed number such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. It follows that there is a sequence $(V_i)_i$ of $l$-spheres of $\bar{\mathbb{R}}^n$ such that $V_i\in\mathcal{F}_l(y_i,u_i)$ and

$$\rho_i=\rho(\bar{B}^n(y_i,u_i)\cap\tau_i L(G),\bar{B}^n(y_i,u_i)\cap V_i)\to 0.$$

We can assume using Lemma 2.55 that $(y_i,u_i)\to(z,w)\in C$ and that $\tau_i\to\omega$ uniformly for some $\omega\in M_{(z,w)}$. We use Lemma 2.61 with $Z=\bar{B}^n(z,w+\kappa)$, where $\kappa>0$ is a fixed number, and assume that the claim of Lemma 2.61 is true for the sequence $(V_i)_i$ itself. Let $W$ be the set given by Lemma 2.61. Observe that $W\cap\bar{B}^n(z,w)\neq\emptyset$. Lemma 2.63 implies the existence of a constant $c_0>0$ such that $\rho_i\geq c_0$ for all large enough $i$. This is a contradiction, and so the quantities defined by (4.9) are greater than some positive constant. □

Recall that we write $\beta^f = \beta$ and $\gamma_l^f = \gamma_l$ for $l \in \{1, 2, \dots, n\}$ if $f \in G$ and $G$ is a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$.

**Theorem 4.10.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $v \in [0, 1[$. Let $p \in \mathbb{R}^n$ be a bounded parabolic fixed point of $G$ of rank $k \in \{1, 2, \dots, n\}$. Let $H_p$ be a horoball of $\mathbb{H}^{n+1}$ based at $p$. Then there are constants $a > 0$, $b > 0$ and $c > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t > 0$ be such that $(x, t) \in H_p$ and that $|x - x'|/t \leq v$ for some $x' \in L(G) \cap \mathbb{R}^n$. Then*

$$b^{-1} \leq \beta(x, t) \leq b, \tag{4.11}$$

$$c^{-1} e^{-d((x,t),\partial H_p)} \leq \gamma_k(x, t) \leq c e^{-d((x,t),\partial H_p)} \tag{4.12}$$

*if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$, and*

$$a^{-1} \leq \gamma_l(x, t) \leq a \tag{4.13}$$

*for every $l \in \{1, 2, \dots, n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* We begin by observing that the upper bounds in (4.11) and (4.13) follow immediately from the definitions (4.1) and (4.2). Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. Assume that $d((x, t), \partial H_p) \leq d_0$, where we have chosen $d_0 > 0$ freely. Lemma 2.56 implies that there is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x, t) \in G_p C$. We note that $\beta^g = \beta$ and $\gamma_l^g = \gamma_l$ for every $l \in \{1, 2, \dots, n\}$ and every $g \in G_p$ and use Theorem 4.5 to deduce that there are constants $c_0 > 0$ and $c_1 > 0$ such that

$$c_0^{-1} \leq \beta(x, t) \leq c_0 \qquad \text{and} \qquad c_1^{-1} \leq \gamma_l(x, t) \leq c_1$$

for every $l \in \{1, 2, \dots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. The estimates (4.11), (4.12) and (4.13) are clearly valid in the considered case. This means that we can make the additional assumption that $d((x, t), \partial H_p) > d_0$ when we consider $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ as in the claim, where we can choose $d_0 > 0$ freely.

Observe next that we can assume that $p = 0$. Let $V_0$ be a fixed $G_p$-invariant $k$-sphere of $\bar{\mathbb{R}}^n$ through $p$ whose existence follows from the claim (i) of Theorem 2.7. Let $\sigma$ be the inversion $y \mapsto y/|y|^2$ in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. Since $p$ is a bounded parabolic fixed point of $G$, the point $\infty$ is a bounded parabolic fixed point of $\sigma G \sigma$. Lemma 2.37 implies that there are constants $u_0 > 0$ and $u_1 > 0$ such that

$$d_{\text{euc}}(y, \sigma V_0) \leq u_0 \tag{4.14}$$

for every $y \in \sigma L(G) \cap \mathbb{R}^n$ and that

$$d_{\text{euc}}(y, \sigma L(G)) \leq u_1 \tag{4.15}$$

for every $y \in \sigma V_0 \cap \mathbb{R}^n$. Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim so that $d((x, t), \partial H_p) > d_0$. We use Lemma 2.35 with $w = (1 - v)/2$ and $y = x'$ to conclude that if $d_0$ is chosen large enough, then (4.14) implies that

$$\frac{d_{\text{euc}}(x, V_0)}{t} \leq c_2 \qquad \text{and} \qquad \frac{d_{\text{euc}}(\bar{B}^n(x, t) \cap V_0)}{t} \geq c_3, \tag{4.16}$$

where $c_2 \in ]0, 1[$ and $c_3 > 0$ are constants. It is the case in particular that $V_0 \in \mathcal{F}_k(x, t)$.

We show next that the lower bound of (4.11) is valid. Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim with $d((x,t), \partial H_p) > d_0$. Assuming that $d_0$ is large enough, (4.16) implies that we can choose $y_1, y_2 \in B^n(x,t) \cap V_0$ and a fixed number $\varepsilon > 0$ such that $\bar{B}^n(y_j, 2\varepsilon t) \subset B^n(x,t)$, $j = 1, 2$, and $\bar{B}^n(y_1, 2\varepsilon t) \cap \bar{B}^n(y_2, 2\varepsilon t) = \emptyset$. We apply Lemma 2.35 with $w = \varepsilon$ and $y = y_j$, $j = 1$ or $j = 2$, and recall (4.15) to deduce that there is $y'_j \in B^n(y_j, \varepsilon t) \cap L(G)$, assuming that $d_0$ is large enough. Now

$$\beta(x,t) \geq \frac{1}{t}|y'_1 - y'_2| \geq 2\varepsilon,$$

so the lower bound of (4.11) is valid.

We turn to the quantities in (4.12) and (4.13). Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim with $d((x,t), \partial H_p) > d_0$. Given $l \in \{1, 2, \ldots, n\}$ and $V \in \mathcal{F}_l(x,t)$, we write

$$\gamma_l^V(x,t) = \frac{1}{t}\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V). \tag{4.17}$$

Recall from Lemma 2.32 that

$$e^{d((x,t),\partial H_p)} = \frac{d_{\mathrm{euc}}(H_p)t}{|x|^2 + t^2}. \tag{4.18}$$

We start with the upper bound of (4.12). Suppose that $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$. Let $x$, $t$ and $x'$ be as above. Recall that $V_0$ denotes a fixed $G_p$-invariant $k$-sphere of $\bar{\mathbb{R}}^n$ through $p = 0$ whose existence is implied by the claim (i) of Theorem 2.7. Recall that $V_0 \in \mathcal{F}_k(x,t)$ by (4.16). The definition (4.2) implies that $\gamma_k(x,t) \leq \gamma_k^V(x,t)$ for every $V \in \mathcal{F}_k(x,t)$. It is natural to suspect that the choice $V = V_0$ gives a good upper bound. We proceed to show that this is indeed the case.

Write $A = \sigma(\bar{B}^n(x,t) \cap L(G)) \cap \mathbb{R}^n$ and $B = \sigma(\bar{B}^n(x,t) \cap V_0) \cap \mathbb{R}^n$. It is the case that $d_{\mathrm{euc}}(y, \sigma V_0) \leq u_0$ and $d_{\mathrm{euc}}(z, \sigma L(G)) \leq u_1$ for every $y \in A$ and $z \in B$ by (4.14) and (4.15). It is not difficult to see that if $u_0$ and $u_1$ are increased appropriately and $d_0$ is assumed to be large enough, then $d_{\mathrm{euc}}(y, B) \leq u_0$ and $d_{\mathrm{euc}}(z, A) \leq u_1$ for every $y \in A$ and $z \in B$. The essential geometric fact to notice here is that (4.16) implies that $V_0$ is not close to being tangential to $S^{n-1}(x,t)$ and so $\sigma V_0$ is not close to being tangential to $\sigma S^{n-1}(x,t)$. Write $u_2 = \max(u_0, u_1)$. Note that

$$|y| \geq d_{\mathrm{euc}}(0, \sigma S^{n-1}(x,t)) = \frac{1}{|x| + t} \tag{4.19}$$

for every $y \in \sigma\bar{B}^n(x,t)$.

Let $y \in \sigma A$. Our reasoning above implies that there is $z_y \in B$ such that $|\sigma(y) - z_y| \leq u_2$. Recall the formula (2.52) and the fact that $|\sigma'(z)| = |z|^{-2}$ for $z \in \mathbb{R}^{n+1} \setminus \{0\}$ (see (2.10)). We use (2.52) and (4.19) to calculate that

$$\begin{aligned}
d_{\mathrm{euc}}(y, \bar{B}^n(x,t) \cap V_0) &\leq d_{\mathrm{euc}}(y, (\bar{B}^n(x,t) \cap V_0) \setminus \{0\}) = d_{\mathrm{euc}}(y, \sigma B) \\
&\leq |y - \sigma(z_y)| = \frac{1}{|\sigma(y)|}\frac{1}{|z_y|}|\sigma(y) - z_y| \\
&\leq (|x| + t)^2|\sigma(y) - z_y| \leq u_2(|x| + t)^2.
\end{aligned}$$

We can use a similar argument to show that the distances $d_{\text{euc}}(z, \bar{B}^n(x,t) \cap L(G))$, $z \in \sigma B$, have an upper bound of the same form. Note that, since $0 \in L(G) \cap V_0$, we have that $d_{\text{euc}}(0, \bar{B}^n(x,t) \cap V_0) = 0 = d_{\text{euc}}(0, \bar{B}^n(x,t) \cap L(G))$ in case $0 \in \bar{B}^n(x,t)$. We can now conclude that

$$\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V_0) \leq u_2(|x|+t)^2.$$

It follows that (recall (4.17))

$$\gamma_k(x,t) \leq \gamma_k^{V_0}(x,t) \leq \frac{u_2}{t}(|x|+t)^2. \tag{4.20}$$

We recall (4.18) and deduce that $e^{d((x,t),\partial H_p)}(|x|+t)^2/t$ is bounded by a constant. The upper bound of (4.12) follows.

We prove next the lower bound of (4.12). We continue to consider $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ as in the claim with $d((x,t),\partial H_p) > d_0$. We start by showing that there are constants $c_4 > 0$ and $c_5 > 0$ such that

$$|y| \geq c_4(|x|+t) \qquad \text{and} \qquad \bar{B}^n(y, c_5 t) \subset B^n(x,t) \tag{4.21}$$

for some $y \in B^n(x,t) \cap V_0$, where $V_0$ is the same $G_p$-invariant $k$-sphere of $\bar{\mathbb{R}}^n$ through $p = 0$ as considered earlier.

If $x = 0$, then (4.21) follows immediately from (4.16). We assume, therefore, that $x \neq 0$ for the moment. Observe that (4.16) implies that there is $y \in (B^n(x,t) \cap V_0) \setminus \{x\}$ such that the cosine of the angle between $x$ and $y - x$ is greater than or equal to some constant $\alpha_0 \in ]-1,1]$. It is easy to see that we can now use Lemma 2.67 and (4.16) to conclude that we can assume that this $y$ satisfies (4.21).

We give up the temporary assumption that $x \neq 0$. Let $y \in B^n(x,t) \cap V_0$ be as in (4.21). We apply Lemma 2.35 to $y$ with $w = c_5$ to conclude that if $d_0$ is large enough, then $\bar{B}^n(\sigma(y), b_0) \subset \sigma B^n(y, c_5 t) \subset \sigma \bar{B}^n(x,t)$, where the number $b_0 > 0$ can be fixed freely.

Recall next that the claim (i) of Theorem 2.7 implies that there is a compact set $K \subset \mathbb{R}^n$ such that $\sigma V_0 \cap \mathbb{R}^n = \hat{G}_\infty K$, where $\hat{G} = \sigma G \sigma$. Let $b_0$ be so large that $B^n(z, b_0/2) \cap L(\hat{G}) \neq \emptyset$ for every $z \in K$. We wish to show that there is a constant $c_6 > 0$ satisfying the following. If $z \in K$ and $V$ is a $k$-sphere of $\bar{\mathbb{R}}^n$, then there is $z_V$ such that either $z_V \in B^n(z, b_0) \cap L(\hat{G})$ and $d_{\text{euc}}(z_V, V) \geq c_6$, or $z_V \in B^n(z, b_0) \cap V$ and $d_{\text{euc}}(z_V, L(\hat{G})) \geq c_6$.

Suppose that a constant $c_6$ as described above does not exist. Then there is a sequence $(z_i)_i$ of points in $K$, a sequence $(V_i)_i$ of $k$-spheres of $\bar{\mathbb{R}}^n$, and a sequence $(\varepsilon_i)_i$ of positive numbers with $\varepsilon_i \to 0$ satisfying the following: if $\zeta \in B^n(z_i, b_0) \cap L(\hat{G})$, then $d_{\text{euc}}(\zeta, V_i) \leq \varepsilon_i$, and if $\zeta \in B^n(z_i, b_0) \cap V_i$, then $d_{\text{euc}}(\zeta, L(\hat{G})) \leq \varepsilon_i$.

We can assume that $z_i \to z$ for some $z \in K$. We can assume also that $V_i$ intersects $\bar{B}^n(z, b_0) \cap \bar{B}^n(z_i, b_0)$ for every $i$. We use Lemma 2.61 with $Z = \bar{B}^n(z, b_0)$ and assume that the claim of Lemma 2.61 is true for the sequence $(V_i)_i$ itself. Let $W$ be the set given by Lemma 2.61. We can now use Lemma 2.63 to conclude that the following holds. If $i$ is large enough, there is a point $\zeta_i$ such that either $\zeta_i \in B^n(z_i, b_0) \cap L(\hat{G})$ and $d_{\text{euc}}(\zeta_i, V_i) \geq a_0$, or $\zeta_i \in B^n(z_i, b_0) \cap V_i$ and $d_{\text{euc}}(\zeta_i, L(\hat{G})) \geq a_0$, where $a_0 > 0$ is a constant. We conclude that the situation is contradictory, and so a constant $c_6$ as described above exists.

Recall that $\sigma V_0 \cap \mathbb{R}^n = \hat{G}_\infty K$ and that the elements in $\hat{G}_\infty$ are euclidean isometries by the claim (iii) of Theorem 2.7. It follows that we can generalize the claim regarding the constant $c_6$ as follows. If $z \in \sigma V_0 \cap \mathbb{R}^n$ and $V$ is a $k$-sphere of $\bar{\mathbb{R}}^n$, then there is a point $z_V$ such that either $z_V \in B^n(z, b_0) \cap L(\hat{G})$ and $d_{\text{euc}}(z_V, V) \geq c_6$, or $z_V \in B^n(z, b_0) \cap V$ and $d_{\text{euc}}(z_V, L(\hat{G})) \geq c_6$.

Let us apply the above to the point $\sigma(y)$ (recall that, at the moment, $y$ is a fixed point satisfying (4.21)). We obtain that the following claim is true. Given $V \in \mathcal{F}_k(x, t)$, we can choose a point $z_V \in \bar{B}^n(x, t)$ in the following way. It is the case that $\sigma(z_V) \in B^n(\sigma(y), b_0) \cap \sigma L(G)$ and $d_{\text{euc}}(\sigma(z_V), \sigma(\bar{B}^n(x, t) \cap V)) \geq c_6$, or $\sigma(z_V) \in B^n(\sigma(y), b_0) \cap \sigma V$ and $d_{\text{euc}}(\sigma(z_V), \sigma(\bar{B}^n(x, t) \cap L(G))) \geq c_6$. Note that, since $|y| \geq c_4(|x| + t)$ by (4.21) so $|\sigma(y)| = |y|^{-1} \leq 1/c_4(|x| + t)$, we have that

$$|\sigma(z_V)| \leq |\sigma(y)| + b_0 \leq \frac{1}{|x| + t}\left(\frac{1}{c_4} + b_0(|x| + t)\right) \leq \frac{c_7}{|x| + t}, \tag{4.22}$$

where $c_7 > 0$ is a suitable constant.

We can now finish the proof of the lower bound of (4.12). Let $V \in \mathcal{F}_k(x, t)$ be arbitrary. Our reasoning above implies that $z_V \in \bar{B}^n(x, t) \cap L(G)$ or $z_V \in \bar{B}^n(x, t) \cap V$. Let us assume that $z_V \in \bar{B}^n(x, t) \cap L(G)$. Recall (2.52), (4.17), (4.22), and the fact that $|\sigma'(z)| = |z|^{-2}$ for $z \in \mathbb{R}^{n+1} \setminus \{0\}$ (see (2.10)). Write $E = \bar{B}^n(x, t) \cap V$. Suppose first that $0 \notin E$. We calculate that

$$\begin{aligned} t\gamma_k^V(x, t) &\geq d_{\text{euc}}(z_V, \bar{B}^n(x, t) \cap V) = d_{\text{euc}}(z_V, E) = \inf_{z \in E} |z_V - z| \\ &= \inf_{z \in \sigma E} \frac{1}{|\sigma(z_V)|} \frac{1}{|z|} |\sigma(z_V) - z| \geq \frac{|x| + t}{c_7} \inf_{z \in \sigma E} \frac{1}{|z|} |\sigma(z_V) - z|. \end{aligned}$$

Let $z \in \sigma E$. If

$$\frac{1}{|z|} \geq \frac{|x| + t}{2c_7}, \qquad \text{then} \qquad \frac{|\sigma(z_V) - z|}{|z|} \geq \frac{c_6(|x| + t)}{2c_7}.$$

Suppose that

$$\frac{1}{|z|} < \frac{|x| + t}{2c_7} \qquad \text{so} \qquad |z| > \frac{2c_7}{|x| + t} \geq 2|\sigma(z_V)|.$$

Now

$$\frac{1}{|z|}|\sigma(z_V) - z| \geq 1 - \frac{|\sigma(z_V)|}{|z|} \geq \frac{1}{2} \geq |x| + t,$$

assuming that $d_0$ is large enough. We see that we have obtained an estimate of the form

$$t\gamma_k^V(x, t) \geq c_8(|x| + t)^2, \tag{4.23}$$

where $c_8 > 0$ is a constant. The estimate (4.23) is valid also if $0 \in E$. To prove this, we only need to observe that (4.22) implies that

$$|z_V| = \frac{1}{|\sigma(z_V)|} \geq \frac{|x| + t}{c_7} \geq \frac{(|x| + t)^2}{c_7},$$

assuming that $d_0$ is large enough.

We assumed above that $z_V \in \bar{B}^n(x, t) \cap L(G)$, but we can use essentially the same argument to derive an estimate of the form (4.23) in case $z_V \in \bar{B}^n(x, t) \cap V$. Since (4.23) is valid for all $V \in \mathcal{F}_k(x,t)$, we conclude that

$$\gamma_k(x,t) \geq \frac{c_8(|x|+t)^2}{t}.$$

We recall (4.18) and conclude that $\gamma_k(x,t)e^{d((x,t),\partial H_p)}$ is larger than some positive constant. It follows that the lower bound of (4.12) is valid.

To finish the proof, we must establish the lower bound of (4.13). Suppose that $l \in \{1,2,\ldots,n\} \setminus \{k\}$ is such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. We continue to assume that $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$ for the time being. Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim with $d((x,t),\partial H_p) > d_0$. Recall that $V_0$ denotes a fixed $G_p$-invariant $k$-sphere of $\bar{\mathbb{R}}^n$ through $p$ whose existence follows from the claim (i) of Theorem 2.7. Let $V \in \mathcal{F}_l(x,t)$. We see that (recall (4.17))

$$\gamma_l^V(x,t) \geq \frac{1}{t}|\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V_0) - \rho(\bar{B}^n(x,t) \cap V_0, \bar{B}^n(x,t) \cap V)|. \tag{4.24}$$

We show that there is a constant $b_1 > 0$ such that

$$\frac{1}{t}\rho(\bar{B}^n(x,t) \cap V_0, \bar{B}^n(x,t) \cap V) \geq b_1. \tag{4.25}$$

By (4.16), we can choose two points $z_1, z_2 \in B^n(x,t) \cap V_0$ and a small fixed number $a_1 > 0$ as follows. It is the case that $\bar{B}^n(z_j, 2a_1t) \subset B^n(x,t)$, $j = 1,2$, and also that $\bar{B}^n(z_1, 2a_1t) \cap \bar{B}^n(z_2, 2a_1t) = \emptyset$. If $V$ is such that $\bar{B}^n(z_j, a_1t) \cap V = \emptyset$ for $j = 1$ or $j = 2$, then (4.25) is trivial. So suppose that $V$ is such that $V$ meets $\bar{B}^n(z_j, a_1t)$ for $j = 1,2$. It follows that $d_{\mathrm{euc}}(V) \geq a_2t$ and $d_{\mathrm{euc}}(x,V) \leq a_3t$ for some constants $a_2 > 0$ and $a_3 \in [0,1[$. The formula (4.25) follows now from Lemma 2.65.

Observe next that

$$\gamma_k^{V_0}(x,t) = \frac{1}{t}\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V_0) \leq b_2 e^{-d((x,t),\partial H_p)}, \tag{4.26}$$

where $b_2 > 0$ is a constant. The estimate (4.26) follows from an argument provided earlier in this proof, see (4.18) and (4.20). So if $d_0$ large enough, we have that

$$\frac{1}{t}\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V_0) \leq \frac{b_1}{2}. \tag{4.27}$$

Combining (4.24), (4.25) and (4.27), we deduce that $\gamma_l^V(x,t) \geq b_1/2$ for every $V \in \mathcal{F}_l(x,t)$. Therefore, the lower bound of (4.13) is true in case $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$.

If $L(G)$ is a $k$-sphere of $\bar{\mathbb{R}}^n$, we can use the same argument we used to prove (4.25) to show that

$$\gamma_l^V(x,t) = \frac{1}{t}\rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V) \geq b_3$$

for every $V \in \mathcal{F}_l(x,t)$, where $b_3 > 0$ is a constant. The lower bound of (4.13) is hence valid also in this case. □

**Theorem 4.28.** *Let $G$, $v$, $p$, $k$ and $H_p$ be as in Theorem 4.10. Let $A \subset$ Möb$(\mathbb{H}^{n+1})$ be a non-empty and compact set of Möbius transformations mapping $H_p$ onto itself. Then there are constants $a > 0$, $b > 0$ and $c > 0$ satisfying the following. Let $f = \lambda \circ \alpha$, where*

$\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ *is a euclidean similarity and* $\alpha \in A$. *Let* $x \in \mathbb{R}^n$ *and* $t > 0$ *be such that* $(x,t) \in fH_p$ *and* $|x - x'|/t \leq \nu$ *for some* $x' \in fL(G) \cap \mathbb{R}^n$. *Then*

$$b^{-1} \leq \beta^f(x,t) \leq b, \tag{4.29}$$

$$c^{-1} e^{-d((x,t),\partial fH_p)} \leq \gamma_k^f(x,t) \leq c e^{-d((x,t),\partial fH_p)} \tag{4.30}$$

*if* $L(G)$ *is not a* $k$*-sphere of* $\bar{\mathbb{R}}^n$, *and*

$$a^{-1} \leq \gamma_l^f(x,t) \leq a \tag{4.31}$$

*for every* $l \in \{1,2,\ldots,n\} \setminus \{k\}$ *such that* $L(G)$ *is not an* $l$*-sphere of* $\bar{\mathbb{R}}^n$.

*Proof.* Let $f = \lambda \circ \alpha$, $x$, $t$ and $x'$ be as in the claim. Suppose first that $d((x,t),\partial fH_p) \leq d_0$, where $d_0 > 0$ is a fixed number. Lemma 2.56 implies that there is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x,t) \in (f \circ g)C$ for some $g \in G_p$. Note that $\beta^{f\circ h} = \beta^f$ and $\gamma_l^{f\circ h} = \gamma_l^f$ for every $h \in G_p$ and every $l \in \{1,2,\ldots,n\}$. We use Theorem 4.5 to conclude that

$$c_0^{-1} \leq \beta^f(x,t) \leq c_0 \qquad \text{and} \qquad c_1^{-1} \leq \gamma_l^f(x,t) \leq c_1 \tag{4.32}$$

for every $l \in \{1,2,\ldots,n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, where $c_0 > 0$ and $c_1 > 0$ are constants. We see that the claim is valid in this situation. Therefore, we can make the additional assumption that $d((x,t),\partial fH_p) > d_0$, where we can choose $d_0 > 0$ freely.

We write next $y = \lambda^{-1}(x)$, $y' = \lambda^{-1}(x')$ and $u = |\lambda'|^{-1}t$. Now $\bar{B}^n(y,u) = \lambda^{-1}\bar{B}^n(x,t)$, $(y,u) \in H_p$ with $d((y,u),\partial H_p) = d((x,t),\partial fH_p) > d_0$, and $y' \in \alpha L(G) \cap \mathbb{R}^n$ so that $|y - y'|/u = |x - x'|/t \leq \nu$. We apply Lemma 2.57 to the present situation and conclude that if $d_0$ is chosen large enough, the following four claims are true. It is the case that $\alpha^{-1}\bar{B}^n(y,u) = f^{-1}\bar{B}^n(x,t) = \bar{B}^n(\hat{x},\hat{t})$ for some $\hat{x} \in \mathbb{R}^n$ and $\hat{t} > 0$. It is also the case that $c_2^{-1}|\lambda'|^{-1}t \leq \hat{t} \leq c_2|\lambda'|^{-1}t$ and $|\hat{x} - f^{-1}(x')|/\hat{t} \leq \hat{\nu}$ for some constants $c_2 > 0$ and $\hat{\nu} \in [0,1[$. It is true that $c_2^{-1} \leq |\alpha'| \leq c_2$ in $\bar{B}^n(\hat{x},\hat{t})$. It is the case that $(\hat{x},\hat{t}) \in H_p$ so that

$$d((x,t),\partial fH_p) - c_3 \leq d((\hat{x},\hat{t}),\partial H_p) \leq d((x,t),\partial fH_p) + c_3, \tag{4.33}$$

where $c_3 > 0$ is a constant.

We observe that we can apply Theorem 4.10 to $\bar{B}^n(\hat{x},\hat{t})$. We conclude that there are constants $c_4 > 0$, $c_5 > 0$ and $c_6 > 0$ such that

$$c_4^{-1} e^{-d((\hat{x},\hat{t}),\partial H_p)} \leq \gamma_k(\hat{x},\hat{t}) \leq c_4 e^{-d((\hat{x},\hat{t}),\partial H_p)} \tag{4.34}$$

if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$,

$$c_5^{-1} \leq \beta(\hat{x},\hat{t}) \leq c_5, \tag{4.35}$$

and

$$c_6^{-1} \leq \gamma_l(\hat{x},\hat{t}) \leq c_6 \tag{4.36}$$

for every $l \in \{1,2,\ldots,n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$.

Let us consider $\beta^f(x,t)$ in detail. We calculate that (recall (2.52))

$$\begin{aligned}\beta^f(x,t) &= \frac{1}{t}\sup_{z_1,z_2\in\bar{B}^n(\hat{x},\hat{t})\cap L(G)}|f(z_1)-f(z_2)|\\ &= \frac{1}{t}\sup_{z_1,z_2\in\bar{B}^n(\hat{x},\hat{t})\cap L(G)}|f'(z_1)|^{1/2}|f'(z_2)|^{1/2}|z_1-z_2|\\ &= \frac{|\lambda'|}{t}\sup_{z_1,z_2\in\bar{B}^n(\hat{x},\hat{t})\cap L(G)}|\alpha'(z_1)|^{1/2}|\alpha'(z_2)|^{1/2}|z_1-z_2|.\end{aligned}$$

We use the fact that $c_2^{-1}\le|\alpha'|\le c_2$ in $\bar{B}^n(\hat{x},\hat{t})$ to obtain that

$$\frac{1}{c_2}\frac{|\lambda'|}{t}d_{\mathrm{euc}}(\bar{B}^n(\hat{x},\hat{t})\cap L(G))\le\beta^f(x,t)\le c_2\frac{|\lambda'|}{t}d_{\mathrm{euc}}(\bar{B}^n(\hat{x},\hat{t})\cap L(G)),$$

and we use the fact that $c_2^{-1}|\lambda'|^{-1}t\le\hat{t}\le c_2|\lambda'|^{-1}t$ to obtain further that

$$c_2^{-2}\beta(\hat{x},\hat{t})\le\beta^f(x,t)\le c_2^2\beta(\hat{x},\hat{t}).$$

The estimate (4.29) follows now from (4.35).

On the other hand, since

$$\gamma_l^f(x,t)=\frac{1}{t}\inf_{V\in\mathcal{F}_l(\hat{x},\hat{t})}\rho(f\bar{B}^n(\hat{x},\hat{t})\cap fL(G),f\bar{B}^n(\hat{x},\hat{t})\cap fV)$$

for every $l\in\{1,2,\dots,n\}$, it is clear that we can use similar reasoning as we did above for $\beta^f(x,t)$ to deduce that

$$c_2^{-2}\gamma_l(\hat{x},\hat{t})\le\gamma_l^f(x,t)\le c_2^2\gamma_l(\hat{x},\hat{t})$$

for every $l\in\{1,2,\dots,n\}$. We see now that (4.36) implies (4.31) and that (4.33) and (4.34) imply (4.30). □

**Theorem 4.37.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $v\in[0,1[$. Let $P\subset\mathbb{R}^n$ be a finite, possibly empty, set of bounded parabolic fixed points of $G$ such that if $p_1$ and $p_2$ are two points in $P$, then $Gp_1\cap Gp_2=\emptyset$. Suppose that there is a pairwise disjoint collection $\{H_p : p\in GP\}$ of horoballs of $\mathbb{H}^{n+1}$ such that $H_p$, $p\in GP$, is based at $p$ and that $gH_p=H_{g(p)}$ for every $g\in G$ and every $p\in GP$. Let $C\subset\mathbb{H}^{n+1}$ be compact.*

*The following is true in this situation. There are constants $a_0>0$, $a_1>0$, $a_2>0$ and $a_3>0$ which satisfy the following. Let $x\in\mathbb{R}^n$ and $t>0$ be such that there is some $x'\in L(G)\cap\mathbb{R}^n$ with $|x-x'|/t\le v$. Suppose that $(x,t)\in GC\cup\bigcup_{p\in GP\cap\mathbb{R}^n}H_p$. Then*

$$a_0^{-1}\le\beta(x,t)\le a_0. \tag{4.38}$$

*Moreover, if $(x,t)\in GC$, then*

$$a_1^{-1}\le\gamma_l(x,t)\le a_1 \tag{4.39}$$

*for every $l\in\{1,2,\dots,n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, and if $(x,t)\in H_p$ for some $p\in GP\cap\mathbb{R}^n$ of rank $k\in\{1,2,\dots,n\}$, then*

$$a_2^{-1}\le\gamma_l(x,t)\le a_2 \tag{4.40}$$

*for every $l \in \{1, 2, \ldots, n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ and*

$$a_3^{-1} e^{-d((x,t),\partial H_p)} \le \gamma_k(x,t) \le a_3 e^{-d((x,t),\partial H_p)} \tag{4.41}$$

*if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. Suppose that $(x,t) \in GC$. This means that $(x,t) \in gC$ for some $g \in G$ and that $x' \in gL(G) \cap \mathbb{R}^n$ since $L(G)$ is $G$-invariant. Now $\beta^g = \beta$ and $\gamma_l^g = \gamma_l$ for every $l \in \{1, 2, \ldots, n\}$, so Theorem 4.5 implies that there exist constants $c_0 > 0$ and $c_1 > 0$ such that

$$c_0^{-1} \le \beta(x,t) \le c_0 \qquad \text{and} \qquad c_1^{-1} \le \gamma_l(x,t) \le c_1$$

for every $l \in \{1, 2, \ldots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. We have proved (4.39) in its totality and (4.38) in the case considered.

Let $q \in P$. Let $k \in \{1, 2, \ldots, n\}$ be the rank of $q$. The collection $\{gH_q : g \in G\}$ of horoballs of $\mathbb{H}^{n+1}$ is pairwise disjoint by assumption. Lemma 2.59 implies that there is a compact set $A \subset \text{Möb}(\mathbb{H}^{n+1})$ satisfying the following two claims. The elements in $A$ map $H_q$ onto itself. If $g \in G$ and $g(q) \in \mathbb{R}^n$, then there is $f \in G_q$ such that $g \circ f = \lambda \circ \alpha$, where $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ is a euclidean similarity mapping $H_q$ onto $H_{g(q)}$ and $\alpha \in A$.

