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JAN CRISTINA

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to Ella and Tycho

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Jan Cristina

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CHAPTER 1

Introduction

In two dimensions the uniformisation theorem classifies oriented Riemann surfaces into conformally elliptic, parabolic and hyperbolic classes based on their universal covers: the Riemann sphere, the complex plane and the hyperbolic disk. The existence of isothermal coordinates in two dimensions says that every surface is locally conformally flat, hence the only obstruction to global flatness is topological.

A map $f : \Omega \to \mathbb{R}^n$ is conformal if it satisfies the n-dimensional Beltrami equation:

$$J_f^{-2/n} Df^t Df = I \tag{1.1}$$

almost everywhere. Liouville's theorem [IM01, Theorem 5.1.1] says that for $n \geq 3$ every conformal map $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ from an open set $\Omega \subset \mathbb{R}^n$ is a Möbius map *i.e.* is of the form

$$x \mapsto b + \frac{\alpha A(x - x_0)}{|x - x_0|^\varepsilon},$$

where $b, x_0 \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^+$, $\varepsilon \in \{0, 2\}$ and $A \in SO(n)$. A subsequent question then arises: can we relax the geometric rigidity of these maps and still get some kind of control on the global topology along the lines of the uniformisation theorem?

If Ω is given a Riemannian metric g represented by a symmetric positive definite matrix, then $f : (\Omega, g) \to \mathbb{R}^n$ is conformal if and only if

$$g = \lambda Df^t Df$$

for some scalar function $\lambda : \Omega \to \mathbb{R}$. If $G = g(\det g)^{-1/n}$, then f is conformal with respect to g if and only if

$$G = J_f^{-2/n} Df^t Df. \tag{1.2}$$

Hence, given a measurable conformal class of metrics specified by a symmetric positive definite matrix field G with determinant 1 almost everywhere, one would like to know whether it locally arises as a scalar multiple of the pull back of the Euclidean metric *i.e.* whether or not G is integrable. If the manifold and metric are sufficiently smooth, this is answered by the Weyl-Schouten theorem [HJ03, P.5.1]: for $n \geq 4$ G is locally conformally flat if and only if the Weyl tensor vanishes and for $n = 3$ G is locally conformally flat if and only if the antisymmetric component of $\nabla(\mathrm{Ric} - \frac{\mathrm{scal}}{2(n-1)} g)$ vanishes [Pet06, Theorem 3.132].

Conformal maps are very rigid. However, we can reduce geometric rigidity and consider *quasiregular maps.* For $1 \leq K < \infty$, a K-*quasiregular* map between n-manifolds is a continuous map f between Riemannian manifolds $(\mathcal{N}, g)$ and $(\mathcal{M}, h)$ such that $f \in W^{1,n}_{\mathrm{loc}}(\mathcal{N}, \mathcal{M})$ and satisfies

$$|Df|^n(x) \leq K J_f(x)$$

for almost every $x \in \mathcal{N}$, where J_f is the Jacobian determinant of f. If f is a homeomorphism, it is said to be *quasiconformal.* If f is Lipschitz and J_f is bounded away from 0, then f is said to be of *bounded length distortion.* If $\mathcal{N}$ and $\mathcal{M}$ are domains in $\mathbb{R}^n$, then conformality is equivalent to

$$J_f^{-2/n} Df^t(x) H(f(x)) Df(x) = G(x), \tag{1.3}$$

for almost every $x \in \mathcal{N}$, where $G = g \det g^{-1/n}$ and $H = h \det h^{-1/n}$. This is the Beltrami system for the *distortion tensors* G *and* H. Another way of saying this is that every quasiconformal map is conformal for the right metric. Hence the existence of quasiconformal maps with a given distortion tensor is equivalent to asking whether the distortion tensor is integrable. The naïve smoothness assumption for the Weyl-Schouten theorem is that g is C^3, but interesting topological behaviour, like branching, only emerges if a solution to (1.3) is *a priori* $W^{1,n}$, in which case G and H are only measurable. It is not clear how the Weyl-Schouten theorem could be applied to such a non-smooth scenario.

If we suppose some given strong geometric conditions are integrable, *e.g.* a given metric with vanishing Weyl tensor, then low regularity will be difficult to handle. Sullivan in [Sul95] suggested that one could stipulate a geometric condition on an object that behaves like the derivative of a coordinate chart on a manifold, in this case a co-frame of one-forms, and apply an approximate integrability condition: that the co-frame's exterior derivative is essentially bounded. Doing so, he constructed maps whose derivatives approximate his geometric condition nicely.

A natural question to ask is, can this be extended to a global setting? Given maps $f : \mathcal{N} \to \mathcal{M}$, what is the natural generalisation of the co-frame that Sullivan used? An answer lies in the concept of an *Ehresmann connection*, which is a sub-bundle $\mathcal{H} \subset T(\mathcal{N} \times \mathcal{M})$ such that the differential of the Cartesian projection $\pi_{\mathcal{N}} : \mathcal{N} \times \mathcal{M} \to \mathcal{N}$,

$$D_{(x,y)} \pi_{\mathcal{N}} | \mathcal{H}_{(x,y)} : \mathcal{H}_{(x,y)} \to T_x \mathcal{N}$$

is an isomorphism for every $(x, y) \in \mathcal{N} \times \mathcal{M}$ [Ehr51]. In Ehresmann's original definition, such a connection was assumed to be *complete*, that is every smooth path $\gamma : [0, 1] \to \mathcal{N}$ has for any $y \in \mathcal{M}$, a lift $\tilde{\gamma} : [0, 1] \to \mathcal{M}$ satisfying

$$\frac{d}{dt}(\gamma(t), \tilde{\gamma}(t)) \in \mathcal{H}_{\gamma(t), \tilde{\gamma}(t)}.$$

and $\tilde{\gamma}(0) = y$.

Ehresmann connections generalise several notions of connection, such as affine and principal connections. A principle G-bundle over a manifold $\mathcal{N}$ has a principal connection locally given by a $\mathfrak{g}$-valued connection one-form $A \in C^\infty(\Omega, \mathfrak{g} \otimes \Lambda^1 \Omega)$. In this case the associated Ehresmann connection is given on the local trivialisation by

$$\mathcal{H}_{x,g} = \{X + g \cdot A(x, X) : X \in T_x \Omega\}$$

where $\Omega \subset \mathcal{N}$.

An Ehresmann connection is said to be integrable if for any $(x_0, y_0) \in \mathcal{N} \times \mathcal{M}$ there exists $f : \mathcal{N} \to \mathcal{M}$ such that

$$\mathcal{H}_{(x,f(x))} = \{X \oplus Df(x) \cdot X : X \in T_x\mathcal{N}\}$$

and $f(x_0) = y_0$.

Whereas a sharp integrability condition for a one-form is that its exterior derivative vanishes, the corresponding integrability condition for an Ehresmann connection $\mathcal{H}$ is Frobenius' integrability condition

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H},$$

originally proven for systems of differential equations by Frobenius in [Fro77]. If this holds then $\mathcal{H}$ is locally given by the tangent planes of the graph of a function $f : \mathcal{N} \to \mathcal{M}$. If $\mathcal{N}$ is simply connected, then the lift along $\mathcal{H}$ of any loop is also a loop (the start and endpoints of the lifted path are the same).

The appropriate regularity conditions for Frobenius' theorem are a natural subject of interest in the context of Sullivan's work. Is there anything like an essentially bounded Frobenius condition? Simić in [Sim96] very elegantly showed that the hyperplane distribution need only be Lipschitz continuous for this integrability condition to be necessary and sufficient. However, his distributions are still continuous, unlike the derivative of a quasiregular map. Nonetheless the theory of commutators of Lipschitz vector fields is interesting in its own right and has been extended further, for instance in [RS07, Ram07].

For $\mathcal{N} = \Omega \subset \mathbb{R}^n$ we consider an *Ehresmann connection form* $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ associated to an Ehresmann connection $\mathcal{H} \subset T(\Omega \times \mathcal{M})$. We say that an Ehresmann connection form ρ is in $\mathcal{A}(\Omega \times \mathcal{M})$ if ρ is essentially bounded, there is a number C such that for almost every $x \in \Omega$ the map $y \mapsto \rho_{x,y}$ is C-Lipschitz, and ρ has an essentially bounded exterior derivative with respect to Ω. If the Ehresmann connection is a principal connection, then this is equivalent to the connection one-form being a Whitney form (i.e. essentially bounded with essentially bounded exterior derivative).

We say an Ehresmann connection form ρ is in $\mathcal{A}_{\text{loc}}(\Omega \times \mathcal{M})$ if there is a number C such that for almost every $x \in \Omega$ the map $y \mapsto \rho_{x,y}$ is C-Lipschitz, the exterior derivative of ρ with respect to Ω is locally essentially bounded, and there is a $U \subset \mathcal{M}$ such that $\rho|(\Omega \times U) \in \mathcal{A}(\Omega \times U)$. See §2.2 for discussion. For this regularity class we can define the curvature of ρ to be a section

$$F_\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^2\Omega$$

by

$$F_\rho(X, Y) = (\nabla_{\rho(X)}\rho)(Y) - (\nabla_{\rho(Y)}\rho)(X) + d_\Omega\rho(X, Y),$$

where $d_\Omega\rho$ is the exterior derivative with respect to the coordinates of Ω and $X, Y \in T\Omega$.

In the event that ρ is a principal connection form, this coincides with the usual curvature

$$F_\rho = \frac{1}{2}[\rho \wedge \rho] + d\rho.$$

Consequently if ρ is smooth (and in fact Lipschitz) then

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \text{ if and only if } F_\rho = 0.$$

If ρ is only Lipschitz continuous then this equivalence only holds almost everywhere.

We can define the holonomy for an Ehresmann connection about a point to be the distance of the start and endpoints of a lift along ρ about a closed loop. If ρ arises from an integrable connection then the holonomy is zero. We prove the following quantitative estimate for the holonomy of an Ehresmann connection form in $\mathcal{A}(\Omega \times U)$ using a similar estimate for smooth connections and an adapted smooth approximation.

Theorem 1.1. *Let Ω and U in $\mathbb{R}^n$ be smooth bounded domains, and let $U' \subset\subset U$ be a domain. Let $\rho \in \mathcal{A}(\Omega \times U)$ be an Ehresmann connection form. Let $r_0 < d(U', \partial U)/(4\|\rho\|_\infty)$. There is a constant $C = C(\rho, r_0)$ such that for every $y \in U'$, and x_0, x_1, $x_2 \in \Omega$, if $\operatorname{dist}(y, \partial U') > 12 r_0 \|\rho\|_\infty$ and $|x_i - x_j| < r_0$ for every $i, j = 0, 1, 2$, then*

$$\mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) \leq C(\rho, r_0) \|F_\rho\|_\infty |\Delta|,$$

where Δ is the triangle given by the convex hull of x_0,x_1 and x_2, $\partial\Delta$ is the boundary of this triangle and $|\Delta|$ its area.