Suppose that $x \in \mathbb{R}^n$, $t > 0$ and $x' \in L(G) \cap \mathbb{R}^n$ are as in the claim and additionally that $(x,t) \in H_{g(q)}$ for some $g \in G$ such that $g(q) \in \mathbb{R}^n$. According to the above, there is $f \in G_q$, $\alpha \in A$ and a euclidean similarity $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ mapping $H_q$ onto $H_{g(q)}$ such that $g \circ f = \lambda \circ \alpha$. Note that $H_{g(q)} = H_{(g \circ f)(q)} = (g \circ f)H_q = (\lambda \circ \alpha)H_q$ and furthermore that $x' \in (g \circ f)L(G) \cap \mathbb{R}^n = (\lambda \circ \alpha)L(G) \cap \mathbb{R}^n$ because $L(G)$ is $G$-invariant. Note also that $\beta^{\lambda \circ \alpha} = \beta^{g \circ f} = \beta$ and $\gamma_l^{\lambda \circ \alpha} = \gamma_l^{g \circ f} = \gamma_l$ for every $l \in \{1, 2, \ldots, n\}$. Using Theorem 4.28, we see that there are constants $c_2 > 0$, $c_3 > 0$ and $c_4 > 0$ such that

$$c_2^{-1} \le \beta(x,t) \le c_2 \qquad \text{and} \qquad c_3^{-1} \le \gamma_l(x,t) \le c_3$$

for every $l \in \{1, 2, \ldots, n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ and that

$$c_4^{-1} e^{-d((x,t),\partial H_{g(q)})} \le \gamma_k(x,t) \le c_4 e^{-d((x,t),\partial H_{g(q)})}$$

if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$. This means that we have proved (4.38), (4.40) and (4.41) for all $(x,t)$ such that $(x,t) \in H_p$ for some $p \in Gq \cap \mathbb{R}^n$. Since $P$ is finite by assumption, we see that (4.38), (4.40) and (4.41) are valid for all $(x,t)$ such that $(x,t) \in H_p$ for some $p \in GP \cap \mathbb{R}^n$, which completes the proof. □

4.2. **Additional results.** We have now proved our main results on the geometry of limit sets of non-elementary Kleinian groups. We proceed to prove two additional results which we will need in Chapter 7 when discussing some variants of the main results of this work.

The additional results involve alternative versions of flatness functions defined as follows. Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $f \in \text{Möb}(\mathbb{H}^{n+1})$ and $l \in \{1, 2, \ldots, n\}$. We define the *alternative flatness function* $\Gamma_l^f$ of $G$ by

$$\Gamma_l^f(x,t) = \frac{1}{t} \inf_{T \in \mathcal{P}_l(x)} \rho(\bar{B}^n(x,t) \cap fL(G), \bar{B}^n(x,t) \cap T) \tag{4.42}$$

for every $x \in fL(G) \cap \mathbb{R}^n$ and $t > 0$, where $\mathcal{P}_l(x)$ is the collection of $l$-planes of $\bar{\mathbb{R}}^n$ through $x$ and $\rho$ is the same metric as in (4.2). We observe that there are two differences between the flatness functions $\gamma_l^f$ defined by (4.2) and $\Gamma_l^f$. The first difference is that $\gamma_l^f(x,t)$, where $f \in \text{Möb}(\mathbb{H}^{n+1})$ and $l \in \{1,2,\dots,n\}$ are fixed, is defined for all $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap fL(G) \neq \emptyset$, but $\Gamma_l^f(x,t)$ is defined only for all $x \in fL(G) \cap \mathbb{R}^n$ and $t > 0$. So if $\Gamma_l^f(x,t)$ is defined, then $\gamma_l^f(x,t)$ is defined, but the converse is not true in general. The other difference is that the collection $\mathcal{F}_l(x,t)$ of (4.2) has been replaced by the simpler collection $\mathcal{P}_l(x)$. Observe that, since $\mathcal{P}_l(x) \subset \mathcal{F}_l(x,t)$, it is the case that

$$\Gamma_l^f(x,t) \geq \gamma_l^f(x,t) \tag{4.43}$$

for every $f \in \text{Möb}(\mathbb{H}^{n+1})$, $l \in \{1,2,\dots,n\}$, $x \in fL(G) \cap \mathbb{R}^n$ and $t > 0$.

Our aim is to prove two results which show that the functions $\Gamma_l^f$ satisfy similar estimates as the functions $\gamma_l^f$. Theorem 4.44 corresponds to Theorem 4.5 and Theorem 4.46 corresponds to Theorem 4.28.

**Theorem 4.44.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $C \subset \mathbb{H}^{n+1}$ be compact. Then there is a constant $c > 0$ such that the following is true. Let $f \in \text{Möb}(\mathbb{H}^{n+1})$. Let $x \in fL(G) \cap \mathbb{R}^n$ and $t > 0$ be such that $(x,t) \in fC$. Then*

$$c^{-1} \leq \Gamma_l^f(x,t) \leq c \tag{4.45}$$

*for every $l \in \{1,2,\dots,n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* Observe first that the upper bound of (4.45) follows immediately from the definition (4.42). The lower bound of (4.45) is implied by (4.43) since the functions $\gamma_l^f$, where $l \in \{1,2,\dots,n\}$ is such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, satisfy an estimate of the form (4.45) by Theorem 4.5. □

**Theorem 4.46.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{H}^{n+1}$. Let $p \in \mathbb{R}^n$ be a bounded parabolic fixed point of $G$ of rank $k \in \{1,2,\dots,n\}$. Let $H_p$ be a horoball of $\mathbb{H}^{n+1}$ based at $p$. Let $A \subset \text{Möb}(\mathbb{H}^{n+1})$ be a non-empty and compact set of Möbius transformations mapping $H_p$ onto itself. Then there are constants $b > 0$ and $c > 0$ satisfying the following. Let $f = \lambda \circ \alpha$, where $\lambda \in \text{Möb}(\mathbb{H}^{n+1})$ is a euclidean similarity and $\alpha \in A$. Let $x \in fL(G) \cap \mathbb{R}^n$ and $t > 0$ be such that $(x,t) \in fH_p$. Then*

$$c^{-1} e^{-d((x,t),\partial f H_p)} \leq \Gamma_k^f(x,t) \leq c e^{-d((x,t),\partial f H_p)} \tag{4.47}$$

*if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$, and*

$$b^{-1} \leq \Gamma_l^f(x,t) \leq b \tag{4.48}$$

*for every $l \in \{1,2,\dots,n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* The upper bound of (4.48) follows immediately from the definition (4.42). The lower bounds of (4.47) and (4.48) are implied by (4.43), since the functions $\gamma_k^f$ and $\gamma_l^f$, where $f$ is as in the claim and $l \in \{1,2,\dots,n\} \setminus \{k\}$ is such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, satisfy lower bounds of the same form by Theorem 4.28. Moreover, if $k = n$, then $\Gamma_k^f(x,t) = \gamma_k^f(x,t)$ for all $f$, $x$ and $t$ as in the claim, and so the upper bound of (4.47) is valid in this case, since $\gamma_k^f(x,t)$ has an upper bound of the same form by Theorem

4.28. We are, therefore, left to prove the upper bound of (4.47) assuming additionally that $k \in \{1, 2, \dots, n-1\}$.

Suppose that $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$. Let $f = \lambda \circ \alpha$, $x$ and $t$ be as in the claim. Assume additionally that $d((x,t), \partial f H_p) \leq d_0$ for some $d_0 > 0$. The definition (4.42) implies that $\Gamma_k^f(x,t) \leq 2$. We see that the upper bound of (4.47) is valid in this situation. We can, therefore, make the additional assumption that $d((x,t), \partial f H_p) > d_0$, where we can choose $d_0 > 0$ freely.

Definition (4.42) implies that

$$\Gamma_k^f(x,t) \leq \frac{1}{t} \rho(\bar{B}^n(x,t) \cap f L(G), \bar{B}^n(x,t) \cap T)$$

for every $T \in \mathcal{P}_k(x)$. Our aim is to show that if $d_0$ is large enough, then $f^{-1}\bar{B}^n(x,t)$ is a euclidean $n$-ball of $\bar{\mathbb{R}}^n$ and there is a $k$-sphere $V$ of $\bar{\mathbb{R}}^n$ through $f^{-1}(x)$ such that $V$ intersects $f^{-1}S^{n-1}(x,t)$ orthogonally and that $fV \in \mathcal{P}_k(x)$ can be used to obtain the desired upper bound for $\Gamma_k^f(x,t)$.

Let us write $y = \lambda^{-1}(x)$ and $u = |\lambda'|^{-1}t$. We now have that $\bar{B}^n(y,u) = \lambda^{-1}\bar{B}^n(x,t)$, $(y,u) \in H_p$, and $d((y,u), \partial H_p) = d((x,t), \partial f H_p) > d_0$. According to Lemma 2.57, the following claim is true assuming that $d_0$ is large enough. It is the case that $\alpha^{-1}\bar{B}^n(y,u) = f^{-1}\bar{B}^n(x,t) = \bar{B}^n(\hat{x}, \hat{t})$ for some $\hat{x} \in \mathbb{R}^n$ and $\hat{t} > 0$ such that $(\hat{x}, \hat{t}) \in H_p$ with

$$d((x,t), \partial f H_p) - c_1 \leq d((\hat{x}, \hat{t}), \partial H_p) \leq d((x,t), \partial f H_p) + c_1, \tag{4.49}$$

where $c_1 > 0$ is a constant.

We study the situation in greater detail. Observe that we can assume that $p = 0$. Write

$$s(y,u) = \sup_{z \in \bar{B}^n(y,u)} |(\alpha^{-1})'(z)| \quad \text{and} \quad i(y,u) = \inf_{z \in \bar{B}^n(y,u)} |(\alpha^{-1})'(z)|. \tag{4.50}$$

If $\alpha(\infty) = \infty = \alpha^{-1}(\infty)$, then $\alpha$ is a euclidean isometry (since $\alpha H_p = H_p$), so $|(\alpha^{-1})'| = 1$ in $\bar{\mathbb{R}}^{n+1}$ in this case. If $\alpha(\infty) \neq \infty \neq \alpha^{-1}(\infty)$, then (2.12) implies that

$$|(\alpha^{-1})'(z)| = \frac{r_\alpha^2}{|z - \alpha(\infty)|^2}$$

for every $z \in \mathbb{R}^{n+1} \setminus \{\alpha(\infty)\}$, where $r_\alpha$ is the euclidean radius of the isometric spheres of $\alpha$ and $\alpha^{-1}$. It follows that if $\alpha(\infty) \neq \infty \neq \alpha^{-1}(\infty)$, then

$$\frac{i(y,u)}{s(y,u)} \geq \left( \frac{|\alpha(\infty)| - (|y| + u)}{|\alpha(\infty)| - (|y| - u)} \right)^2, \tag{4.51}$$

since $p = 0$. Since $A$ is compact and the elements in $A$ fix 0, it is true that $|\alpha(\infty)|$ is greater than some positive constant. Lemma 2.31 implies that the set $\{z \in H_p : d(z, \partial H_p) > d_0\}$ is a horoball of $\mathbb{H}^{n+1}$ based at $p$ whose euclidean radius decreases to 0 as $d_0 \to \infty$. It follows that if $d_0$ is large enough, then $|y| + u \leq \varepsilon$, where $\varepsilon > 0$ is any given fixed number. Therefore, combining these results, we see that

$$\frac{i(y,u)}{s(y,u)} \to 1 \tag{4.52}$$

as $d_0 \to \infty$.

Note next that

$$\begin{aligned}\hat{t} &= \frac{1}{2}d_{\mathrm{euc}}(\alpha^{-1}\bar{B}^n(y,u)) = \frac{1}{2}\sup_{z_1,z_2\in\bar{B}^n(y,u)}|\alpha^{-1}(z_1)-\alpha^{-1}(z_2)| \\ &= \frac{1}{2}\sup_{z_1,z_2\in\bar{B}^n(y,u)}|(\alpha^{-1})'(z_1)|^{1/2}|(\alpha^{-1})'(z_2)|^{1/2}|z_1-z_2|\end{aligned}$$

by (2.52), so $i(y,u)u \le \hat{t} \le s(y,u)u$. Similarly, writing $\hat{y} = \alpha^{-1}(y) = f^{-1}(x) \in L(G)$, we have that

$$d_{\mathrm{euc}}(\hat{y}, S^{n-1}(\hat{x},\hat{t})) = d_{\mathrm{euc}}(\alpha^{-1}(y), \alpha^{-1}S^{n-1}(y,u)) \ge i(y,u)u \ge \frac{i(y,u)}{s(y,u)}\hat{t},$$

so

$$|\hat{y}-\hat{x}| = \hat{t} - d_{\mathrm{euc}}(\hat{y}, S^{n-1}(\hat{x},\hat{t})) \le \left(1-\frac{i(y,u)}{s(y,u)}\right)\hat{t}. \tag{4.53}$$

The formulae (4.52) and (4.53) imply that there is a constant $c_2 > 0$ such that $\bar{B}^n(\hat{y}, c_2\hat{t}) \subset B^n(\hat{x},\hat{t})$, assuming that $d_0$ is large enough.

Let us introduce a fixed $G_p$-invariant $k$-sphere $V_0$ of $\bar{\mathbb{R}}^n$ through $p = 0$ as described by the claim (i) of Theorem 2.7. Denote by $\sigma$ the inversion $z \mapsto z/|z|^2$ in the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$. The point $\infty$ is a bounded parabolic fixed point of $\sigma G\sigma$ of rank $k$, so the claim (ii) of Lemma 2.37 implies that there is a constant $c_3 > 0$ such that $d_{\mathrm{euc}}(z, \sigma V_0) \le c_3$ for every $z \in \sigma L(G) \cap \mathbb{R}^n$. Note that

$$|z| \ge d_{\mathrm{euc}}(0, \sigma S^{n-1}(\hat{x},\hat{t})) = \frac{1}{|\hat{x}|+\hat{t}} \tag{4.54}$$

for every $z \in \sigma\bar{B}^n(\hat{x},\hat{t})$.

Recall that $\hat{y} = f^{-1}(x) \in L(G)$, so we can apply the above to $\hat{y}$. If $\hat{y} \neq 0$, let $\hat{y}^* \in V_0 \setminus \{0\}$ be such that $|\sigma(\hat{y}) - \sigma(\hat{y}^*)| \le c_3$. If $\hat{y} = 0$, choose $\hat{y}^* = 0$. The existence of the constants $c_2$ and $c_3$ implies that we can use Lemma 2.35 to deduce that $\hat{y}^* \in B^n(\hat{y}, c_2\hat{t})$, assuming that $d_0$ is large enough. Recall (2.52), (4.54) and the fact that $|\sigma'(z)| = |z|^{-2}$ for $z \in \mathbb{R}^{n+1} \setminus \{0\}$ (see (2.10)). If $\hat{y} \neq 0$, we calculate that

$$|\hat{y}-\hat{y}^*| = |\sigma'(\sigma(\hat{y}))|^{1/2}|\sigma'(\sigma(\hat{y}^*))|^{1/2}|\sigma(\hat{y})-\sigma(\hat{y}^*)| = \frac{|\sigma(\hat{y})-\sigma(\hat{y}^*)|}{|\sigma(\hat{y})||\sigma(\hat{y}^*)|} \le c_3(|\hat{x}|+\hat{t})^2.$$

Lemma 2.32 implies that

$$e^{d((\hat{x},\hat{t}),\partial H_p)} = \frac{d_{\mathrm{euc}}(H_p)\hat{t}}{|\hat{x}|^2+\hat{t}^2}. \tag{4.55}$$

We conclude that

$$|\hat{y}-\hat{y}^*| \le c_4\hat{t}e^{-d((\hat{x},\hat{t}),\partial H_p)}, \tag{4.56}$$

where $c_4 > 0$ is a constant.

We can finally find a suitable $T \in \mathcal{P}_k(x)$ to estimate $\Gamma_k^f(x,t)$. Recall that $\mathcal{F}_k(\hat{x},\hat{t})$ denotes the collection of all $k$-spheres of $\bar{\mathbb{R}}^n$ intersecting $\bar{B}^n(\hat{x},\hat{t})$. Let $V_1 \in \mathcal{F}_k(\hat{x},\hat{t})$ be the $k$-sphere of $\bar{\mathbb{R}}^n$ through $\hat{y}$ such that $V_1$ is obtained from $V_0$ using the translation $z \mapsto z + (\hat{y}-\hat{y}^*)$. If $V_1$ is a euclidean $k$-sphere of $\bar{\mathbb{R}}^n$, let $V_2 \in \mathcal{F}_k(\hat{x},\hat{t})$ be the $k$-plane of $\bar{\mathbb{R}}^n$ such that $V_2$ is tangential to $V_1$ at $\hat{y}$ so that $V_2$ and $V_1$ are contained in a $(k+1)$-dimensional plane of $\bar{\mathbb{R}}^n$. (Recall that we made the additional assumption that $k < n$ at the beginning of this proof.) If $V_1$ is a $k$-plane of $\bar{\mathbb{R}}^n$, let $V_2 = V_1$. If $V_2$ intersects $S^{n-1}(\hat{x}, \hat{t})$ orthogonally, let $V_3 = V_2$. If $V_2$ does not intersect $S^{n-1}(\hat{x}, \hat{t})$ orthogonally, let $V_3 \in \mathcal{F}_k(\hat{x}, \hat{t})$ be a euclidean $k$-sphere of $\bar{\mathbb{R}}^n$ such that $V_2$ is tangential to $V_3$ at $\hat{y}$, that $V_2$ and $V_3$ are contained in a $(k+1)$-plane of $\bar{\mathbb{R}}^n$, and that $V_3$ intersects $S^{n-1}(\hat{x}, \hat{t})$ orthogonally. Now $fV_3 \in \mathcal{P}_k(x)$ since $\hat{y} = f^{-1}(x)$. We will show that we can use $fV_3$ to prove the upper bound of (4.47).

Let us write $\bar{B}^n(x,t) = \bar{B}_1$, $\bar{B}^n(y,u) = \bar{B}_2$ and $\bar{B}^n(\hat{x}, \hat{t}) = \bar{B}_3$. We use Lemma 2.57 to deduce that there is a constant $c_5 > 0$ such that $c_5^{-1} \leq |\alpha'| \leq c_5$ in $\bar{B}_3$ and that $c_5^{-1}u \leq \hat{t} \leq c_5 u$. We can now estimate that (observe that the sixth step of the following estimate follows from the fact that $|\alpha(z_1) - \alpha(z_2)| = |\alpha'(z_1)|^{1/2}|\alpha'(z_2)|^{1/2}|z_1 - z_2| \leq c_5|z_1 - z_2|$ for every $z_1, z_2 \in \bar{B}_3$ by (2.52))

$$\begin{aligned}
\Gamma_k^f(x,t) &\leq \frac{1}{t}\rho(\bar{B}_1 \cap fL(G), \bar{B}_1 \cap fV_3) = \frac{1}{t}\rho(\lambda\bar{B}_2 \cap \lambda(\alpha L(G)), \lambda\bar{B}_2 \cap \lambda(\alpha V_3)) \\
&= \frac{1}{u}\rho(\bar{B}_2 \cap \alpha L(G), \bar{B}_2 \cap \alpha V_3) \leq \frac{c_5}{\hat{t}}\rho(\bar{B}_2 \cap \alpha L(G), \bar{B}_2 \cap \alpha V_3) \\
&= \frac{c_5}{\hat{t}}\rho(\alpha\bar{B}_3 \cap \alpha L(G), \alpha\bar{B}_3 \cap \alpha V_3) \leq \frac{c_5^2}{\hat{t}}\rho(\bar{B}_3 \cap L(G), \bar{B}_3 \cap V_3) \\
&\leq \frac{c_5^2}{\hat{t}}(\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2) + \rho(\bar{B}_3 \cap V_2, \bar{B}_3 \cap V_1) \\
&\quad + \rho(\bar{B}_3 \cap V_1, \bar{B}_3 \cap V_0) + \rho(\bar{B}_3 \cap V_0, \bar{B}_3 \cap L(G))).
\end{aligned}$$

We can use the argument we used to prove the upper bound of (4.12) in the proof of Theorem 4.10 to show that $\rho(\bar{B}_3 \cap V_0, \bar{B}_3 \cap L(G))e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is bounded from above by a constant, see the reasoning starting from page 59. On the other hand, since $V_1$ is obtained from $V_0$ using the translation $z \mapsto z + (\hat{y} - \hat{y}^*)$ and (4.56) is valid, it is not difficult to see that $\rho(\bar{B}_3 \cap V_1, \bar{B}_3 \cap V_0)e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is also bounded from above by a constant. (To make this easier to see, assume that $d_0$ is large so that $d_{\text{euc}}(V_0)/\hat{t} = d_{\text{euc}}(V_1)/\hat{t}$ is large and $|\hat{y} - \hat{y}^*|/\hat{t}$ is small. Then $V_0$ and $V_1$ are close to $k$-planes of $\bar{\mathbb{R}}^n$ with respect to the scale of $\hat{t}$ which are not close to being tangential to $S^{n-1}(\hat{x}, \hat{t})$.) Our next task is to show that the quantities

$$\frac{\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2)e^{d((\hat{x},\hat{t}),\partial H_p)}}{\hat{t}} \qquad \text{and} \qquad \frac{\rho(\bar{B}_3 \cap V_2, \bar{B}_3 \cap V_1)e^{d((\hat{x},\hat{t}),\partial H_p)}}{\hat{t}}$$

are bounded from above by constants. Once this has been accomplished, we see that $\Gamma_k^f(x,t)e^{d((\hat{x},\hat{t}),\partial H_p)}$ is bounded from above by a constant. The upper bound of (4.47) follows then immediately from (4.49).

We show first that $\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2)e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is bounded from above by a constant. We begin by showing that $d_{\text{euc}}(V_3)$ is larger than some positive constant. This claim is trivial if $V_3$ is a $k$-plane of $\bar{\mathbb{R}}^n$, so assume for the moment that $V_3$ is a euclidean $k$-sphere of $\bar{\mathbb{R}}^n$ with center $\zeta$ and euclidean radius $r$. This assumption implies that $\hat{y} \neq \hat{x}$ and $\alpha(\infty) \neq \infty \neq \alpha^{-1}(\infty)$. It is not difficult to see that $d_{\text{euc}}(V_3)$ is minimal in case $\zeta = \hat{y} + r(\hat{y} - \hat{x})/|\hat{y} - \hat{x}|$, so let us assume this momentarily. The assumption that $V_3$ intersects

$S^{n-1}(\hat{x},\hat{t})$ orthogonally implies that $\hat{t}^2 + r^2 = |\hat{x} - \zeta|^2$. We note that $|\hat{x} - \zeta| = |\hat{y} - \hat{x}| + r$ and use (4.53) to conclude that

$$r = \frac{\hat{t}^2 - |\hat{y} - \hat{x}|^2}{2|\hat{y} - \hat{x}|} \geq \frac{1 - \left(1 - \frac{i(y,u)}{s(y,u)}\right)^2}{2\left(1 - \frac{i(y,u)}{s(y,u)}\right)}\hat{t}. \tag{4.57}$$

We combine (4.57) with (4.52) and deduce that the following is true. It is the case that $d_{\text{euc}}(V_3)$ is larger than some positive constant if we show that $(1 - i(y,u)/s(y,u))/\hat{t}$ is smaller than some constant. We write $\kappa = |\alpha(\infty)|$ and calculate that (recall (4.51))

$$1 - \frac{i(y,u)}{s(y,u)} \leq 1 - \left(\frac{\kappa - (|y| + u)}{\kappa - (|y| - u)}\right)^2 = \frac{4(\kappa - |y|)}{(\kappa - (|y| - u))^2}u.$$

We showed following (4.51) that $\kappa$ is larger than some positive constant, say $\kappa_0 > 0$, and that $|y| + u \leq \varepsilon$, where $\varepsilon > 0$ is any given fixed number, assuming that $d_0$ is large enough. It follows that if $d_0$ is large enough, then $\kappa - (|y| - u) \geq \kappa/2$. Recall that $u \leq c_5\hat{t}$. We conclude that

$$1 - \frac{i(y,u)}{s(y,u)} \leq \frac{4\kappa u}{(\kappa/2)^2} = \frac{16u}{\kappa} \leq \frac{16c_5}{\kappa_0}\hat{t}.$$

It follows that $(1 - i(y,u)/s(y,u))/\hat{t}$ is bounded from above by a constant. We showed above that this implies that there is a constant $c_6 > 0$ such that $d_{\text{euc}}(V_3) \geq c_6$.

We continue with our proof to show that $\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2)e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is smaller than some constant. The case $V_3 = V_2$ is trivial, so assume that $V_3 \neq V_2$. This implies that $V_3$ is a euclidean $k$-sphere of $\bar{\mathbb{R}}^n$.

If $z_3 \in \bar{B}_3 \cap V_3$, let $\hat{z}_3 \in V_2$ be the orthogonal projection of $z_3$ to $V_2$. Let $z_2 \in \bar{B}_3 \cap V_2$. Assuming that $d_0$ is large enough, the existence of $c_2$ and $c_6$ implies that there are two points, say $\zeta_2$ and $\zeta_2'$, in $V_3$ such that $z_2$ is the orthogonal projection of these points to $V_2$. Suppose that $\zeta_2$ is closer to $z_2$ than $\zeta_2'$ and write $\hat{z}_2 = \zeta_2$. Note that if $d_0$ is large enough, then $\zeta_2' \notin \bar{B}_3$.

Observe next that there is a constant $c_7 > 0$ such that $d_{\text{euc}}(z_3, \bar{B}_3 \cap V_2) \leq c_7|z_3 - \hat{z}_3|$ for every $z_3 \in \bar{B}_3 \cap V_3$ and that $d_{\text{euc}}(z_2, \bar{B}_3 \cap V_3) \leq c_7|z_2 - \hat{z}_2|$ for every $z_2 \in \bar{B}_3 \cap V_2$. To make the existence of $c_7$ easier to see, assume that $d_0$ is so large that $V_3$ is close to a $k$-plane of $\bar{\mathbb{R}}^n$ with respect to the scale of $\hat{t}$ and note that $V_2$ and $V_3$ are not close to being tangential to $S^{n-1}(\hat{x},\hat{t})$. In order to estimate $\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2)$, we find suitable upper bounds for $|z_2 - \hat{z}_2|$ and $|z_3 - \hat{z}_3|$, where $z_2 \in \bar{B}_3 \cap V_2$ and $z_3 \in \bar{B}_3 \cap V_3$ are arbitrary.

Recall that $\zeta$ denotes the center of $V_3$ and $r$ the euclidean radius of $V_3$. Let $\xi \in \mathbb{S}^{n-1} = \partial\mathbb{B}^n$ be such that $\zeta = \hat{y} + r\xi$. Let $z \in \bar{B}_3 \cap V_2$. Let $z' \in \mathbb{S}^{n-1}$ and $s \geq 0$ be such that $\hat{y} + s\hat{t}z' = z$. Note that there is a constant $s_0 > 0$ such that $s \leq s_0$. Write $\omega = |z - \hat{z}|$. Now

$$|(\hat{y} + s\hat{t}z' + \omega\xi) - (\hat{y} + r\xi)|^2 = (s\hat{t})^2 + (\omega - r)^2 = r^2,$$

so $\omega = r - \sqrt{r^2 - (s\hat{t})^2}$. Since $r \geq c_6/2$ and $s \leq s_0$, it follows that

$$\frac{\omega}{|\hat{x}|^2 + \hat{t}^2} = \frac{s^2}{r + \sqrt{r^2 - (s\hat{t})^2}}\frac{\hat{t}^2}{|\hat{x}|^2 + \hat{t}^2}$$

is smaller than some constant, assuming that $d_0$ is large enough. We use (4.55) to deduce that $\omega \leq c_8 \hat{t} e^{-d((\hat{x},\hat{t}),\partial H_p)}$, where $c_8 > 0$ is a constant.

We have established that $|z_2 - \hat{z}_2| \leq c_8 \hat{t} e^{-d((\hat{x},\hat{t}),\partial H_p)}$ for every $z_2 \in \bar{B}_3 \cap V_2$. We can repeat the above argument with $z \in \bar{B}_3 \cap V_3$, $z' \in \mathbb{S}^{n-1}$ and $s \geq 0$ such that $\hat{y} + s\hat{t}z' = \hat{z}$ to show that $|z_3 - \hat{z}_3| \leq c_8 \hat{t} e^{-d((\hat{x},\hat{t}),\partial H_p)}$ for every $z_3 \in \bar{B}_3 \cap V_3$, assuming that $s_0$ and $c_8$ are increased appropriately.

We conclude that $d_{\text{euc}}(z_2, \bar{B}_3 \cap V_3)/\hat{t} \leq c_9 e^{-d((\hat{x},\hat{t}),\partial H_p)}$ for every $z_2 \in \bar{B}_2 \cap V_2$ and that $d_{\text{euc}}(z_3, \bar{B}_3 \cap V_2)/\hat{t} \leq c_9 e^{-d((\hat{x},\hat{t}),\partial H_p)}$ for every $z_3 \in \bar{B}_3 \cap V_3$, where $c_9 > 0$ is a constant. It follows that $\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2) e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is smaller than some constant.

Recall, finally, that in order to finish the proof we need to establish that the quantity $\rho(\bar{B}_3 \cap V_2, \bar{B}_3 \cap V_1) e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$ is smaller than some constant. It is easy to see that to do this we can use essentially the same argument as in the case of $\rho(\bar{B}_3 \cap V_3, \bar{B}_3 \cap V_2) e^{d((\hat{x},\hat{t}),\partial H_p)}/\hat{t}$. Indeed, the argument is considerably shorter since we do not need to give a proof for the trivial fact that the fixed quantity $d_{\text{euc}}(V_1) = d_{\text{euc}}(V_0)$ is larger than some positive constant. The proof is complete. $\square$

## 5. Geometric measure constructions

We introduce in this chapter the basic versions of our modifications of the standard covering and packing measure constructions. We will show in Chapter 6 that these modified constructions can be used to construct measures which are identical to Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups up to multiplicative constants. It is possible to define variants of the basic versions of the modifications just like in the case of the standard constructions. We will consider some of them and their relation to Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups in Chapter 7.

This chapter is similar to Chapters 3 and 4 in that we consider here a more general situation than the one in which we will actually be applying the main results of this chapter. In particular, it is natural to define the modified measure constructions without reference to Kleinian groups and their conformal measures. Indeed, it may turn out that the modified constructions presented here or some variants of them prove useful in some other contexts than that of Kleinian groups.

Let us discuss the modified constructions in detail. We fix a non-empty subset $X$ of $\mathbb{R}^n$. The set $X$ is the *base set* with respect to which the modified constructions will be applied. In Chapter 6, we will choose $X = L(G) \cap \mathbb{R}^n$ when discussing a non-elementary geometrically finite Kleinian group $G$ acting on $\mathbb{H}^{n+1}$.

We define *flatness functions* $\tau_l$, $l \in \{1, 2, \ldots, n\}$, with respect to $X$ which are similar to the flatness functions discussed in Chapter 4. Given $l \in \{1, 2, \ldots, n\}$, we set that

$$\tau_l(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap X, \bar{B}^n(x,t) \cap V) \tag{5.1}$$

for every $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap X \neq \emptyset$, where $\mathcal{F}_l(x,t)$ is the collection of all $l$-spheres of $\bar{\mathbb{R}}^n$ meeting $\bar{B}^n(x,t)$ and $\rho$ is the Hausdorff pseudometric defined with respect to the euclidean metric in the collection of non-empty and bounded subsets of $\mathbb{R}^n$. (That is, $\rho(A, B)$ is defined by (4.3) for all non-empty and bounded subsets $A$ and $B$ of $\mathbb{R}^n$.) We define also the *diameter function* $d$ with respect to $X$ by

$$d(x,t) = d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap X) \tag{5.2}$$

for every $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap X \neq \emptyset$.

We discuss next the functions which we will use to define the gauge functions of the modified constructions. Fix $\eta > 0$ and $\eta_1, \eta_2, \dots, \eta_n \in \mathbb{R}$. Given $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap X \neq \emptyset$, we define that

$$\alpha(x,t) = d(x,t)^\eta \tag{5.3}$$

and

$$\omega(x,t) = \prod_{l=1}^{n} \tau_l(x,t)^{\eta_l}. \tag{5.4}$$

The gauge functions of the modified constructions will feature quantities of the form $\alpha(x,t)\omega(x,t)$, where $x$ and $t$ are as above. Let us show that if we introduce proper conventions, these quantities are always well-defined. We start with the following lemma.

**Lemma 5.5.** *Suppose that $x \in \mathbb{R}^n$ and $t > 0$ are such that $B^n(x,t) \cap X \neq \emptyset$ and that $d(x,t) > 0$. Suppose that $\tau_{l_0}(x,t) = 0$ for some $l_0 \in \{1, 2, \dots, n\}$. Then $\tau_l(x,t) > 0$ for all $l \in \{1, 2, \dots, n\} \setminus \{l_0\}$.*

*Proof.* We write $B^n(x,t) = B$ and $\bar{B}^n(x,t) = \bar{B}$ in this proof. It is sufficient to prove the claim for a fixed $l \in \{1, 2, \dots, n\} \setminus \{l_0\}$. Let $V \in \mathcal{F}_l(x,t)$. Our aim is to show that there is a constant $c > 0$ such that $\rho(\bar{B} \cap X, \bar{B} \cap V) \geq c$.

Since $B \cap X \neq \emptyset$ and $d(x,t) > 0$, we can choose $z_1 \in B \cap X$ and $z_2 \in \bar{B} \cap X$ such that $z_1 \neq z_2$. Let $\varepsilon > 0$ be a small number such that $\bar{B}^n(z_1, 2\varepsilon) \subset B$ and $\bar{B}^n(z_1, 2\varepsilon) \cap \bar{B}^n(z_2, 2\varepsilon) = \emptyset$. If $\bar{B} \cap V$ does not meet $\bar{B}^n(z_j, \varepsilon)$, $j = 1, 2$, we can choose $c = \varepsilon$. We assume, therefore, that $(\bar{B} \cap V) \cap \bar{B}^n(z_j, \varepsilon) \neq \emptyset$ for $j = 1, 2$.

Let $W \in \mathcal{F}_{l_0}(x,t)$ be such that $(\bar{B} \cap W) \cap \bar{B}^n(z_j, \varepsilon) \neq \emptyset$ for $j = 1, 2$. We can use Lemma 2.65 to conclude that there is a constant $c_0 > 0$ such that $\rho(\bar{B} \cap W, \bar{B} \cap V) \geq c_0$. Since $\tau_{l_0}(x,t) = 0$, we can assume that $W$ is such that $\rho(\bar{B} \cap X, \bar{B} \cap W) \leq c_0/2$. We can now deduce that

$$\rho(\bar{B} \cap X, \bar{B} \cap V) \geq |\rho(\bar{B} \cap X, \bar{B} \cap W) - \rho(\bar{B} \cap W, \bar{B} \cap V)| \geq \frac{c_0}{2},$$

and so we can choose $c = c_0/2$ in this case. The proof is complete. □

We can now introduce the conventions which guarantee that the quantities $\alpha(x,t)\omega(x,t)$ are well-defined, where $x \in \mathbb{R}^n$ and $t > 0$ are such that $B^n(x,t) \cap X \neq \emptyset$. Note first that $\tau_l(x,t) \in [0, 2]$ for every $l \in \{1, 2, \dots, n\}$. We introduce the convention that if $\eta_l = 0$ for some $l \in \{1, 2, \dots, n\}$, then $\tau_l(x,t)^{\eta_l} = 1$. Suppose that $d(x,t) > 0$. Then Lemma 5.5 implies that $\tau_l(x,t) = 0$ for at most one $l \in \{1, 2, \dots, n\}$. Hence $\tau_l(x,t)^{\eta_l} \in \{0, \infty\}$ for at most one $l \in \{1, 2, \dots, n\}$, and so we see that $\alpha(x,t)\omega(x,t) \in [0, \infty]$ is well-defined if $d(x,t) > 0$. On the other hand, if $d(x,t) = 0$, we set that $\alpha(x,t)\omega(x,t) = 0$ by convention.

We conclude that the quantity $\alpha(x,t)\omega(x,t)$ is well-defined for every $x \in \mathbb{R}^n$ and every $t > 0$ such that $B^n(x,t) \cap X \neq \emptyset$.

We have shown that the quantities which we will be using in the gauge functions of our modified constructions are always well-defined. We proceed to give the explicit definitions of the constructions themselves.

We define first the *modified covering outer measure m*. Let $A \subset X$. Our aim is to define $m(A)$. Let $\varepsilon > 0$ and $\nu \in ]0,1[$. We say that a countable collection $\mathcal{T}$ of closed balls $\bar{B}^n(x,t)$ of $\mathbb{R}^n$ is an $(\varepsilon,\nu)$-*covering* of $A$ if the union of the balls in $\mathcal{T}$ covers $A$, $x \in \mathbb{R}^n$, $t \in ]0,\varepsilon]$ , and there is $x' \in B^n(x,t) \cap X$ with $|x - x'|/t \leq \nu$. Observe that the collection containing the balls $\bar{B}^n(x,\varepsilon)$ is an $(\varepsilon,\nu)$-covering of $X$, where $x \in \mathbb{R}^n$ is such that the coordinates of $x$ are rational numbers and there is $x' \in X$ with $|x - x'| \leq \nu\varepsilon$. It is the case, therefore, that $A$ has $(\varepsilon,\nu)$-coverings. We define the preliminary quantity

$$m_\varepsilon^\nu(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t), \tag{5.6}$$

where $\mathcal{T}$ varies in the collection of all $(\varepsilon,\nu)$-coverings of $A$. It is the case that every $(\varepsilon',\nu')$-covering of $A$ is an $(\varepsilon,\nu)$-covering of $A$ if $\varepsilon' \in ]0,\varepsilon]$ and $\nu' \in ]0,\nu]$, so $m_{\varepsilon'}^{\nu'}(A) \geq m_\varepsilon^\nu(A)$ in this case. This means that it is natural to define the *m-measure* of $A$ to be

$$m(A) = \sup_{\varepsilon>0,\nu\in]0,1[} m_\varepsilon^\nu(A). \tag{5.7}$$

We define next the *modified packing outer measure p*. Let again $A \subset X$. Our aim is to define $p(A)$, but we need to define first a preliminary quantity $p^*(A)$ as follows. If $A = \emptyset$, we set $p^*(A) = 0$. Assume that $A \neq \emptyset$ for the moment. Let $\varepsilon > 0$ and $\nu \in ]0,1[$. We say that a countable collection $\mathcal{T}$ of closed balls $\bar{B}^n(x,t)$ of $\mathbb{R}^n$ is an $(\varepsilon,\nu)$-*packing* of $A$ if the balls in $\mathcal{T}$ are pairwise disjoint, $x \in \mathbb{R}^n$, $t \in ]0,\varepsilon]$ and there is $x' \in B^n(x,t) \cap A$ with $|x - x'|/t \leq \nu$. Define

$$p_\varepsilon^\nu(A) = \sup_{\mathcal{T}} \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t), \tag{5.8}$$

where $\mathcal{T}$ varies in the collection of all $(\varepsilon,\nu)$-packings of $A$. If $\varepsilon' \in ]0,\varepsilon]$ and $\nu' \in ]0,\nu]$, then any $(\varepsilon',\nu')$-packing of $A$ is an $(\varepsilon,\nu)$-packing of $A$, which implies that $p_{\varepsilon'}^{\nu'}(A) \leq p_\varepsilon^\nu(A)$. This motivates us to define that

$$p^*(A) = \inf_{\varepsilon>0,\nu\in]0,1[} p_\varepsilon^\nu(A). \tag{5.9}$$

The set function $p^*$ is not countably subadditive in general. However, as we will see, countable subadditivity can be obtained using the following standard procedure. We define that the *p-measure* of any $A \subset X$ is

$$p(A) = \inf_{\mathcal{D}} \sum_{D\in\mathcal{D}} p^*(D), \tag{5.10}$$

where the infimum is taken over all countable collections $\mathcal{D}$ of subsets of $A$ such that $\bigcup \mathcal{D} = A$.

It is clear that the above constructions are straightforward modifications of the standard covering and packing constructions. (The packing construction is probably somewhat less well-known than the covering construction. The definition of the packing construction in the standard situation is discussed, for example, in [TT1985], and the basic properties of packing measures are studied, for instance, in [Mattila1995]. We give the definitions of the standard constructions on page 102.) The main difference between the standard constructions and the modified constructions is that the modified constructions take explicitly into account certain geometric properties of the sets of the form $\bar{B}^n(x,t) \cap X$ appearing in the constructions: the quantity $\tau_l(x,t)$, $l \in \{1, 2, \dots, n\}$, describes how closely $X$ resembles an $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$ and the term $\omega(x,t)$ in the gauge function $(x,t) \mapsto \alpha(x,t)\omega(x,t)$ is determined by the quantities $\tau_l(x,t)$. In contrast, the gauge functions used in the standard constructions usually depend explicitly only on $t$, the most basic gauge function being $(x,t) \mapsto t^s$, where $s \geq 0$ is fixed. Note that the term $\alpha(x,t)$ of our gauge function performs a very similar role as the terms of the form $t^s$ appearing in the gauge functions of the standard constructions. Indeed, we could replace the definition (5.3) by $\alpha(x,t) = t^\eta$ or $\alpha(x,t) = (2t)^\eta$ and the main results of this work would remain true. The reason for using the definition (5.3) is that we want the quantity $\alpha(x,t)$ to be intrinsically connected to the base set $X$. The purpose of the parameter $\nu$ is to guarantee that the sets $\bar{B}^n(x,t) \cap X$ considered contain enough of the set $X$ in order for the geometric properties of $\bar{B}^n(x,t) \cap X$ to appear. We will discuss the parameter $\nu$ more at the end of this chapter.

Let us verify that the above constructions construct outer measures of $X$ whose $\sigma$-algebras of measurable sets contain all Borel sets of $X$. It is natural that the arguments needed to do this are easy modifications of the standard arguments (see, for example, [Mattila1995] or basically any standard textbook on measure theory). Most of the following reasoning is very easy, but we include all the details because doing so allows us to refer conveniently to the proofs in this chapter when we discuss some variants of our modified constructions in Chapter 7.

**Theorem 5.11.** *The set functions $m$ and $p$ defined by (5.7) and (5.10) are outer measures of $X$. Every Borel set of $X$ is measurable with respect to $m$ and $p$.*

*Proof.* We establish first that $m$ and $p$ are outer measures of $X$. We begin by showing that $m(\emptyset) = 0 = p(\emptyset)$. It is natural to allow that $\emptyset$ is an $(\varepsilon, \nu)$-covering of $\emptyset$ for every $\varepsilon > 0$ and $\nu \in ]0,1[$. Hence $m^\nu_\varepsilon(\emptyset) = 0$ for every $\varepsilon > 0$ and $\nu \in ]0,1[$, and so $m(\emptyset) = 0$. (Alternatively, we could simply define that $m(\emptyset) = 0$.) On the other hand, $p^*(\emptyset) = 0$ by definition, so $p(\emptyset) = 0$ since $p(\emptyset) \leq p^*(\emptyset)$.

We prove that $m$ and $p$ are monotonic. Let $A_1 \subset A_2 \subset X$. Given $\varepsilon > 0$ and $\nu \in ]0,1[$, every $(\varepsilon, \nu)$-covering of $A_2$ is an $(\varepsilon, \nu)$-covering of $A_1$. It follows that $m^\nu_\varepsilon(A_1) \leq m^\nu_\varepsilon(A_2)$, and so $m(A_1) \leq m(A_2)$. On the other hand, if $A_1 = \emptyset$, then $0 = p^*(A_1) \leq p^*(A_2)$. Suppose that $A_1 \neq \emptyset$ for the moment. Then every $(\varepsilon, \nu)$-packing of $A_1$ is an $(\varepsilon, \nu)$-packing of $A_2$ for every $\varepsilon > 0$ and $\nu \in ]0,1[$, which implies that $p^\nu_\varepsilon(A_1) \leq p^\nu_\varepsilon(A_2)$. We conclude that $p^*(A_1) \leq p^*(A_2)$ whether or not $A_1$ is empty. Let next $\mathcal{D}$ be a countable collection of subsets of $A_2$ whose union is $A_2$. Then $\{D \cap A_1 : D \in \mathcal{D}\}$ is a countable collection of subsets of $A_1$ whose union is $A_1$. The first part of the argument implies that

$$p(A_1) \leq \sum_{D \in \mathcal{D}} p^*(D \cap A_1) \leq \sum_{D \in \mathcal{D}} p^*(D).$$

Since $\mathcal{D}$ was arbitrary, we see that $p(A_1) \leq p(A_2)$.

We need to show that $m$ and $p$ are countably subadditive to complete the proof of the claim that $m$ and $p$ are outer measures of $X$. Let $A_1, A_2, \ldots \subset X$. We prove that

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i) \qquad \text{and} \qquad p\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} p(A_i).$$

We consider $m$ first. Fix $u > 0$. Let $\varepsilon > 0$ and $v \in ]0, 1[$. Given $i \in \{1, 2, \ldots\}$, let $\mathcal{T}_i$ be an $(\varepsilon, v)$-covering of $A_i$ such that

$$\sum_{\bar{B}^n(x,t) \in \mathcal{T}_i} \alpha(x, t)\omega(x, t) \leq m_\varepsilon^v(A_i) + \frac{u}{2^i}.$$

The collection $\bigcup_{i=1}^{\infty} \mathcal{T}_i$ is an $(\varepsilon, v)$-covering of $\bigcup_{i=1}^{\infty} A_i$. We conclude that

$$\begin{aligned} m_\varepsilon^v\left(\bigcup_{i=1}^{\infty} A_i\right) &\leq \sum_{i=1}^{\infty} \sum_{\bar{B}^n(x,t) \in \mathcal{T}_i} \alpha(x, t)\omega(x, t) \leq \sum_{i=1}^{\infty} \left(m_\varepsilon^v(A_i) + \frac{u}{2^i}\right) \\ &= \sum_{i=1}^{\infty} m_\varepsilon^v(A_i) + u \leq \sum_{i=1}^{\infty} m(A_i) + u. \end{aligned}$$

We take the supremum over $\varepsilon > 0$ and $v \in ]0, 1[$ and obtain that

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i) + u, \text{ which implies that } m\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$$

since $u > 0$ was arbitrary. We have proved that $m$ is countably subadditive.

Consider next $p$. Fix again $u > 0$. Suppose that $\mathcal{D}_i$ is a countable collection of subsets of $A_i$ whose union is $A_i$ for $i \in \{1, 2, \ldots\}$. We can choose $\mathcal{D}_i$ so that

$$\sum_{D \in \mathcal{D}_i} p^*(D) \leq p(A_i) + \frac{u}{2^i}$$

for every $i \in \{1, 2, \ldots\}$. It is true that $\bigcup_{i=1}^{\infty} \mathcal{D}_i$ is a countable collection of subsets of $\bigcup_{i=1}^{\infty} A_i$ whose union is $\bigcup_{i=1}^{\infty} A_i$. Thus

$$p\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sum_{D \in \mathcal{D}_i} p^*(D) \leq \sum_{i=1}^{\infty} \left(p(A_i) + \frac{u}{2^i}\right) = u + \sum_{i=1}^{\infty} p(A_i),$$

and the countable subadditivity of $p$ follows immediately.

We have shown that $m$ and $p$ are outer measures of $X$. We prove next that every Borel set of $X$ is measurable with respect to $m$ and $p$. It is a well-known theorem of general measure theory that if $(Y, d_Y)$ is a metric space and $\mu$ is an outer measure of $Y$, then every Borel set of $Y$ is $\mu$-measurable if and only if $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ for every non-empty $A_1, A_2 \subset Y$ such that $d_Y(A_1, A_2) > 0$. (This result is mentioned, for example, in

[Mattila1995] as Theorem 1.7, but no detailed proof is given there.) Accordingly, we need to show that $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ and that $p(A_1 \cup A_2) = p(A_1) + p(A_2)$ for every non-empty $A_1, A_2 \subset X$ such that $d_{\text{euc}}(A_1, A_2) > 0$.

Consider $m$ first. Let $A_1, A_2 \subset X$ be non-empty sets such that $d = d_{\text{euc}}(A_1, A_2) > 0$. Suppose that $\varepsilon \in ]0, d/2[$ and $\nu \in ]0, 1[$. Let $\mathcal{T}$ be an $(\varepsilon, \nu)$-covering of $A_1 \cup A_2$. Let $\mathcal{T}_j = \{\bar{B}^n(x,t) \in \mathcal{T} : \bar{B}^n(x,t) \cap A_j \neq \emptyset\}$ for $j = 1, 2$. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint and $\mathcal{T}_j$ is an $(\varepsilon, \nu)$-covering of $A_j$ for $j = 1, 2$. We obtain that

$$\sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t) \geq \sum_{\bar{B}^n(x,t)\in\mathcal{T}_1} \alpha(x,t)\omega(x,t) + \sum_{\bar{B}^n(x,t)\in\mathcal{T}_2} \alpha(x,t)\omega(x,t) \geq m_\varepsilon^\nu(A_1) + m_\varepsilon^\nu(A_2).$$

It follows that

$$m(A_1 \cup A_2) \geq m_\varepsilon^\nu(A_1 \cup A_2) \geq m_\varepsilon^\nu(A_1) + m_\varepsilon^\nu(A_2)$$

for every $\varepsilon \in ]0, d/2[$ and $\nu \in ]0, 1[$. This implies easily that $m(A_1 \cup A_2) \geq m(A_1) + m(A_2)$. Since we obtain that $m(A_1 \cup A_2) \leq m(A_1) + m(A_2)$ from the countable subadditivity of $m$, we see that $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ and, therefore, every Borel set of $X$ is $m$-measurable.

We consider $p$ next. Let $A_1$ and $A_2$ be as above. Suppose that $D_j \subset A_j$ is non-empty for $j = 1, 2$. Assume that $\varepsilon \in ]0, d/4[$ and $\nu \in ]0, 1[$. Let $\mathcal{T}_j$ be an $(\varepsilon, \nu)$-packing of $D_j$ for $j = 1, 2$. The collections $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is an $(\varepsilon, \nu)$-packing of $D_1 \cup D_2$. Hence

$$p_\varepsilon^\nu(D_1 \cup D_2) \geq \sum_{\bar{B}^n(x,t)\in\mathcal{T}_1} \alpha(x,t)\omega(x,t) + \sum_{\bar{B}^n(x,t)\in\mathcal{T}_2} \alpha(x,t)\omega(x,t).$$

It clearly follows that $p_\varepsilon^\nu(D_1 \cup D_2) \geq p_\varepsilon^\nu(D_1) + p_\varepsilon^\nu(D_2) \geq p^*(D_1) + p^*(D_2)$. We deduce that $p^*(D_1 \cup D_2) \geq p^*(D_1) + p^*(D_2)$. Note that this estimate is trivial if $D_1 = \emptyset$ or $D_2 = \emptyset$. Let $\mathcal{D}$ be a countable collection of subsets of $A_1 \cup A_2$ whose union is $A_1 \cup A_2$. Let $\mathcal{D}_j = \{A_j \cap D : D \in \mathcal{D}\}$ for $j = 1, 2$. Then $\mathcal{D}_j$ is a countable collection of subsets of $A_j$ whose union is $A_j$ for $j = 1, 2$. We conlude that

$$\begin{aligned} \sum_{D\in\mathcal{D}} p^*(D) &= \sum_{D\in\mathcal{D}} p^*((A_1 \cap D) \cup (A_2 \cap D)) \geq \sum_{D\in\mathcal{D}} (p^*(A_1 \cap D) + p^*(A_2 \cap D)) \\ &= \sum_{D\in\mathcal{D}_1} p^*(D) + \sum_{D\in\mathcal{D}_2} p^*(D) \geq p(A_1) + p(A_2). \end{aligned}$$

This shows that $p(A_1 \cup A_2) \geq p(A_1) + p(A_2)$, since $\mathcal{D}$ was arbitrary. Like in the case of $m$, the opposite inequality follows from the countable subadditivity of $p$. We obtain that every Borel set of $X$ is $p$-measurable. □

We have proved that the set functions $m$ and $p$ defined by (5.7) and (5.10) are outer measures of $X$ such that every Borel set of $X$ is measurable with respect to $m$ and $p$. We denote the measures of $X$ corresponding to $m$ and $p$ by the same symbols.

We prove next the following theorem, Theorem 5.12, which shows that the measures $m$ and $p$ satisfy a transformation rule resembling closely the $s$-conformality condition (2.14) of conformal measures of Kleinian groups. We need to make distinctions in the claim and proof of Theorem 5.12 between measures constructed using the above modified constructions with respect to different base sets. Accordingly, we introduce the convention that the (outer) measures constructed when the modified constructions are applied to the base set $X \subset \mathbb{R}^n$ are denoted by $m^X$ and $p^X$. The corresponding flatness and diameter functions are denoted by $\tau_l^X$, $l \in \{1, 2, \ldots, n\}$, and $d^X$. We also denote the quantities defined by (5.3) and (5.4) by $\alpha^X(x, t)$ and $\omega^X(x, t)$. Note that the parameters $\eta, \eta_1, \eta_2, \ldots, \eta_n$ are the same for all base sets considered. Finally, if $\mathcal{T}$ is a covering defined with respect to the parameters $\varepsilon > 0$ and $\nu \in ]0, 1[$ and the base set $X$, we say that $\mathcal{T}$ is an $(\varepsilon, \nu)^X$-covering. The definition of a packing does not refer explicitly to the base set so there is no need for a change in the notation for packings. The proof we will give for Theorem 5.12 is again a straightforward modification of proofs given in the case of the standard constructions.

**Theorem 5.12.** *Let $X \subset \mathbb{R}^n$ be non-empty. Let $g \in \mathrm{M\ddot{o}b}(n)$ be such that $gX \cap \mathbb{R}^n \neq \emptyset$. Then*

$$m^{gX \cap \mathbb{R}^n}(gA) = \int_A |g'|^\eta dm^X \qquad \text{and} \qquad p^{gX \cap \mathbb{R}^n}(gA) = \int_A |g'|^\eta dp^X \tag{5.13}$$

*for every Borel set $A \subset X \setminus \{g^{-1}(\infty)\}$ of $X$.*

*Proof.* We start by proving the first formula of (5.13). Recall that if $g$ fixes $\infty$, then $|g'|$ is a finite and positive constant in $\bar{\mathbb{R}}^n$, and that if $g$ does not fix $\infty$, then

$$|g'(x)| = \frac{r_g^2}{|x - g^{-1}(\infty)|^2}$$

for $x \in \mathbb{R}^n \setminus \{g^{-1}(\infty)\}$, where $r_g$ is the euclidean radius of the isometric sphere of $g$, see (2.12). Let $A \subset X \setminus \{g^{-1}(\infty)\}$ be a non-empty Borel set of $X$. We can divide $A$ into countably many pairwise disjoint non-empty Borel sets of $X$ such that $|g'|$ is between two positive constants in a small neighbourhood of each of these sets. The countable additivity of $m^X$ and $m^{gX \cap \mathbb{R}^n}$ implies that it is sufficient to prove the first formula of (5.13) separately for each of these subsets of $A$. We can thus assume that $M^{-1} \leq |g'| \leq M$ in $A(a_0)$ for some constants $M > 0$ and $a_0 > 0$, where $A(a_0) = \{x \in \mathbb{R}^n : d_{\mathrm{euc}}(x, A) < a_0\}$.

Let $\lambda > 0$ be small. Divide $A$ into pairwise disjoint non-empty Borel sets $A_1, A_2, \ldots, A_{k_\lambda}$ of $X$ in the following way. Let $M_k$ and $m_k$ denote the supremum and infimum of $|g'|$ over $A_k(a_1) \subset A(a_0)$ for $k \in \{1, 2, \ldots, k_\lambda\}$, where $a_1 > 0$ is a number depending on $\lambda$ and $A_k(a_1) = \{x \in \mathbb{R}^n : d_{\mathrm{euc}}(x, A_k) < a_1\}$. We require that the division $A_1, A_2, \ldots, A_{k_\lambda}$ corresponding to $\lambda$ be such that $M_k/m_k \leq \sigma$ for every $k \in \{1, 2, \ldots, k_\lambda\}$, where $\sigma = \sigma(\lambda) \geq 1$ and $\sigma \to 1$ as $\lambda \to 0$.

We consider a fixed $A_k$, $k \in \{1, 2, \ldots, k_\lambda\}$. Let $\varepsilon > 0$ and $\nu \in ]0, 1[$. Let $\mathcal{T}$ be an $(\varepsilon, \nu)^X$-covering of $A_k$. We assume that $\bar{B} \cap A_k \neq \emptyset$ for all $\bar{B} \in \mathcal{T}$. We assume also that $\varepsilon$ is so small, say $\varepsilon \leq \varepsilon_0$ for some fixed $\varepsilon_0 > 0$, that $\bar{B} \subset A_k(a_1)$ for every $\bar{B} \in \mathcal{T}$.

The mapping $g$ maps the balls in $\mathcal{T}$ onto euclidean balls. We write $\bar{B}^n(y, u) = g\bar{B}^n(x, t)$ if $\bar{B}^n(x, t) \in \mathcal{T}$. We see that (recall (2.52))

$$u = \frac{1}{2} \sup_{z_1, z_2 \in \bar{B}^n(x,t)} |g(z_1) - g(z_2)| = \frac{1}{2} \sup_{z_1, z_2 \in \bar{B}^n(x,t)} |g'(z_1)|^{1/2} |g'(z_2)|^{1/2} |z_1 - z_2|$$

for every $\bar{B}^n(x,t) \in \mathcal{T}$. It follows that $u \in [m_g(\bar{B})t, M_g(\bar{B})t]$ for every $\bar{B} = \bar{B}^n(x,t) \in \mathcal{T}$, where

$$m_g(\bar{B}) = \inf_{z\in\bar{B}} |g'(z)| \geq m_k \qquad \text{and} \qquad M_g(\bar{B}) = \sup_{z\in\bar{B}} |g'(z)| \leq M_k.$$

We have that $M_g(\bar{B})/m_g(\bar{B}) \leq \theta$ for every $\bar{B} \in \mathcal{T}$, where $\theta = \theta(\varepsilon) \geq 1$ and $\theta \to 1$ as $\varepsilon \to 0$. Recall next that, given $\bar{B}^n(x,t) \in \mathcal{T}$, there is $x' \in B^n(x,t) \cap X$ such that $|x - x'|/t \leq v$. Write $y' = g(x')$ for every $\bar{B}^n(x,t) \in \mathcal{T}$. We apply (2.52) again and deduce that

$$d_{\mathrm{euc}}(y', S^{n-1}(y,u)) \geq m_g(\bar{B}) d_{\mathrm{euc}}(x', S^{n-1}(x,t)) \geq m_g(\bar{B})(1-v)t \geq \frac{m_g(\bar{B})}{M_g(\bar{B})}(1-v)u \geq \frac{1-v}{\theta}u$$

for every $\bar{B} = \bar{B}^n(x,t) \in \mathcal{T}$. We have shown that the collection $g\mathcal{T} = \{g\bar{B} : \bar{B} \in \mathcal{T}\}$ is an $(M\varepsilon, 1-(1-v)/\theta)^{gX\cap\mathbb{R}^n}$-covering of $gA_k$.

We turn to the quantities appearing in the gauge functions. We apply (2.52) once again to see that

$$\alpha^{gX\cap\mathbb{R}^n}(y,u) = d_{\mathrm{euc}}(\bar{B}^n(y,u) \cap gX)^\eta \geq m_k^\eta d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap X)^\eta = m_k^\eta \alpha^X(x,t)$$

and, similarly, that $\alpha^{gX\cap\mathbb{R}^n}(y,u) \leq M_k^\eta \alpha^X(x,t)$ for every $\bar{B}^n(x,t) \in \mathcal{T}$. Furthermore, continuing to use (2.52) and noting that $V \in \mathcal{F}_l(x,t)$ if and only if $gV \in \mathcal{F}_l(y,u)$ for every $l \in \{1,2,\ldots,n\}$ and every $\bar{B}^n(x,t) \in \mathcal{T}$, we calculate that

$$\begin{aligned}
\tau_l^{gX\cap\mathbb{R}^n}(y,u) &= \frac{1}{u} \inf_{V\in\mathcal{F}_l(y,u)} \rho(\bar{B}^n(y,u) \cap gX, \bar{B}^n(y,u) \cap V) \\
&= \frac{1}{u} \inf_{V\in\mathcal{F}_l(x,t)} \rho(g\bar{B}^n(x,t) \cap gX, g\bar{B}^n(x,t) \cap gV) \\
&\geq \frac{m_k}{M_k t} \inf_{V\in\mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap X, \bar{B}^n(x,t) \cap V) \\
&\geq \sigma^{-1}\tau_l^X(x,t)
\end{aligned}$$

for every $l \in \{1,2,\ldots,n\}$ and every $\bar{B}^n(x,t) \in \mathcal{T}$. We obtain similarly that $\tau_l^{gX\cap\mathbb{R}^n}(y,u) \leq \sigma\tau_l^X(x,t)$ for every $l \in \{1,2,\ldots,n\}$ and every $\bar{B}^n(x,t) \in \mathcal{T}$. We deduce that there is $\chi = \chi(\lambda) \geq 1$ such that $\chi \to 1$ as $\lambda \to 0$ and that

$$\omega^{gX\cap\mathbb{R}^n}(y,u) = \prod_{l=1}^n \tau_l^{gX\cap\mathbb{R}^n}(y,u)^{\eta_l} \in [\chi^{-1}\omega^X(x,t), \chi\omega^X(x,t)]$$

for every $\bar{B}^n(x,t) \in \mathcal{T}$.

We continue to consider a fixed $A_k$, $k \in \{1,2,\ldots,k_\lambda\}$. We combine the facts established above and conclude that

$$(m^{gX\cap\mathbb{R}^n})_{M\varepsilon}^{1-\frac{1-v}{\theta}}(gA_k) \leq \sum_{\bar{B}^n(y,u)\in g\mathcal{T}} \alpha^{gX\cap\mathbb{R}^n}(y,u)\omega^{gX\cap\mathbb{R}^n}(y,u) \leq \chi M_k^\eta \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha^X(x,t)\omega^X(x,t).$$

The inequality between the first and the third quantity in the above estimate is valid for every $(\varepsilon, v)^X$-covering $\mathcal{T}$ of $A_k$ and not just for those $\mathcal{T}$ whose elements meet $A_k$. We can, therefore, take the infimum over the $(\varepsilon, v)^X$-coverings $\mathcal{T}$ of $A_k$ and conclude that

$$(m^{gX\cap\mathbb{R}^n})^{1-\frac{1-v}{\theta}}_{M\varepsilon}(gA_k) \leq \chi M_k^\eta (m^X)^v_\varepsilon(A_k) \leq \chi\left(\frac{M_k}{m_k}\right)^\eta m_k^\eta m^X(A_k).$$

Recall that we have considered above arbitrary $\varepsilon \in ]0, \varepsilon_0]$ and $v \in ]0, 1[$, where $\varepsilon_0 > 0$ is fixed. Recall also that $\theta = \theta(\varepsilon) \geq 1$ and that $\theta \to 1$ as $\varepsilon \to 0$. It is not difficult to see that

$$\sup_{\beta>0, w\in]0,1[} (m^{gX\cap\mathbb{R}^n})^w_\beta(gA_k) = \sup_{\varepsilon\in]0,\varepsilon_0], v\in]0,1[} (m^{gX\cap\mathbb{R}^n})^{1-\frac{1-v}{\theta}}_{M\varepsilon}(gA_k). \tag{5.14}$$

The quantity on the left hand side of (5.14) is $m^{gX\cap\mathbb{R}^n}(gA_k)$ by definition, so

$$m^{gX\cap\mathbb{R}^n}(gA_k) \leq \chi\left(\frac{M_k}{m_k}\right)^\eta m_k^\eta m^X(A_k) \leq \chi\sigma^\eta \int_{A_k} |g'|^\eta dm^X. \tag{5.15}$$

Recall that the sets $A_k$, $k \in \{1, 2, \ldots, k_\lambda\}$, are pairwise disjoint and that their union is $A$. This means that we can sum with respect to $k$ in (5.15) to obtain that

$$m^{gX\cap\mathbb{R}^n}(gA) \leq \chi\sigma^\eta \int_A |g'|^\eta dm^X.$$

Let $\lambda \to 0$. It follows that $\chi \to 1$ and $\sigma \to 1$. We can conclude that

$$m^{gX\cap\mathbb{R}^n}(gA) \leq \int_A |g'|^\eta dm^X. \tag{5.16}$$

We have proved the first half of the first formula of (5.13).

Our next claim is that (5.16) implies that

$$\int_{gA} |h'|^\eta dm^{gX\cap\mathbb{R}^n} \leq \int_A (|h'|^\eta \circ g)|g'|^\eta dm^X \tag{5.17}$$

for every $h \in$ Möb$(n)$ such that $|h'|$ is between two positive constants in $gA$. The formula (5.17) is obvious if $m^X(A) = \infty$. Suppose that $m^X(A) < \infty$ for the time being. Let $s > 0$ be arbitrary. We can divide $A$ into pairwise disjoint and non-empty Borel sets $A_1, A_2, \ldots, A_{k_s}$ such that $\sup_{gA_k}|h'|^\eta - \inf_{gA_k}|h'|^\eta \leq s$ for every $k \in \{1, 2, \ldots, k_s\}$. We can use (5.16) to estimate that (recall that $|g'| \leq M$ in $A$ by assumption)

$$\begin{aligned}
\int_{gA} |h'|^\eta dm^{gX\cap\mathbb{R}^n} &\leq \sum_{k=1}^{k_s} (\sup_{gA_k} |h'|^\eta) m^{gX\cap\mathbb{R}^n}(gA_k) \leq \sum_{k=1}^{k_s} (\sup_{A_k}(|h'|^\eta \circ g)) \int_{A_k} |g'|^\eta dm^X \\
&\leq \int_A ((|h'|^\eta \circ g) + s)|g'|^\eta dm^X \leq \int_A (|h'|^\eta \circ g)|g'|^\eta dm^X + sM^\eta m^X(A).
\end{aligned}$$

Since $s > 0$ is arbitrary and $m^X(A)$ is assumed to be finite for the moment, the formula (5.17) follows.

Observe next that $gA$ is a non-empty Borel set of $gX \cap \mathbb{R}^n$ such that $M^{-1} \leq |(g^{-1})'| \leq M$ in $gA(a_0') = \{z \in \mathbb{R}^n : d_{\text{euc}}(z, gA) < a_0'\}$ for some constant $a_0' > 0$. We can, therefore, apply the reasoning used to prove (5.16) to $gA$ and $g^{-1}$. The formula (5.17) becomes

$$\int_A |h'|^\eta dm^X \leq \int_{gA} (|h'|^\eta \circ g^{-1})|(g^{-1})'|^\eta dm^{gX\cap\mathbb{R}^n}$$

in this case, where $h \in \text{Möb}(n)$ is such that $|h'|$ is between two positive constants in $A$. We choose $h = g$ and obtain that

$$\int_A |g'|^\eta dm^X \leq \int_{gA} (|g'|^\eta \circ g^{-1})|(g^{-1})'|^\eta dm^{gX\cap\mathbb{R}^n} = m^{gX\cap\mathbb{R}^n}(gA). \tag{5.18}$$

We see that the first formula of (5.13) is valid.