With this and a homotopy lifting lemma (Lemma 2.52) we can prove the following theorem for connections with zero curvature.

Theorem 1.2. *Let $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold, and $\Omega \subset \mathbb{R}^n$ a connected and simply connected domain, and let $\rho \in \mathcal{A}(\Omega \times \mathcal{M})$ be an Ehresmann connection form with zero curvature, that is*

$$F_\rho = 0 \text{ almost everywhere.}$$

Then for every $y \in \mathcal{M}$ and $x_0 \in \Omega$, there is a unique Lipschitz map $\gamma_y : \Omega \to \mathcal{M}$ such that

$$D_x \gamma_y = \rho_{x, \gamma_y(x)}$$
$$\gamma_y(x_0) = y.$$

Frobenius' theorem for Lipschitz distributions follows from this in Corollary 2.54.

Ehresmann connections arise naturally also in the case when they are completely non-integrable, that is, if their curvature is of maximal rank. In this case they provide interesting examples of sub-Riemannian geometries [Mon02, Chapter 11]. Naturally one would like to extend these regularity properties to connections whose curvature has maximal rank almost everywhere in some sense. This is an interesting potential subject of research to which the methods developed herein could be extended.

We return now to Sullivan's investigations into the smoothability of Lipschitz manifolds. Sullivan supposed that a co-frame $\rho = (\rho^1, \ldots, \rho^n)$, called a *Cartan–Whitney* presentation, was given by forms $\rho^i \in L^\infty(\Omega, \Lambda^1\Omega)$ satisfying

$$\operatorname{essinf} \star \rho^1 \wedge \cdots \wedge \rho^n > 0.$$

Furthermore, he supposed the approximate integrability condition $d\rho^i \in L^\infty(\Omega, \Lambda^2\Omega)$. He showed that associated to every Cartan–Whitney presentation ρ there is an upper semicontinuous *local degree* function $\deg_\rho : \Omega \to \mathbb{Z}$ which depends continuously on the L^∞-norm of ρ.

He further showed that if a Lipschitz manifold has a measurable vector bundle isomorphism from its measurable tangent bundle to a Lipschitz vector bundle given locally by a Cartan–Whitney presentation ρ with local degree 1 everywhere then the manifold has a smooth structure provided that $d\rho = A \wedge \rho$, where $A :\in L^\infty(\Omega, \mathfrak{so}_n \otimes$

$\Lambda^1\Omega)$ is an antisymmetric-matrix valued one-form with essentially bounded exterior derivative.

Heinonen and Sullivan subsequently applied Cartan–Whitney presentations to investigate metric gauges, that is topological spaces considered with a family of bi-Lipschitz–equivalent metrics [HS02]. Under suitable topological assumptions these can be characterised with Cartan–Whitney presentations.

Heinonen and Keith then continued this topic of investigation and discovered a more analytic condition for the smoothability of a Cartan–Whitney presentation [HK11]; see also [HK00].

It is not surprising that Heinonen chose to investigate generalisations of Cartan–Whitney presentations to the *quasiconformal category*. Similar to the notion of a metric gauge, there is the concept of a *conformal gauge* [Hei01, Chapter 15], that is a topological space $\mathcal{X}$ with a family of metrics such that for any two metrics d and d' the map

$$\mathrm{Id}_{\mathcal{X}} : (\mathcal{X}, d) \to (\mathcal{X}, d')$$

is an η-quasisymmetric map in the sense of Tukia and Väisälä [Hei01, Chapter 10].

Along with co-authors Pankka and Rajala in [HPR10] Heinonen introduced the notion of a *quasiconformal frame* (*caveat lector* in this thesis we refer to the same objects as quasiconformal co-frames). A quasiconformal frame on a domain Ω is an n-tuple of one-forms $\rho = (\rho^1, \ldots, \rho^n)$, satisfying for some $p > n/2$ and some $K \geq n^{n/2}$

$$\rho^i \in L^n(\Omega, \Lambda^1\Omega), \quad d\rho^i \in L^p(\Omega, \Lambda^2\Omega)$$

for $i = 1, \ldots, n$ and

$$|\rho|^n \leq \star K \rho^1 \wedge \cdots \wedge \rho^n$$

almost everywhere, where $|\cdot|$ is the non-normalised Hilbert–Schmidt norm. Under suitable geometric conditions, the authors derived a local degree for the frame. They were, however, unable to find an approximate quasiregular function which would potentially allow for the characterisation of locally Euclidean (and possibly branched-Euclidean) conformal gauges.

Pankka and Rajala in [PR11] continued investigations into quasiconformal frames, and using variational methods, constructed interesting examples. In particular for $\Omega = B(0, r') \setminus \overline{B}(0, r)$ they managed to show that minimisers of the energy functional

$$\int_{\Omega} |d\rho|^q \, dx$$

exist in the class of all K-quasiconformal frames ρ for which

$$\rho|_{B(0,r)} = d\mathbf{x} \text{ and } \rho|_{\mathbb{R}^n \setminus B(0,r')} = df$$

for some fixed quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$. Furthermore, they showed a lower bound for the minimiser based on the degree of the function f.

By considering the variation $\rho \mapsto (1 + h)\rho$ where $h \in C_0^\infty(\Omega)$, they showed that minimisers also satisfy a weak reverse Hölder inequality.

In this vein, we seek to examine quasiconformal co-frames minimising the p-integral of their exterior derivative. Let $1 < p < \infty$, and let $\rho_0 \in W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^k\Omega)$ (*cf.* §1.1.2) be a quasiconformal co-frame. The space $\mathcal{CO}^p_{\rho_0}(\Omega)$ is called the space of *quasiconformal co-frames with conformal class* ρ_0, and is defined to be

$$\begin{aligned}\mathcal{CO}^p_{\rho_0}(\Omega) := \{\varrho \in W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) \cap L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) :\\ \varrho - \rho_0 \in W^{d,p}_T(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega),\ \text{there exists}\\ A : \Omega \to CO^+_0(n) \text{ measurable such that } \varrho = A\rho_0\}.\end{aligned}$$

The space $\mathcal{SO}^p_{\rho_0}(\Omega)$ is called the space of *quasiconformal co-frames with orthogonal class* ρ_0 and is defined to be

$$\begin{aligned}\mathcal{SO}^p_{\rho_0}(\Omega) := \{\varrho \in W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) \cap L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) :\\ \varrho - \rho_0 \in W^{d,p}_T(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega),\ \text{there exists}\\ R : \Omega \to SO(n) \text{ measurable such that } \varrho = R\rho_0\}.\end{aligned}$$

Intuitively one could say that these are the spaces of forms which are a multiples of ρ_0, respectively by conformal and orthogonal matrix fields, with the same boundary values. Indeed one can construct simple non-trivial examples by starting with an exact frame df for quasiregular map f, then taking any essentially bounded map $s \in W^{1,n}(\Omega, \mathcal{M}_{n\times n})$ with determinant bounded away from 0. Then set $\rho_0 := sdf$. It follows that any for any $\sigma \in W^{1,n}(\Omega, SO(n))$ with σ equal to I on the boundary of Ω in the trace sense satisfies $\sigma\rho_0 \in \mathcal{SO}^{n/2}_{\rho_0}(\Omega)$. If $p > n/2$ and p^* denotes the Sobolev conjugate of p, then for any $\sigma \in W^{1,p^*}(\Omega, CO^+_0)$ equal to I on the boundary in the trace sense, $\sigma\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$. Whether all such co-frames can be given in such a manner is an interesting question, relating to weighted Sobolev spaces and differential inclusions.

We define the *exterior energy* of ρ to be

$$\mathcal{E}_p(\rho) := \int_\Omega |d\rho|^p\, dx. \tag{1.4}$$

We are able to show that minimisers for the exterior energy exist in these classes.

THEOREM 1.3. *Let $p > n/2$, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that the space of harmonic 1-fields with vanishing tangential component $\mathcal{H}_T(\Omega, \Lambda^1\Omega)$ is trivial* i.e. *$\mathcal{H}_T(\Omega, \Lambda^1\Omega) = \{0\}$, and let ρ_0 be a quasiconformal co-frame in $W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) \cap L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. Then there is a minimiser of $\mathcal{E}_p$ in the space $\mathcal{CO}^p_{\rho_0}(\Omega)$.*

Nota bene the condition $\mathcal{H}_T(\Omega, \Lambda^1\Omega) = 0$ is equivalent to the topological condition that $H^1(\Omega, \partial\Omega) = 0$ see [DS52, Theorem 3]. The proof of the theorem is an application of the compensated compactness theorem [IL93, Theorem 5.1]. With a small modification of the originial compensated compactness theorem we are able to extend the proof in the orthogonal class. If we assume that $\rho_0 \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ for some $p > n$, then we can construct a minimiser of $\mathcal{E}_q$, for some q below the critical exponent of integrability for two-forms, $n/2$.

THEOREM 1.4. *Let $p > n$ and $q > np/((n+1)p - n(n-1))$. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $\rho_0 \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be a quasiconformal co-frame. Suppose $d\rho_0 \in L^q(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$. Then there exists a minimiser of $\mathcal{E}_q$ in $\mathcal{SO}^q_{\rho_0}(\Omega)$.*

In particular, $np/((n+1)p - n(n-1)) < n/2$, that is below the critical exponent of $n/2$.

We say that $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ is a local minimiser of $\mathcal{E}_p$ if there is an $\varepsilon > 0$ such that for any $\varrho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ satisfying $\|\rho - \varrho\|_{\max\{p,n\}} + \|d\rho - d\varrho\|_p \le \varepsilon$ it holds that $\mathcal{E}_p(\rho) \le \mathcal{E}_p(\varrho)$. Naturally local minimisers satisfy Euler–Lagrange equations.

THEOREM 1.5. *Let, $1 < p < \infty$. If $\rho \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ is a local minimiser of $\mathcal{E}_p : \mathcal{CO}^p_{\rho_0}(\Omega) \to \mathbb{R}$, then it satisfies the Euler-Lagrange equations*

$$\int_\Omega \langle |d\rho|^{p-2} d\rho, d(\lambda\rho)\rangle \, dx = 0 \tag{1.5}$$

and

$$\int_\Omega \langle |d\rho|^{p-2} d\rho, du \wedge \rho\rangle \, dx = 0, \tag{1.6}$$

where $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$ and $\lambda \in C_0^\infty(\Omega)$.

We call (1.5) the *scalar Euler–Lagrange equations with exponent* p and (1.6) the *orthogonal Euler–Lagrange equations with exponent* p.

A combination of Theorems 1.3, 1.4, 1.5 and the higher integrability result of Pankka and Rajala [PR11, Corollary 7.8] yields the following nice existence theorem

THEOREM 1.6. *Let $p > n/2$ and let Ω be a bounded smooth domain with $\mathcal{H}_T(\Omega, \Lambda^1\Omega) = 0$. Suppose $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ is a K-quasiconformal co-frame and $d\rho_0 \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$ then there exists a $q_0 = q_0(n, K) < n/2$ such that for every $q > q_0$ there is a $\rho \in \mathcal{CO}^q_{\rho_0}(\Omega)$ satisfying (1.6) with exponent q.*

Quasiconformal maps have interesting morphism properties for so called $\mathcal{A}$-harmonic equations, *cf.* §3.2.2 and [HKM06, §14.35]. That is, if $u \in W^{1,n}(\Omega')$ satisfies the $\mathcal{A}$-harmonic equation

$$\operatorname{div}(\mathcal{A}(x, du)) = 0$$

or in weak form

$$\int_{\Omega'} \langle \mathcal{A}(x, du(x)), dv(x)\rangle \, dx = 0$$

for every $v \in C_0^\infty(\Omega')$, and $f : \Omega \to \Omega'$ is quasiconformal, then $u \circ f$ satisfies another $\mathcal{A}$-harmonic equation

$$\int_{\Omega} \langle \mathcal{A}'(x, d(u \circ f(x))), dv(x)\rangle \, dx = 0$$

for every $v \in C_0^\infty(\Omega)$. In particular, if u is n-harmonic, then $u \circ f$ is $\mathcal{A}$-harmonic [HKM06, Theorem 14.39].

Equation (1.6) can be written in divergence form

$$\operatorname{div}(\mathcal{A}(x, A(x)) \det \rho^{1/n}(x)) = 0$$

where $A : \Omega \to \mathfrak{so}_n \otimes \Lambda^1\Omega$ is the essentially unique measurable map satisfying

$$d\rho = A \wedge (\det \rho)^{-1/n} \rho.$$

and $\mathcal{A} : \Omega \times \mathfrak{so}_n \otimes \Lambda^1\Omega \to \mathfrak{so}_n \otimes \Lambda^1\Omega$ is a monotone map of growth p. (*cf.* §3.2.2. By the quasiconformality of ρ, and Proposition 3.2 $A \in L^p(\Omega, \mathfrak{so}_n \otimes \Lambda^1\Omega)$.

It is tempting to ask if the Euler–Lagrange equations have a similar $\mathcal{A}$-harmonic morphism property under quasiconformal maps. Indeed if we examine quasiconformal frames in the class of an exact frame, we get the following theorem.

THEOREM 1.7. *Let $f : \Omega \to \Omega'$ be a quasiconformal map and $\sigma : \Omega \to SO(n)$ be a measurable map such that $\tilde{\sigma} = \sigma \circ f^{-1} \in W^{1,1}(\Omega', SO(n))$. If $\sigma df \in W^{d,n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ then $\tilde{\sigma} \in W^{1,n/2}(\Omega', SO(n))$. If σdf is a solution to (1.6) for $p = n/2$, then there is a monotone $\mathcal{A} : \Omega' \times \mathfrak{so}_n \otimes \mathbb{R}^n \to \mathfrak{so}_n \otimes \mathbb{R}^n$, of growth $n/2$ such that $\tilde{\sigma}$ satisfies the $\mathcal{A}$-harmonic equation*

$$\int_{\Omega'} \langle \mathcal{A}(y, \tilde{\sigma}^{-1} d\tilde{\sigma}), \tilde{\sigma}^{-1} du \tilde{\sigma} \rangle \, dy = 0 \tag{1.7}$$

for all $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$.

The monotone function $\mathcal{A}$ is given by (3.11).

In analogy to the distinction between the Lagrangian description (reference configuration) and Eulerian description (current configuration) in elasticity theorey [MH94] Theorem 1.7 illustrates how a different configuration (i.e. coordinate frame) can critically simplify the the Euler–Lagrange equations.

In Section 3.4 we examine a modest modification to our functional to yield an even simpler equation. We consider an energy functional $\mathcal{E}'_p : \mathcal{CO}^p_{\rho_0}(\Omega) \to \mathbb{R}$,

$$\mathcal{E}'_p(\rho) = \int_\Omega |\mathfrak{A}_\rho(d\rho)|^p \, dx,$$

where $\mathfrak{A}_\rho$ satisfies

$$C^{-1}|d\rho| \leq |\mathfrak{A}_\rho d\rho| \leq C|d\rho|.$$

Analogues of Theorems 1.3 and 1.4 hold for this energy. Crucially, the $\mathcal{A}$-harmonic morphism behaviour simplifies nicely, yielding the following analogue to Theorem 1.7.

THEOREM 1.8. *Let $f : \Omega \to \Omega'$ be a quasiconformal map with inverse $h : \Omega' \to \Omega$. Let $\sigma : \Omega \to SO(n)$ be a measurable map satisfying $\tilde{\sigma} = \sigma \circ h \in W^{1,1}(\Omega', SO(n))$. Suppose $d(\sigma df) \in L^{n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$, then $\tilde{\sigma}$ is in $W^{1,n/2}(\Omega', SO(n))$. Furthermore there is a monotone map $\mathcal{A} : \mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n \to \mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n$ of growth $n/2$ such that if σdf is a local minimiser of $\mathcal{E}'_{n/2}$ then $\tilde{\sigma}$ satisfies the equation*

$$\int_{\Omega'} \langle \mathcal{A}(D_L \tilde{\sigma}), \tilde{\sigma}^{-1} du \tilde{\sigma} \rangle \, dy = 0 \tag{1.8}$$

for all $u \in C_0^\infty(\Omega', SO(n))$.

Equation (1.8) can be written in divergence form

$$\operatorname{div}(\tilde{\sigma} \mathcal{A}(D_L \tilde{\sigma}) \tilde{\sigma}^{-1}) = 0$$

In particular the monotone function $\mathcal{A}$ is independent of $y \in \Omega'$ hence equation (1.8) is the Euler–Lagrange equation for a functional with C^1 integrand which is proportional to the Dirichlet $n/2$-energy for maps $\sigma \in W^{1,n/2}(\Omega, SO(n))$. As such, existing higher regularity theory [HL87] can be applied, yielding the following corollary.

COROLLARY 1.9. *Let σ be as in Theorem 1.8. Then there is a set $\Sigma \subset \Omega'$ of Hausdorff dimension less than $\lceil n/2 \rceil - 1$, such that $\sigma \circ f^{-1} \in C^{1,\alpha}_{\mathrm{loc}}(\Omega' \setminus \Sigma)$.*

Proof. This follows by applying Theorem 1.8 and Corollary 3.10 □

1.1. Preliminaries

We work on oriented C^∞ Riemannian manifolds possibly with boundary. When working on domains $\Omega \subset \mathbb{R}^n$, they will be considered as smooth submanifolds with boundary of $\mathbb{R}^n$, unless otherwise stated. We use the Einstein summation convention, where if an index is repeated as both a subscript and a superscript, then summation over the appropriate dimensions is implied, unless otherwise stated.

Much of the material deals with differential forms on domains, which are sections of the exterior algebra of the cotangent bundle: $\alpha : \Omega \to \Lambda^k\Omega$. Because Ω is a domain, $T^*\Omega$ can be identified with $\Omega \times \mathbb{R}^n$. Consequently $\Lambda^k\Omega$ can be identified with $\Omega \times \Lambda^k\mathbb{R}^n$, and sections of $\Lambda^k\Omega$ can be identified with functions $\Omega \to \Lambda^k\mathbb{R}^n$.

Differential forms are equipped with the wedge product

$$\cdot \wedge \cdot : \Lambda^l\Omega \times \Lambda^k\Omega \to \Lambda^{l+k}\Omega.$$

For $\alpha \in \Lambda^l\Omega$ and $\beta \in \Lambda^k\Omega$ it satisfies

$$\alpha \wedge \beta = (-1)^{lk}\beta \wedge \alpha.$$

For every $X \in \mathbb{R}^n$ and $\alpha \in \Lambda^k\Omega$, we can define the interior product $X \llcorner \alpha \in \Lambda^{k-1}\Omega$ by

$$(X \llcorner \alpha)(X_1, \dots, X_{k-1}) := \alpha(X, X_1, \dots, X_{k-1}),$$

for $X_1, \dots, X_{k-1} \in \mathbb{R}^n$. For $k \geq 0$, we can equip $\Lambda^k\Omega$ with the following inner product: for $I = \{i_1, \dots, i_k\}$, $i_1 < i_2 < \cdots < i_k$, and $J = \{j_1, \dots, j_k\}$, $j_1 < j_2 < \cdots < j_k$,

$$\langle dx^{i_1} \wedge \cdots \wedge dx^{i_k}, dx^{j_1} \wedge \cdots \wedge dx^{j_k} \rangle = \begin{cases} 1, & I = J \\ 0, & \text{otherwise.} \end{cases}$$

Let $0 \leq k \leq n$. The *Hodge star* is a map

$$\star : \Lambda^k\Omega \to \Lambda^{n-k}\Omega$$

defined by

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle dx^1 \wedge \cdots \wedge dx^n.$$

The *exterior derivative* d is a linear map

$$d : C^\infty(\Omega, \Lambda^k\Omega) \to C^\infty(\Omega, \Lambda^{k+1}\Omega).$$

It is defined for $f \in C^\infty(\Omega)$ by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i,$$

and extended to higher order forms via the relations

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta \text{ and } d^2\alpha = 0$$

for $\alpha \in C^\infty(\Omega, \Lambda^k\Omega)$ and $\beta \in C^\infty(\Omega, \Lambda^l\Omega)$.

The *co-exterior derivative* is the linear map

$$d^* : C^\infty(\Omega, \Lambda^k\Omega) \to C^\infty(\Omega, \Lambda^{k-1}\Omega)$$

given by $d^* = (-1)^{nk+n+1} \star d\star$.

Definition 1.10. A finite dimensional smooth manifold G is a *Lie group*, if there exists a group structure on G such that the maps

$$G \times G \to G, \quad (g,h) \mapsto gh$$

and

$$G \to G, \quad g \mapsto g^{-1}$$

are smooth.

The tangent space T_eG of a Lie group G at the identity element $e \in G$ is called a *Lie algebra*.

A vector field $X : G \to TG$ is said to be *left-invariant* if for every $g \in G$, the map $l_g : G \to G$, $h \mapsto gh$ fixes X, that is $Dl_g(h)(X(h)) = X(gh)$. The value of a left-invariant vector field at the identity specifies the value at any other point by $X(g) = Dl_g(e)(X(e))$. For $X, Y \in T_eG$, let Dl_gX and Dl_gY denote the corresponding left-invariant vector fields. Their commutator $[Dl_gX, Dl_gY]$ is a left-invariant vector field. This defines a *Lie* bracket on T_eG, by

$$[X,Y] := [Dl_gX, Dl_gY](e).$$

This is antisymmetric and satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for every $X, Y, Z \in T_eG$ [Lee03, Chap. 15].

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The *adjoint action* of G on $\mathfrak{g}$ is a map

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$$

given by

$$\mathrm{Ad}(g,v) = D\Phi_g(e)(v),$$

where $\Phi_g : G \to G$ is the map $h \mapsto g^{-1}hg$ and $e \in G$ is the identity element. The adjoint action $\mathrm{Ad}(g,v)$ is denoted by $\mathrm{Ad}_g(v)$.