We turn to the second formula of (5.13). Let $A \subset X \setminus \{g^{-1}(\infty)\}$ be a non-empty Borel set of $X$. We can assume that $M^{-1} \leq |g'| \leq M$ in $A(a_0)$, where $M$, $a_0$ and $A(a_0)$ are as in the case of the first formula of (5.13). Furthermore, we can choose the parameter $\lambda$ and the division $A_1, A_2, \ldots, A_{k_\lambda}$ of $A$ as before. Let the symbols $\sigma$, $a_1$, $A_k(a_1)$, $m_k$ and $M_k$, $k \in \{1, 2, \ldots, k_\lambda\}$, have the same meanings as before.

We consider a fixed $A_k$, $k \in \{1, 2, \ldots, k_\lambda\}$. Let $\mathcal{D}$ be a countable collection of subsets of $gA_k$ whose union is $gA_k$. Consider a non-empty $D \in \mathcal{D}$. Let $\varepsilon > 0$ and $v \in ]0, 1[$. Let $\mathcal{T}$ be an $(\varepsilon, v)$-packing of $g^{-1}D$. We assume that $\varepsilon$ is so small, say $\varepsilon \leq \varepsilon_0$ for some fixed $\varepsilon_0 > 0$, that $\bar{B} \subset A_k(a_1)$ for every $\bar{B} \in \mathcal{T}$. We write $g\bar{B}^n(x,t) = \bar{B}^n(y,u)$ if $\bar{B}^n(x,t) \in \mathcal{T}$. We can argue as in the first part of this proof to obtain that $g\mathcal{T} = \{g\bar{B} : \bar{B} \in \mathcal{T}\}$ is an $(M\varepsilon, 1 - (1-v)/\theta)$-packing of $D$, where $\theta = \theta(\varepsilon) \geq 1$ and $\theta \to 1$ as $\varepsilon \to 0$. We obtain also that

$$m_k^\eta \alpha^X(x,t) \leq \alpha^{gX\cap\mathbb{R}^n}(y,u) \leq M_k^\eta \alpha^X(x,t)$$

and

$$\chi^{-1}\omega^X(x,t) \leq \omega^{gX\cap\mathbb{R}^n}(y,u) \leq \chi\omega^X(x,t)$$

for every $\bar{B}^n(x,t) \in \mathcal{T}$, where $\chi = \chi(\lambda) \geq 1$ is such that $\chi \to 1$ as $\lambda \to 0$.

We conclude that

$$(p^{gX\cap\mathbb{R}^n})^{1-\frac{1-v}{\theta}}_{M\varepsilon}(D) \geq \sum_{\bar{B}^n(y,u)\in g\mathcal{T}} \alpha^{gX\cap\mathbb{R}^n}(y,u)\omega^{gX\cap\mathbb{R}^n}(y,u) \geq \chi^{-1}m_k^\eta \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha^X(x,t)\omega^X(x,t).$$

We take the supremum over the $(\varepsilon, v)$-packings $\mathcal{T}$ of $g^{-1}D$ and deduce that

$$(p^{gX\cap\mathbb{R}^n})^{1-\frac{1-v}{\theta}}_{M\varepsilon}(D) \geq \chi^{-1}\left(\frac{m_k}{M_k}\right)^\eta M_k^\eta (p^X)^v_\varepsilon(g^{-1}D) \geq \chi^{-1}\sigma^{-\eta}M_k^\eta (p^X)^*(g^{-1}D).$$

Recall that $\varepsilon \in ]0, \varepsilon_0]$ and $v \in ]0, 1[$, where $\varepsilon_0 > 0$ is fixed. Recall also that $\theta = \theta(\varepsilon) \geq 1$ and $\theta \to 1$ as $\varepsilon \to 0$. It is not difficult to see that

$$\inf_{\beta>0, w\in]0,1[} (p^{gX\cap\mathbb{R}^n})^w_\beta(D) = \inf_{\varepsilon\in]0,\varepsilon_0], v\in]0,1[} (p^{gX\cap\mathbb{R}^n})^{1-\frac{1-v}{\theta}}_{M\varepsilon}(D). \tag{5.19}$$

The left hand side of (5.19) is $(p^{gX\cap\mathbb{R}^n})^*(D)$ by definition, so

$$(p^{gX\cap\mathbb{R}^n})^*(D) \geq \chi^{-1}\sigma^{-\eta}M_k^\eta (p^X)^*(g^{-1}D).$$

This estimate is trivially true if $D = \emptyset$. Since $\{g^{-1}D : D \in \mathcal{D}\}$ is a countable collection of subsets of $A_k$ whose union is $A_k$, we obtain that

$$\begin{aligned}\sum_{D\in\mathcal{D}}(p^{gX\cap\mathbb{R}^n})^*(D) &\geq \chi^{-1}\sigma^{-\eta}M_k^\eta\sum_{D\in\mathcal{D}}(p^X)^*(g^{-1}D)\\ &\geq \chi^{-1}\sigma^{-\eta}M_k^\eta p^X(A_k) \geq \chi^{-1}\sigma^{-\eta}\int_{A_k}|g'|^\eta dp^X.\end{aligned}$$

The collection $\mathcal{D}$ was arbitrary, so we can take the infimum over $\mathcal{D}$ and deduce that

$$p^{gX\cap\mathbb{R}^n}(gA_k) \geq \chi^{-1}\sigma^{-\eta}\int_{A_k}|g'|^\eta dp^X. \tag{5.20}$$

Recall that the sets $A_k$, $k \in \{1, 2, \ldots, k_\lambda\}$, are pairwise disjoint and that their union is $A$. It follows that we can sum with respect to $k$ in (5.20) to obtain that

$$p^{gX\cap\mathbb{R}^n}(gA) \geq \chi^{-1}\sigma^{-\eta}\int_{A}|g'|^\eta dp^X.$$

Let $\lambda \to 0$. This implies that $\chi \to 1$ and $\sigma \to 1$. We conclude that

$$p^{gX\cap\mathbb{R}^n}(gA) \geq \int_{A}|g'|^\eta dp^X.$$

We see that we have proved the first half of the second formula of (5.13). It is easy to see that we can use an analogous argument as in the case of the first formula of (5.13) to prove the second half of the second formula of (5.13), see (5.17) and (5.18). We omit the details since the situations are so similar. The proof is complete. □

We end this chapter with the following discussion regarding the parameter $v$ introduced in the modified measure constructions. Recall that we explained following the definitions of the constructions that the main point in introducing $v$ is to guarantee that the sets $\bar{B}^n(x,t) \cap X$ considered in the constructions contain enough of the set $X$ for the geometric properties of $\bar{B}^n(x,t) \cap X$ to appear. Of course, if we set that $x \in X$ (i.e. that $x \in A$ in case $\bar{B}^n(x,t)$ is contained in a packing of a set $A \subset X$), we could guarantee that the sets $\bar{B}^n(x,t) \cap X$ contain enough of the set $X$ without having to introduce the parameter $v$. Indeed, if we change the constructions in this way, we still obtain a measure of $X$ from both constructions, which is straightforward to verify. But it seems that these alternative measures do not satisfy the claim of Theorem 5.12, or if they do, the above proof, which seems very natural, is not applicable. Let us study this in detail.

Let $g \in \text{Möb}(n)$ and $A \subset X \setminus \{g^{-1}(\infty)\}$ be as in Theorem 5.12. Let $\mathcal{T}$ be a covering or packing of a set $E \subset A$ as considered in the proof of Theorem 5.12. Write $g\bar{B}^n(x,t) = \bar{B}^n(y,u)$ for every $\bar{B}^n(x,t) \in \mathcal{T}$. The essential fact in the current situation is that the collection $g\mathcal{T} = \{g\bar{B} : \bar{B} \in \mathcal{T}\}$ is a suitable covering or packing of $gE$. This implies that the guantities of the form $\alpha^{gX\cap\mathbb{R}^n}(y,u)\omega^{gX\cap\mathbb{R}^n}(y,u)$ can be used when the measure of $gE$ is determined, and the proof of Theorem 5.12 shows that these quantities are naturally related to the quantities of the form $\alpha^X(x,t)\omega^X(x,t)$. (Another essential fact is that $V \in \mathcal{F}_l(x,t)$ if and only if $gV \in \mathcal{F}_l(y,u)$ for every $\bar{B}^n(x,t) \in \mathcal{T}$ and every $l \in \{1, 2, \ldots, n\}$.)

Suppose that we use the alternative definitions mentioned above, i.e. we assume that $x \in X$ in case $\bar{B}^n(x,t)$ is in a covering of some set $A \subset X$ and that $x \in A$ in case $\bar{B}^n(x,t)$ is in a packing of $A$. Let $g$, $A$, $E$ and $\mathcal{T}$ be as above. Assume that $g$ is not a euclidean similarity. Now it is the case that $g\mathcal{T}$ is not necessarily a covering or packing of $gE$. The natural choice is to replace $g\mathcal{T}$ by $\mathcal{T}' = \{\bar{B}^n(g(x), u + |g(x) - y|) : \bar{B}^n(x,t) \in \mathcal{T}\}$ if $\mathcal{T}$ is a covering, and by $\mathcal{T}'' = \{\bar{B}^n(g(x), d_{\text{euc}}(g(x), S^{n-1}(y,u))) : \bar{B}^n(x,t) \in \mathcal{T}\}$ if $\mathcal{T}$ is a packing. Now it seems that although the collections $g\mathcal{T}$ and $\mathcal{T}'$ are close to one another in the sense of euclidean metric, the geometric properties of the sets in the collections $\{g\bar{B} \cap gX : \bar{B} \in \mathcal{T}\}$ and $\{\bar{B}' \cap gX : \bar{B}' \in \mathcal{T}'\}$ can differ in a significant way. The same is true of $g\mathcal{T}$ and $\mathcal{T}''$. So it seems that there is not necessarily a similar relation between the quantities appearing in the gauge functions as in the case featuring the parameter $v$, or if there is, one cannot use the rather natural argument given in the proof of Theorem 5.12 to establish it.

We conclude that it seems natural to introduce the parameter $v$ into the constructions. We will discuss in Chapter 7 variants of the constructions which do not employ the parameter $v$ and which use flatness functions of the form defined by (4.42) instead of those defined by (5.1). As we will see, the above discussion applies to these variants.

## 6. Equivalence results for conformal measures

The topic of this chapter is the equivalence of conformal measures of a Kleinian group. We define that two measures $\mu_1$ and $\mu_2$ of $\bar{\mathbb{R}}^{n+1}$ are *equivalent* if $\mu_1$ and $\mu_2$ have the same measurable sets and $\mu_1 = c\mu_2$, where $c > 0$ is a constant. Note that this is a non-standard definition for the equivalence of measures. This is not a problem, however, since we do not use the standard definitions in this work.

In the first section of this chapter, we study the equivalence of conformal measures of a non-elementary Kleinian group in a general setting. The main result of this section gives a general sufficient condition guaranteeing that two conformal measures of a non-elementary Kleinian group are equivalent.

In the second section of this chapter, we apply the sufficient equivalence condition established in the first section in the context of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups. We show that, given a Patterson-Sullivan measure $\mu$ of a non-elementary geometrically finite Kleinian group $G$, we can use the modified covering construction and the modified packing construction introduced in Chapter 5 to construct measures $m$ and $p$, respectively, such that all of the measures $\mu$, $m$ and $p$ are equivalent to each other. We present the basic version of this equivalence result in this chapter and discuss some of its variants in Chapter 7.

6.1. **A general equivalence result.** Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu_1$ and $\mu_2$ be two $s$-conformal measures of $G$ for some $s > 0$. The purpose of this section is to establish a condition guaranteeing that $\mu_1 = c\mu_2$ for some constant $c > 0$, i.e. that $\mu_1$ and $\mu_2$ are equivalent. This condition will be formulated in Theorem 6.9. The main results of this section are well-known in the theory of conformal measures of Kleinian groups, see [Nicholls1989] or [Sullivan1979], for example, for standard accounts. Our exposition is based on unpublished lecture notes by P. Tukia.

We begin with a well-known result stating an interesting dichotomy in the theory of conformal measures of Kleinian groups. Recall that $L_c(G)$ denotes the set of conical limit points of a Kleinian group $G$ (see page 18 for the definition of a conical limit point). Recall also that, by definition, a conformal measure $\mu$ of a Kleinian group $G$ acting on $\mathbb{X}^{n+1}$ is supported by the limit set $L(G)$ and that the $\sigma$-algebra of $\mu$-measurable sets is the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$.

**Theorem 6.1.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Then it is the case that either $\mu(L_c(G)) = 0$ or $\mu(L(G) \setminus L_c(G)) = 0$.*

*Proof.* We begin by verifying that $L_c(G)$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$, so the claim of the theorem is well-defined. We assume that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ for the moment. Given $r > 0$, we define the set $L_c(G, r)$ as follows. A point $x \in \mathbb{R}^n$ is in $L_c(G, r)$ if there are $g_1, g_2, \ldots \in G$ such that $g_i(e_{n+1}) \to x$ with $d(g_i(e_{n+1}), L_x) \leq r$, where $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{H}^{n+1}$ and $L_x$ is the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x$ and $\infty$. We claim that $L_c(G) \cap \mathbb{R}^n = \bigcup_{r>0} L_c(G, r)$.

It is trivial that $L_c(G) \cap \mathbb{R}^n \subset \bigcup_{r>0} L_c(G, r)$, so let $x \in L_c(G, r)$ for some $r > 0$. Let $y \in \mathbb{H}^{n+1}$ and let $L$ be a hyperbolic line of $\mathbb{H}^{n+1}$ with $x$ as one of its endpoints. Let $g_1, g_2, \ldots \in G$ and $L_x$ be as in the definition of $L_c(G, r)$. Observe that if $z \in L_x$ approaches $x$, then $d(z, L)$ converges to 0. It is clear that since the elements in $G$ are hyperbolic isometries of $\mathbb{H}^{n+1}$ and $g_i(e_{n+1}) \to x$ so that $d(g_i(e_{n+1}), L_x) \leq r$ for every $i \in \{1, 2, \ldots\}$, it is the case that $g_i(y) \to x$ and there is $r' \geq 0$ such that $d(g_i(y), L) \leq r'$ for every $i \in \{1, 2, \ldots\}$. It is true, therefore, that $x \in L_c(G)$, and so $L_c(G) \cap \mathbb{R}^n = \bigcup_{r>0} L_c(G, r)$.

To prove that $L_c(G)$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$, it is sufficient to show that $L_c(G, r)$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$ for a fixed $r > 0$. Let $\bar{D}(y, r)$ be the closed hyperbolic ball of $\mathbb{H}^{n+1}$ with center $y \in \mathbb{H}^{n+1}$ and radius $r$. Now $g\bar{D}(y, r) = \bar{D}(g(y), r)$ for every $y \in \mathbb{H}^{n+1}$ and every $g \in G$. Denote by $P(\bar{D}(y, r))$ the orthogonal projection of $\bar{D}(y, r)$ into $\mathbb{R}^n$ for every $y \in \mathbb{H}^{n+1}$. Given $k \in \{1, 2, \ldots\}$, write $G_k = \{g \in G : g(e_{n+1})_{n+1} \in ]0, k^{-1}[\}$. Now

$$L_c(G, r) = \bigcap_{k=1}^{\infty} \bigcup_{g \in G_k} P(\bar{D}(g(e_{n+1}), r)), \tag{6.2}$$

so $L_c(G, r)$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$. Hence $L_c(G)$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$.

We turn to the claim of the theorem. Suppose first that the claim is valid if $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $\mu(\infty) = 0$. We show that we can reduce the general case to this special case.

Suppose that $\mathbb{X}^{n+1} = \mathbb{B}^{n+1}$. We can choose $h \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ such that $\mu(h^{-1}(\infty)) = 0$ since $\mu$ is finite by definition and $L(G)$ is uncountable. Now $h^s_*\mu$ is an $s$-conformal measure of $hGh^{-1} = G_h$ such that $h^s_*\mu(\infty) = 0$. By our assumption, either $h^s_*\mu(L_c(G_h)) = 0$ or $h^s_*\mu(L(G_h) \setminus L_c(G_h)) = 0$. Suppose that $h^s_*\mu(L_c(G_h)) = 0$. Assume that $\mu(L_c(G)) > 0$. Since $h(\infty) \neq \infty$, we see that $|h'| > 0$ in $L_c(G) \subset \mathbb{S}^n = \partial\mathbb{B}^{n+1}$ by (2.12). It follows that there is $\varepsilon > 0$ such that $\mu(A_\varepsilon) > 0$, where $A_\varepsilon = \{x \in L_c(G) : |h'(x)| \geq \varepsilon\}$. Now

$$h^s_*\mu(L_c(G_h)) = h^s_*\mu(hL_c(G)) \geq h^s_*\mu(hA_\varepsilon) = \int_{A_\varepsilon} |h'|^s d\mu \geq \varepsilon^s \mu(A_\varepsilon) > 0,$$

which is contradictory. We obtain that $\mu(L_c(G)) = 0$. Similar reasoning gives that if $h^s_*\mu(L(G_h) \setminus L_c(G_h)) = 0$, then $\mu(L(G) \setminus L_c(G)) = 0$.

Suppose next that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $\mu(\infty) \neq 0$. By the definition of $\mu$, there is $h \in$ Möb$(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and an $s$-conformal measure $\nu$ of $h^{-1}Gh = G_{h^{-1}}$ such that $\mu = h^s_*\nu$. According to the case considered above, either $\nu(L_c(G_{h^{-1}})) = 0$ or $\nu(L(G_{h^{-1}})\setminus L_c(G_{h^{-1}})) = 0$. The definition (2.17) implies immediately that either $\mu(L_c(G)) = 0$ or $\mu(L(G) \setminus L_c(G)) = 0$, respectively.

We obtain from above that we need to prove the claim of the theorem assuming that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $\mu(\infty) = 0$. Let us assume that $\mu(L(G) \setminus L_c(G)) > 0$. Our aim is to show that $\mu(L_c(G)) = 0$.

Recall that $L_c(G) \cap \mathbb{R}^n = \bigcup_{r>0} L_c(G, r)$. Our aim is to show that $L_c(G, r)$ contains no $\mu$-density points with respect to $L_c(G)$ for any fixed $r > 0$, which implies that $L_c(G) \cap \mathbb{R}^n$ contains no $\mu$-density points with respect to $L_c(G)$. This combined with the assumption $\mu(\infty) = 0$ implies that $\mu(L_c(G)) = 0$ (see, for example, Corollaries 1.11 and 2.14 of [Mattila1995]).

By definition, there is $h \in$ Möb$(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ such that $\mu = h^s_*\nu$ for some $s$-conformal measure $\nu$ of $h^{-1}Gh$. Write $G_{h^{-1}} = h^{-1}Gh$. Define the measure $\hat{\nu}$ by $\hat{\nu}(A) = \nu(A \setminus L_c(G_{h^{-1}}))$ for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. The set $L_c(G_{h^{-1}})$ is $G_{h^{-1}}$-invariant, so given $g \in G_{h^{-1}}$, we have that

$$\hat{\nu}(gA) = \nu(g(A \setminus L_c(G_{h^{-1}}))) = \int_{A\setminus L_c(G_{h^{-1}})} |g'|^s \nu = \int_A |g'|^s d\hat{\nu}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. We assumed that $\mu(L(G) \setminus L_c(G)) > 0$, which implies immediately that $\nu(L(G_{h^{-1}}) \setminus L_c(G_{h^{-1}})) = \hat{\nu}(L(G_{h^{-1}})) > 0$. We conclude that $\hat{\nu}$ is an $s$-conformal measure of $G_{h^{-1}}$, and so $\hat{\mu} = h^s_*\hat{\nu}$ is an $s$-conformal measure of $G$. It is clear that $\hat{\mu}(A) = \mu(A \setminus L_c(G))$ for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$.

Next, fix $r > 0$ and $x \in L_c(G, r)$. By the above, the proof is finished once we show that $x$ is not a $\mu$-density point with respect to $L_c(G)$. We obtain from (6.2) that there is a sequence $(g_i)_i$ of elements in $G$ and a sequence $(t_i)_i$ of positive numbers such that $(x, t_i) \in g_i\bar{D}(e_{n+1}, r)$ for every $i \in \{1, 2, \ldots\}$ and that $t_i \to 0$. The set $\bar{D}(e_{n+1}, r)$ is compact, so we can use the formula (2.46) and Theorem 3.1 to deduce the existence of constants $c_0 > 0$ and $c_1 > 0$ such that

$$c_0^{-1}u^s \le \mu(\bar{B}^n(y, u)) \le c_0u^s \qquad \text{and} \qquad c_1^{-1}u^s \le \hat{\mu}(\bar{B}^n(y, u)) \le c_1u^s$$

for every $y \in L(G) \cap \mathbb{R}^n$ and $u > 0$ such that $(y, u) \in G\bar{D}(e_{n+1}, r)$. Applying this to the points $(x, t_i)$, $i \in \{1, 2, \ldots\}$, we obtain that

$$\begin{aligned}\frac{\mu(\bar{B}^n(x, t_i) \cap L_c(G))}{\mu(\bar{B}^n(x, t_i))} &= \frac{\mu(\bar{B}^n(x, t_i)) - \mu(\bar{B}^n(x, t_i) \setminus L_c(G))}{\mu(\bar{B}^n(x, t_i))} = \frac{\mu(\bar{B}^n(x, t_i)) - \hat{\mu}(\bar{B}^n(x, t_i))}{\mu(\bar{B}^n(x, t_i))}\\ &= 1 - \frac{\hat{\mu}(\bar{B}^n(x, t_i))}{\mu(\bar{B}^n(x, t_i))} \le 1 - \frac{c_1^{-1}t_i^s}{c_0t_i^s} = 1 - \frac{1}{c_0c_1},\end{aligned}$$

so we see that $x$ is not a $\mu$-density point with respect to $L_c(G)$. According to the discussion given earlier, the proof is complete. □

The dichotomy expressed by Theorem 6.1 is important in the theory of conformal measures of Kleinian groups. The fact that makes the dichotomy important is that it can be formulated in equivalent but theoretically very different ways, which unifies the theory considerably. What is very remarkable is that one of the equivalent formulations is actually trivial. Namely, if $G$ is a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$ and $\mu$ is an $s$-conformal measure of $G$ for some $s > 0$, then the dichotomy that either $\mu(L_c(G)) = 0$ or $\mu(L(G) \setminus L_c(G)) = 0$ is equivalent to the trivial dichotomy that either every Poincaré series $P_s(x, y)$, $x, y \in \mathbb{X}^{n+1}$, of $G$ converges or every one of these series diverges (for the definition of $P_s(x, y)$, see (2.8)). It is relatively easy to show that if $P_s(x, y)$ converges for all $x, y \in \mathbb{X}^{n+1}$, then $\mu(L_c(G)) = 0$, but the converse claim is much more difficult to prove. Proofs for the converse claim are given, for example, in [Nicholls1989] and [Tukia1994b]. Other conditions expressing the same dichotomy are discussed, for instance, in [Nicholls1989] and [Sullivan1979].

We prove next the following theorem related to Theorem 6.1 (the claim of the theorem is included in Theorem 21 of [Sullivan1979]).

**Theorem 6.3.** *Let $G$, $\mu$ and $s$ be as in Theorem 6.1. Then $\mu(x) = 0$ for every $x \in L_c(G)$.*

*Proof.* Suppose first that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $x \in L_c(G) \cap \mathbb{R}^n$. We can argue as in the last paragraph of the proof of Theorem 6.1 to see that there is a sequence $(t_i)_i$ of positive numbers with $t_i \to 0$ such that $(x, t_i) \in GC$ for every $i \in \{1, 2, \ldots\}$, where $C \subset \mathbb{H}^{n+1}$ is compact. We obtain also that $\mu(\bar{B}^n(x, t_i)) \leq c_0 t_i^s$ for every $i \in \{1, 2, \ldots\}$, where $c_0 > 0$ is a constant. Since $\mu(x) \leq \mu(\bar{B}^n(x, t_i))$ for every $i \in \{1, 2, \ldots\}$, it follows that $\mu(x) = 0$.

Suppose next that $\mathbb{X}^{n+1} = \mathbb{B}^{n+1}$. Let $x \in L_c(G)$. Choose any $h \in$ Möb$(n + 1)$ which maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ so that $h(x) \neq \infty$. Now $h^s_*\mu$ is an $s$-conformal measure of $hGh^{-1}$ and $h(x) \in L_c(hGh^{-1}) \cap \mathbb{R}^n$. By our above argument, $h^s_*\mu(h(x)) = |h'(x)|^s\mu(x) = 0$. Since $|h'(x)|^s \in ]0, \infty[$ (see (2.12)), we obtain that $\mu(x) = 0$.

Suppose finally that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and $x = \infty \in L_c(G)$. Suppose that $h \in$ Möb$(n + 1)$ maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and that $\nu$ is an $s$-conformal measure of $h^{-1}Gh$ such that $\mu = h^s_*\nu$. Now $h^{-1}(x) \in L_c(h^{-1}Gh)$ and $\nu(h^{-1}(x)) = 0$ by the above reasoning, so $\mu(x) = 0$. □

Theorem 6.1 has as a consequence the following standard ergodicity theorem of conformal measures of Kleinian groups.

**Theorem 6.4.** *Let $G$, $\mu$ and $s$ be as in Theorem 6.1. We assume that $\mu(L(G) \setminus L_c(G)) = 0$. Then either $\mu(A) = 0$ or $\mu(L(G) \setminus A) = 0$ for every $G$-invariant Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$.*

*Proof.* The proof of this theorem is very similar to the proof of Theorem 6.1. Like in the proof of Theorem 6.1, we can assume that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$ and that $\mu(\infty) = 0$. Let $A$ be a $G$-invariant Borel set of $\bar{\mathbb{R}}^{n+1}$. We can assume that $A \subset L_c(G)$. Let us suppose that $\mu(L(G) \setminus A) > 0$. Our aim is to show that now $\mu(A) = 0$.

Given $r > 0$, define the set $L_c(G, r)$ like in the proof of Theorem 6.1. We show that $A \cap L_c(G, r)$ contains no $\mu$-density points with respect to $A$ for any fixed $r > 0$. Since $A \cap \mathbb{R}^n = \bigcup_{r>0}(A \cap L_c(G, r))$, this gives that $A \cap \mathbb{R}^n$ contains no $\mu$-density points with respect to $A$. This result and the assumption $\mu(\infty) = 0$ imply that $\mu(A) = 0$.

Let $h \in$ Möb$(n+1)$ be a Möbius transformation mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and $\nu$ an $s$-conformal measure of $h^{-1}Gh = G_{h^{-1}}$ such that $\mu = h^s_* \nu$. Define $\hat{\nu}$ by $\hat{\nu}(B) = \nu(B \setminus h^{-1}A)$ for every Borel set $B$ of $\bar{\mathbb{R}}^{n+1}$. Since $h^{-1}A$ is $G_{h^{-1}}$-invariant and $\mu(L(G) \setminus A) > 0$, we can argue as in the proof of Theorem 6.1 to show that $\hat{\nu}$ is an $s$-conformal measure of $G_{h^{-1}}$. The measure $\hat{\mu} = h^s_* \hat{\nu}$ satisfying $\hat{\mu}(B) = \mu(B \setminus A)$ for every Borel set $B$ of $\bar{\mathbb{R}}^{n+1}$ is thus an $s$-conformal measure of $G$.

Fix $r > 0$. Let $x \in A \cap L_c(G, r)$. It is evident that we can use a similar estimation argument as in the proof of Theorem 6.1 to show that $x$ is not a $\mu$-density point with respect to $A$. We omit the details of this argument in order to avoid unnecessary repetition. We see that the proof is complete. □

To proceed closer to the main result of this section, we introduce an auxiliary notion. Let $G$ be a Kleinian group acting on $\mathbb{X}^{n+1}$ and let $A \subset \bar{\mathbb{R}}^{n+1}$ be $G$-invariant. We say that $f : A \to [0, \infty[$ is *$G$-automorphic* if $f(g(x)) = f(x)$ for every $x \in A$ and every $g \in G$. As one would expect, the ergodicity property described by Theorem 6.4 implies the following result on $G$-automorphic functions.

**Theorem 6.5.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Let us assume that $\mu(L(G) \setminus L_c(G)) = 0$. Let $A$ be a $G$-invariant Borel set of $\bar{\mathbb{R}}^{n+1}$. Let $f : A \to [0, \infty[$ be $G$-automorphic and $\mu$-measurable. Then $f$ is constant in $A$ outside a $\mu$-nullset.*

*Proof.* We may assume that $A \subset L_c(G)$ and that $\mu(A) > 0$. Given $t \in [0, \infty[$, let $F(t) = f^{-1}[0, t]$. Since $f$ is $\mu$-measurable and $G$-automorphic, $F(t)$ is a $G$-invariant Borel set of $\bar{\mathbb{R}}^{n+1}$ for every $t \in [0, \infty[$ (recall that the $\mu$-measurable sets are exactly the Borel sets of $\bar{\mathbb{R}}^{n+1}$). By Theorem 6.4, we have that $\mu(F(t)) = 0$ or $\mu(L(G) \setminus F(t)) = 0$ for every $t \in [0, \infty[$. Since $\mu(A) > 0$, there is $t \in [0, \infty[$ such that $\mu(L(G) \setminus F(t)) = 0$. Define

$$t_0 = \inf\{t \in [0, \infty[ : \mu(L(G) \setminus F(t)) = 0\}.$$

The definition of $t_0$ implies immediately that $\mu(L(G) \setminus F(t)) = 0$ for every $t > t_0$ and that $\mu(F(t)) = 0$ for every $t \in [0, t_0[$. We conclude that $\mu(L(G) \setminus f^{-1}(t_0)) = 0$. Our claim follows. □

We need to prove the following lemma before we can prove the main result of this section.

**Lemma 6.6.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{B}^{n+1}$. Let $\mu$ be an $s$-conformal measure of $G$ for some $s > 0$. Let $\phi : \bar{\mathbb{R}}^{n+1} \to [0, \infty[$ be $\mu$-measurable. (It is sufficient that $\phi$ is defined outside a $\mu$-nullset.) Then*

$$\int_{gA} \phi \, d\mu = \int_A (\phi \circ g)|g'|^s d\mu \tag{6.7}$$

*for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$.*

*Proof.* We prove (6.7) assuming first that $A = \bar{\mathbb{R}}^{n+1} = gA$, where $g \in G$ is fixed. Given $B \subset \bar{\mathbb{R}}^{n+1}$, we let $\chi_B$ denote the characteristic function of $B$. (That is, $\chi_B(x) = 1$ if $x \in B$ and $\chi_B(x) = 0$ if $x \notin B$.) Suppose that $S = \sum_{i=1}^k c_i \chi_{B_i}$ is a simple function such that $S \leq \phi$, which means that $\{B_1, B_2, \ldots, B_k\}$ is a pairwise disjoint division of $\bar{\mathbb{R}}^{n+1}$ into non-empty Borel sets and $c_1, c_2, \ldots, c_k \in [0, \infty[$ are $k$ distinct numbers. Now

$$\begin{aligned}\int_{\bar{\mathbb{R}}^{n+1}} S\,d\mu &= \sum_{i=1}^{k} c_i \mu(B_i) = \sum_{i=1}^{k} c_i \int_{g^{-1}B_i} |g'|^s d\mu = \sum_{i=1}^{k} \int_{\bar{\mathbb{R}}^{n+1}} c_i \chi_{g^{-1}B_i} |g'|^s d\mu \\ &= \int_{\bar{\mathbb{R}}^{n+1}} \left( \sum_{i=1}^{k} c_i \chi_{g^{-1}B_i} \right) |g'|^s d\mu = \int_{\bar{\mathbb{R}}^{n+1}} \left( \left( \sum_{i=1}^{k} c_i \chi_{B_i} \right) \circ g \right) |g'|^s d\mu \\ &= \int_{\bar{\mathbb{R}}^{n+1}} (S \circ g) |g'|^s d\mu \leq \int_{\bar{\mathbb{R}}^{n+1}} (\phi \circ g) |g'|^s d\mu.\end{aligned}$$

We take the supremum over the simple functions $S$ such that $S \leq \phi$ and conclude that

$$\int_{\bar{\mathbb{R}}^{n+1}} \phi d\mu \leq \int_{\bar{\mathbb{R}}^{n+1}} (\phi \circ g) |g'|^s d\mu. \tag{6.8}$$

Let next $S = \sum_{i=1}^{k} c_i \chi_{B_i}$ be a simple function such that $S \leq (\phi \circ g)|g'|^s$. Note that we can assume that $(\phi \circ g)|g'|^s$ is a function from $\bar{\mathbb{R}}^{n+1}$ into $[0, \infty[$ because $\mu(g^{-1}(\infty)) = 0$ and because $|g'(x)|^s = \infty$ if and only if $g(\infty) \neq \infty$ and $x = g^{-1}(\infty)$. We can argue as in the proof of (6.8) to see that

$$\begin{aligned}\int_{\bar{\mathbb{R}}^{n+1}} S\,d\mu &\leq \int_{\bar{\mathbb{R}}^{n+1}} (((\phi \circ g)|g'|^s) \circ g^{-1}) |(g^{-1})'|^s d\mu \\ &= \int_{\bar{\mathbb{R}}^{n+1}} \phi (|g'|^s \circ g^{-1}) |(g^{-1})'|^s d\mu = \int_{\bar{\mathbb{R}}^{n+1}} \phi d\mu,\end{aligned}$$

where the validity of the application of the chain rule in the third step is guaranteed by the fact that $\mu(\infty) = 0 = \mu(g(\infty))$. The converse of (6.8) follows. We have proved (6.7) in case $A = \bar{\mathbb{R}}^{n+1} = gA$ for a fixed $g \in G$.