1.1.1. **Vector-valued forms.** Let V be a finite dimensional vector space. A function $\alpha : \Omega \to V \otimes \Lambda^k\Omega$ is called a *V-valued form on Ω.*

Suppose U and W are also finite dimensional vector spaces, and $B : U \times V \to W$ a bilinear map. Let $k, l \in \{1, \dots, n\}$, and let $\alpha : \Omega \to U \otimes \Lambda^k\Omega$ and $\beta : \Omega \to V \otimes \Lambda^l\Omega$ be U- and V- valued forms, respectively. Then define $B(\alpha \wedge \beta) : \Omega \to W \otimes \Lambda^{k+l}\Omega$ by

$$B(\alpha\wedge\beta)(p)(X_1, \dots, X_{k+l}) = \sum_{\sigma \in \Sigma_{n,k}} \mathrm{sign}(\sigma) B(\alpha(p)(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \beta(p)(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})).$$

for $p \in \Omega$ and vectors $X_1, \dots, X_{k+l} \in \mathbb{R}^n$, where $\Sigma_{k,n}$ is the set of permutations on n elements preserving the order of the first k elements and the last $n-k$ elements. In particular for $u \in U$, $v \in V$, $\alpha \in \Lambda^k\Omega$ and $\beta \in \Lambda^l\Omega$,

$$B(u \otimes \alpha \wedge v \otimes \beta) = B(u,v) \otimes \alpha \wedge \beta.$$

For example, let $A : \Omega \to \mathcal{M}_{m\times n} \otimes \Lambda^k\Omega$ be an $(m \times n)$-matrix-valued k-form over Ω, and let $\rho : \Omega \to \mathbb{R}^n \otimes \Lambda^l\Omega$ be an $\mathbb{R}^n$-valued l-form over Ω. Then

$$A = \begin{pmatrix} A^{11} & \cdots & A^{1n} \\ \vdots & & \vdots \\ A^{m1} & \cdots & A^{mn} \end{pmatrix} \qquad \text{and} \qquad \rho = \begin{pmatrix} \rho^1 \\ \vdots \\ \rho^n \end{pmatrix},$$

where ρ^i and A^{ij} are l- and k-forms, respectively. In this case $A \wedge \rho : \Omega \to \mathbb{R}^m \otimes \Lambda^{k+l}\Omega$ is an $\mathbb{R}^m$-valued $(k+l)$-form given by

$$A \wedge \rho = \begin{pmatrix} \sum_{j=1}^n A^{1j} \wedge \rho^j \\ \vdots \\ \sum_{j=1}^n A^{mj} \wedge \rho^j \end{pmatrix}.$$

We can extend the Hodge star $\star : \Lambda^k\Omega \to \Lambda^{n-k}\Omega$ to a map $\star : V \otimes \Lambda^k\Omega \to V \otimes \Lambda^{n-k}\Omega$, by identifying $\star$ with $\mathrm{Id}_V \otimes \star$. If V has an inner product $\langle \cdot, \cdot \rangle_V$, then define an inner product on $V \otimes \Lambda^k\Omega$ by

$$\langle \alpha, \beta \rangle = \star \langle \alpha \wedge \star\beta \rangle_V, \tag{1.9}$$

where $\alpha, \beta \in V \otimes \Lambda^k\Omega$.

Let $\alpha, \beta \in \mathbb{R}^n \otimes \Lambda^1\Omega$, then

$$\alpha = \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta^1 \\ \vdots \\ \beta^n \end{pmatrix},$$

where $\alpha^i, \beta^i \in \Lambda^1\Omega$. Let

$$\alpha^i = A^i_j dx^j \text{ and } \beta^i = B^i_j dx^j.$$

We can identify α with the matrix A whose elements are given by A^i_j, and β can be identified with B whose elements are given by B^i_j. In this way $\alpha, \beta \in \mathbb{R}^n \otimes \Lambda^1\Omega$ are identified with $(n \times n)$-matrices.

Proposition 1.11. *Let $\alpha, \beta \in \mathbb{R}^n \otimes \Lambda^1\Omega$ and let A and B denote the corresponding matrices, then the inner product on $\mathbb{R}^n \otimes \Lambda^1\Omega$ given by*

$$\langle \alpha, \beta \rangle = \star \langle \alpha \wedge \star\beta \rangle_{\mathbb{R}^n}$$

satisfies

$$\langle \alpha, \beta \rangle = \mathrm{tr}\,(A^t B).$$

Proof. We calculate

$$\begin{aligned} \langle \alpha, \beta \rangle &= \star \langle \alpha \wedge \star\beta \rangle_{\mathbb{R}^n} \\ &= \star \left(\sum_{i=1}^n \alpha^i \wedge \star\beta^i \right) \\ &= \sum_{i=1}^n \star(\alpha^i \wedge \star\beta^i) \\ &= \sum_{i=1}^n \langle \alpha^i, \beta^i \rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n A^i_j B^i_j \\ &= \mathrm{tr}\,(A^t B). \end{aligned}$$

□

Proposition 1.12. *Let $(V, \langle \cdot, \cdot \rangle_V)$ be a finite dimensional inner product space. Let $A : V \to V$ be an antisymmetric linear map on V. Then the induced map $A' := A \otimes \mathrm{Id} : V \otimes \Lambda^k\Omega \to V \otimes \Lambda^k\Omega$, is antisymmetric for the inner product defined in (1.9), that is for every $\alpha, \beta \in V \otimes \Lambda^k\Omega$*

$$\langle \alpha, A'\beta \rangle = -\langle A'\alpha, \beta \rangle. \tag{1.10}$$

In particular

$$\langle \alpha, A\alpha \rangle = 0,$$

for every $\alpha \in V \otimes \Lambda^k\Omega$.

Proof. It is sufficient to examine this for simple elements of the form $v \otimes \alpha$ where $v \in V$ and $\alpha \in \Lambda^k\Omega$. Let $\alpha, \beta \in \Lambda^k\Omega$, and let v, w be elements of V. Then

$$\begin{aligned}\langle v \otimes \alpha, A'(w \otimes \beta) \rangle &= \star\langle v, A(w) \rangle_V (\alpha \wedge \star\beta) \\ &= - \star \langle A(v), w \rangle_V (\alpha \wedge \star\beta) \\ &= -\langle A'(v \otimes \alpha), w \otimes \beta \rangle.\end{aligned}$$

□

Proposition 1.13. *Let $R \in SO(n)$. Let $R' = R \otimes \mathrm{Id} : \mathbb{R}^n \otimes \Lambda^k\Omega \to \mathbb{R}^n \otimes \Lambda^k\Omega$. Then, for every $\alpha \in \mathbb{R}^n \otimes \Lambda^k\Omega$*

$$|R'\alpha| = |\alpha|. \tag{1.11}$$

Proof. It is sufficient to check for simple elements $v \otimes \beta$ where $v \in \mathbb{R}^n$ and $\beta \in \Lambda^k\Omega$. We have

$$\begin{aligned}|R'(v \otimes \beta)|^2 &= \langle R'(v \otimes \beta), R'(v \otimes \beta) \rangle \\ &= \langle R(v) \otimes \beta, R(v) \otimes \beta \rangle \\ &= \langle R(v), R(v) \rangle_{\mathbb{R}^n} \star (\beta \wedge \star\beta) \\ &= |v|^2 |\beta|^2 \\ &= |v \otimes \beta|^2.\end{aligned}$$

□

In the future if it is unambiguous, we will denote $A \otimes \mathrm{Id}$ and $R \otimes \mathrm{Id}$ by A and R respectively.

We call an $\mathbb{R}^n$-valued one-form $\rho : \Omega \to \mathbb{R}^n \otimes \Lambda^1\Omega$ a *co-frame* on Ω. More concretely, a co-frame on Ω is a vector

$$\rho = \begin{pmatrix} \rho^1 \\ \vdots \\ \rho^n \end{pmatrix}$$

where the elements ρ^i are one-forms. The *determinant* of ρ is

$$\det \rho := \star\rho^1 \wedge \cdots \wedge \rho^n.$$

We call the frame $d\mathbf{x} : \Omega \to \mathbb{R}^n \otimes \Lambda^1\Omega$ given by

$$d\mathbf{x} = \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix}$$

the *standard Cartesian co-frame*. It satisfies $|d\mathbf{x}| = |I| = \sqrt{n}$.

Given an $(n \times n)$-matrix A, we define $A^{\#} : \Lambda^k\Omega \to \Lambda^k\Omega$ by

$$(A^{\#}\alpha)(X_1, \dots, X_k) = \alpha(AX_1, \dots, AX_k),$$

for $\alpha \in \Lambda^k\Omega$ and $X_1, \dots, X_k \in \mathbb{R}^n$. *N.b.* for $\lambda \in \mathbb{R}$, $(\lambda A)^{\#} : \Lambda^k\Omega \to \Lambda^k\Omega$ is equal to $\lambda^k(A^{\#})$.

Let V be a finite dimensional vector space. Given an $(n\times n)$-matrix A, we identify $\mathrm{Id}_V \otimes A^{\#}$ with $A^{\#}$. We also extend the exterior derivative to V-valued forms, by identifying $C^\infty(\Omega, V \otimes \Lambda^k\Omega)$ with $V \otimes C^\infty(\Omega, \Lambda^k\Omega)$, and identifying d with $\mathrm{Id}_V \otimes d$. In this way we get a map

$$d : C^\infty(\Omega, V \otimes \Lambda^k\Omega) \to C^\infty(\Omega, V \otimes \Lambda^{k+1}\Omega)$$

cf. [MT97, §16,17]. We define $d^* : C^\infty(\Omega, V \otimes \Lambda^k\Omega) \to C^\infty(\Omega, V \otimes \Lambda^{k-1}\Omega)$ similarly by identifying d^* with $\mathrm{Id}_V \otimes d^*$.

1.1.2. **L^p spaces of differential forms.** We say a form $\alpha : \Omega \to V \otimes \Lambda^k\Omega$ is measurable if for every open subset $U \subset V \otimes \Lambda^k\Omega$, $\alpha^{-1}(U)$ is a Lebesgue measurable set in Ω.

Given an inner product space V and induced inner product and norm on $V \otimes \Lambda^k\Omega$, we can define L^p-spaces of V-valued forms for $1 \le p < \infty$ by

$$L^p(\Omega, V \otimes \Lambda^k\Omega) = \{\alpha : \Omega \to \Lambda^k\Omega |\ \alpha \text{ is measurable };\ \int_\Omega |\alpha|^p \, dx < \infty\}.$$

For $p = \infty$ we define

$$L^\infty(\Omega, V \otimes \Lambda^k\Omega) = \{\alpha : \Omega \to V \otimes \Lambda^k\Omega |\ \alpha \text{ measurable, } \operatorname{ess\,sup} |\alpha| < \infty\}.$$

These are equivalent to saying that the coefficients are in $L^p(\Omega)$ for $1 \le p \le \infty$.

We say forms are equal almost everywhere if they are equal outside of a set of measure zero. After passing to equivalence classes as usual for $1 \le p \le \infty$ the spaces $(L^p(\Omega, V \otimes \Lambda^k\Omega), \|\cdot\|_p)$ are Banach spaces where

$$\|\alpha\|_p = \left(\int_\Omega |\alpha|^p \, dx\right)^{1/p} \text{ for } 1 \le p < \infty \text{ and } \|\alpha\|_\infty = \operatorname{ess\,sup} |\alpha|.$$

The local L^p-spaces $L^p_{\mathrm{loc}}(\Omega, V \otimes \Lambda^k\Omega)$ are defined to be the set of measurable functions $\alpha : \Omega \to V \otimes \Lambda^k V$ for which for every $x \in \Omega$ there is an open set $U \subset\subset \Omega$ containing x such that $\alpha|_U \in L^p(U, V \otimes \Lambda^k\Omega)$. As such, for $1 \le p \le q \le \infty$ we have

$$L^q_{\mathrm{loc}}(\Omega, V \otimes \Lambda^k\Omega) \subset L^p_{\mathrm{loc}}(\Omega, V \otimes \Lambda^k\Omega).$$

Let $\alpha \in L^1_{\mathrm{loc}}(\Omega, V \otimes \Lambda^k\Omega)$, $\varphi \in C^\infty_0(\Omega, V \otimes \Lambda^k\Omega)$, and $\eta \in C^\infty_0(\Omega, V \otimes \Lambda^{n-k}\Omega)$. Then we can define evaluations

$$(\alpha, \varphi) := \int_\Omega \langle \alpha, \varphi \rangle \, dx \text{ and } (\alpha \wedge \eta) := \int_\Omega \langle \alpha \wedge \eta \rangle_V.$$

These evaluations make α a linear functional on the spaces $C^\infty_0(\Omega, V \otimes \Lambda^k\Omega)$ and $C^\infty_0(\Omega, V \otimes \Lambda^{n-k}\Omega)$. We call continuous linear functionals

$$C^\infty_0(\Omega, V \otimes \Lambda^k\Omega) \to \mathbb{R}$$

distributional V-valued k-forms on Ω, and we denote the space of these functionals by $\mathscr{D}'(\Omega, V \otimes \Lambda^k\Omega)$. For $\alpha \in \mathscr{D}'(\Omega, V \otimes \Lambda^k\Omega)$ and $\varphi \in C^\infty_0(\Omega, V \otimes \Lambda^k\Omega)$, we denote the evaluation of α at φ by (α, φ).

We extend d and $d^\star$ to the space of distributional vector valued forms: for $\alpha \in \mathscr{D}'(\Omega, V \otimes \Lambda^k\Omega)$

$$(d\alpha, \varphi) := (\alpha, d^*\varphi) \text{ and } (d^*\alpha, \psi) := (\alpha, d\psi)$$

for every $\varphi \in C_0^\infty(\Omega, V \otimes \Lambda^{k+1}\Omega)$ and $\psi \in C_0^\infty(\Omega, V \otimes \Lambda^{k-1}\Omega)$.

Proposition 1.14. *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $p' = p/(p-1)$ and $q' = q/(q-1)$. Let U, V, and W be finite dimensional inner-product spaces and let $B : V \times W \to U$ be a bilinear map. Suppose $\alpha \in L^p_{\mathrm{loc}}(\Omega, V \otimes \Lambda^k\Omega)$ $\beta \in L^q_{\mathrm{loc}}(\Omega, W \otimes \Lambda^l\Omega)$, $d\alpha \in L^{q'}_{\mathrm{loc}}(\Omega, V \otimes \Lambda^{k+1}\Omega)$, and $d\beta \in L^{p'}_{\mathrm{loc}}(\Omega, W \otimes \Lambda^{l+1}\Omega)$. If $p \geq q'$, then*

$$d(B(\alpha \wedge \beta)) = B(d\alpha \wedge \beta) + (-1)^k B(\alpha \wedge d\beta)$$

and $d(B(\alpha \wedge \beta)) \in L^1_{\mathrm{loc}}(\Omega, U \otimes \Lambda^{l+k+1}\Omega)$.

Proof. This is a standard application of smooth approximations [Eva98],[ISS]. □

We define the *exterior Sobolev space* $W^{d,p}(\Omega, V \otimes \Lambda^k\Omega)$ (also called the *partial Sobolev space*), *cf.* [ISS], to be

$$W^{d,p}(\Omega, V \otimes \Lambda^k\Omega) := \{\alpha \in L^p(\Omega, V \otimes \Lambda^k\Omega) : d\alpha \in L^p(\Omega, V \otimes \Lambda^{k+1}\Omega)\}.$$

We equip this space with the norm

$$\|\alpha\|_{W^{d,p}} := \|\alpha\|_p + \|d\alpha\|_p.$$

The subspace of $W^{d,p}(\Omega, V \otimes \Lambda^k\Omega)$ consisting of forms $\alpha \in W^{d,p}(\Omega, V \otimes \Lambda^k\Omega)$ satisfying

$$\int_\Omega \langle \alpha, d^*\varphi\rangle \, dx = \int_\Omega \langle d\alpha, \varphi\rangle \, dx$$

for every $\varphi \in C^\infty(\Omega, V \otimes \Lambda^{k+1}\Omega)$ is denoted $W^{d,p}_T(\Omega, V \otimes \Lambda^k\Omega)$.

Similarly we define $W^{d^*,p}(\Omega, V \otimes \Lambda^k\Omega)$ to be

$$W^{d^*,p}(\Omega, V \otimes \Lambda^k\Omega) = \{\alpha \in L^p(\Omega, V \otimes \Lambda^k\Omega) : d^*\alpha \in L^p(\Omega, V \otimes \Lambda^{k-1}\Omega\}.$$

We equip this space with the norm

$$\|\alpha\|_{W^{d^*,p}} = \|\alpha\|_p + \|d^*\alpha\|_p.$$

The subspace of $W^{d^*,p}(\Omega, V \otimes \Lambda^k\Omega)$ consisting of forms $\alpha \in W^{d^*,p}$ satisfying

$$\int_\Omega \langle \alpha, d\varphi\rangle \, dx = \int_\Omega \langle d^*\alpha, \varphi\rangle \, dx$$

for every $\varphi \in C^\infty(\Omega, V \otimes \Lambda^{k-1}\Omega)$ is denoted $W^{d^*,p}_N(\Omega, V \otimes \Lambda^k\Omega)$.

It follows from elementary algebra of the Hodge star operator that

$$\star : W^{d,p}(\Omega, V \otimes \Lambda^k\Omega) \to W^{d^*,p}(\Omega, V \otimes \Lambda^{n-k}\Omega)$$

is an isometry.

Proposition 1.15. *The norms $\|\cdot\|_{W^{d,p}}$ and $\|\cdot\|_{W^{d^*,p}}$ make $W^{d,p}(\Omega, V \otimes \Lambda^k\Omega)$ and $W^{d^*,p}(\Omega, V \otimes \Lambda^{n-k}\Omega)$, respectively Banach spaces. Furthermore $W^{d,p}_T(\Omega, V \otimes \Lambda^k\Omega)$ and $W^{d^*,p}_N(\Omega, V \otimes \Lambda^k)$ are, respectively closed subspaces. Furthermore the subspace $C^\infty(\Omega, V \otimes \Lambda^k\Omega)$ is dense in $W^{d,p}(\Omega, V \otimes \Lambda^k\Omega)$ and $W^{d^*,p}(\Omega, V \otimes \Lambda^k\Omega)$, while $C_0^\infty(\Omega, V \otimes \Lambda^k\Omega)$ is dense in $W^{d,p}_T(\Omega, V \otimes \Lambda^k\Omega)$ and $W^{d^*,p}_N(\Omega, V \otimes \Lambda^k\Omega)$.*

Proof. This is proven for $W^{d,p}(\Omega, \Lambda^k\Omega)$ and $W^{d^*,p}(\Omega, \Lambda^k\Omega)$ in [ISS, Corollaries 3.6-3.8]. It is elementary to extend the proof to vector valued forms, by fixing a basis in V, and considering forms taking values in the span of each basis element. □

We define the spaces

$$W^{1,p}(\Omega,\Lambda^k\Omega) := \{\alpha \in L^p(\Omega,\Lambda^k\Omega) : \partial_i\alpha \in L^p(\Omega,\Lambda^k\Omega) \text{ for } i = 1,\dots,n\}.$$

And

$$W^{1,p}_N(\Omega,\Lambda^k\Omega) := W^{1,p}(\Omega,\Lambda^k\Omega) \cap W^{d^*,p}_N(\Omega,\Lambda^k\Omega),$$
$$W^{1,p}_T(\Omega,\Lambda^k\Omega) = W^{1,p}(\Omega,\Lambda^k\Omega) \cap W^{d,p}_T(\Omega,\Lambda^k\Omega),$$
$$\mathcal{H}^p(\Omega,\Lambda^k\Omega) := \{\alpha \in L^p(\Omega,\Lambda^k\Omega) : d\alpha = 0,\ d^*\alpha = 0\},$$
$$\mathcal{H}_T(\Omega,\Lambda^k\Omega) = \mathcal{H}^1(\Omega,\Lambda^k\Omega) \cap W^{d,1}_T(\Omega,\Lambda^k\Omega),$$

and

$$\mathcal{H}_N(\Omega,\Lambda^k\Omega) = \mathcal{H}^1(\Omega,\Lambda^k\Omega) \cap W^{d^*,1}_N(\Omega,\Lambda^k\Omega).$$

A significant point is that provided Ω is a smooth bounded domain, $\mathcal{H}_N(\Omega,\Lambda^k\Omega)$ and $\mathcal{H}_T(\Omega,\Lambda^k\Omega)$ are finite dimensional spaces of forms whose derivatives to all orders are continuous up to the boundary, while $\mathcal{H}^p(\Omega,\Lambda^k\Omega)$ is a space of forms which are smooth on the interior of Ω.

An important tool that we make use of is the Hodge decomposition for differential forms [ISS, (1.2)]. It says that for a smooth bounded domain $\Omega \subset \mathbb{R}^n$

$$L^p(\Omega,\Lambda^k\Omega) = d(W^{1,p}(\Omega,\Lambda^{k-1}\Omega)) \oplus d^*(W^{1,p}_N(\Omega,\Lambda^k\Omega)) \oplus \mathcal{H}_T(\Omega,\Lambda^k\Omega),$$
$$L^p(\Omega,\Lambda^k\Omega) = d(W^{1,p}_T(\Omega,\Lambda^{k-1}\Omega)) \oplus d^*(W^{1,p}(\Omega,\Lambda^k\Omega)) \oplus \mathcal{H}_N(\Omega,\Lambda^k\Omega),$$
$$L^p(\Omega,\Lambda^k\Omega) = d(W^{1,p}_T(\Omega,\Lambda^{k-1}\Omega)) \oplus d^*(W^{1,p}_N(\Omega,\Lambda^k\Omega)) \oplus \mathcal{H}^p(\Omega,\Lambda^k\Omega).$$

We extend this to vector valued forms by identifying $L^p(\Omega, V \otimes \Lambda^k\Omega) = V \otimes L^p(\Omega,\Lambda^k\Omega)$.