Let $A$ be a Borel set of $\bar{\mathbb{R}}^{n+1}$ and $g \in G$. We can use the first part of this proof to conclude that

$$\begin{aligned}\int_{gA} \phi d\mu &= \int_{\bar{\mathbb{R}}^{n+1}} \phi \chi_{gA} d\mu = \int_{\bar{\mathbb{R}}^{n+1}} (\phi \circ g)(\chi_{gA} \circ g) |g'|^s d\mu \\ &= \int_{\bar{\mathbb{R}}^{n+1}} (\phi \circ g) \chi_A |g'|^s d\mu = \int_A (\phi \circ g) |g'|^s d\mu.\end{aligned}$$

□

We are finally in a position to prove the main result of this section. We stress that the result is a standard result in the theory of conformal measures of Kleinian groups, see, for example, [Nicholls1989] or [Sullivan1979].

**Theorem 6.9.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{B}^{n+1}$. Let $\mu_1$ and $\mu_2$ be two $s$-conformal measures of $G$ for some $s > 0$. Suppose that $\mu_1(L(G) \setminus L_c(G)) = 0 = \mu_2(L(G) \setminus L_c(G))$. Then there is a constant $c > 0$ such that $\mu_1 = c\mu_2$. Furthermore, if $h \in$ Möb$(n+1)$ maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$, then $h_*^s \mu_1 = c h_*^s \mu_2$.*

*Proof.* Let us assume for the moment that there is a constant $c > 0$ such that $\mu_1 = c\mu_2$ in case $\mu_1$ is absolutely continuous with respect to $\mu_2$. Let $\mu_1$ and $\mu_2$ be any measures as in the claim. Now $\nu = \mu_1 + \mu_2$ is clearly an $s$-conformal measure of $G$ such that $\mu_1$ is absolutely continuous with respect to $\nu$. Furthermore, $\nu(L(G) \setminus L_c(G)) = 0$. Our temporary assumption implies that $\mu_1 = c\nu$ for some constant $c > 0$. We have that $c \neq 1$, since otherwise $\mu_2 = 0$. We conclude that $\mu_1 = (c/(1-c))\mu_2$.

We see that we need to prove our claim assuming additionally that $\mu_1$ is absolutely continuous with respect to $\mu_2$. Let $\phi$ be the Radon-Nikodym derivative of $\mu_1$ with respect to $\mu_2$ ($\phi$ is defined up to a $\mu_2$-nullset). Let us show that $\phi$ is $G$-automorphic outside a $\mu_2$-nullset.

Let $g \in G$. Let $A$ be a Borel set of $\bar{\mathbb{R}}^{n+1}$. It is now the case that (we use (6.7) in the second step of the following calculation; note that if $g(\infty) \neq \infty$, the first step of the calculation is valid since $\mu_2(\infty) = 0 = \mu_2(g^{-1}(\infty))$)

$$\begin{aligned}\int_A (\phi \circ g) d\mu_2 &= \int_A (\phi \circ g)(|(g^{-1})'|^s \circ g)|g'|^s d\mu_2 \\ &= \int_{gA} \phi |(g^{-1})'|^s d\mu_2 = \int_{gA} |(g^{-1})'|^s d\mu_1 \\ &= \mu_1(g^{-1}(gA)) = \mu_1(A).\end{aligned}$$

We conclude that $\phi \circ g = \phi$ outside a $\mu_2$-nullset, say $U_g$. Note that $\mu_2(fU_g) = 0$ for every $f \in G$ by the $s$-conformality of $\mu_2$. Recall that $G$ is countable. We obtain that $\phi$ is $G$-automorphic in the complement of the $G$-invariant $\mu_2$-nullset $\bigcup_{g \in G} GU_g$. Theorem 6.5 implies thus that $\phi$ is constant outside a $\mu_2$-nullset. It follows that there is a constant $c > 0$ such that $\mu_1 = c\mu_2$.

Finally, it is evident that if $\mu_1 = c\mu_2$ for some constant $c > 0$, where $\mu_1$ and $\mu_2$ are any measures as in the claim, and $h \in \text{Möb}(n+1)$ maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$, then $h_*^s\mu_1 = ch_*^s\mu_2$. $\square$

The discussion following Theorem 6.1 motivates the following remarks. Let $G$ be a non-elementary Kleinian group acting on $\mathbb{B}^{n+1}$ and let $s > 0$. The discussion following Theorem 6.1 implies that if $\mu$ is an $s$-conformal measure of $G$, then $\mu(L(G) \setminus L_c(G)) = 0$ if and only if every Poincaré series $P_s(x, y)$, $x, y \in \mathbb{B}^{n+1}$, diverges. It is true, therefore, that if $\mu(L(G) \setminus L_c(G)) = 0$ for *some* $s$-conformal measure $\mu$ of $G$, then $\nu(L(G) \setminus L_c(G)) = 0$ for *any* $s$-conformal measure $\nu$ of $G$. This means that Theorem 6.9 can be strengthened to claim that if $G$ has an $s$-conformal measure $\mu$ such that $\mu(L(G) \setminus L_c(G)) = 0$, then $\nu = c_\nu\mu$ for some number $c_\nu > 0$ depending on $\nu$, where $\nu$ is an arbitrary $s$-conformal measure of $G$. However, as indicated by our discussion following Theorem 6.1, it is relatively difficult to prove that if $\mu(L(G) \setminus L_c(G)) = 0$ for some $s$-conformal measure $\mu$ of $G$, then $\nu(L(G) \setminus L_c(G)) = 0$ for any $s$-conformal measure $\nu$ of $G$. Consequently, we prove explicitly only the weaker Theorem 6.9, since it is sufficient for our later needs.

6.2. **The main equivalence result.** Our goal in this section is to prove the basic version of the main result of this work. We will show that if $\mu$ is a Patterson-Sullivan measure of a non-elementary geometrically finite Kleinian group $G$, then we can use the modified covering construction and the modified packing construction introduced in Chapter 5 to construct measures $m$ and $p$, respectively, such that $c_m m = \mu = c_p p$, where $c_m > 0$ and $c_p > 0$ are constants, i.e. such that $\mu$, $m$ and $p$ are equivalent to each other. We will consider some variants of this result in Chapter 7.

In order to get into the technical details of the main result, we need to give the explicit definition for non-elementary geometrically finite Kleinian groups. Hyperbolic convex hulls, horoballs, bounded parabolic fixed points and conical limit points will play a central role in the following discussion. These objects were defined on page 18.

Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. We say that the group $G$ is *geometrically finite* if the following conditions are satisfied. There is a finite, possibly empty, set $P$ of bounded parabolic fixed points of $G$ such that the set of parabolic fixed points of $G$ is $GP = \{g(p) : g \in G, p \in P\}$ and the orbits $Gp_1$ and $Gp_2$ are disjoint for every two points $p_1, p_2 \in P$. Additionally, there is a collection $\{H_p : p \in GP\}$ of horoballs of $\mathbb{X}^{n+1}$ called a *complete collection of horoballs* of $G$ which satisfies the following. The horoball $H_p$, $p \in GP$, is based at $p$; the horoballs in the collection have pairwise disjoint closures; if $p \in GP$ and $g \in G$, then $gH_p = H_{g(p)}$. The final condition in the definition of geometric finiteness of $G$ is that there is a compact set $C \subset \mathbb{X}^{n+1}$ such that

$$H(G) \setminus \bigcup_{p \in GP} H_p = GC, \tag{6.10}$$

where $H(G)$ is the hyperbolic convex hull of $G$.

Geometrically finite Kleinian groups have several equivalent definitions. These are discussed, for example, in Chapter 4 of [Apanasov2000], Sections 3 and 4 of [Bowditch1993] and Chapter 12 of [Ratcliffe2006] in the general situation. The classical situation, i.e. the case of Kleinian groups acting on $\mathbb{X}^2$ or $\mathbb{X}^3$ and containing only orientation preserving elements, is considered, for instance, in Chapter VI of [Maskit1988] and Chapter 3 of [MT1998]. We emphasize that our definition for geometric finiteness of Kleinian groups is not among those definitions which are stated explicitly in the above sources. However, it is not difficult to see that our definition is equivalent to the definitions considered in the sources. Let us take a look at this in detail.

The book [Ratcliffe2006] discusses an equivalent condition for geometric finiteness of Kleinian groups which is very close to our definition. In order to state this condition, we need to introduce some terminology of [Ratcliffe2006].

Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Let $p \in \partial\mathbb{X}^{n+1}$ be a parabolic fixed point of $G$. A horoball $H_p$ of $\mathbb{X}^{n+1}$ based at $p$ is said to be a *horocusped region* of $G$ based at $p$ if the collection $\{gH_p : g \in G\}$ of horoballs of $\mathbb{X}^{n+1}$ is pairwise disjoint. A horocusped region of $G$ is called *proper* if it is not maximal in size. Let $M = \mathbb{X}^{n+1}/G$ be the standard quotient space of $G$ and let $\pi : \mathbb{X}^{n+1} \to M$ be the quotient map. Let $H_p$ be a (proper) horocusped region of $G$ based at $p \in \partial\mathbb{X}^{n+1}$. We say that the open set $\pi H_p \subset M$ is the *(proper) horocusp* of $M$ corresponding to $H_p$. Furthermore, the *hyperbolic convex core* $C(M)$ of $M$ is the quotient set $H(G)/G$.

The equivalent condition for geometric finiteness of Kleinian groups very close to our definition is the following. The group $G$ satisfies this condition, say Condition (1), if there is a finite, possibly empty, union $V$ of proper horocusps of $M$ with pairwise disjoint closures such that $C(M) \setminus V$ is compact.

According to Theorem 12.4.5 of [Ratcliffe2006], Condition (1) is equivalent to the following condition that we call Condition (2). The group $G$ satisfies Condition (2) in case $L(G)$ is the disjoint union of the set $L_c(G)$ of conical limit points of $G$ and the set of bounded parabolic fixed points of $G$.

Condition (2) is a common equivalent condition for geometric finiteness of Kleinian groups and it or some variant of it appears in every one of the sources mentioned above. (In the variants of Condition (2), conical limit points of $G$ or bounded parabolic fixed points of $G$ are replaced by limit points of $G$ of special type with equivalent definitions.) So once we establish that our definition for geometric finiteness of Kleinian groups is equivalent to Condition (2), we see that our notion of a geometrically finite Kleinian group refers to the same class of Kleinian groups as do the standard definitions appearing in the literature.

We prove next Theorem 6.11 which states that our definition of geometric finiteness of Kleinian groups is equivalent to Condition (2). Theorem 6.11 states also that the condition (6.10) can be somewhat strengthened (cf. Lemma B of [Tukia1984]). Let us make the following remarks before going into Theorem 6.11. We will give a self-contained proof for the fact that geometric finiteness implies Condition (2). When proving the converse, we will need to refer to results in [Ratcliffe2006]. Now we could shorten considerably the proof of the claim that geometric finiteness implies Condition (2) if we referred to the same results in [Ratcliffe2006]. However, since this result will be important in later developments, we prefer to give a self-contained proof. We remark also that many papers, e.g. [SV1995], [Sullivan1984], [Tukia1984] and [Tukia1994c], use (usually without explicit proof) the fact that the standard definitions for geometrically finite Kleinian groups imply our definition without considering the converse claim.

**Theorem 6.11.** *Let $G$ be a non-elementary Kleinian group acting on $\mathbb{X}^{n+1}$. Then the following claims are true. (i) $G$ is geometrically finite if and only if $G$ satisfies Condition (2), i.e. $L(G)$ is the disjoint union of the set $L_c(G)$ of conical limit points of $G$ and the set of bounded parabolic fixed points of $G$. (ii) If $G$ is geometrically finite and $c \geq 0$, then there is a compact set $C \subset \mathbb{X}^{n+1}$ such that*

$$\bar{N}(H(G), c) \setminus \bigcup_{p \in GP} H_p = GC, \tag{6.12}$$

*where $\bar{N}(H(G), c) = \{z \in \mathbb{X}^{n+1} : d(z, H(G)) \leq c\}$ and the (possibly empty) set $P$ containing bounded parabolic fixed points of $G$ and the complete collection $\{H_p : p \in GP\}$ of horoballs of $\mathbb{X}^{n+1}$ are as in the definition of a non-elementary geometrically finite Kleinian group on page 89.*

*Proof.* We consider first the claim (i). Suppose first that $G$ is geometrically finite. Let $P$ and $\{H_p : p \in GP\}$ be as in the definition of geometric finiteness on page 89. Let $x \in L(G)$. Our aim is to show that $x$ is either a bounded parabolic fixed point of $G$ or a conical limit point of $G$. The claim is clearly conjugation invariant, so we can assume that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$, that $\infty \in L(G)$, and that $x \neq \infty$.

Suppose that $x$ is a parabolic fixed point of $G$. By the definition of geometric finiteness, we have that $x = g(p)$ for some $g \in G$ and $p \in P$. Since $p$ is a bounded parabolic fixed point of $G$, $x$ is also a bounded parabolic fixed point of $G$.

We show next that $x$ is not a conical limit point of $G$. Assume that $x$ is a conical limit point of $G$. Let $L$ be the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x$ and $\infty$. Choose a point $z_0 \in \mathbb{H}^{n+1} \setminus \bigcup_{p\in GP} H_p$. By the definition of a conical limit point (see page 18), there are $g_1, g_2, \ldots \in G$ and $r \geq 0$ such that $g_i(z_0) \to x$ with $d(g_i(z_0), L) \leq r$. The set $T(r) = \{z \in \mathbb{H}^{n+1} : d(z, L) \leq r\}$ is a cone of $\mathbb{R}^{n+1}$ of infinite height with apex $x$ and axis $L$. There is $u > 0$ such that $T(r) \cap B^{n+1}(x, u) \subset H_x$. Since $z_0 \in \mathbb{H}^{n+1} \setminus \bigcup_{p\in GP} H_p$ and the elements in $G$ map the horoballs in $\{H_p : p \in GP\}$ onto each other, we see that $g_i(z_0) \notin T(r) \cap B^{n+1}(x, u)$ for every $i \in \{1, 2, \ldots\}$, which is contradictory. So $x$ is not a conical limit point of $G$.

Suppose next that the limit point $x$ of $G$ is not a parabolic fixed point of $G$. We show that $x$ is a conical limit point of $G$. Let again $L$ be the hyperbolic line of $\mathbb{H}^{n+1}$ with endpoints $x$ and $\infty$. Recall that we have assumed that $\infty \in L(G)$, which implies that $L \subset H(G)$. Let $C \subset \mathbb{H}^{n+1}$ be a compact set as in (6.10). Since $x$ is not in $GP$ and the horoballs in the collection $\{H_p : p \in GP\}$ have pairwise disjoint closures, there are $t_1 > t_2 > \ldots > 0$ with $t_i \to 0$ such that $(x, t_i) \notin H_p$ for any $p \in GP$. This means that $(x, t_i) = g_i(y_i)$ for every $i \in \{1, 2, \ldots\}$, where $g_i \in G$ and $y_i \in C$. Since $C$ is a compact set of $\mathbb{H}^{n+1}$, $C$ has a finite hyperbolic diameter $d(C)$. Fix $z_1 \in C$. Now $d(g_i(z_1), L) \leq d(g_i(z_1), g_i(y_i)) = d(z_1, y_i) \leq d(C)$ for every $i \in \{1, 2, \ldots\}$, and $g_i(z_1) \to x$. Next, let $L'$ be any hyperbolic line of $\mathbb{H}^{n+1}$ with $x$ as one of its endpoints and let $z_2 \in \mathbb{H}^{n+1}$ be arbitrary. Observe that if $z$ approaches $x$ along $L$, the distance $d(z, L')$ converges to 0. It is now clear that, since the elements in $G$ are hyperbolic isometries of $\mathbb{H}^{n+1}$, it is true that $g_i(z_2) \to x$ and that $d(g_i(z_2), L') \leq r'$ for some $r' > 0$ and every $i \in \{1, 2, \ldots\}$. We have shown that $x$ is a conical limit point of $G$. We conclude that $G$ satisfies Condition (2).

Suppose next that $G$ satisfies Condition (2). Our aim is to show that $G$ is geometrically finite. As stated earlier, $G$ satisfies now Condition (1) by Theorem 12.4.5 of [Ratcliffe2006]. There is thus a finite, possibly empty, collection $\{U_1, U_2, \ldots, U_m\}$ of proper horocusps of $M = \mathbb{X}^{n+1}/G$ with pairwise disjoint closures such that $C(M) \setminus \bigcup_{l=1}^m U_l$ is compact, where $C(M) = H(G)/G$ is the hyperbolic convex core of $M$.

Given $l \in \{1, 2, \ldots, m\}$, let $p_l \in \partial\mathbb{X}^{n+1}$ be a parabolic fixed point of $G$ and $H_{p_l}$ a proper horocusped region of $G$ based at $p_l$ such that $\pi H_{p_l} = U_l$, where $\pi : \mathbb{X}^{n+1} \to M$ is the quotient map. Write $P = \{p_1, p_2, \ldots, p_m\}$. Write also $H_{g(p)} = gH_p$ for every $g \in G$ and every $p \in P$. We show that $P$ is a set of bounded parabolic fixed points of $G$ and $\{H_p : p \in GP\}$ a collection of horoballs of $\mathbb{X}^{n+1}$ as required by the definition of geometric finiteness.

We start by showing that there is a compact set $C \subset \mathbb{X}^{n+1}$ such that $H(G) \setminus \bigcup_{p\in GP} H_p = GC$. Fix $x_0 \in \mathbb{X}^{n+1}$ and denote by $C_i$, $i \in \{1, 2, \ldots\}$, the closed hyperbolic ball of $\mathbb{X}^{n+1}$ with center $x_0$ and radius $i$. We claim that there is $i_0 \in \{1, 2, \ldots\}$ such that $H(G) \setminus \bigcup_{p\in GP} H_p \subset GC_{i_0}$. Suppose that no such $i_0$ exists. This means that there is $y_i \in \mathbb{X}^{n+1}$ for every $i \in \{1, 2, \ldots\}$ such that $y_i \in H(G) \setminus \bigcup_{p\in GP} H_p$ but $y_i \notin GC_i$. Write $C(M) \setminus \bigcup_{l=1}^m U_l = D$. Now $\pi(y_i) \in D$ for every $i \in \{1, 2, \ldots\}$, and since the set $D$ is compact, we can assume that $\pi(y_i) \to \hat{y} \in D$. Let $y \in \pi^{-1}(\hat{y})$. Let $i_1 \in \{1,2,\ldots\}$ be such that $B^{n+1}(y,\varepsilon) \subset C_{i_1}$, where $\varepsilon > 0$ is a fixed small number. We obtain that $\pi(y_i) \in \pi B^{n+1}(y,\varepsilon) \subset \pi C_{i_1}$ for all large enough $i$. On the other hand, we have that $\pi(y_i) \notin \pi C_i$ for all $i \in \{1,2,\ldots\}$. Since $\pi C_{i_1} \subset \pi C_i$ for all $i \geq i_1$, the situation is contradictory. We conclude that there is $i_0 \in \{1,2,\ldots\}$ such that $H(G) \setminus \bigcup_{p\in GP} H_p \subset GC_{i_0}$. Recalling that $H(G)$ is $G$-invariant and closed in $\mathbb{X}^{n+1}$, we see that $H(G) \setminus \bigcup_{p\in GP} H_p = GC$, where $C = C_{i_0} \cap (H(G) \setminus \bigcup_{p\in GP} H_p)$ is compact.

Observe next that, since the horocusps $U_1, U_2, \ldots, U_m$ have pairwise disjoint closures and these horocusps are proper, the collection $\{H_p : p \in GP\}$ is such that $H_p$, $p \in GP$, is based at $p$, that the horoballs in the collection have pairwise disjoint closures, and that $gH_p = H_{g(p)}$ for every $g \in G$ and every $p \in GP$.

To finish the proof of the geometric finiteness of $G$, we need to show that the points in $P$ are bounded parabolic fixed points of $G$ and that $GP$ is the set of parabolic fixed points of $G$. Note that we can use a similar argument as in the last paragraph of the first part of this proof to show that every point in $L(G) \setminus GP$ is a conical limit point of $G$. According to Theorem 12.6.3 of [Ratcliffe2006], a parabolic fixed point of a Kleinian group is not a conical limit point of the group. We conclude that $GP$ is the set of parabolic fixed points of $G$. And since $G$ satisfies Condition (2), it follows that the points in $P$ are bounded parabolic fixed points of $G$. Therefore, the group $G$ is geometrically finite. We have proved the claim (i).

We consider the claim (ii). Let $G$ be geometrically finite and $c \geq 0$. Suppose that $P$ and $\{H_p : p \in GP\}$ are as in the definition of geometric finiteness on page 89. Our goal is to prove (6.12). If $A \subset \mathbb{X}^{n+1}$ is non-empty and $a \geq 0$, we employ the notation $\bar{N}(A,a) = \{y \in \mathbb{X}^{n+1} : d(y,A) \leq a\}$.

Fix $p \in P$ for the time being. Let $H'_p \subset H_p$ be the horoball of $\mathbb{X}^{n+1}$ based at $p$ such that $d(\partial H_p \cap \mathbb{X}^{n+1}, \partial H'_p \cap \mathbb{X}^{n+1}) = c$, see Lemma 2.31. Write $H'_{g(p)} = gH'_p$ for $g \in G$. By Lemma 2.38, there is a compact set $C_p \subset \mathbb{X}^{n+1}$ such that if $z \in (\bar{H}_{g(p)} \setminus H'_{g(p)}) \cap H(G)$ for some $g \in G$, then $z \in GC_p$. Let $C \subset \mathbb{X}^{n+1}$ be a compact set as in (6.10).

Let $z \in \bar{N}(H(G),c) \setminus \bigcup_{p\in GP} H_p$. Let $\hat{z} \in H(G)$ be such that $d(z,\hat{z}) \leq c$. If $\hat{z} \in H(G) \setminus \bigcup_{p\in GP} H_p$, then $\hat{z} \in GC$. Therefore, $z \in G\bar{N}(C,c)$ in this case. Suppose that $\hat{z} \in H_p$ for some $p \in GP$. Since $z \notin H_p$, we see that $\hat{z} \in \bar{H}_p \setminus H'_p$. Let $p' \in P$ be such that $p \in Gp'$. We deduce from the above that $\hat{z} \in GC_{p'}$. We see hence that $z \in G\bar{N}(C_{p'},c)$ in this case. Let

$$\hat{C} = \left(\bar{N}(C,c) \cup \bigcup_{p\in P} \bar{N}(C_p,c)\right) \cap \left(\bar{N}(H(G),c) \setminus \bigcup_{p\in GP} H_p\right).$$

It is easy to see that

$$\bar{N}(H(G),c) \setminus \bigcup_{p\in GP} H_p = G\hat{C},$$

and so (6.12) is valid. We have proved the claim (ii). □

Our next task is to adapt the results of Chapters 3 and 4 to the context of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups. Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$. Let $P$ be a (possibly empty) set of bounded parabolic fixed points of $G$ and $\{H_p : p \in GP\}$ a complete collection of horoballs of $\mathbb{H}^{n+1}$ for $G$ as described on page 89. Let $\delta$ be the exponent of convergence of $G$, see (2.9) for the definition. Let $\mu$ be a Patterson-Sullivan measure of $G$. This means simply that $\mu$ is a $\delta$-conformal measure of $G$ such that $\mu = h^{\delta}_{*}\nu$, where $h \in \text{Möb}(n+1)$ maps $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and $\nu$ is a $\delta$-conformal measure of $h^{-1}Gh$ obtained from the construction of Patterson mentioned in Chapter 2. Before proceeding, we fix two constants $t_0 > 0$ and $v_0 \in ]0,1[$. We will prove a series of results in this setting. We begin with the following two auxiliary results.

**Lemma 6.13.** *Let $G$ be the non-elementary geometrically finite Kleinian group and $t_0$ and $v_0$ the constants introduced above. Then there is a compact set $C \subset \mathbb{H}^{n+1}$ such that the following is true. If $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ are such that $|x - x'|/t \le v_0$ for some $x' \in L(G) \cap \mathbb{R}^n$, then $(x,t) \in GC \cup \bigcup_{p \in GP \cap \mathbb{R}^n} H_p$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. Lemma 2.34 implies that there is a constant $c_0 > 0$ such that $d((x,t), H(G)) \le c_0$. By the claim (ii) of Theorem 6.11, there is a compact set $K \subset \mathbb{H}^{n+1}$ such that

$$\bar{N}(H(G), c_0) \setminus \bigcup_{p \in GP} H_p = GK,$$

where $\bar{N}(H(G), c_0) = \{y \in \mathbb{H}^{n+1} : d(y, H(G)) \le c_0\}$. Observe that if $\infty \in GP$, then there is a horoball $H'_\infty \subset H_\infty$ of $\mathbb{H}^{n+1}$ based at $\infty$ such that $(x,t) \in \bar{H}_\infty \setminus H'_\infty$ in case $(x,t) \in H_\infty$. By Lemma 2.38, there is a compact set $K' \subset \mathbb{H}^{n+1}$ such that $\bar{N}(H(G), c_0) \cap (\bar{H}_\infty \setminus H'_\infty) = G_\infty K'$ if $\infty \in GP$. We conclude that if $\infty \notin GP$, we can choose $C = K$, and that if $\infty \in GP$, we can choose $C = K \cup K'$. □

**Theorem 6.14.** *Let $G$ be the non-elementary geometrically finite Kleinian group introduced before Lemma 6.13. Recall that $\delta$ is the exponent of convergence of $G$. Now the Hausdorff dimension of $L(G)$ is equal to $\delta$. Furthermore, if $L(G)$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ for some $l \in \{1, 2, \dots, n\}$, then the rank of every parabolic fixed point of $G$ is $l$.*

*Proof.* There is no space for a self-contained proof of the claim that the Hausdorff dimension of the limit set of a non-elementary geometrically finite Kleinian group equals the exponent of convergence of the group. For a proof of this claim, see Chapter 9 of [Nicholls1989] or Section 6 of [Sullivan1979] and Section 6 of [Sullivan1984].

Suppose that $L(G)$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ for some $l \in \{1, 2, \dots, n\}$. We can assume that $L(G) = \bar{\mathbb{R}}^l$. It is clear that we can regard $G$ as a Kleinian group acting on $\mathbb{H}^{l+1}$, which means that we can assume that $l = n$. Suppose that $p \in L(G)$ is a parabolic fixed point of $G$ of rank $k \in \{1, 2, \dots, n\}$. We can assume that $p = \infty$. Let $V \subset \mathbb{R}^n$ be a $G_\infty$-invariant $k$-plane as described in the claim (i) of Theorem 2.7. The claim (ii) of Lemma 2.37 implies that the distances $d_{\text{euc}}(x, V)$, $x \in L(G) \cap \mathbb{R}^n = \mathbb{R}^n$, are bounded by a constant, which is possible only if $k = n$. □

We discuss next the geometry of $L(G)$ using the results of Chapter 4. Like in Chapter 4, we define the $l$-flatness function $\gamma_l$ of $G$, $l \in \{1, 2, \ldots, n\}$, by

$$\gamma_l(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}^n(x,t) \cap L(G), \bar{B}^n(x,t) \cap V) \tag{6.15}$$

for all $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap L(G) \neq \emptyset$, where $\mathcal{F}_l(x,t)$ is the collection of all $l$-spheres of $\bar{\mathbb{R}}^n$ meeting $\bar{B}^n(x,t)$ and $\rho$ is the Hausdorff metric in the collection of all non-empty compact subsets of $\mathbb{R}^n$ defined using the euclidean metric (cf. page 55). We define also the normalized diameter function $\beta$ by

$$\beta(x,t) = \frac{1}{t} d_{\text{euc}}(\bar{B}^n(x,t) \cap L(G)) \tag{6.16}$$

for all $x \in \mathbb{R}^n$ and $t > 0$ such that $B^n(x,t) \cap L(G) \neq \emptyset$. The following theorem states the main results of Chapter 4 in the considered situation.

**Theorem 6.17.** *Let $G$ be the non-elementary geometrically finite Kleinian group and $t_0$ and $v_0$ the constants introduced before Lemma 6.13. Then there are constants $b_0 > 0$, $c_0 > 0$ and $c_1 > 0$ satisfying the following. Let $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ be such that there is $x' \in L(G) \cap \mathbb{R}^n$ with $|x - x'|/t \leq v_0$. Then*

$$b_0^{-1} \leq \beta(x,t) \leq b_0. \tag{6.18}$$

*If, additionally, $(x,t) \notin H_p$ for any $p \in GP$, then*

$$c_0^{-1} \leq \gamma_l(x,t) \leq c_0 \tag{6.19}$$

*for all $l \in \{1, 2, \ldots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$, and if $(x,t) \in H_p$ for some $p \in GP$ of rank $k \in \{1, 2, \ldots, n\}$, then (6.19) is valid for all $l \in \{1, 2, \ldots, n\} \setminus \{k\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ and*

$$c_1^{-1} e^{-d((x,t),\partial H_p)} \leq \gamma_k(x,t) \leq c_1 e^{-d((x,t),\partial H_p)} \tag{6.20}$$

*if $L(G)$ is not a $k$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. There is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x,t) \in GC \cup \bigcup_{p \in GP \cap \mathbb{R}^n} H_p$ by Lemma 6.13. Note that, since $t \in ]0, t_0[$, it is the case that $d((x,t), \partial H_\infty) \leq a_0$ if $\infty \in GP$ and $(x,t) \in H_\infty$, where $a_0 > 0$ is a constant. It is easy to see that we can use Theorem 4.37 to prove the claim. □

To discuss the interpretation of Theorem 6.17, we assume, for the moment, that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \ldots, n\}$, since otherwise the situation is obvious. Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in Theorem 6.17. We see that if $(x,t)$ is deep in some horoball $H_p$, where $p \in GP$ is of rank $k \in \{1, 2, \ldots, n\}$, then $L(G)$ resembles closely a $k$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$. On the other hand, if $(x,t)$ is not deep in any of the horoballs $H_p$, $p \in GP$, then $L(G)$ does not resemble closely any $l$-sphere of $\bar{\mathbb{R}}^n$ in $\bar{B}^n(x,t)$ for any $l \in \{1, 2, \ldots, n\}$. The natural interpretation associated with $\{H_p : p \in GP\}$ is that $(x,t)$ is close to $p \in GP$ in a natural sense if and only if $(x,t)$ is deep in $H_p$. (There is no space to discuss the technical details of this interpretation here. Accounts of varying detail on the intuitions connected to geometrically finite Kleinian groups can be found, for example, in the sources mentioned after (6.10).) The geometric property of $L(G)$ stated by Theorem

6.17 can, therefore, be succinctly described by saying that $L(G)$ resembles a $k$-sphere of $\bar{\mathbb{R}}^n$ for some $k \in \{1, 2, \ldots, n\}$ close to parabolic fixed points of $G$ of rank $k$ and no $l$-sphere of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \ldots, n\}$ otherwise. Let us now give up the temporary assumption that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \ldots, n\}$.

We will next apply the results of Chapter 3 to the Patterson-Sullivan measure $\mu$ of $G$. Theorem 6.17 shows how the results on the geometry of limit sets in Chapter 4 combined with the geometric finiteness of $G$ can be used to obtain a uniform description on some aspects of the geometry of $L(G)$. Similarly, combining the estimation results proved in Chapter 3 with the geometric finiteness of $G$, we will prove for $\mu$ the following estimation theorem, Theorem 6.22, of global nature. Before proving Theorem 6.22, we establish that $\mu$ has no atoms. Recall that $\mu$ is supported by $L(G)$ and that $\mu$ is $\delta$-conformal, where $\delta$ is the exponent of convergence of $G$, see (2.9) for the definition. Recall also that $\delta > 0$, since $G$ is non-elementary.