REMARK 1.16. Similarly for $1 \le p \le \infty$ we define the spaces $L^p(\mathcal{N}, E \otimes \Lambda^k\mathcal{N})$ and $L^p_{\mathrm{loc}}(\mathcal{N}, E \otimes \Lambda^k\mathcal{N})$ of vector bundle valued forms, where $E \to \mathcal{N}$ is a vector bundle over the Riemannian manifold $(\mathcal{N}, \langle\cdot,\cdot\rangle_{\mathcal{N}})$ with metric $\langle\cdot,\cdot\rangle_E$. In this case $E \otimes \Lambda^k\mathcal{N})$ has the Riemannian metric $\langle\cdot,\cdot\rangle_E \otimes \langle\cdot,\cdot\rangle_{\mathcal{N}}$; *cf.* [MT97, §16].

1.1.3. **Bundle-valued forms.**

DEFINITION 1.17. Let $E \to \mathcal{X}$ and $F \to \mathcal{Y}$ be vector bundles over distinct manifolds $\mathcal{X}$ and $\mathcal{Y}$ with fibres E_x and F_y at the points $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively. By $E \oplus F \to \mathcal{X} \times \mathcal{Y}$ we mean the vector bundle over $\mathcal{X} \times \mathcal{Y}$ whose total space is given by

$$E \oplus F = \{(x,y,u,v) \in \mathcal{X} \times \mathcal{Y} \times E \times F : x \in \mathcal{X},\ y \in \mathcal{Y},\ u \in E_x, v \in F_y\},$$

and whose projection map is given by the Cartesian projection onto the first two coordinates. The fibres $(E \oplus F)_{x,y}$ are naturally isomorphic to $E_x \oplus F_y$.

If $E \to \mathcal{X}$ is a smooth vector bundle, we denote the smooth sections of E by $\Gamma(E)$. The set of such sections which are compactly supported is denoted $\Gamma_0(E)$.

Using this notation the bundle $T(\Omega \times \mathcal{M})$ is canonically isomorphic to $T\Omega \oplus T\mathcal{M}$ via the natural inclusion maps: for $y \in \mathcal{M}$ $T\Omega \to T\Omega \times \{y\} \subset T(\Omega \times \mathcal{M})$, and for $x \in \Omega$, $T\mathcal{M} \to \{x\} \times T\mathcal{M} \subset T(\Omega \times \mathcal{M})$.

For a given topological space $\mathcal{X}$ we define the 0-*bundle over* $\mathcal{X}$, $0_{\mathcal{X}} \to \mathcal{X}$ with total space

$$0_{\mathcal{X}} = \{0\} \times \mathcal{X},$$

and projection given by the Cartesian projection onto $\mathcal{X}$.

Consequently $T(\Omega \times \mathcal{M})$ is isomorphic to $(T\Omega \oplus 0_{\mathcal{M}}) \oplus (0_\Omega \oplus T_{\mathcal{M}})$. In what follows we identify these two bundles. Let $\Pi_{\mathcal{M}} : T(\Omega \times \mathcal{M}) \to 0_\Omega \oplus T\mathcal{M}$ and $\Pi_\Omega : T(\Omega \times \mathcal{M}) \to T\Omega \oplus 0_{\mathcal{M}}$ denote the projections under this direct sum decomposition.

DEFINITION 1.18. The vector bundle $T\mathcal{M} \otimes \Lambda^k\Omega \to \mathcal{M} \times \Omega$ is defined to be the bundle with total space

$$T\mathcal{M} \otimes \Lambda^k\Omega := \bigcup_{y\in\mathcal{M}} (T_y\mathcal{M}) \otimes \Lambda^k\Omega,$$

and natural projection $T\mathcal{M} \otimes \Lambda^k\Omega \to \mathcal{M} \times \Omega$. For $y \in \mathcal{M}$ and $x \in \Omega$, it has fibre

$$(T\mathcal{M} \otimes \Lambda^k\Omega)_{(y,x)} = T_y\mathcal{M} \otimes \Lambda^k_x\Omega \cong T_y\mathcal{M} \otimes \Lambda^k\mathbb{R}^n.$$

CONVENTION. For notational convenience we identify the bundles $T\mathcal{M} \otimes \Lambda^k\Omega \to \mathcal{M} \times \Omega$ and $T\mathcal{M} \otimes \Lambda^k\Omega \to \Omega \times \mathcal{M}$ with the map $(x, y) \mapsto (y, x)$. For every $(x, y) \in \Omega \times \mathcal{M}$ we note that $(T\mathcal{M} \otimes \Lambda^k\Omega)_{(x,y)} = T_y\mathcal{M} \otimes \Lambda^k_x\Omega$.

Let $\langle\cdot,\cdot\rangle_y$ denote the metric tensor of $\mathcal{M}$ at y. Then it is an inner product on $T_y\mathcal{M}$, and we define

$$\langle\xi, \chi\rangle = \star\langle\xi \wedge \star\chi\rangle_y \text{ and } |\xi| = \sqrt{\langle\xi, \xi\rangle}$$

for $\xi, \chi \in T_y\mathcal{M} \otimes \Lambda^k\Omega$.

DEFINITION 1.19. Let $\alpha \in \Gamma(T\mathcal{M} \otimes \Lambda^k\Omega)$. For every $x \in \Omega$, define the section $\alpha|_x : \mathcal{M} \to T\mathcal{M} \otimes \Lambda^k_x\Omega$ by

$$\alpha|_x : y \mapsto \alpha(x, y).$$

For every $y \in \mathcal{M}$, define the section $\alpha|^y : \Omega \to T_y\mathcal{M} \otimes \Lambda^k\Omega$ by

$$x \mapsto \alpha(x, y).$$

The *exterior derivative* $d_\Omega : \Gamma(T\mathcal{M} \otimes \Lambda^k\Omega) \to \Gamma(T\mathcal{M} \otimes \Lambda^{k+1}\Omega)$ is the linear map which takes $\alpha \in \Gamma(T\mathcal{M} \otimes \Lambda^k\Omega)$ to the section

$$d_\Omega\alpha : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^{k+1}\Omega, \quad (x, y) \mapsto d(\alpha|^y)(x).$$

The *co-exterior derivative* $d^*_\Omega : \Gamma(T\mathcal{M} \otimes \Lambda^k\Omega) \to \Gamma(T\mathcal{M} \otimes \Lambda^{k-1}\Omega)$ is the linear map which takes $\alpha \in \Gamma(T\mathcal{M} \otimes \Lambda^k\Omega)$ to the section

$$d^*_\Omega\alpha : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^{k-1}\Omega, \quad (x, y) \mapsto d^*(\alpha|^y)(x).$$

Let ∇ denote the Levi-Civita connection for the product Riemannian metric:

$$\nabla : \Gamma(\Omega \times \mathcal{M}) \to \Lambda^1(\Omega \times \mathcal{M}) \otimes \Gamma(\Omega \times \mathcal{M}).$$

The bundles $0_\mathcal{M} \oplus T\mathcal{M}$ and $T\Omega \oplus 0_\mathcal{M}$ are parallel sub-bundles under the connection (because of the product structure), so for any vector field $X \in \Gamma(\Omega \times \mathcal{M})$

$$\nabla_X(\Gamma(T\Omega \oplus 0_\mathcal{M})) \subset \Gamma(T\Omega \oplus 0_\mathcal{M}) \quad \text{and} \quad \nabla_X(\Gamma(0_\Omega \oplus T\mathcal{M})) \subset \Gamma(0_\Omega \oplus T\mathcal{M}).$$

Let $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$. For any vector $X \in T(\Omega \times \mathcal{M})$, the covariant derivative of ρ, denoted by $\nabla_X\rho$, is

$$(\nabla_X\rho)(Y) = \nabla_X(\rho(Y)) - \rho(\nabla_X Y),$$

where $Y \in \Gamma(T\Omega \oplus 0_\mathcal{M})$. This is well defined because $\nabla_X Y \in \Gamma(T\Omega \oplus 0_\mathcal{M})$ and $\nabla_X(\rho(Y)) \in \Gamma(0_\Omega \oplus T\mathcal{M})$, so $\nabla_X\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$.

DEFINITION 1.20. We define the *vertical covariant derivative of* ρ $\nabla^\mathcal{M}\rho$ to be a section $\Omega \times \mathcal{M} \to (T\mathcal{M} \otimes \Lambda^1\Omega)) \otimes \Lambda^1(\Omega \times \mathcal{M})$

$$\nabla^\mathcal{M}_X\rho = \nabla_{(\Pi_\mathcal{M} X)}\rho,$$

where $X \in T(\Omega \times \mathcal{M})$.

We can also describe the exterior derivative of ρ using the covariant derivative:

$$d_\Omega \rho(X,Y) = (\nabla_X \rho)(Y) - (\nabla_Y \rho)(X),$$

for X and Y in $T\Omega \oplus 0_{\mathcal{M}}$. This can be easily verified using the fact that for a one-form θ, and vector fields X and Y,

$$(\nabla_X \theta)(Y) := X(\theta(Y)) - \theta(\nabla_X Y).$$

Let $\mathcal{N} = \Omega \times \mathcal{M}$. Then $T\mathcal{M} \otimes \Lambda^k \Omega$ can be identified with a subset of $T\mathcal{N} \otimes \Lambda^k \mathcal{N}$ as an isometric embedding. Consequently, for $1 \leq p \leq \infty$ we define the space $L^p(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^k \Omega)$ to be the subset of $L^p(\mathcal{N}, T\mathcal{N} \otimes \Lambda^k \mathcal{N})$ taking values almost everywhere in $T\mathcal{M} \otimes \Lambda^k \Omega$, where $L^p(\mathcal{N}, T\mathcal{N} \otimes \Lambda^k \Omega)$ was defined in Remark 1.16. We define the space $L^p_{\text{loc}}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^k \Omega)$ similarly.

Similarly we define distributions $\mathscr{D}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^k \Omega)$ to be the set of $\alpha \in \mathscr{D}(\mathcal{N}, T\mathcal{N} \otimes \Lambda^k \mathcal{N})$ such that

$$(\alpha, \beta) = 0$$

for all $\beta \in \Gamma((T\mathcal{M} \otimes \Lambda^k \Omega)^\perp)$. We have similarly, that

$$L^1_{\text{loc}}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^k \Omega) \subset \mathscr{D}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^k \Omega).$$

For $\rho \in L^1_{\text{loc}}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^1 \Omega)$, if there is a constant $C \geq 0$ such that for every triple of smooth compactly supported vector fields $X \in \Gamma_0(0_{\mathcal{M}} \oplus T\Omega)$ and $Z, Y \in \Gamma_0(T\mathcal{M} \oplus 0_\Omega)$

$$\left| \int_{\Omega \times \mathcal{M}} \langle \rho(X), \nabla_Z Y \rangle + \langle \rho(\nabla_Z X), Y \rangle \, dxdy \right| \leq C \|X\|_\infty \|Z\|_\infty \|Y\|_1.$$

The smallest such C for which this holds is defined to be $\|\nabla^{\mathcal{M}} \rho\|_\infty$.

Definition 1.21. Let A be a finite dimensional vector space, $U \subset \mathbb{R}^m$ a domain, y^j the coordinates of $\mathbb{R}^m$, and $f : \Omega \times U \to A$ a smooth compactly supported function. Define the differential operator ∇^U by $\nabla^U f : \Omega \times U \to A \otimes \mathbb{R}^m$,

$$\nabla^U f = \sum_{i=1}^m \partial_{y^i} f \otimes dy^i.$$

We define the formal adjoint of ∇^U to be the operator $(\nabla^U)^* : C^\infty(\Omega \times U, A \otimes \mathbb{R}^m) \to C^\infty(\Omega \times U, A)$ for which

$$\int_{\Omega \times U} \langle (\nabla^U)^* \phi, \theta \rangle \, dxdy = \int_{\Omega \times U} \langle \phi, \nabla^U \theta \rangle \, dxdy$$

for all C^∞_0 test functions $\theta : \Omega \times U \to A$.

Let $\theta \in \mathscr{D}'(\Omega \times U, A \otimes \Lambda^1 \Omega)$ and let $\phi \in C^\infty(\Omega \times U, A \otimes \Lambda^1 \Omega \otimes \mathbb{R}^m)$. Then $\nabla^U \theta \in \mathscr{D}'(\Omega \times U, A \otimes \Lambda^1 \Omega \otimes \mathbb{R}^m)$ is given by the evaluation

$$(\nabla^U \theta, \phi) := (\theta, (\nabla^U)^* \phi).$$

CHAPTER 2

THE CURVATURE OF NON-SMOOTH CONNECTIONS

Ehresmann first developed his notion of connection in [Ehr51]. His connections generalise both affine and principal connections. Although Ehresmann connections can be specified for more general fibre bundles, we will only be interested in the local case, that is, a product $\Omega \times \mathcal{M}$, where $\mathcal{M}$ is a smooth m-dimensional manifold and $\Omega \subset \mathbb{R}^n$ is a smooth domain.

DEFINITION 2.1. Let $\mathcal{X}$ be a smooth m-manifold. A sub-bundle of $T\mathcal{X}$ $\mathcal{H}$ is a *hyperplane distribution of rank* k on $\mathcal{X}$ if it is a rank k sub-bundle of the tangent bundle. That is $\mathcal{H}_p := \mathcal{H} \cap T_p\mathcal{X}$ is a k-dimensional subspace of $T_p\mathcal{X}$ for every $p \in \mathcal{X}$. We say that $\mathcal{H}$ is *smooth* if it is a smooth manifold and the inclusion map $\mathcal{H} \hookrightarrow T\mathcal{X}$ is a smooth vector bundle homomorphism.

DEFINITION 2.2. Let $\mathcal{X}$ be a smooth m-manifold, and $\mathcal{H}$ a smooth rank k hyperplane distribution. The distribution $\mathcal{H}$ is said to be *integrable* about a point $x \in \mathcal{X}$ if there is a neighbourhood U of x, and a diffeomorphism $\phi : U \to V \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$ such that $D\phi^{-1}_{\phi(p)}|_{\mathbb{R}^k \times \{0\}}$ is an isomorphism onto $\mathcal{H}_p$ for all $p \in U$. The distribution $\mathcal{H}$ is said to be *completely integrable* if it is integrable about every point.

We denote by $\Gamma(\mathcal{H})$ the set of smooth vector fields $v : X \to T\mathcal{X}$ such that $v(p) \in \mathcal{H}_p$ for every $p \in \mathcal{X}$.

THEOREM 2.3 (Frobenius' integrability condition). *Let $\mathcal{X}$ be a smooth manifold and $\mathcal{H}$ a smooth hyperplane distribution in $\mathcal{X}$. Then $\mathcal{H}$ is completely integrable if and only if for every pair of smooth vector fields $X, Y \in \Gamma(\mathcal{H})$, we have*

$$[X, Y] \in \Gamma(\mathcal{H}).$$

Theorem 2.3 is proved in many differential geometry text books *cf.* [Lee03, Theorem 14.5]. Subsequently this condition will be expressed as:

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}. \tag{2.1}$$

DEFINITION 2.4. An *Ehresmann connection* $\mathcal{H}$ on the product $\Omega \times \mathcal{M}$ over Ω is a smooth vector sub-bundle of $T(\Omega \times \mathcal{M})$ satisfying

$$T(\Omega \times \mathcal{M}) = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{V}$ is the bundle $\mathcal{V} = 0_\Omega \oplus T\mathcal{M}$.

We now make more explicit some facts about Ehresmann connections. Let π_Ω denote the Cartesian projection onto Ω, and let $\pi_{\mathcal{M}}$ denote the projection onto $\mathcal{M}$:

$$\pi_\Omega : \Omega \times \mathcal{M} \to \Omega \quad \text{and} \quad \pi_{\mathcal{M}} : \Omega \times \mathcal{M} \to \mathcal{M}.$$

LEMMA 2.5. *Let $\mathcal{H}$ be a smooth n-dimensional hyperplane distribution on $\Omega \times \mathcal{M}$. Then $\mathcal{H}$ is a smooth Ehresmann connection if and only if for every $(x, y) \in \Omega \times \mathcal{M}$, $D\pi_\Omega|_{\mathcal{H}_{x,y}} : \mathcal{H}_{x,y} \to T_x\Omega$ is a linear isomorphism.*

Proof. Assume $\mathcal{H}$ is an Ehresmann connection; $\mathcal{H}$ is a smooth distribution such that $\mathcal{H} \oplus \mathcal{V} = T(\Omega \times \mathcal{M})$. The map $\pi_\Omega : \Omega \times \mathcal{M} \to \Omega$ is a submersion, that is, the differential of the map $D_{x,y}\pi_\Omega : T_{x,y}(\Omega \times \mathcal{M}) \to T_x\Omega$ is a surjection for every $x \in \Omega$ and $y \in \mathcal{M}$, and $\mathcal{V} = \ker D\pi_\Omega$. So $D\pi|\mathcal{H}$ is a fibrewise isomorphism from $\mathcal{H}_{x,y}$ to $T_x\Omega$.

For the converse we observe that, since $\mathcal{V} = \ker D\pi_\Omega$, and $\mathcal{H} \cap \ker D\pi_\Omega = 0_{\Omega\times\mathcal{M}}$, we have that $\mathcal{H} \oplus \mathcal{V} = T(\Omega \times \mathcal{M})$. □

A section

$$\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$$

is called an *Ehresmann connection form.*

LEMMA 2.6. *Let $\mathcal{H} \subset T(\Omega \times \mathcal{M})$ be an n-dimensional hyperplane distribution. Then $\mathcal{H}$ is a smooth Ehresmann connection if and only if there is a smooth Ehresmann connection form $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ such that*

$$\mathcal{H} = (\mathrm{id} + \rho)(T\Omega \oplus 0_{\mathcal{M}}).$$

Proof. First, to clarify, we identify $T\mathcal{M} \otimes \Lambda^1\Omega$ with $\mathrm{Hom}(T\Omega \oplus 0_{\mathcal{M}}, 0_\Omega \oplus T\mathcal{M})$ canonically. Now given an Ehresmann connection we have, by Lemma 2.5, that $D\pi_\Omega|_{\mathcal{H}} : \mathcal{H} \to T\Omega$ is an isomorphism on fibres. We define the bundle map $\Pi_\Omega : \mathcal{H} \to T\Omega \oplus 0_{\mathcal{M}}$ by

$$\Pi_\Omega(V) = (D_{x,y}\pi_\Omega(V), 0),$$

for $V \in \mathcal{H}_{x,y}$ and $(x, y) \in \Omega \times \mathcal{M}$. Because $\mathcal{H}$ is a smooth Ehresmann connection, the map Π_Ω is a smooth vector bundle isomorphism from $\mathcal{H}$ to $T\Omega \oplus 0_{\mathcal{M}}$. Similarly we define a map $\Pi_{\mathcal{M}} : \mathcal{H} \to 0_\Omega \oplus T\mathcal{M}$ by

$$\Pi_{\mathcal{M}}(V) = (0, D_{x,y}\pi_{\mathcal{M}}(V)),$$

where $V \in \mathcal{H}_{x,y}$. We note that, in general, this map is not an isomorphism.

Let $\rho := \Pi_{\mathcal{M}} \circ \Pi_\Omega^{-1}$. Since $0_\Omega \oplus T\mathcal{M}$ and $T\Omega \oplus 0_{\mathcal{M}}$ form a direct sum decomposition of $T(\Omega \times \mathcal{M})$, we have that for any $V \in \mathcal{H}$,

$$\begin{aligned} V &= \Pi_\Omega(V) + \Pi_{\mathcal{M}}(V) \\ &= \Pi_\Omega(V) + \Pi_{\mathcal{M}} \circ \Pi_\Omega^{-1}(\Pi_\Omega(V)) \\ &= \Pi_\Omega(V) + \rho(\Pi_\Omega(V)). \end{aligned}$$

In particular we have that $\mathcal{H}$ is of the form $W + \rho(W)$ for W in $T\Omega \oplus 0_{\mathcal{M}}$;

$$\mathcal{H} = \{W + \rho(W) \in T(\Omega \times \mathcal{M}) : W \in T\Omega \oplus 0_{\mathcal{M}}\}.$$

Given a section $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ such that $\mathcal{H} = (\mathrm{id} + \rho)(T\Omega \oplus 0_{\mathcal{M}})$, we have that the natural projection $D\pi_\Omega|_{\mathcal{H}} : \mathcal{H} \to T\Omega$ is a fibrewise isomorphism, since $\rho(T\Omega \oplus 0_{\mathcal{M}}) \subset 0_\Omega \oplus T\mathcal{M} = \ker D\pi_\Omega$. Hence $\mathcal{H}$ is an Ehresmann connection. □

2.1. Smooth connections and curvature

We define the *curvature* of $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$ to be the section of $T\mathcal{M} \otimes \Lambda^2\Omega$ given by:

$$F(X,Y) := d_\Omega \rho(X,Y) + (\nabla_{\rho(X)}\rho)(Y) - (\nabla_{\rho(Y)}\rho)(X), \tag{2.2}$$

where $X, Y \in T\Omega \oplus 0_{\mathcal{M}}$.