**Theorem 6.21.** *Let $G$ be the non-elementary geometrically finite Kleinian group and $t_0$ and $v_0$ the constants introduced before Lemma 6.13. Recall that $\mu$ is a given Patterson-Sullivan measure of $G$ and that $\delta$ is the exponent of convergence of $G$. It is the case now that $\mu(x) = 0$ for all $x \in L(G)$.*

*Proof.* The claim (i) of Theorem 6.11 implies that $L(G)$ is the disjoint union of $L_c(G)$ (the set of conical limit points of $G$) and $GP$. Theorem 6.3 implies that $\mu(x) = 0$ for every $x \in L_c(G)$. It is also true that $\mu(x) = 0$ for every $x \in GP$, but a detailed proof of this claim is rather long and there is no space for such a proof here. A proof for the claim can be found, for example, in Section 3.5 of [Nicholls1989], Section 2 of [Patterson1987] or Section 2 of [Sullivan1984]. Let us remark that a bounded parabolic fixed point of a non-elementary Kleinian group is never an atom of a Patterson-Sullivan measure of the group. The group does not have to be geometrically finite. □

**Theorem 6.22.** *Let $G$ be the non-elementary geometrically finite Kleinian group and $t_0$ and $v_0$ the constants introduced before Lemma 6.13. Recall that $\mu$ is a given Patterson-Sullivan measure of $G$ and that $\delta$ is the exponent of convergence of $G$. In this situation, there is a constant $c > 0$ such that the following holds. Let $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ be such that $|x - x'|/t \leq v_0$ for some $x' \in L(G) \cap \mathbb{R}^n$. Then*

$$c^{-1} t^{\delta} \leq \mu(\bar{B}^n(x, t)) \leq c t^{\delta} \tag{6.23}$$

*if $(x, t) \notin H_p$ for any $p \in GP$, and*

$$c^{-1} t^{\delta} e^{d((x,t),\partial H_p)(k-\delta)} \leq \mu(\bar{B}^n(x, t)) \leq c t^{\delta} e^{d((x,t),\partial H_p)(k-\delta)} \tag{6.24}$$

*if $(x, t) \in H_p$ for some $p \in GP$ of rank $k \in \{1, 2, \ldots, n\}$.*

*Proof.* Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. By Lemma 6.13, there is a compact set $C \subset \mathbb{H}^{n+1}$ such that $(x, t) \in GC \cup \bigcup_{p \in GP \cap \mathbb{R}^n} H_p$. Note that, since $t \in ]0, t_0[$, it is the case that $d((x, t), \partial H_\infty) \leq a_0$ if $\infty \in GP$ and $(x, t) \in H_\infty$, where $a_0 > 0$ is a constant. Note also that $\mu(p) = 0$ for every $p \in GP$ by Theorem 6.21. It is now easy to use Theorem 3.14 to prove the claim. □

We will next combine Theorems 6.17 and 6.22 into Theorem 6.25 which states a global estimation formula (6.26) estimating quantities of the form $\mu(\bar{B}^n(x,t))$ using geometric properties of $\bar{B}^n(x,t) \cap L(G)$.

**Theorem 6.25.** *Let $G$ be the non-elementary geometrically finite Kleinian group and $t_0$ and $v_0$ the constants introduced before Lemma 6.13. Recall that $\mu$ is a given Patterson-Sullivan measure of $G$ and that $\delta$ is the exponent of convergence of $G$. In this situation, there is a constant $c > 0$ such that the following is true. Let $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ be such that there is $x' \in L(G) \cap \mathbb{R}^n$ with $|x - x'|/t \leq v_0$. Then*

$$c^{-1}\alpha(x,t)\omega(x,t) \leq \mu(\bar{B}^n(x,t)) \leq c\alpha(x,t)\omega(x,t), \tag{6.26}$$

*where*

$$\alpha(x,t) = d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^{\delta} \tag{6.27}$$

*and*

$$\omega(x,t) = \prod_{l=1}^{n} \gamma_l(x,t)^{\delta - l}. \tag{6.28}$$

*Proof.* Let $c > 0$ be the constant given by Theorem 6.22. We can express the estimates (6.23) and (6.24) as one estimate, namely

$$c^{-1}t^{\delta}\tau(x,t) \leq \mu(\bar{B}^n(x,t)) \leq ct^{\delta}\tau(x,t), \tag{6.29}$$

where

$$\tau(x,t) = \prod_{l=1}^{n} \exp((l-\delta)\max\{d_{H_p}((x,t),\partial H_p) : p \in GP, r(p) = l\}), \tag{6.30}$$

where $r(p) \in \{1, 2, \ldots, n\}$ denotes the rank of $p \in GP$ and $d_{H_p}((x,t),\partial H_p)$ is equal to $d((x,t),\partial H_p)$ if $(x,t) \in H_p$ and 0 otherwise; if $G$ has no parabolic fixed points of some rank $l \in \{1, 2, \ldots, n\}$, we set that the term corresponding to $l$ on the right hand side of (6.30) equals 1. Observe next that (6.18) can be rewritten as

$$b_0^{-1}t \leq d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G)) \leq b_0 t. \tag{6.31}$$

Suppose first that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$ for any $l \in \{1, 2, \ldots, n\}$. We can use (6.19), (6.20) and (6.31) to see that if the constant $c$ in (6.29) is adjusted accordingly, we can rewrite (6.29) as (6.26). Suppose next that $L(G)$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ for some $l \in \{1, 2, \ldots, n\}$. Theorem 6.14 implies that now $\delta = l$ and that if $p \in \bar{\mathbb{R}}^n$ is a parabolic fixed point of $G$, then the rank of $p$ is $l$. It follows that

$$c^{-1}t^l \leq \mu(\bar{B}^n(x,t)) \leq ct^l. \tag{6.32}$$

Since we employ the convention that $\gamma_l(x,t)^{\delta - l} = \gamma_l(x,t)^0 = 1$, we see that (6.26) is valid in this case as well. □

We will next adapt the geometric measure constructions introduced in Chapter 5 to the present setting. We choose the base set $X \subset \mathbb{R}^n$ used in the constructions to be the set $L(G) \cap \mathbb{R}^n$, where $G$ is the non-elementary geometrically finite Kleinian group introduced before Lemma 6.13 with the exponent of convergence $\delta$. The formulae (6.27) and (6.28) imply that the parameters $\eta > 0$ and $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{R}$ used in the constructions (see (5.3) and (5.4)) are $\eta = \delta$ and $\eta_l = \delta - l$ for $l \in \{1, 2, \ldots, n\}$.

Let $\bar{m}$ be the outer measure of $L(G) \cap \mathbb{R}^n$ obtained when the covering construction introduced in Chapter 5 is applied to the present setting. Let $\bar{p}$ be the outer measure of $L(G) \cap \mathbb{R}^n$ obtained when the packing construction introduced in Chapter 5 is applied to the present setting. Every Borel set of $L(G) \cap \mathbb{R}^n$ is measurable with respect to $\bar{m}$ and $\bar{p}$ by Theorem 5.11. We can, therefore, define measures $m$ and $p$ by setting that

$$m(A) = \bar{m}(A \cap (L(G) \cap \mathbb{R}^n)) \quad \text{and} \quad p(A) = \bar{p}(A \cap (L(G) \cap \mathbb{R}^n)) \tag{6.33}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$.

Our aim is to show that $m$ and $p$ are atomless $\delta$-conformal measures of $G$. We begin our discussion on $m$ and $p$ by showing that these measures do not have atoms and that they satisfy a transformation rule of the form (2.14) with $s = \delta$. The constants $t_0 > 0$ and $v_0 \in ]0, 1[$ appearing in the proof of the following result are the ones introduced before Lemma 6.13.

**Theorem 6.34.** *Let $m$ and $p$ be the measures defined by (6.33). Then $m(x) = 0 = p(x)$ for every $x \in \bar{\mathbb{R}}^{n+1}$. Furthermore,*

$$m(gA) = \int_A |g'|^\delta dm \qquad \textit{and} \qquad p(gA) = \int_A |g'|^\delta dp \tag{6.35}$$

*for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$.*

*Proof.* It is trivial that $m(x) = 0 = p(x)$ for every $x \in \bar{\mathbb{R}}^{n+1} \setminus (L(G) \cap \mathbb{R}^n)$. Let $x \in L(G) \cap \mathbb{R}^n$ be arbitrary. We prove that $m(x) = 0 = p(x)$ by proving the equivalent claim that $\bar{m}(x) = 0 = \bar{p}(x)$.

We consider first $\bar{m}$. Let $\varepsilon_0 \in ]0, t_0[$ be fixed for the time being. Let $\varepsilon \in ]0, \varepsilon_0[$ and $v \in ]0, v_0[$. Now $\{\bar{B}^n(x, \varepsilon)\}$ is an $(\varepsilon, v)$-covering of $\{x\}$. The definition (5.6) and the estimate (6.26) imply that

$$\bar{m}_\varepsilon^v(x) \leq \alpha(x, \varepsilon)\omega(x, \varepsilon) \leq c_0\mu(\bar{B}^n(x, \varepsilon)) \leq c_0\mu(\bar{B}^n(x, \varepsilon_0))$$

for some constant $c_0 > 0$. It follows that $\bar{m}(x) \leq c_0\mu(\bar{B}^n(x, \varepsilon_0))$. Theorem 6.21 implies that $\lim_{\varepsilon_0 \to 0} \mu(\bar{B}^n(x, \varepsilon_0)) = \mu(x) = 0$, and so $\bar{m}(x) = 0$.

We consider $\bar{p}$. Let again $\varepsilon_0 \in ]0, t_0[$ be fixed for the time being. Let $\varepsilon \in ]0, \varepsilon_0[$ and $v \in ]0, v_0[$. Let $\mathcal{T}$ be an $(\varepsilon, v)$-packing of $\{x\}$. Then $\mathcal{T} = \{\bar{B}^n(y, u)\}$, where $\bar{B}^n(y, u)$ is such that $y \in \mathbb{R}^n$, $u \in ]0, \varepsilon]$, and $|y - x|/u \leq v$. Recall that we denote by $c_0$ the constant of (6.26) in this proof. We estimate that

$$\alpha(y, u)\omega(y, u) \leq c_0\mu(\bar{B}^n(y, u)) \leq c_0\mu(\bar{B}^n(x, 2\varepsilon_0)).$$

Since $\mathcal{T}$ was an arbitrary $(\varepsilon, v)$-packing of $\{x\}$, we can deduce that $\bar{p}^*(x) \leq \bar{p}_\varepsilon^v(x) \leq c_0\mu(\bar{B}^n(x, 2\varepsilon_0))$. We can argue as in the case of $\bar{m}$ to obtain that $\bar{p}^*(x) = 0$. The definition (5.10) implies that $\bar{p}(x) \leq \bar{p}^*(x) = 0$.

Assume next that $A$ is a non-empty Borel set of $\bar{\mathbb{R}}^{n+1}$ and that $g \in G$. Let us prove the first formula of (6.35). The argument needed to do this is essentially the same as in the proof of Theorem 5.12. First, since $m$ is atomless and supported by $L(G)$, we can assume that $A \subset L(G) \setminus \{\infty, g^{-1}(\infty)\}$. Like in the proof of Theorem 5.12, we can assume that $M^{-1} \leq |g'| \leq M$ in $A(a_0)$, where $M > 0$ and $a_0 > 0$ are constants and $A(a_0) = \{z \in \mathbb{R}^n : d_{\text{euc}}(z, A) < a_0\}$. We proceed to introduce the same parameter $\lambda > 0$ and the same division $A_1, A_2, \ldots, A_{k_\lambda}$ of $A$ corresponding to $\lambda$ as in the proof of Theorem 5.12. Let the symbols $M_k$ and $m_k$, $k \in \{1, 2, \ldots, k_\lambda\}$, have the same meanings as in the proof of Theorem 5.12. We can continue to argue as in the proof of Theorem 5.12 to obtain the formula corresponding to (5.15), namely

$$\bar{m}(gA_k) \leq \chi \left( \frac{M_k}{m_k} \right)^\delta m_k^\delta \bar{m}(A_k), \tag{6.36}$$

where $k \in \{1, 2, \ldots, k_\lambda\}$ is fixed and $\chi \geq 1$ is a number determined by $\lambda$ such that $\chi \to 1$ as $\lambda \to 0$; observe that the outer measure $\bar{m}$ appears on both sides of (6.36) because $L(G)$ is $G$-invariant. Since $\bar{m}(gA_k) = m(gA_k)$ and

$$m_k^\delta \bar{m}(A_k) = m_k^\delta m(A_k) \leq \int_{A_k} |g'|^\delta dm,$$

it is clear that we can reason as in the proof of Theorem 5.12 to prove that

$$m(gA) \leq \int_A |g'|^\delta dm.$$

The second half of the first formula of (6.35) can be proved by applying the corresponding argument in the proof of Theorem 5.12 to the present situation. The second formula of (6.35) is proved in a similar way. □

We continue our discussion on $m$ and $p$ by proving Theorem 6.38 which shows that $m$ and $p$ are non-trivial and that they give finite measures to bounded Borel sets of $\mathbb{R}^{n+1}$. The classical Besicovitch covering theorem will be needed in the proof of Theorem 6.38 and we state it as Theorem 6.37. The proof of Theorem 6.38 consists mostly of straightforward modifications of arguments which can be found in Section 8 of [Sullivan1984].

**Theorem 6.37.** *Given $k \in \{1, 2, \ldots\}$, there are constants $b_c > 0$ and $b_p > 0$ satisfying the following. Let $A \subset \mathbb{R}^k$ be non-empty and bounded and let $\mathcal{S}$ be any collection of closed balls of $\mathbb{R}^k$ such that every $x \in A$ is the center of some $\bar{B} \in \mathcal{S}$. Then there is a countable subcollection $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}'$ is a covering of $A$ whose multiplicity is bounded by $b_c$, i.e. the number of elements in $\mathcal{S}'$ containing any given $x \in A$ is smaller than or equal to $b_c$. Furthermore, it is the case that $\mathcal{S}' = \mathcal{S}'_1 \cup \mathcal{S}'_2 \cup \ldots \cup \mathcal{S}'_{i_0}$, where $i_0 \in \{1, 2, \ldots, b_p\}$ and the balls in $\mathcal{S}'_i$ are pairwise disjoint for every $i \in \{1, 2, \ldots, i_0\}$.*

*Proof.* As we mentioned above, this is a classical result. An explicit proof is given in [Mattila1995], for example, where the result is stated as Theorem 2.7. □

**Theorem 6.38.** *Let $m$ and $p$ be the measures defined by (6.33). Let $B_0$ be an open $n$-ball of $\mathbb{R}^n$ such that $A = B_0 \cap L(G) \neq \emptyset$. Then $m(A), p(A) \in ]0, \infty[$.*

*Proof.* Let $b_c > 0$ and $b_p > 0$ be the constants obtained when Theorem 6.37 is applied to the case $k = n$. Let $\mu$ be the Patterson-Sullivan measure of the non-elementary geometrically finite Kleinian group $G$ introduced before Lemma 6.13. Note that $\mu(A) > 0$ by

Lemma 2.47 and that $\mu(C) < \infty$ for every compact subset $C$ of $\mathbb{R}^n$ by Theorem 2.27. Let $t_0 > 0$ and $v_0 \in ]0,1[$ be the constants introduced before Lemma 6.13. We denote in this proof by $c_0$ the positive constant appearing in (6.26).

We show first that $m(A) \in ]0,\infty[$. Let $\varepsilon \in ]0,t_0[$ and $v \in ]0,v_0[$. Let $\mathcal{T}$ be an $(\varepsilon, v)$-covering of $A$. Now

$$0 < \mu(A) \le \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \mu(\bar{B}^n(x,t)) \le c_0 \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t)$$

by (6.26). We can take the infimum over $\mathcal{T}$ to obtain that $\bar{m}^v_\varepsilon(A) \ge \mu(A)/c_0$. It follows that

$$m(A) = \bar{m}(A) = \sup_{\varepsilon\in]0,t_0[,v\in]0,v_0[} \bar{m}^v_\varepsilon(A) \ge \frac{\mu(A)}{c_0} > 0.$$

To prove that $m(A) < \infty$, let again $\varepsilon \in ]0,t_0[$. According to Theorem 6.37, the collection $\{\bar{B}^n(x,\varepsilon) : x \in A\}$ has a countable subcollection $\mathcal{T}_\varepsilon$ such that $\mathcal{T}_\varepsilon$ is a covering of $A$ whose multiplicity is bounded by $b_c$. Note that $\mathcal{T}_\varepsilon$ is an $(\varepsilon, v)$-covering of $A$ for any fixed $v \in ]0,v_0[$. We conclude using (6.26) that

$$\begin{aligned}\bar{m}^v_\varepsilon(A) &\le \sum_{\bar{B}^n(x,t)\in\mathcal{T}_\varepsilon} \alpha(x,t)\omega(x,t) \le c_0 \sum_{\bar{B}^n(x,t)\in\mathcal{T}_\varepsilon} \mu(\bar{B}^n(x,t)) \\ &\le b_c c_0 \mu\left(\bigcup_{\bar{B}^n(x,t)\in\mathcal{T}_\varepsilon} \bar{B}^n(x,t)\right) \le b_c c_0 \mu(K) < \infty,\end{aligned}$$

where $K \subset \mathbb{R}^n$ is a suitable large enough compact set. We obtain that

$$m(A) = \bar{m}(A) = \sup_{\varepsilon\in]0,t_0[,v\in]0,v_0[} \bar{m}^v_\varepsilon(A) \le b_c c_0 \mu(K) < \infty.$$

We have proved that $m(A) \in ]0,\infty[$.

We consider next $p$. In order to show that $p(A) < \infty$, let $\mathcal{T}$ be an $(\varepsilon, v)$-packing of $A$, where $\varepsilon \in ]0,t_0[$ and $v \in ]0,v_0[$. Then

$$\sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t) \le c_0 \mu\left(\bigcup_{\bar{B}^n(x,t)\in\mathcal{T}} \bar{B}^n(x,t)\right) \le c_0 \mu(K') < \infty$$

by (6.26), where $K' \subset \mathbb{R}^n$ is a suitable large enough compact set. We take the supremum over $\mathcal{T}$ to obtain that $\bar{p}^v_\varepsilon(A) \le c_0\mu(K')$. We now have that $p(A) = \bar{p}(A) \le \bar{p}^*(A) \le \bar{p}^v_\varepsilon(A) \le c_0\mu(K') < \infty$.

Our final task is to prove that $p(A) > 0$. Let $\mathcal{D}$ be any countable collection of subsets of $A$ whose union is $A$. Consider a non-empty $D \in \mathcal{D}$. Since $D$ is not necessarily a Borel set of $\bar{\mathbb{R}}^{n+1}$, the quantity $\mu(D)$ is not necessarily well-defined. To solve this problem, we introduce the natural outer measure $\mu^*$ of $\bar{\mathbb{R}}^{n+1}$ extending $\mu$. That is, if $U \subset \bar{\mathbb{R}}^{n+1}$, we define that

$$\mu^*(U) = \inf\{\mu(E) : U \subset E \text{ and } E \text{ is a Borel set of } \bar{\mathbb{R}}^{n+1}\}. \tag{6.39}$$

It is evident that $\mu^*(\emptyset) = 0$, that $\mu^*(U_1) \le \mu^*(U_2)$ if $U_1 \subset U_2 \subset \bar{\mathbb{R}}^{n+1}$, and that $\mu^*(U) = \mu(U)$ if $U$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$. So to verify that $\mu^*$ is an outer measure of $\bar{\mathbb{R}}^{n+1}$, we need to show that $\mu^*$ is countably subadditive. Let $U_1, U_2, \ldots \subset \bar{\mathbb{R}}^{n+1}$. Let $\kappa > 0$. Given $i \in \{1, 2, \ldots\}$, choose a Borel set $E_i$ of $\bar{\mathbb{R}}^{n+1}$ such that $E_i$ contains $U_i$ and that $\mu(E_i) \leq \mu^*(U_i) + \kappa/2^i$. Now $\bigcup_{i=1}^\infty E_i$ is a Borel set of $\bar{\mathbb{R}}^{n+1}$ containing $\bigcup_{i=1}^\infty U_i$, so

$$\mu^*\left(\bigcup_{i=1}^\infty U_i\right) \leq \mu\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty \mu(E_i) \leq \kappa + \sum_{i=1}^\infty \mu^*(U_i).$$

Since $\kappa > 0$ was arbitrary, the countable subadditivity of $\mu^*$ follows. Hence $\mu^*$ is indeed an outer measure of $\bar{\mathbb{R}}^{n+1}$ extending $\mu$.

To proceed with the argument, let $\varepsilon \in ]0, t_0[$ and $\nu \in ]0, \nu_0[$. Let $\mathcal{T}_\varepsilon$ be a countable subcollection of $\{\bar{B}^n(x, \varepsilon) : x \in D\}$ given by Theorem 6.37. That is, $\mathcal{T}_\varepsilon$ is a covering of $D$ and $\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^1 \cup \mathcal{T}_\varepsilon^2 \cup \ldots \cup \mathcal{T}_\varepsilon^{i(D,\varepsilon)}$, where $i(D, \varepsilon) \in \{1, 2, \ldots, b_p\}$ and the balls contained in $\mathcal{T}_\varepsilon^i$ are pairwise disjoint for every $i \in \{1, 2, \ldots, i(D, \varepsilon)\}$. It is clear that, since $\mathcal{T}_\varepsilon$ covers $D$ and $\mathcal{T}_\varepsilon$ is the union of the subcollections $\mathcal{T}_\varepsilon^i$, $i \in \{1, 2, \ldots, i(D, \varepsilon)\}$, there is $i_0 \in \{1, 2, \ldots, i(D, \varepsilon)\}$ such that

$$\mu^*\left(\bigcup_{\bar{B}^n(x,t) \in \mathcal{T}_\varepsilon^{i_0}} \bar{B}^n(x, t)\right) \geq \frac{\mu^*(D)}{i(D, \varepsilon)} \geq \frac{\mu^*(D)}{b_p}.$$

The collection $\mathcal{T}_\varepsilon^{i_0}$ is an $(\varepsilon, \nu)$-packing of $D$, and thus

$$\begin{aligned} \bar{p}_\varepsilon^\nu(D) &\geq \sum_{\bar{B}^n(x,t) \in \mathcal{T}_\varepsilon^{i_0}} \alpha(x, t)\omega(x, t) \geq \frac{1}{c_0} \sum_{\bar{B}^n(x,t) \in \mathcal{T}_\varepsilon^{i_0}} \mu(\bar{B}^n(x, t)) \\ &= \frac{1}{c_0}\mu^*\left(\bigcup_{\bar{B}^n(x,t) \in \mathcal{T}_\varepsilon^{i_0}} \bar{B}^n(x, t)\right) \geq \frac{\mu^*(D)}{b_p c_0} \end{aligned}$$

by (6.26). Hence $\bar{p}^*(D) = \inf_{\varepsilon \in ]0,t_0[, \nu \in ]0,\nu_0[} \bar{p}_\varepsilon^\nu(D) \geq \mu^*(D)/b_p c_0$. This estimate is trivial if $D = \emptyset$. Therefore,

$$\sum_{D \in \mathcal{D}} \bar{p}^*(D) \geq \frac{1}{b_p c_0} \sum_{D \in \mathcal{D}} \mu^*(D) \geq \frac{\mu^*(A)}{b_p c_0} = \frac{\mu(A)}{b_p c_0} > 0.$$

Taking the infimum over the collections $\mathcal{D}$, we arrive at the conclusion that $p(A) = \bar{p}(A) \geq \mu(A)/b_p c_0 > 0$. The proof is complete. $\square$

We can now prove that $m$ and $p$ are atomless $\delta$-conformal measures of $G$ giving full measure to the set $L_c(G)$ of conical limit points of $G$.

**Theorem 6.40.** *Let $G$ be the non-elementary geometrically finite Kleinian group with the exponent of convergence $\delta$ introduced before Lemma 6.13. Let $m$ and $p$ be the measures defined by (6.33). Let $f \in$ Möb$(n+1)$ map $\mathbb{H}^{n+1}$ onto $\mathbb{B}^{n+1}$. Then $f_*^\delta m = m_f$ and $f_*^\delta p = p_f$ are atomless $\delta$-conformal measures of $fGf^{-1} = G_f$ such that $m_f(L(G_f) \setminus L_c(G_f)) = 0 = p_f(L(G_f) \setminus L_c(G_f))$. Moreover, $m = (f^{-1})_*^\delta m_f$ and $p = (f^{-1})_*^\delta p_f$, so $m$ and $p$ are atomless $\delta$-conformal measures of $G$ such that $m(L(G) \setminus L_c(G)) = 0 = p(L(G) \setminus L_c(G))$.*

*Proof.* The definition (6.33) and Theorems 6.34 and 6.38 imply that the following claims are true. The measures $m$ and $p$ are measures of $\bar{\mathbb{R}}^{n+1}$ with the Borel sets of $\bar{\mathbb{R}}^{n+1}$ as measurable sets. The measures $m$ and $p$ are supported by $L(G)$. The measures $m$ and $p$ are non-trivial and they give finite measures to all bounded Borel sets of $\mathbb{R}^{n+1}$. The measures $m$ and $p$ are atomless. The measures $m$ and $p$ satisfy the transformation rules

$$m(gA) = \int_A |g'|^\delta dm \qquad \text{and} \qquad p(gA) = \int_A |g'|^\delta dp$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$. Theorem 2.29 implies now that $m_f$ and $p_f$ are atomless $\delta$-conformal measures of $G_f$ and that $m = (f^{-1})^\delta_* m_f$ and $p = (f^{-1})^\delta_* p_f$ are $\delta$-conformal measures of $G$. Note that it is well known that since $G$ is non-elementary, no point in $\bar{\mathbb{R}}^{n+1}$ is fixed by every element of $G$, see, for example, Theorem 2T of [Tukia1994a]. Recall that, by the claim (i) of Theorem 6.11, the limit set $L(G)$ is the disjoint union of $L_c(G)$ and $GP$. Since $m$ is atomless and $GP$ is countable, we obtain that $m(L(G) \setminus L_c(G)) = m(GP) = 0$. The corresponding claims for $p$, $m_f$ and $p_f$ are proved using the same argument. □

We are now in a position to prove the basic version of the main result of this work.

**Theorem 6.41.** *Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be a Patterson-Sullivan measure of $G$. Then either one of the measure constructions introduced in Chapter 5 can be used to construct a measure $\nu$ such that $\mu = c\nu$, where $c > 0$ is a constant.*

*Proof.* Denote by $\delta$ the exponent of convergence of $G$. Assume first that $\mathbb{X}^{n+1} = \mathbb{B}^{n+1}$. Let $h \in \text{Möb}(n+1)$ map $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$. Let $m$ and $p$ be the $\delta$-conformal measures of $hGh^{-1}$ obtained when the covering and packing constructions introduced in Chapter 5 are applied to $hGh^{-1}$, see (6.33) and Theorem 6.40. Theorem 6.40 implies that $(h^{-1})^\delta_* m = m_{h^{-1}}$ and $(h^{-1})^\delta_* p = p_{h^{-1}}$ are $\delta$-conformal measures of $G$ such that $m_{h^{-1}}(L(G) \setminus L_c(G)) = 0 = p_{h^{-1}}(L(G) \setminus L_c(G))$. Theorem 6.21 implies that $\mu(L(G) \setminus L_c(G)) = 0$ (recall that $L(G) \setminus L_c(G)$ is a countable set of bounded parabolic fixed points of $G$ by the claim (i) of Theorem 6.11). We obtain now from Theorem 6.9 that there are constants $c_m > 0$ and $c_p > 0$ such that $c_m m_{h^{-1}} = \mu = c_p p_{h^{-1}}$.

Suppose next that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$. By definition, there is $h \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and a Patterson-Sullivan measure $\mu_{h^{-1}}$ of $h^{-1}Gh = G_{h^{-1}}$ such that $\mu = h^\delta_* \mu_{h^{-1}}$. Let $m$ and $p$ be the $\delta$-conformal measures of $G$ obtained when the covering and packing constructions introduced in Chapter 5 are applied to $G$, see (6.33) and Theorem 6.40. Write $(h^{-1})^\delta_* m = m_{h^{-1}}$ and $(h^{-1})^\delta_* p = p_{h^{-1}}$. We obtain from Theorem 6.40 that $m_{h^{-1}}$ and $p_{h^{-1}}$ are $\delta$-conformal measures of $G_{h^{-1}}$ such that $m_{h^{-1}}(L(G_{h^{-1}}) \setminus L_c(G_{h^{-1}})) = 0 = p_{h^{-1}}(L(G_{h^{-1}}) \setminus L_c(G_{h^{-1}}))$. We obtain as above that $\mu_{h^{-1}}(L(G_{h^{-1}}) \setminus L_c(G_{h^{-1}})) = 0$. Theorem 6.9 implies that there are constants $c_m > 0$ and $c_p > 0$ such that $c_m m_{h^{-1}} = \mu_{h^{-1}} = c_p p_{h^{-1}}$. According to Theorem 6.40, we have that $m = h^\delta_* m_{h^{-1}}$ and $p = h^\delta_* p_{h^{-1}}$. Theorem 6.9 implies that $c_m m = \mu = c_p p$. □

## 7. Variants of the main result

We present in this chapter some variants of the measure constructions introduced in Chapter 5 and show that each one of them satisfies an equivalence result similar to Theorem 6.41 or some related weaker theorem.

Recall that many of the the main considerations of this work are modifications and generalizations of similar considerations presented in [Sullivan1984] by D. Sullivan. We begin this chapter by taking a somewhat closer look at the relevant results of [Sullivan1984].

After having discussed the relevant results of Sullivan, we will consider simple variants of the covering and packing constructions introduced in Chapter 5 which satisfy an equivalence theorem similar to Theorem 6.41.

The third section of this chapter presents variants of the measure constructions of Chapter 5 which employ considerably simpler flatness functions than the original constructions. The price to pay for this simplification is that these variants satisfy a weaker result related to equivalence than the original constructions: if $G$ is a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$ and $\mu$ is a Patterson-Sullivan measure of $G$, then any of the variants can be used to construct a measure $\nu$ such that $c^{-1}\nu \leq \mu \leq c\nu$, where $c > 0$ is a constant.

Recall that the measure constructions introduced in Chapter 5 construct measures supported by subsets of $\mathbb{R}^n$. We show in the fourth and final section of this chapter that these constructions can be modified so that they construct measures supported by subsets of $\mathbb{S}^n$. This implies that we can prove an alternative version of Theorem 6.41 for non-elementary geometrically finite Kleinian groups acting on $\mathbb{B}^{n+1}$.

### 7.1. **The relevant results of Sullivan.**

Let us discuss the results of [Sullivan1984] which are important from the point of view of our work. These results consider the question asking when is a Patterson-Sullivan measure of a non-elementary geometrically finite Kleinian group equivalent to a measure obtained from the standard covering construction or the standard packing construction. Let us recall the definitions of these constructions.

Let $s \geq 0$. The standard covering outer measure $m_s$ of $\bar{\mathbb{R}}^{n+1}$ obtained using the gauge function $t \mapsto t^s$ and coverings of closed euclidean balls of $\bar{\mathbb{R}}^{n+1}$ is defined as follows. Given $A \subset \mathbb{R}^{n+1}$, we set that

$$m_s(A) = \lim_{\varepsilon \to 0} \inf_{\mathcal{T}} \sum_{\bar{B}^{n+1}(x,t) \in \mathcal{T}} t^s, \tag{7.1}$$

where $\varepsilon > 0$ and $\mathcal{T}$ varies in the collection of all $\varepsilon$-coverings of $A$, i.e. $\mathcal{T}$ is a countable collection of closed balls of $\mathbb{R}^{n+1}$ with radii bounded by $\varepsilon$ such that $A \subset \bigcup \mathcal{T}$. Additionally, we set that $m_s(\infty) = 0$ if $s > 0$ and that $m_0(\infty) = 1$. We use $m_s$ to denote the corresponding measure of $\bar{\mathbb{R}}^{n+1}$ as well.

The standard packing outer measure $p_s$ of $\bar{\mathbb{R}}^{n+1}$ obtained using the gauge function $t \mapsto t^s$ and packings of closed euclidean balls of $\bar{\mathbb{R}}^{n+1}$ is defined as follows. Given $A \subset \mathbb{R}^{n+1}$, we define that

$$p_s(A) = \inf_{\mathcal{D}} \sum_{D \in \mathcal{D}} p_s^*(D), \tag{7.2}$$

where the infimum is taken over all countable collections $\mathcal{D}$ of subsets of $A$ such that $\bigcup \mathcal{D} = A$ and the set function $p_s^*$ is defined as follows. Set $p_s^*(\emptyset) = 0$, and if $D \neq \emptyset$, set

$$p_s^*(D) = \lim_{\varepsilon \to 0} \sup_{\mathcal{T}} \sum_{\bar{B}^{n+1}(x,t) \in \mathcal{T}} t^s, \tag{7.3}$$

where $\varepsilon > 0$ and $\mathcal{T}$ varies in the collection of all $\varepsilon$-packings of $D$, i.e. $\mathcal{T}$ is a countable collection of pairwise disjoint closed balls of $\mathbb{R}^{n+1}$ with centers in $D$ and radii bounded by $\varepsilon$. To obtain an outer measure of $\bar{\mathbb{R}}^{n+1}$, we set that $p_s(\infty) = 0$ if $s > 0$ and that $p_0(\infty) = 1$. We use $p_s$ to denote the corresponding measure of $\bar{\mathbb{R}}^{n+1}$ as well.

We remark that the definition given in [Sullivan1984] for the packing construction differs from the definition given above. But since the relevant results of [Sullivan1984] are independent of which definition is used, we prefer to use the above standard definition. We remark also that only classical Kleinian groups, i.e. groups acting on $\mathbb{X}^3$ and containing only orientation preserving elements, are treated explicitly in [Sullivan1984]. However, it is not difficult to generalize the relevant results of [Sullivan1984].

Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\delta > 0$ be the exponent of convergence of $G$. If $G$ contains parabolic elements, denote by $k_{\max}$ and $k_{\min}$ the maximum and minimum over the ranks of parabolic fixed points of $G$. Let $\mu$ be a Patterson-Sullivan measure of $G$. We define the measures $m_\delta^G$ and $p_\delta^G$ by setting that

$$m_\delta^G(A) = m_\delta(A \cap L(G)) \qquad \text{and} \qquad p_\delta^G(A) = p_\delta(A \cap L(G)) \tag{7.4}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. The following theorem, Theorem 7.5, states the main equivalence result of [Sullivan1984]. The core of the proof is the same as in the proof of Theorem 6.41. The crucial difference is that the estimation results proved in Chapter 3 suffice, i.e. there is no need to consider the geometry of $L(G)$ as in Chapter 4. We will provide enough details to indicate how the situation considered in [Sullivan1984] relates to our considerations, but we will omit most of the details for the sake of brevity.

**Theorem 7.5.** *Let $G$, $\delta$, $k_{\max}$, $k_{\min}$, $\mu$, $m_\delta^G$ and $p_\delta^G$ be as above. Then the following claims are true. If $G$ contains no parabolic elements, then $\mu$, $m_\delta^G$ and $p_\delta^G$ are all equivalent to one another, i.e. they are identical up to multiplicative constants. If $G$ contains parabolic elements, then $\mu$ is equivalent to $m_\delta^G$ in case $\delta \geq k_{\max}$, and $\mu$ is equivalent to $p_\delta^G$ in case $\delta \leq k_{\min}$.*

*Proof.* We consider the claims concerning $m_\delta^G$. We assume, therefore, that either $G$ contains no parabolic elements or $G$ contains parabolic elements and $\delta \geq k_{\max}$.

Let first $\mathbb{X}^{n+1} = \mathbb{B}^{n+1}$. Recall that $L(G)$ is the disjoint union of the set $L_c(G)$ of conical limit points of $G$ and the (possibly empty) set $P(G)$ of parabolic fixed points of $G$ by the claim (i) of Theorem 6.11. The measure $m_\delta^G$ is atomless, so $m_\delta^G(P(G)) = 0$. Similarly, $\mu(P(G)) = 0$ by Theorem 6.21. Theorem 6.9 implies that if we show that $m_\delta^G$ is a $\delta$-conformal measure of $G$, then we obtain that $\mu$ and $m_\delta^G$ are equivalent. It is well known that $m_\delta^G$ satisfies a transformation rule of the form (2.14) with $s = \delta$, so to prove that $m_\delta^G$ is a $\delta$-conformal measure of $G$, we need to show that $m_\delta^G$ is non-trivial and finite.

Suppose first that $L(G) = \mathbb{S}^n = \partial\mathbb{B}^{n+1}$. Theorem 6.14 implies that $\delta = n$ and that every parabolic fixed point of $G$ is of rank $n$ if such points exist. (Note that the condition $\delta \geq k_{\max}$ is satisfied if $G$ contains parabolic elements.) It is well known that now $m_\delta^G = c_0 L_{\mathbb{S}^n}$, where $c_0 > 0$ is a constant and $L_{\mathbb{S}^n}$ is the Lebesgue measure of $\mathbb{S}^n$ restricted to the $\sigma$-algebra of Borel sets of $\bar{\mathbb{R}}^{n+1}$. It follows that $m_\delta^G$ is non-trivial and finite in this case.

Suppose next that $L(G) \neq \mathbb{S}^n$. Choose $h \in \text{Möb}(n+1)$ such that $h\mathbb{B}^{n+1} = \mathbb{H}^{n+1}$ and $h^{-1}(\infty) \notin L(G)$. Write $\hat{G} = hGh^{-1}$ and $\hat{\mu} = h_*^\delta \mu$. Note that $L(\hat{G})$ is a bounded subset of $\mathbb{R}^n$. It is not difficult to see that $h_*^\delta m_\delta^G = m_\delta^{\hat{G}}$. We will prove that $m_\delta^{\hat{G}}$ is non-trivial and finite. It follows then easily that $m_\delta^G = (h^{-1})_*^\delta m_\delta^{\hat{G}}$ is non-trivial and finite.

Fix constants $t_0 > 0$ and $v_0 \in ]0, 1[$. Let $c_1 > 0$ be a constant such that Theorem 6.22 is valid for $\hat{G}$ and $\hat{\mu}$ with respect to $t_0$, $v_0$ and $c_1$.

We show first that $m_\delta^{\hat{G}}(L(\hat{G})) > 0$. Let $\varepsilon \in ]0, t_0/2[$ and let $\mathcal{T}$ be an $\varepsilon$-covering of $L(\hat{G})$. Denote by $\mathcal{T}'$ the collection of all $\bar{B} \in \mathcal{T}$ such that $\bar{B} \cap L(\hat{G}) \neq \emptyset$. Given $\bar{B} \in \mathcal{T}'$, choose $z_{\bar{B}} \in \bar{B} \cap L(\hat{G})$. Recall our assumption that either $G$ does not contain parabolic elements or $G$ contains parabolic elements and $\delta \geq k_{\max}$. This assumption and Theorem 6.22 imply that $\hat{\mu}(\bar{B}^n(y, u)) \leq c_1 u^\delta$ for every $y \in L(\hat{G})$ and $u \in ]0, t_0[$. It is now the case that

$$0 < \hat{\mu}(L(\hat{G})) \leq \sum_{\bar{B}^{n+1}(x,t) \in \mathcal{T}'} \hat{\mu}(\bar{B}^n(z_{\bar{B}^{n+1}(x,t)}, 2t)) \leq 2^\delta c_1 \sum_{\bar{B}^{n+1}(x,t) \in \mathcal{T}} t^\delta.$$

We take the infimum over $\mathcal{T}$ and then the limit $\varepsilon \to 0$ and conclude that

$$m_\delta^{\hat{G}}(L(\hat{G})) = m_\delta(L(\hat{G})) \geq \frac{\hat{\mu}(L(\hat{G}))}{2^\delta c_1} > 0.$$

We prove next that $m_\delta^{\hat{G}}(L(\hat{G})) < \infty$. Let $\varepsilon \in ]0, t_0[$. Let $x \in L_c(\hat{G})$. Since the horoballs in the given (possibly empty) complete collection of $\hat{G}$ have pairwise disjoint closures, it follows that there is $t_x \in ]0, \varepsilon]$ such that the point $(x, t_x)$ is not contained in any of the horoballs in the collection. Theorem 6.22 implies that $\hat{\mu}(\bar{B}^{n+1}(x, t_x)) = \hat{\mu}(\bar{B}^n(x, t_x)) \geq c_1^{-1} t_x^\delta$ for every $x \in L_c(\hat{G})$. According to Theorem 6.37, the collection $\{\bar{B}^{n+1}(x, t_x) : x \in L_c(\hat{G})\}$ has a countable subcollection $\mathcal{T}_\varepsilon$ which is an $\varepsilon$-covering of $L_c(\hat{G})$ with a multiplicity bounded by a constant $c_2 > 0$. We conclude that

$$\sum_{\bar{B}^{n+1}(x,t) \in \mathcal{T}_\varepsilon} t^\delta \leq c_1 \sum_{\bar{B} \in \mathcal{T}_\varepsilon} \hat{\mu}(\bar{B}) \leq c_1 c_2 \hat{\mu}\left( \bigcup_{\bar{B} \in \mathcal{T}_\varepsilon} \bar{B} \right) \leq c_1 c_2 \hat{\mu}(K) < \infty,$$

where $K \subset \mathbb{R}^{n+1}$ is a fixed large enough compact set. It follows that $m_\delta^{\hat{G}}(L_c(\hat{G})) = m_\delta(L_c(\hat{G})) \leq c_1 c_2 \hat{\mu}(K) < \infty$. Since $L(\hat{G}) \setminus L_c(\hat{G})$ is the possibly empty set of parabolic fixed points of $\hat{G}$ and $m_\delta^{\hat{G}}$ is atomless, we obtain that $m_\delta^{\hat{G}}(L(\hat{G})) < \infty$. We have shown that $m_\delta^{\hat{G}}$ is non-trivial and finite. Our earlier discussion implies that $\mu$ and $m_\delta^G$ are equivalent.

Suppose next that $\mathbb{X}^{n+1} = \mathbb{H}^{n+1}$. By definition, there is a Möbius transformation $f \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ and a Patterson-Sullivan measure $\nu$ of $f^{-1}Gf$ such that $\mu = f_*^\delta \nu$. The above reasoning implies that there is a constant $c > 0$ such that $\nu = c m_\delta^{f^{-1}Gf}$. Theorem 6.9 implies that $\mu = c f_*^\delta m_\delta^{f^{-1}Gf}$. Our claim follows since $f_*^\delta m_\delta^{f^{-1}Gf} = m_\delta^G$.

The proofs of the claims concerning $p^G_\delta$ and our arguments regarding the modified packing construction are related in a similar way as the above proofs of the claims concerning $m^G_\delta$ and our arguments regarding the modified covering construction. These proofs tell nothing essentially new about the relation between this work and the results in [Sullivan1984], and so we omit them. □

The following complement of Theorem 7.5 is also contained in [Sullivan1984]. Suppose that the situation is as in Theorem 7.5 and that $G$ contains parabolic elements. Then it is true that if $\delta < k_{\max}$, then $\mu$ is not equivalent to $m^G_\delta$, and if $\delta > k_{\min}$, then $\mu$ is not equivalent to $p^G_\delta$. The proof of this claim is not very difficult, but we omit it nonetheless for the sake of brevity.

We conclude that if $\mu$ is a Patterson-Sullivan measure of a non-elementary geometrically finite Kleinian group $G$, one can sometimes use the standard covering construction or the standard packing construction to construct a measure $\nu$ such that $\mu = c\nu$, where $c > 0$ is a constant; sometimes one can use either of the two constructions and sometimes neither of the constructions can be used to construct a suitable measure $\nu$.

It seems, in fact, that Sullivan's other results strengthen the negative part of the above conclusion. More specifically, Sullivan claims on page 261 of [Sullivan1984] that a deeper analysis than the one given in [Sullivan1984] shows that if the standard covering construction or Sullivan's version of the packing construction cannot be used to construct a measure $\nu$ such as above, then this conclusion continues to hold even if the gauge function $t \mapsto t^\delta$ is replaced by an arbitrary gauge function (Sullivan does not give the explicit definition of such functions). Without giving further details, Sullivan refers to his paper [Sullivan1983]. Sullivan's claim is indeed considered in Section 10 of [Sullivan1983] although only the covering construction is treated explicitly. Moreover, the given argument is not as detailed as we would hope, and the exact scope of Sullivan's negative equivalence results remains unclear to us. Indeed, it seems that the issue remains open, since the paper [Stratmann2006] contains the conjecture that if $\mu$ is a Patterson-Sullivan measure of a non-elementary geometrically finite Kleinian group $G$ acting on $\mathbb{B}^3$ and containing only orientation preserving elements, if $G$ has parabolic fixed points of rank 1 and 2, and if the exponent of convergence $\delta$ of $G$ is strictly between 1 and 2, then $\mu$ is equivalent to a measure obtained from the standard covering construction using the gauge function

$$t \mapsto t^\delta \exp\left(\frac{2-\delta}{2(\delta-1)}\left(\log\log\frac{1}{t} + \log\log\log\log\frac{1}{t}\right)\right). \tag{7.6}$$

We have not been able to verify or falsify this conjecture.

It seems in any case that Sullivan's negative results suggest that the solution proposed in this work to the characterization problem of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups may well be natural in the following sense. Our solution is sufficient since it provides an answer in every possible case, and our solution may also be necessary since perhaps no such complete solution can be reached using standard constructions. As the final remark in our discussion on Sullivan's results, we note that the papers [SV1995] and [Tukia1994c] also consider negative equivalence results such as above for standard covering constructions employing gauge functions of the form $t \mapsto t^\delta(\log t^{-1})^\alpha$, where $\alpha > 0$ is a suitable constant.

7.2. **Simple variants.** We will discuss next some simple variants of the measure constructions introduced in Chapter 5. These variants satisfy an equivalence result similar to Theorem 6.41.

We begin this discussion in the same general setting as in Chapter 5. That is, we fix a non-empty base set $X \subset \mathbb{R}^n$ and define the flatness functions $\tau_l$, $l \in \{1, 2, \ldots, n\}$, and the diameter function $d$ connected to $X$ as in (5.1) and (5.2). Moreover, we introduce the numbers $\eta > 0$ and $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{R}$ and define the functions $\alpha$ and $\omega$ used in the gauge functions as in (5.3) and (5.4).

We consider here one variant of each of the original constructions. (It is naturally possible that there are other interesting variants with similar equivalence properties.) The main changes are the following. Let $\varepsilon > 0$ and $\nu \in ]0,1[$. Let $\mathcal{T}$ be an $(\varepsilon,\nu)$-covering or an $(\varepsilon,\nu)$-packing of a set $A \subset X$ as in the definitions given in Chapter 5. This means that if $\bar{B}^n(x,t) \in \mathcal{T}$, then $|x - x'|/t \le \nu$, where $x'$ is some point contained in $X$ in case $\mathcal{T}$ is a covering and in $A$ in case $\mathcal{T}$ is a packing. We change the definitions of coverings and packings for the variants discussed here as follows. We set that if $\bar{B}^n(x,t) \in \mathcal{T}$, then $|x - x'|/t \le \nu$, where $x'$ is some point contained in $A$ in case $\mathcal{T}$ is a covering and in $X$ in case $\mathcal{T}$ is a packing; additionally, we require that $B^n(x,t) \cap A \neq \emptyset$ in case $\mathcal{T}$ is a packing.

We will denote the new covering (outer) measure by $m'$ and the new packing (outer) measure by $p'$. The definition of $p'$ is the same as the definition of our original packing outer measure on page 73 except for the change in the definition of packings. We will not, therefore, write down the details of the definition of $p'$. On the other hand, the change in the definition of coverings implies further changes in the definition of the covering outer measure. Let us give the exact definition of $m'$.

Let $A \subset X$. Before we can define $m'(A)$, we need to define a preliminary quantity $(m')^*(A)$. If $A = \emptyset$, we set that $(m')^*(A) = 0$. Suppose that $A \neq \emptyset$ for the moment. The definition of $(m')^*(A)$ is analogous to the definition of $m(A)$ on page 73, i.e. we define first a preliminary quantity

$$(m')^\nu_\varepsilon(A) = \inf_{\mathcal{T}} \sum_{\bar{B}^n(x,t)\in\mathcal{T}} \alpha(x,t)\omega(x,t) \tag{7.7}$$

for every $\varepsilon > 0$ and $\nu \in ]0,1[$, where $\mathcal{T}$ varies in the collection of all $(\varepsilon,\nu)$-coverings of $A$ satisfying the altered definition of a covering. Like in the case of $m$, we can then set that

$$(m')^*(A) = \sup_{\varepsilon>0,\nu\in]0,1[} (m')^\nu_\varepsilon(A). \tag{7.8}$$

The set function $(m')^*$ is not necessarily monotonic. We guarantee that $m'$ is monotonic by adding the following step. We define that the $m'$-*measure* of any set $A \subset X$ is

$$m'(A) = \sup\{(m')^*(E) : E \subset A\}. \tag{7.9}$$

Let us verify that the set functions $m'$ and $p'$ are outer measures of $X$ such that every Borel set of $X$ is measurable with respect to $m'$ and $p'$.

**Theorem 7.10.** *The set functions $m'$ and $p'$ defined above are outer measures of $X$. Every Borel set of $X$ is measurable with respect to $m'$ and $p'$.*

*Proof.* The proofs of the claims concerning $p'$ are essentially the same as the proofs of the corresponding claims concerning our original packing outer measure, see the proof of Theorem 5.11. There is no reason to repeat the necessary arguments. Additionally, the proofs of the claims concerning $m'$ are simplified by the fact that, since $(m')^*$ and $m$ as defined on page 73 have analogous definitions, these two set functions have a number of properties in common.

Let us show that $m'$ is an outer measure of $X$. The definition of $m'$ implies immediately that $m'(\emptyset) = 0$. To prove that $m'$ is monotonic, let $A_1 \subset A_2 \subset X$. Let $E \subset A_1$. Then $E \subset A_2$, and so $(m')^*(E) \leq m'(A_2)$. Since $E$ is an arbitrary subset of $A_1$, it follows that $m'(A_1) \leq m'(A_2)$, i.e. that $m'$ is monotonic. To finish the proof that $m'$ is an outer measure of $X$, we need to show that $m'$ is countably subadditive. Observe that we can argue as in the case of $m$ in the proof of Theorem 5.11 to establish that $(m')^*$ is countably subadditive. Let $A_1, A_2, \ldots \subset X$. Suppose that $E \subset \bigcup_{i=1}^{\infty} A_i$. It is the case that $E \cap A_i \subset A_i$ for every $i \in \{1, 2, \ldots\}$, so $(m')^*(E \cap A_i) \leq m'(A_i)$. Thus,

$$\sum_{i=1}^{\infty} m'(A_i) \geq \sum_{i=1}^{\infty} (m')^*(E \cap A_i) \geq (m')^* \left( \bigcup_{i=1}^{\infty} (E \cap A_i) \right) = (m')^*(E).$$

Since $E$ was an arbitrary subset of $\bigcup_{i=1}^{\infty} A_i$, it follows that $\sum_{i=1}^{\infty} m'(A_i) \geq m'\left(\bigcup_{i=1}^{\infty} A_i\right)$. We conclude that $m'$ is countably subadditive and hence an outer measure of $X$.

We prove next that every Borel set of $X$ is $m'$-measurable. We use the same method as in the proof of Theorem 5.11. That is, we assume that $A_1, A_2 \subset X$ are non-empty sets such that $d_{\mathrm{euc}}(A_1, A_2) > 0$ and show that $m'(A_1 \cup A_2) = m'(A_1) + m'(A_2)$.

Let $A_1$ and $A_2$ be as above. We can use essentially the same argument as we did in the proof of Theorem 5.11 in the case of $m$ to show that $(m')^*(A_1 \cup A_2) \geq (m')^*(A_1) + (m')^*(A_2)$. Let $E_j \subset A_j$ for $j = 1, 2$. Suppose that $E_1 \neq \emptyset \neq E_2$. Then $d_{\mathrm{euc}}(E_1, E_2) > 0$, and thus

$$(m')^*(E_1) + (m')^*(E_2) \leq (m')^*(E_1 \cup E_2) \leq m'(A_1 \cup A_2)$$

since $E_1 \cup E_2 \subset A_1 \cup A_2$. This estimate is valid also if one or both of the sets $E_1$ and $E_2$ are empty. It follows easily that $m'(A_1) + m'(A_2) \leq m'(A_1 \cup A_2)$. Since the converse inequality follows from the countable subadditivity of $m'$, we obtain that every Borel set of $X$ is $m'$-measurable. □

Our next task is to show that $m'$ and $p'$ satisfy similar transformation rules as the measures constructed by our original constructions, see Theorem 5.12. We use here the same notational conventions as we did in the context of Theorem 5.12. Note that the present definition of a covering does not refer to the base set used in the covering construction and hence we can talk about $(\varepsilon, \nu)$-coverings instead of $(\varepsilon, \nu)^X$-coverings.

**Theorem 7.11.** *Let $X \subset \mathbb{R}^n$ be non-empty. Let $g \in$ Möb$(n)$ be such that $gX \cap \mathbb{R}^n \neq \emptyset$. Then*

$$(m')^{gX\cap\mathbb{R}^n}(gA) = \int_A |g'|^\eta d(m')^X \qquad \text{and} \qquad (p')^{gX\cap\mathbb{R}^n}(gA) = \int_A |g'|^\eta d(p')^X \tag{7.12}$$

*for every Borel set $A \subset X \setminus \{g^{-1}(\infty)\}$ of $X$.*

*Proof.* The argument needed to establish the claim for $p'$ is essentially the same as the one we used for $p$ in the proof of Theorem 5.12. Therefore, we omit the details of the proof of the latter formula of (7.12). Due to the similarities in the definitions of the set functions $(m')^*$ and $m$, we will be able to adapt parts of the proof of Theorem 5.12 regarding $m$ to the present situation.

Let us prove the first formula of (7.12). Let $A \subset X \setminus \{g^{-1}(\infty)\}$ be a non-empty Borel set of $X$. Arguing as in the first paragraph of the proof of Theorem 5.12, we see that we can make the following additional assumption. There are constants $M > 0$ and $a_0 > 0$ such that $M^{-1} \leq |g'| \leq M$ in $A(a_0) = \{x \in \mathbb{R}^n : d_{\mathrm{euc}}(x, A) < a_0\}$.

We introduce the same conventions as in the second paragraph of the proof of Theorem 5.12. That is, given a small $\lambda > 0$, we divide $A$ into pairwise disjoint non-empty Borel sets $A_1, A_2, \ldots, A_{k_\lambda}$ of $X$ as follows. Let $M_k$ and $m_k$ denote the supremum and infimum of $|g'|$ over $A_k(a_1) \subset A(a_0)$ for $k \in \{1, 2, \ldots, k_\lambda\}$, where $a_1 > 0$ is a number depending on $\lambda$ and $A_k(a_1) = \{x \in \mathbb{R}^n : d_{\mathrm{euc}}(x, A_k) < a_1\}$. We require that the division $A_1, A_2, \ldots, A_{k_\lambda}$ corresponding to $\lambda$ be such that $M_k/m_k \leq \sigma$ for every $k \in \{1, 2, \ldots, k_\lambda\}$, where $\sigma = \sigma(\lambda) \geq 1$ and $\sigma \to 1$ as $\lambda \to 0$.

We consider a fixed $A_k$, $k \in \{1, 2, \ldots, k_\lambda\}$. Fix a non-empty subset $E$ of $gA_k$. Let $\varepsilon > 0$ and $v \in ]0, 1[$. Suppose that $\mathcal{T}$ is an $(\varepsilon, v)$-covering of $g^{-1}E$. Assume that $\varepsilon \leq \varepsilon_0$ for some constant $\varepsilon_0 > 0$ so that $\bar{B} \subset A_k(a_1)$ for every $\bar{B} \in \mathcal{T}$. Given $\bar{B}^n(x, t) \in \mathcal{T}$, write $g\bar{B}^n(x, t) = \bar{B}^n(y, u)$. We can reason as in the case of $m$ in the proof of Theorem 5.12 to establish the following facts. The collection $g\mathcal{T} = \{g\bar{B} : \bar{B} \in \mathcal{T}\}$ is an $(M\varepsilon, 1 - (1 - v)/\theta)$-covering of $E$, where $\theta = \theta(\varepsilon) \geq 1$ and $\theta \to 1$ as $\varepsilon \to 0$. Furthermore,

$$\alpha^{gX \cap \mathbb{R}^n}(y, u) \in [m_k^\eta \alpha^X(x, t), M_k^\eta \alpha^X(x, t)]$$

and

$$\omega^{gX \cap \mathbb{R}^n}(y, u) \in [\chi^{-1} \omega^X(x, t), \chi \omega^X(x, t)]$$

for all $\bar{B}^n(x, t) \in \mathcal{T}$, where $\chi = \chi(\lambda) \geq 1$ and $\chi \to 1$ as $\lambda \to 0$.

We now obtain that

$$((m')^{gX \cap \mathbb{R}^n})_{M\varepsilon}^{1 - \frac{1-v}{\theta}}(E) \leq \sum_{\bar{B}^n(y,u) \in g\mathcal{T}} \alpha^{gX \cap \mathbb{R}^n}(y, u) \omega^{gX \cap \mathbb{R}^n}(y, u) \leq \chi M_k^\eta \sum_{\bar{B}^n(x,t) \in \mathcal{T}} \alpha^X(x, t) \omega^X(x, t).$$

We take the infimum over the $(\varepsilon, v)$-coverings $\mathcal{T}$ of $g^{-1}E$ and conclude that

$$((m')^{gX \cap \mathbb{R}^n})_{M\varepsilon}^{1 - \frac{1-v}{\theta}}(E) \leq \chi M_k^\eta ((m')^X)_\varepsilon^v (g^{-1}E) \leq \chi M_k^\eta ((m')^X)^* (g^{-1}E).$$

Observe that

$$\sup_{\beta > 0, w \in ]0,1[} ((m')^{gX \cap \mathbb{R}^n})_\beta^w (E) = \sup_{\varepsilon \in ]0, \varepsilon_0], v \in ]0,1[} ((m')^{gX \cap \mathbb{R}^n})_{M\varepsilon}^{1 - \frac{1-v}{\theta}}(E). \tag{7.13}$$

The quantity on the left hand side of (7.13) is $((m')^{gX \cap \mathbb{R}^n})^*(E)$ by definition (7.8). We see that

$$((m')^{gX \cap \mathbb{R}^n})^*(E) \leq \chi M_k^\eta ((m')^X)^* (g^{-1}E) \leq \chi M_k^\eta (m')^X (A_k).$$

Since $E$ was an arbitrary non-empty subset of $gA_k$, we can take the supremum over $E$ and deduce that

$$(m')^{gX\cap\mathbb{R}^n}(gA_k) \le \chi\left(\frac{M_k}{m_k}\right)^{\eta} m_k^{\eta}(m')^X(A_k) \le \chi\sigma^{\eta}\int_{A_k}|g'|^{\eta}d(m')^X. \tag{7.14}$$

The estimate (7.14) is of the same form as the estimate (5.15) in the proof of Theorem 5.12. It is clear that we can finish the proof of the first formula of (7.12) using a similar argument as in the proof Theorem 5.12. □

We move next into the context of geometrically finite Kleinian groups. Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$. Let $\delta > 0$ be the exponent of convergence of $G$. We apply the measure constructions considered presently to the case where $X = L(G) \cap \mathbb{R}^n$, $\eta = \delta$, and $\eta_l = \delta - l$ for $l \in \{1, 2, \dots, n\}$. Denote by $\bar{m}'_G$ the measure obtained from the covering construction and by $\bar{p}'_G$ the measure obtained from the packing construction. We define new measures $m'_G$ and $p'_G$ by setting that

$$m'_G(A) = \bar{m}'_G(A \cap (L(G) \cap \mathbb{R}^n)) \quad\text{and}\quad p'_G(A) = \bar{p}'_G(A \cap (L(G) \cap \mathbb{R}^n)) \tag{7.15}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$. Definition (7.15) corresponds to (6.33) in the discussion of Chapter 6. Let us show that, like the measures defined by (6.33), the measures $m'_G$ and $p'_G$ are atomless $\delta$-conformal measures of $G$ giving full measure to the set $L_c(G)$ of conical limit points of $G$.

**Theorem 7.16.** *Let $G$, $\delta$, $m'_G$ and $p'_G$ be as above. Let $f \in$ Möb$(n+1)$ map $\mathbb{H}^{n+1}$ onto $\mathbb{B}^{n+1}$. Then $f^{\delta}_* m'_G$ and $f^{\delta}_* p'_G$ are atomless $\delta$-conformal measures of $fGf^{-1} = G_f$ such that $f^{\delta}_* m'_G(L(G_f) \setminus L_c(G_f)) = 0 = f^{\delta}_* p'_G(L(G_f) \setminus L_c(G_f))$. Moreover, $m'_G = (f^{-1})^{\delta}_*(f^{\delta}_* m'_G)$ and $p'_G = (f^{-1})^{\delta}_*(f^{\delta}_* p'_G)$, so $m'_G$ and $p'_G$ are atomless $\delta$-conformal measures of $G$ such that $m'_G(L(G) \setminus L_c(G)) = 0 = p'_G(L(G) \setminus L_c(G))$.*

*Proof.* This theorem corresponds to Theorem 6.40 in the discussion of Chapter 6. It follows that in order to prove the theorem, we need to establish the facts corresponding to Theorems 6.34 and 6.38 with respect to $m'_G$ and $p'_G$. That is, we need to prove that $m'_G(x) = 0 = p'_G(x)$ for every $x \in \bar{\mathbb{R}}^{n+1}$, that

$$m'_G(gA) = \int_A |g'|^{\delta} dm'_G \qquad\text{and}\qquad p'_G(gA) = \int_A |g'|^{\delta} dp'_G \tag{7.17}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$ and every $g \in G$, and that if $B_0$ is an open $n$-ball of $\mathbb{R}^n$ such that $D = B_0 \cap L(G) \neq \emptyset$, then $m'_G(D), p'_G(D) \in ]0, \infty[$.

We can use essentially the same arguments as in the proof of Theorem 6.34 to prove that $m'_G$ and $p'_G$ are atomless and that they satisfy the formulae in (7.17). The required modifications are very easy and we omit the details in order to avoid repetition. Moreover, we can use essentially the same argument as in the proof of Theorem 6.38 to show that $p'_G(D) \in ]0, \infty[$, where $D = B_0 \cap L(G)$ for some fixed open $n$-ball $B_0$ of $\mathbb{R}^n$ such that $D \neq \emptyset$. Again, we omit the details.

We are left to show that $m'_G(D) \in ]0, \infty[$. Observe that we can use the same argument as in the beginning of the proof of Theorem 6.38 to show that $(\bar{m}'_G)^*(D) > 0$. Since $m'_G(D) = \bar{m}'_G(D) \ge (\bar{m}'_G)^*(D)$, we obtain that $m'_G(D) > 0$. Let $E$ be an arbitrary non-empty subset of $D$. It is evident that we can use the argument given in the second part of the proof of Theorem 6.38 concerning the covering measure to show that $(\bar{m}'_G)^*(E)$ is smaller than some finite constant. It follows that $m'_G(D) < \infty$. □

We obtain now a simple alternative version of Theorem 6.41.

**Theorem 7.18.** *Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be a Patterson-Sullivan measure of $G$. Then either one of the measure constructions introduced in this section can be used to construct a measure $\nu$ such that $\mu = c\nu$, where $c > 0$ is a constant.*

*Proof.* The proof is exactly analogous to the proof of Theorem 6.41. □

7.3. **Variants employing alternative flatness functions.** We turn to the third topic of this chapter. We consider variants of the measure constructions introduced in Chapter 5 which use simpler flatness functions than the original constructions and which satisfy the following result which is similar to Theorem 6.41 but weaker: If $\mu$ is a Patterson-Sullivan measure of a non-elementary geometrically finite Kleinian group $G$ acting on $\mathbb{X}^{n+1}$, then we can use any of the variants discussed presently to construct a measure $\nu$ such that $c^{-1}\nu \leq \mu \leq c\nu$, where $c > 0$ is a constant.

We define the variants of the measure constructions as follows. Let again $X \subset \mathbb{R}^n$ be a non-empty base set. Given $l \in \{1, 2, \ldots, n\}$, we define the *alternative $l$-flatness function* $\tau_l$ with respect to $X$ by setting that

$$\tau_l(x,t) = \frac{1}{t} \inf_{T \in \mathcal{P}_l(x)} \rho(\bar{B}^n(x,t) \cap X, \bar{B}^n(x,t) \cap T) \tag{7.19}$$

for every $x \in X$ and $t > 0$, where $\mathcal{P}_l(x)$ is the collection of $l$-planes of $\bar{\mathbb{R}}^n$ through $x$ and $\rho$ is as in (5.1). We define the diameter function $d$ associated with $X$ as in (5.2), although we consider only $d(x,t)$ such that $x \in X$ in the present context. Similarly, we define the functions $\alpha$ and $\omega$ as in (5.3) and (5.4) but using the flatness functions defined by (7.19) and the restricted diameter function $d$. Observe that the alternative flatness functions satisfy a result corresponding to Lemma 5.5, and the alternative gauge functions are, therefore, well-defined.

It follows that we can modify the covering and packing constructions of Chapter 5 in the following way. The definitions of coverings and packings are modified so that the parameter $\nu$ is eliminated: we assume that if $\mathcal{T}$ is a covering or a packing of some $A \subset X$, then $x \in X$ in case $\mathcal{T}$ is a covering and $x \in A$ in case $\mathcal{T}$ is a packing for every $\bar{B}^n(x,t) \in \mathcal{T}$; the definitions of coverings and packings remain unchanged otherwise. The only additional change needed in the constructions is that the original gauge functions are replaced by the alternative versions defined above. We do not write down the explicit definitions of the variants since these definitions should be obvious. (Note that Theorem 6.37 implies easily that every $A \subset X$ has $\varepsilon$-coverings for every $\varepsilon > 0$.)

It is easy to verify that the set functions constructed by these variants are outer measures of $X$ whose $\sigma$-algebras of measurable sets contain all Borel sets of $X$, see the proof of Theorem 5.11. But note that it seems that the outer measures constructed by the variants do not satisfy transformation rules as described in Theorem 5.12, or that if they do, the straightforward arguments given in the proof of Theorem 5.12 cannot be used to establish this. This point was discussed in greater detail following the proof of Theorem 5.12.

We move on to consider the present variants of the measure constructions in the context of geometrically finite Kleinian groups. Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$. Let $P$ be a possibly empty set of bounded parabolic fixed points of $G$ and let $\{H_p : p \in GP\}$ be a complete collection of horoballs for $G$ as in the definition of geometric finiteness on page 89. Let $\gamma_l$, $l \in \{1, 2, \ldots, n\}$, be the standard flatness functions of $G$ as defined by (6.15). Denote by $\Gamma_l$, $l \in \{1, 2, \ldots, n\}$, the alternative flatness functions of $G$ obtained from (7.19) when we set that $X = L(G) \cap \mathbb{R}^n$. Note that the results on the more general alternative flatness functions proved in Chapter 4 are applicable to the functions $\Gamma_l$, see the discussion starting on page 65. We define the normalized diameter function $\beta$ associated to $G$ as in (6.16). Furthermore, given $l \in \{1, 2, \ldots, n\}$ such that there are parabolic fixed points of $G$ of rank $l$, we set that

$$\lambda_l(x, t) = \exp(-\max\{d_{H_p}((x, t), \partial H_p) : p \in GP, \text{ the rank of } p \text{ is } l\}) \tag{7.20}$$

for every $x \in \mathbb{R}^n$ and $t > 0$, where $d_{H_p}((x, t), \partial H_p) = d((x, t), \partial H_p)$ if $(x, t) \in H_p$ and $d_{H_p}((x, t), \partial H_p) = 0$ otherwise; if $G$ does not have parabolic fixed points of some rank $l \in \{1, 2, \ldots, n\}$, we set that $\lambda_l(x, t) = 1$ for every $x \in \mathbb{R}^n$ and $t > 0$. Finally, fix two constants $t_0 > 0$ and $v_0 \in ]0, 1[$. Let us prove that the following two auxiliary results are true in the present setting.

**Theorem 7.21.** *Let $G$, $\gamma_l$ and $\Gamma_l$, $l \in \{1, 2, \ldots, n\}$, and $t_0$ and $v_0$ be as above. Then there are constants $a > 0$, $b > 0$ and $c > 0$ which satisfy the following. Let $x \in \mathbb{R}^n$ and $t \in ]0, t_0[$ be such that $|x - x'|/t \leq v_0$ for some $x' \in L(G) \cap \mathbb{R}^n$. Then it is the case that*

$$c^{-1} \leq \frac{\Gamma_l(x', 2t)}{\gamma_l(x, t)} \leq c \qquad \text{and} \qquad c^{-1} \leq \frac{\Gamma_l(x', (1 - v_0)t)}{\gamma_l(x, t)} \leq c \tag{7.22}$$

*for every $l \in \{1, 2, \ldots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. Moreover,*

$$b^{-1} \leq \frac{d_{\text{euc}}(\bar{B}^n(x', 2t) \cap L(G))}{d_{\text{euc}}(\bar{B}^n(x, t) \cap L(G))} \leq b \tag{7.23}$$

*and*

$$b^{-1} \leq \frac{d_{\text{euc}}(\bar{B}^n(x', (1 - v_0)t) \cap L(G))}{d_{\text{euc}}(\bar{B}^n(x, t) \cap L(G))} \leq b. \tag{7.24}$$

*Finally, if $x' = x$, then*

$$a^{-1} \leq \frac{\Gamma_l(x, t)}{\gamma_l(x, t)} \leq a \tag{7.25}$$

*for every $l \in \{1, 2, \ldots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$.*

*Proof.* It is sufficient to prove (7.22) and (7.25) for a fixed $l \in \{1, 2, \ldots, n\}$ such that $L(G)$ is not an $l$-sphere of $\bar{\mathbb{R}}^n$. Let $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ be as in the claim. It is clear that there is a constant $c_0 > 0$ such that

$$d((x, t), (x', 2t)) \leq c_0 \qquad \text{and} \qquad d((x, t), (x', (1 - v_0)t)) \leq c_0. \tag{7.26}$$

We can use (7.20) and (7.26) to deduce that

$$c_1^{-1} \le \frac{\lambda_l(x', 2t)}{\lambda_l(x,t)} \le c_1 \qquad \text{and} \qquad c_1^{-1} \le \frac{\lambda_l(x', (1-\nu_0)t)}{\lambda_l(x,t)} \le c_1, \tag{7.27}$$

where $c_1 > 0$ is a constant. Observe next that Theorem 6.17 implies that there is a constant $c_2 > 0$ such that

$$c_2^{-1} \le \frac{\gamma_l(x,t)}{\lambda_l(x,t)} \le c_2. \tag{7.28}$$

Since the flatness function $\Gamma_l$ satisfies estimation results of the same form as $\gamma_l$, compare Theorems 4.5 and 4.44 and Theorems 4.28 and 4.46, we deduce that if $c_2$ is adjusted accordingly, we have that

$$c_2^{-1} \le \frac{\Gamma_l(x', 2t)}{\lambda_l(x', 2t)} \le c_2 \qquad \text{and} \qquad c_2^{-1} \le \frac{\Gamma_l(x', (1-\nu_0)t)}{\lambda_l(x', (1-\nu_0)t)} \le c_2. \tag{7.29}$$

The formulae in (7.22) follow immediately from (7.27), (7.28) and (7.29). The formula (7.25) is proved using a similar argument.

We prove next (7.23) and (7.24). We continue to consider $x \in \mathbb{R}^n$, $t \in ]0, t_0[$ and $x' \in L(G) \cap \mathbb{R}^n$ as in the claim. Recall the definition of $\beta$ from (6.16) and note that Theorem 6.17 implies that there is a constant $c_3 > 0$ such that

$$\beta(x,t), \beta(x', 2t), \beta(x', (1-\nu_0)t) \in [c_3^{-1}, c_3].$$

It is evident that (7.23) and (7.24) are valid. The proof is complete. □

**Theorem 7.30.** *Let $G$ be as in Theorem 7.21. Let $m$ and $p$ be the covering outer measure and the packing outer measure obtained when the measure constructions introduced in Chapter 5 are applied in the situation where the base set is $X = L(G) \cap \mathbb{R}^n$ and the parameters are $\eta = \delta$ and $\eta_l = \delta - l$ for $l \in \{1, 2, \ldots, n\}$, where $\delta$ is the exponent of convergence of $G$. Let $m''$ and $p''$ be the covering outer measure and the packing outer measure obtained when the above variants of the measure constructions of Chapter 5 are applied in the same situation. Then there are constants $c_m > 0$ and $c_p > 0$ such that $c_m^{-1} m \le m'' \le c_m m$ and $c_p^{-1} p \le p'' \le c_p p$.*

*Proof.* Recall that if $L(G)$ is an $l$-sphere of $\bar{\mathbb{R}}^n$ for some $l \in \{1, 2, \ldots, n\}$, then $\delta = l$ (see Theorem 6.14) and we employ the convention

$$\gamma_l^{\delta - l} = 1 = \Gamma_l^{\delta - l}. \tag{7.31}$$

Let us prove the existence of $c_m$. Let the constants $t_0 > 0$ and $\nu_0 \in ]0, 1[$ be as in Theorem 7.21. Let $A \subset L(G) \cap \mathbb{R}^n$. Let $\varepsilon \in ]0, t_0[$. Let $\mathcal{T}$ be an $\varepsilon$-covering of $A$ as in the construction of $m''$. Then $\mathcal{T}$ is an $(\varepsilon, \nu)$-covering of $A$ as in the construction of $m$ for any fixed $\nu \in ]0, \nu_0[$. We use (7.25) and (7.31) to deduce that there is a constant $c_0 > 0$ such that

$$\begin{aligned} m_\varepsilon^\nu(A) &\le \sum_{\bar{B}^n(x,t) \in \mathcal{T}} d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^\delta \prod_{l=1}^n \gamma_l(x,t)^{\delta - l} \\ &\le c_0 \sum_{\bar{B}^n(x,t) \in \mathcal{T}} d_{\mathrm{euc}}(\bar{B}^n(x,t) \cap L(G))^\delta \prod_{l=1}^n \Gamma_l(x,t)^{\delta - l}. \end{aligned}$$

We can take the infimum over $\mathcal{T}$ and conclude that $m^{\nu}_{\varepsilon}(A) \leq c_0(m'')_{\varepsilon}(A) \leq c_0 m''(A)$, and so $m(A) \leq c_0 m''(A)$ (naturally, $(m'')_{\varepsilon}(A)$ is the quantity corresponding to (5.6) in the construction of $m''$).

Let $\varepsilon \in ]0, t_0[$ and $\nu \in ]0, \nu_0[$. Let $\mathcal{T}$ be an $(\varepsilon, \nu)$-covering of $A$ as in the construction of $m$. Given $\bar{B}^n(x, t) \in \mathcal{T}$, let $x' \in L(G) \cap \mathbb{R}^n$ be such that $|x - x'|/t \leq \nu$. Now it is true that $\{\bar{B}^n(x', 2t) : \bar{B}^n(x, t) \in \mathcal{T}\}$ is a $2\varepsilon$-covering of $A$ as in the construction of $m''$. Using the first formula of (7.22) and the formulae (7.23) and (7.31), we see that there is a constant $c_1 > 0$ such that

$$\begin{aligned} (m'')_{2\varepsilon}(A) &\leq \sum_{\bar{B}^n(x,t)\in\mathcal{T}} d_{\mathrm{euc}}(\bar{B}^n(x', 2t) \cap L(G))^{\delta} \prod_{l=1}^{n} \Gamma_l(x', 2t)^{\delta-l} \\ &\leq c_1 \sum_{\bar{B}^n(x,t)\in\mathcal{T}} d_{\mathrm{euc}}(\bar{B}^n(x, t) \cap L(G))^{\delta} \prod_{l=1}^{n} \gamma_l(x, t)^{\delta-l}. \end{aligned}$$

Take the infimum over $\mathcal{T}$ and deduce that $(m'')_{2\varepsilon}(A) \leq c_1 m^{\nu}_{\varepsilon}(A) \leq c_1 m(A)$. It follows that $m''(A) \leq c_1 m(A)$. We have proved the existence of the constant $c_m$.

It is evident that we can prove the existence of the constant $c_p$ by using a similar argument. The two essential observations in the argument are the following. If $\mathcal{T}$ is an $\varepsilon$-packing of a fixed non-empty $A \subset L(G) \cap \mathbb{R}^n$ as in the construction of $p''$ for some $\varepsilon \in ]0, t_0[$, then $\mathcal{T}$ is an $(\varepsilon, \nu)$-packing of $A$ as in the construction of $p$ for every $\nu \in ]0, \nu_0[$. And if $\mathcal{T}$ is an $(\varepsilon, \nu)$-packing of the set $A$ as in the construction of $p$ for some $\varepsilon \in ]0, t_0[$ and $\nu \in ]0, \nu_0[$ such that an arbitrary $\bar{B}^n(x, t) \in \mathcal{T}$ contains $x' \in A$ with $|x - x'|/t \leq \nu$, then $\{\bar{B}^n(x', (1 - \nu_0)t) : \bar{B}^n(x, t) \in \mathcal{T}\}$ is a $(1 - \nu_0)\varepsilon$-packing of $A$ as in the construction of $p''$. It is clear how to complete the argument using estimates employing the second of the estimates in (7.22) and the formulae (7.24), (7.25) and (7.31). We omit the details in order to avoid repetition and conclude that the proof is finished. □

We obtain now easily our main result concerning the present variants of the measure constructions introduced in Chapter 5.

**Theorem 7.32.** *Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{X}^{n+1}$. Let $\mu$ be a Patterson-Sullivan measure of $G$. Then we can use either of the variants of the measure constructions of Chapter 5 discussed above to construct a measure $\nu$ such that $c^{-1}\nu \leq \mu \leq c\nu$, where $c > 0$ is a constant.*

*Proof.* According to Theorem 6.41, we can use either of the measure constructions introduced in Chapter 5 to construct a measure $\nu_0$ such that $\mu = c_0\nu_0$, where $c_0 > 0$ is a constant. It is clear by the proofs of Theorems 6.41 and 7.30 that we can use the present variant of the construction used to construct $\nu_0$ to construct a measure $\nu_1$ such that $c_1^{-1}\nu_1 \leq \nu_0 \leq c_1\nu_1$, where $c_1 > 0$ is a constant. Our claim follows. □

We end the third section of this chapter with the following remarks. We can define similar variants for the measure constructions introduced in the second section of this chapter as we did above for the measure constructions of Chapter 5. It is easy to see that the covering construction obtained in this way satisfies a theorem corresponding to

Theorem 7.32. The argument is essentially the same as in the case considered above. It seems, however, that the same is not necessarily true for the corresponding packing construction. The problem is the following. Let $\varepsilon > 0$ and $\nu \in ]0,1[$ and let $\mathcal{T}$ be an $(\varepsilon,\nu)$-packing of some non-empty subset $A$ of the base set $X$ as in the definition of the packing construction of the second section of this chapter. This means that, given $\bar{B}^n(x,t) \in \mathcal{T}$, it is true that $B^n(x,t) \cap A \neq \emptyset$ and that $|x - x'|/t \leq \nu$ for some $x' \in X$. Now the collection $\{\bar{B}^n(x',(1-\nu)t) : \bar{B}^n(x,t) \in \mathcal{T}\}$ is not necessarily a packing as defined for the variant of the packing construction, since it is possible that $B^n(x',(1-\nu)t) \cap A = \emptyset$ for some $\bar{B}^n(x,t) \in \mathcal{T}$. This implies that the obvious modification of the argument given in the last paragraph of the proof of Theorem 7.30 does not work, and hence it is not guaranteed that the variant of the packing construction satisfies a result corresponding to Theorem 7.32. Since these considerations do not add anything essential to the claim of Theorem 7.32, we will not discuss further details.

We remark also that the supervisor of this work, P. Tukia, conjectured the preliminary hypothesis according to which the covering construction defined at the beginning of this section would satisfy Theorem 6.41. This hypothesis was of paramount importance since it was the starting point of this work. However, as we have seen, if the conjecture is indeed true, its proof seems to require considerably more complicated methods than used in this work.

7.4. **Variants for Kleinian groups acting on $\mathbb{B}^{n+1}$.** We have reached the fourth and final section of this chapter. Our goal in this section is to show that the measure constructions introduced in Chapter 5 can be modified to construct measures which are supported by base sets contained in $\mathbb{S}^n$. This result implies that the main equivalence theorem, Theorem 6.41, has an alternative version for non-elementary geometrically finite Kleinian groups acting on $\mathbb{B}^{n+1}$.

Let $X \subset \mathbb{S}^n = \partial\mathbb{B}^{n+1}$ be non-empty. As stated above, our intention is to show that the constructions in Chapter 5 can be transformed into constructions constructing measures of $X$. We introduce the following notation. If $x \in \mathbb{S}^n$ and $t > 0$, we write $B_n(x,t) = B^{n+1}(x,t) \cap \mathbb{S}^n$. The symbols $\bar{B}_n(x,t)$ and $S_{n-1}(x,t)$ have similar meanings.

Fix a small constant $t_0 > 0$. We define the new flatness functions as follows. Given $l \in \{1,2,\ldots,n\}$, define that

$$\tau_l(x,t) = \frac{1}{t} \inf_{V \in \mathcal{F}_l(x,t)} \rho(\bar{B}_n(x,t) \cap X, \bar{B}_n(x,t) \cap V) \tag{7.33}$$

for every $x \in \mathbb{S}^n$ and $t \in ]0,t_0[$ such that $B_n(x,t) \cap X \neq \emptyset$, where $\mathcal{F}_l(x,t)$ is this time the collection of all $l$-spheres of $\mathbb{S}^n$ meeting $\bar{B}_n(x,t)$ and $\rho$ is the Hausdorff pseudometric in the collection of all non-empty subsets of $\mathbb{S}^n$ defined with respect to the euclidean metric. Similarly, we replace the original diameter function defined by (5.2) by the function defined by

$$d(x,t) = d_{\text{euc}}(\bar{B}_n(x,t) \cap X) \tag{7.34}$$

for every $x \in \mathbb{S}^n$ and $t \in ]0,t_0[$ such that $B_n(x,t) \cap X \neq \emptyset$. The numbers $\eta > 0$ and $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{R}$ and the quantities $\alpha(x,t)$ and $\omega(x,t)$, where $x \in \mathbb{S}^n$ and $t \in ]0,t_0[$ are such that $B_n(x,t) \cap X \neq \emptyset$, are defined like in Chapter 5 using the new flatness and diameter functions, see (5.3) and (5.4). It is easy to see that a result corresponding to Lemma 5.5 is valid in the present setting, and hence we obtain that quantities of the form $\alpha(x,t)\omega(x,t)$ are all well-defined, where $x \in \mathbb{S}^n$ and $t \in ]0,t_0[$ are such that $B_n(x,t) \cap X \neq \emptyset$, see the discussion following the proof of Lemma 5.5.

We can now define the new covering and packing constructions. It is natural that the definitions are formally the same as the original. We need only to replace the functions in the definitions by the versions defined above and modify the definitions of coverings and packings as follows. If $A \subset X$, $\varepsilon \in ]0,t_0[$ and $\nu \in ]0,1[$, we say that a countable collection $\mathcal{T}$ of closed balls $\bar{B}_n(x,t)$, where $x \in \mathbb{S}^n$ and $t \in ]0,\varepsilon]$, is an $(\varepsilon,\nu)$-covering of $A$ if $A \subset \bigcup \mathcal{T}$ and there is $x' \in B_n(x,t) \cap X$ with $|x-x'|/t \leq \nu$ for every $\bar{B}_n(x,t) \in \mathcal{T}$. The notion of an $(\varepsilon,\nu)$-packing of a non-empty $A \subset X$ is defined in an analogous way.

We will use in the following the notation that indicates the base sets used in the constructions, see our remarks preceding Theorem 5.12. We will denote by $m^X$ the set function obtained from the covering construction defined above and by $p^X$ the set function obtained from the packing construction defined above. As stated by the following theorem, these set functions are outer measures of $X$, and we will use the same symbols to denote the corresponding measures of $X$.

**Theorem 7.35.** *The set functions $m^X$ and $p^X$ defined above are outer measures of $X$ such that every Borel set of $X$ is measurable with respect to $m^X$ and $p^X$. Moreover, if $g \in \text{Möb}(n+1)$ maps $\mathbb{S}^n$ onto itself, then*

$$m^{gX}(gA) = \int_A |g'|^\eta dm^X \qquad \text{and} \qquad p^{gX}(gA) = \int_A |g'|^\eta dp^X \tag{7.36}$$

*for every Borel set $A$ of $X$.*

*Proof.* It is easy to adapt the argument of the proof of Theorem 5.11 to prove the first assertion of the present theorem. Let us fix $g \in \text{Möb}(n+1)$ mapping $\mathbb{S}^n$ onto itself and a non-empty Borel set $A$ of $X$. Our aim is to prove (7.36). Let us consider the first of the formulae in (7.36).

The required argument is an easy modification of the corresponding argument in the proof of Theorem 5.12. Since $g^{-1}(\infty) \notin \mathbb{S}^n$, there are constants $M > 0$ and $a_0 > 0$ such that $M^{-1} \leq |g'| \leq M$ in $A(a_0)$, where $A(a_0) = \{x \in \mathbb{R}^{n+1} : d_{\text{euc}}(x,A) < a_0\}$, see (2.12). Similarly as in the proof of Theorem 5.12, we let $A_1, A_2, \ldots, A_{k_\lambda}$ be a division of $A$ into pairwise disjoint non-empty Borel sets of $X$ corresponding to $\lambda > 0$ as follows. Let $M_k$ and $m_k$ denote the supremum and infimum of $|g'|$ over $A_k(a_1) \subset A(a_0)$ for $k \in \{1,2,\ldots,k_\lambda\}$, where $a_1 > 0$ is a number depending on $\lambda$ and $A_k(a_1) = \{x \in \mathbb{R}^{n+1} : d_{\text{euc}}(x,A_k) < a_1\}$. We assume that $A_1, A_2, \ldots, A_{k_\lambda}$ are such that $M_k/m_k \leq \sigma$ for every $k \in \{1,2,\ldots,k_\lambda\}$, where $\sigma = \sigma(\lambda) \geq 1$ and $\sigma \to 1$ as $\lambda \to 0$.

Let us fix $k \in \{1,2,\ldots,k_\lambda\}$. Let $\varepsilon \in ]0,t_0[$ and $\nu \in ]0,1[$. Let $\mathcal{T}$ be an $(\varepsilon,\nu)^X$-covering of $A_k$ defined as above. We assume that $\bar{B} \cap A_k \neq \emptyset$ for all $\bar{B} \in \mathcal{T}$. We assume also that $\varepsilon$ is so small that if $\bar{B}_n(x,t) = \bar{B}^{n+1}(x,t) \cap \mathbb{S}^n \in \mathcal{T}$, then $\bar{B}^{n+1}(x,t) \subset A_k(a_1)$. Given $\bar{B}_n(x,t) \in \mathcal{T}$, define

$$m_g(x,t) = \inf_{z\in\bar{B}^{n+1}(x,t)} |g'(z)| \geq m_k \qquad \text{and} \qquad M_g(x,t) = \sup_{z\in\bar{B}^{n+1}(x,t)} |g'(z)| \leq M_k.$$

It is the case that $M_g(x,t)/m_g(x,t) \leq \theta$ for every $\bar{B}_n(x,t) \in \mathcal{T}$, where $\theta = \theta(\varepsilon) \geq 1$ and $\theta \to 1$ as $\varepsilon \to 0$.

It is not difficult to see that we can write $g\bar{B}_n(x,t) = \bar{B}_n(y,u)$ for every $\bar{B}_n(x,t) \in \mathcal{T}$, where $y \in \mathbb{S}^n$ and $u \in [\phi_0^{-1} m_g(x,t)t, \phi_0 M_g(x,t)t]$, where $\phi_0 = \phi_0(\varepsilon) \geq 1$ with $\phi_0 \to 1$ as $\varepsilon \to 0$ (recall (2.52)). We can write also that $g\bar{B}^{n+1}(x,t) = \bar{B}^{n+1}(y',u')$ for every $\bar{B}_n(x,t) \in \mathcal{T}$, where $y' \in \mathbb{R}^{n+1}$ and $u' \in [m_g(x,t)t, M_g(x,t)t]$ (we use (2.52) again). Note next that $|y - g(x')| \leq \phi_1 |y' - g(x')|$ for every $\bar{B}_n(x,t) \in \mathcal{T}$, where $x' \in B_n(x,t)$ is such that $|x - x'|/t \leq v$ and $\phi_1 = \phi_1(\varepsilon) \geq 1$ so that $\phi_1 \to 1$ as $\varepsilon \to 0$. We obtain furthermore that

$$\begin{aligned}
|y' - g(x')| &= u' - d_{\mathrm{euc}}(g(x'), S^n(y',u')) \leq M_g(x,t)t - m_g(x,t) d_{\mathrm{euc}}(x', S^n(x,t)) \\
&\leq (M_g(x,t) - (1-v)m_g(x,t))t \leq (M_g(x,t) - (1-v)m_g(x,t))\phi_0 u / m_g(x,t) \\
&= (M_g(x,t)/m_g(x,t) - (1-v))\phi_0 u \leq (\theta - 1 + v)\phi_0 u
\end{aligned}$$

for every $\bar{B}_n(x,t) \in \mathcal{T}$. We conclude that $g\mathcal{T} = \{g\bar{B} : \bar{B} \in \mathcal{T}\}$ is a $(\phi_0 M\varepsilon, (\theta-1+v)\phi_0\phi_1)^{gX}$-covering of $gA_k$.

Arguing similarly as in the proof of Theorem 5.12, we obtain that

$$\alpha^{gX}(y,u) \in [m_k^{\eta} \alpha^X(x,t), M_k^{\eta} \alpha^X(x,t)]$$

and

$$\tau_l^{gX}(y,u) \in [\sigma^{-1}\phi_0^{-1}\tau_l^X(x,t), \sigma\phi_0\tau_l^X(x,t)]$$

for every $l \in \{1,2,\ldots,n\}$ and every $\bar{B}_n(x,t) \in \mathcal{T}$. It follows that

$$\omega^{gX}(y,u) \in [\chi^{-1}\Delta^{-1}\omega^X(x,t), \chi\Delta\omega^X(x,t)]$$

for every $\bar{B}_n(x,t) \in \mathcal{T}$, where $\chi = \chi(\lambda) \geq 1$ is such that $\chi \to 1$ as $\lambda \to 0$ and $\Delta = \Delta(\varepsilon) \geq 1$ is such that $\Delta \to 1$ as $\varepsilon \to 0$. We conclude that

$$(m^{gX})^{(\theta-1+v)\phi_0\phi_1}_{\phi_0 M\varepsilon}(gA_k) \leq \sum_{\bar{B}_n(y,u)\in g\mathcal{T}} \alpha^{gX}(y,u)\omega^{gX}(y,u) \leq \chi\Delta M_k^{\eta} \sum_{\bar{B}_n(x,t)\in\mathcal{T}} \alpha^X(x,t)\omega^X(x,t).$$

Compare this estimate with the corresponding estimate in the proof of Theorem 5.12 on page 78. It is not difficult to see that we can finish the proof of the first formula of (7.36) using essentially the same argument as in the proof of Theorem 5.12. The second formula of (7.36) is proved in the same way. □

We move into the context of Patterson-Sullivan measures of non-elementary geometrically finite Kleinian groups. Let $G$ be a non-elementary geometrically finite Kleinian group acting on $\mathbb{B}^{n+1}$. Let $\mu$ be a Patterson-Sullivan measure of $G$. Denote by $\delta$ the exponent of convergence of $G$. Recall that $\mu$ is a $\delta$-conformal measure of $G$. Recall also that we fixed a small constant $t_0 > 0$ when defining the new variants of the flatness functions in (7.33). We fix another constant $v_0 \in ]0,1[$. Define the functions $\alpha$ and $\omega$ as in (6.27) and (6.28) using the diameter and flatness functions of the present discussion. Let us prove that $\mu$ satisfies a formula of the form (6.26).

**Theorem 7.37.** *Let $G$, $\mu$, $\delta$, $t_0$, $v_0$, $\alpha$ and $\omega$ be as above. Then there is a constant $c > 0$ such that the following is true. If $x \in \mathbb{S}^n$ and $t \in ]0, t_0[$ are such that there is $x' \in L(G)$ with $|x - x'|/t \leq v_0$, then*

$$c^{-1}\alpha(x,t)\omega(x,t) \leq \mu(\bar{B}_n(x,t)) \leq c\alpha(x,t)\omega(x,t). \tag{7.38}$$

*Proof.* We can fix two mappings $h_1, h_2 \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ as follows. If $x \in \mathbb{S}^n$, $t \in ]0, t_0[$ and $x' \in L(G)$ are as in the claim, then we can choose $h \in \{h_1, h_2\}$ such that $\bar{B}^{n+1}(x,t)$ does not meet a fixed neighbourhood of $h^{-1}(\infty)$. (If $L(G) \neq \mathbb{S}^n$ and $t_0$ is made smaller if necessary, we need only one $h \in \text{Möb}(n+1)$ mapping $\mathbb{B}^{n+1}$ onto $\mathbb{H}^{n+1}$ with $h^{-1}(\infty) \in \mathbb{S}^n \setminus L(G)$.)

Let $x \in \mathbb{S}^n$, $t \in ]0, t_0[$ and $x' \in L(G)$ be as in the claim. Let $h \in \{h_1, h_2\}$ be as above. It is the case that $M^{-1} \leq |h'| \leq M$ in $\bar{B}^{n+1}(x,t)$, where $M > 0$ is a constant, see (2.12). Since $x \in \mathbb{S}^n$ and $t \in ]0, t_0[$, there is a constant $c_0 > 0$ such that $2t/c_0 \leq d_{\text{euc}}(\bar{B}_n(x,t)) \leq 2c_0 t$. The mapping $h$ maps $\bar{B}_n(x,t)$ onto a closed $n$-ball of $\mathbb{R}^n$, say $\bar{B}^n(y,u)$ for some $y \in \mathbb{R}^n$ and $u > 0$. Since

$$u = \frac{1}{2}d_{\text{euc}}(\bar{B}^n(y,u)) = \frac{1}{2}d_{\text{euc}}(h\bar{B}_n(x,t)),$$

we can use (2.52) to estimate that $t/Mc_0 \leq u \leq Mc_0 t$. Similarly,

$$d_{\text{euc}}(h(x'), S^{n-1}(y,u)) \geq \frac{1}{M}d_{\text{euc}}(x', S^n(x,t)) \geq \frac{1-v_0}{M}t \geq \frac{1-v_0}{c_0M^2}u.$$

Write $\hat{G} = hGh^{-1}$ and $\hat{\mu} = h_*^\delta\mu$. Moreover, define that $\hat{t}_0 = Mc_0t_0 > 0$ and that $\hat{v}_0 = 1 - (1-v_0)/c_0M^2 \in ]0,1[$. We see that we can apply to $\hat{G}$ and $\hat{\mu}$ the discussion on a non-elementary geometrically finite Kleinian group acting on $\mathbb{H}^{n+1}$ and its Patterson-Sullivan measure which begins after the proof of Theorem 6.11. Therefore, defining the functions $\hat{\alpha}$ and $\hat{\omega}$ for $\hat{G}$ as in (6.27) and (6.28), we obtain from (6.26) that

$$c_1^{-1}\hat{\alpha}(y,u)\hat{\omega}(y,u) \leq \hat{\mu}(\bar{B}^n(y,u)) \leq c_1\hat{\alpha}(y,u)\hat{\omega}(y,u), \tag{7.39}$$

where $c_1 > 0$ is a constant. We use (2.20) to conclude that

$$\mu(\bar{B}_n(x,t)) = \int_{\bar{B}_n(x,t)} (|(h^{-1})'|^\delta \circ h)|h'|^\delta d\mu = \int_{\bar{B}^n(y,u)} |(h^{-1})'|^\delta d\hat{\mu}.$$

Since $M^{-1} \leq |(h^{-1})'| \leq M$ in $\bar{B}^n(y,u)$, we obtain that

$$M^{-\delta}\hat{\mu}(\bar{B}^n(y,u)) \leq \mu(\bar{B}_n(x,t)) \leq M^\delta\hat{\mu}(\bar{B}^n(y,u)). \tag{7.40}$$

Applying (2.52) again, we see that

$$M^{-\delta}\alpha(x,t) \leq \hat{\alpha}(y,u) \leq M^\delta\alpha(x,t). \tag{7.41}$$

We fix $l \in \{1, 2, \ldots, n\}$ for the moment. Denote by $\hat{\gamma}_l$ the flatness function of $\hat{G}$ as defined by (6.15). We denote in this proof by $\hat{\mathcal{F}}_l(y,u)$ the collection of all $l$-spheres of $\bar{\mathbb{R}}^n$ meeting $\bar{B}^n(y,u)$. Recall that $\gamma_l$ denotes the $l$-flatness function of $G$ and $\mathcal{F}_l(x,t)$ the collection of all $l$-spheres of $\mathbb{S}^n$ meeting $\bar{B}_n(x,t)$. Observe that $V \in \mathcal{F}_l(x,t)$ if and only if $hV \in \hat{\mathcal{F}}_l(y,u)$. We deduce that

$$\begin{aligned} \hat{\gamma}_l(y,u) &= \frac{1}{u} \inf_{\hat{V} \in \hat{\mathcal{F}}_l(y,u)} \rho(\bar{B}^n(y,u) \cap L(\hat{G}), \bar{B}^n(y,u) \cap \hat{V}) \\ &= \frac{1}{u} \inf_{V \in \mathcal{F}_l(x,t)} \rho(h\bar{B}_n(x,t) \cap hL(G), h\bar{B}_n(x,t) \cap hV). \end{aligned}$$

We use (2.52) and recall that $t/Mc_0 \le u \le Mc_0 t$ to conclude that

$$(c_0 M^2)^{-1} \gamma_l(x,t) \le \hat{\gamma}_l(y,u) \le c_0 M^2 \gamma_l(x,t)$$

for every $l \in \{1, 2, \ldots, n\}$. It follows that there is a constant $c_2 > 0$ such that

$$c_2^{-1} \omega(x,t) \le \hat{\omega}(y,u) \le c_2 \omega(x,t). \tag{7.42}$$

We observe that (7.38) follows from (7.39), (7.40), (7.41) and (7.42). □

We can now prove a new version of Theorem 6.41 for non-elementary geometrically finite Kleinian groups acting on $\mathbb{B}^{n+1}$. We continue to consider a non-elementary geometrically finite Kleinian group $G$ acting on $\mathbb{B}^{n+1}$ and a Patterson-Sullivan measure $\mu$ of $G$. Let us apply the considered variants of our measure constructions to $G$. That is, we perform the constructions with $X = L(G)$, $\eta = \delta$ and $\eta_l = \delta - l$ for $l \in \{1, 2, \ldots, n\}$, where $\delta$ is the exponent of convergence of $G$. Denote by $\bar{m}_G$ and $\bar{p}_G$ the outer measures of $L(G)$ constructed by the covering construction and packing construction, respectively. We define measures $m_G$ and $p_G$ by

$$m_G(A) = \bar{m}_G(A \cap L(G)) \qquad \text{and} \qquad p_G(A) = \bar{p}_G(A \cap L(G)) \tag{7.43}$$

for every Borel set $A$ of $\bar{\mathbb{R}}^{n+1}$.

**Theorem 7.44.** *Let $G$, $\mu$, $\delta$, $m_G$ and $p_G$ be as above. Then there are constants $c_m > 0$ and $c_p > 0$ such that $c_m m_G = \mu = c_p p_G$.*

*Proof.* According to the claim (i) of Theorem 6.11, the limit set $L(G)$ is the disjoint union of the set $L_c(G)$ of conical limit points of $G$ and the set $P(G)$ of parabolic fixed points of $G$. Theorem 6.21 implies that $\mu(P(G)) = 0$. We obtain from Theorem 6.9 that our claim follows if we show that $m_G$ and $p_G$ are $\delta$-conformal measures of $G$ such that $m_G(P(G)) = 0 = p_G(P(G))$.

We can use essentially the same argument as in the proof of Theorem 6.34 to show that $m_G$ and $p_G$ have no atoms, so $m_G(P(G)) = 0 = p_G(P(G))$, and that $m_G$ and $p_G$ satisfy a transformation rule of the form (2.14) with $s = \delta$. Similarly, an obvious modification of the argument given in the proof of Theorem 6.38 establishes that $m_G$ and $p_G$ are non-trivial and finite. It follows that $m_G$ and $p_G$ are $\delta$-conformal measures of $G$, and hence the claim of the theorem is true. □

It is not difficult to see that one can prove a theorem similar to Theorem 7.44 if one modifies the measure constructions considered in the second section of this chapter to construct measures supported by subsets of $\mathbb{S}^n$ like the constructions of Chapter 5 were modified above. We will not go into the details of that discussion, since the discussion adds nothing essential to what has been proved above.

We have reached the end of our exposition.

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