Remark 2.7. The curvature is equivalent to the following expansion

$$F(X,Y) = (\nabla_{X+\rho(X)}\rho)(Y) - (\nabla_{Y+\rho(Y)}\rho)(X). \tag{2.3}$$

This can be verified by expanding the exterior derivative of ρ.

Lemma 2.8. *Let $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ be a smooth Ehresmann connection form. Let X and Y be smooth vector fields on $\Omega \times \mathcal{M}$, taking values in $T\Omega \oplus 0_{\mathcal{M}}$. The curvature $F(X,Y)$ is the unique vector field valued in $\mathcal{V}$, such that*

$$[X + \rho(X), Y + \rho(Y)] - F(X,Y) \in \Gamma(\mathcal{H}). \tag{2.4}$$

Proof. Denote by W the commutator

$$W := [X + \rho(X), Y + \rho(Y)].$$

First we show uniqueness. Because $\mathcal{V} \oplus \mathcal{H} = T(\Omega \times \mathcal{M})$, we have that there is a unique direct sum decomposition of W

$$W = W_{\mathcal{H}} + W_{\mathcal{V}},$$

where $W_{\mathcal{H}} \in \mathcal{H}$ and $W_{\mathcal{V}} \in \mathcal{V}$. Thus there is a unique vector field $W_{\mathcal{V}}$ such that $W - W_{\mathcal{V}} \in \mathcal{H}$.

Now we must show that $F(X,Y) = W_{\mathcal{V}}$. We may write

$$W = H + V$$

where $H \in T\Omega \oplus 0_{\mathcal{M}}$ and $V \in 0_\Omega \oplus T\mathcal{M} = \mathcal{V}$. Then

$$W = H + \rho(H) - \rho(H) + V.$$

Now $H + \rho(H) \in \mathcal{H}$ and $\rho(H) \in \mathcal{V}$ so $V - \rho(H) \in \mathcal{V}$, as such

$$W_{\mathcal{H}} = H + \rho(H) \quad \text{and} \quad W_{\mathcal{V}} = V - \rho(H).$$

In order to prove the result we must show that $F(X,Y) = V - \rho(H)$.

We use the torsion-free property of the Levi–Civita connection to write

$$W = \nabla_{X+\rho(X)}(Y + \rho(Y)) - \nabla_{Y+\rho(Y)}(X + \rho(X)).$$

Since ∇ is a Levi-Civita connection for the product metric, for any vector fields $X \in \Gamma(T\Omega \oplus 0_{\mathcal{M}})$ and $U \in \Gamma(0_\Omega \oplus T\mathcal{M})$, and vector $Z \in T(\Omega \times \mathcal{M})$

$$\nabla_Z X \in T\Omega \oplus 0_{\mathcal{M}} \quad \nabla_Z U \in 0_\Omega \oplus T\mathcal{M}.$$

Consequently

$$H = \nabla_{X+\rho(X)}Y - \nabla_{Y+\rho(Y)}X, \quad \text{and} \quad V = \nabla_{X+\rho(X)}(\rho(Y)) - \nabla_{Y+\rho(Y)}(\rho(X)).$$

Then

$$\begin{aligned} V - \rho(H) &= \nabla_{X+\rho(X)}(\rho(Y)) - \rho(\nabla_{X+\rho(X)}Y) \\ &\quad - \nabla_{Y+\rho(Y)}(\rho(X)) + \rho(\nabla_{Y+\rho(Y)}X) \\ &= (\nabla_{X+\rho(X)}\rho)(Y) - (\nabla_{Y+\rho(Y)}\rho)(X) \\ &= F(X,Y), \end{aligned}$$

and $F(X,Y)$ statisfies the claim of the Lemma. $\square$

Corollary 2.9. *Let $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ be a smooth Ehresmann connection form and $F : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^2\Omega$ the curvature of ρ. Then $\mathrm{id}_\Omega \oplus \rho(T\Omega \oplus 0_{\mathcal{M}})$ is integrable if and only if*

$$F(X,Y) = 0 \tag{2.5}$$

for all smooth vector fields $X, Y \in \Gamma(T\Omega \oplus 0_{\mathcal{M}})$.

This is essentially a restatement of the classical Frobenius theorem in the context of Ehresmann connections

Proof. Let $\mathcal{H} = \mathrm{id}_\Omega \oplus \rho(T\Omega \oplus 0_{\mathcal{M}})$. Let X and Y be smooth vector fields taking values in $\mathcal{H}$. Then by Lemma 2.8 $[X + \rho(X), Y + \rho(Y)] \in \mathcal{H}$ if and only if $F(X,Y) = 0$. Subsequently $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$ if and only if $F(X,Y) = 0$ for all X and Y in $\Gamma(T\Omega \oplus 0_{\mathcal{M}})$. Consequently by Theorem 2.3, $\mathcal{H}$ is integrable if and only if $F(X,Y) = 0$. $\square$

Remark 2.10. Because of Lemma 2.8, the curvature is independent of the choice of torsion-free covariant derivative. As such it is also independent of the metric. See [MT97, §17] for more details.

The curvature of ρ can be expressed in coordinates as well. Let $y : U \subset \mathcal{M} \to V \subset \mathbb{R}^m$ be a coordinate chart. Let $x = (x^1, \dots, x^n)$ denote the standard coordinates in Ω. In this case, using the summation convention and using the expressions $X = X^i\partial_{x^i}$ and $Y = Y^i\partial_{x^i}$, we have

$$\rho = \rho^a_j \partial_{y^a} \otimes dx^j, \quad \rho(X) = \rho^a_j X^j \partial_{y^a} \quad \text{and} \quad \rho(Y) = \rho^a_j Y^j \partial_{y^a},$$

and the curvature is given by

$$F = \left(\frac{\partial \rho^b_k}{\partial x^i} - \frac{\partial \rho^b_i}{\partial x^k} + \rho^a_i \frac{\partial \rho^b_k}{\partial y^a} - \rho^a_k \frac{\partial \rho^b_i}{\partial y^a} \right) \partial_{y^b} \otimes dx^i \wedge dx^k. \tag{2.6}$$

Definition 2.11. Let $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$. Define $|\nabla^{\mathcal{M}}\rho| : \Omega \times \mathcal{M} \to \mathbb{R}$ by

$$(x,y) \mapsto \sup\{|\nabla_Y \rho(X)| : Y \in T_y\mathcal{M}, X \in \mathbb{R}^n, |Y| \le 1, |X| \le 1\}.$$

Let $c : \Omega \to \mathbb{R}$ be a function. We say ρ is *vertically Lipschitz with respect to c*, if for every $(x,y) \in \Omega \times \mathcal{M}$

$$|\nabla^{\mathcal{M}}\rho|(x,y) \le c(x).$$

Definition 2.12. Let ρ be a smooth Ehresmann connection and $\sigma : [0,1] \to \Omega$ an absolutely continuous curve starting at x_0 and ending at x_1. A *lift of σ along ρ starting at $y_0 \in \mathcal{M}$* is an absolutely continuous curve $\gamma : [0,1] \to \Omega \times \mathcal{M}$ for which $\gamma(0) = (x_0, y_0)$, $\pi_\Omega \circ \gamma = \sigma$, and

$$\dot\gamma(t) = \dot\sigma(t) + \rho_{\sigma(t),\gamma(t)} \cdot \dot\sigma(t)$$

for almost every $t \in [0,1]$.

Remark 2.13. If $\gamma : [0,1] \to \Omega \times \mathcal{M}$ is a lift of $\sigma : [0,1] \to \Omega$ starting at y_0, then γ is a solution to the *initial value problem*

$$\dot\gamma = \dot\sigma + \rho(\dot\sigma)$$

and

$$\gamma(0) = (\sigma(0), y_0).$$

To minimise notation, we identify the lift $\gamma : [0,1] \to \Omega \times \mathcal{M}$ with it's projection onto $\mathcal{M}$ where convenient. If $\gamma : [0,1] \to \mathcal{M}$ is said to be a lift of $\sigma : [0,1] \to \Omega$, then it is to be understood that

$$t \mapsto (\sigma(t), \gamma(t))$$

is a lift of σ in the sense previously defined.

REMARK 2.14. Let $c : \Omega \to \mathbb{R}$. If $\mathcal{M}$ is complete and ρ is vertically Lipschitz with respect to c, then for any C^1 curve σ for which $c \circ \sigma$ is locally integrable and any starting point y_0 there exists a unique lift of γ starting at y_0. This is because γ is the unique solution of an initial value problem [Lev55].

LEMMA 2.15. *Let $V : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a measurable family of Lipschitz vector fields with uniform Lipschitz constant C, that is $V_t = V(t, \cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is C-Lipschitz for almost every t. Suppose for $y \in \mathbb{R}^m$, $\gamma_y : [0,1] \to \mathbb{R}^m$ is a solution of the initial value problem*

$$\partial_t \gamma_y(t) = V_t(\gamma_y(t))$$
$$\gamma_y(0) = y.$$

Then

$$e^{-Ct}|y - y'| \le |\gamma_y(t) - \gamma_{y'}(t)| \le e^{Ct}|y - y'|.$$

In particular for $t \le 1$,

$$e^{-C}|y - y'| \le |\gamma_y(t) - \gamma_{y'}(t)| \le e^{C}|y - y'|.$$

Proof. Let $y, y' \in \mathbb{R}^m$. Since γ_y and $\gamma_{y'}$ are absolutely continuous, $t \mapsto |\gamma_y(t) - \gamma_{y'}(t)|$ is differentiable a.e. and

$$\begin{aligned} |\partial_t |\gamma_y(t) - \gamma_{y'}(t)|| &\le |\partial_t \gamma_y(t) - \partial_t \gamma_{y'}(t)| \\ &= |V_t(\gamma_y(t)) - V_t(\gamma_{y'}(t))| \\ &\le C|\gamma_y(t) - \gamma_{y'}(t)| \text{ for } a.e.\ t \in [0,1]. \end{aligned}$$

Thus multiplying by e^{-Ct} yields

$$e^{-Ct}\partial_t |\gamma_y(t) - \gamma_{y'}(t)| - Ce^{-Ct}|\gamma_y(t) - \gamma_{y'}(t)| \le 0,$$

and we obtain

$$\partial_t \left(e^{-Ct}|\gamma_y(t) - \gamma_{y'}(t)| \right) \le 0.$$

When we integrate both sides from 0 to τ, we arrive at

$$e^{-C\tau}|\gamma_y(\tau) - \gamma_{y'}(\tau)| - |\gamma_y(0) - \gamma_{y'}(0)| \le 0.$$

Rearranging we obtain

$$\begin{aligned} |\gamma_y(\tau) - \gamma_{y'}(\tau)| &\le e^{C\tau}|\gamma_y(0) - \gamma_{y'}(0)| \\ &\le e^{C\tau}|y - y'|. \end{aligned}$$

Similarly,

$$\partial_t e^{Ct}|\gamma_y(t) - \gamma_{y'}(t)| \ge 0.$$

We can again integrate to obtain

$$e^{-C\tau}|y - y'| \le |\gamma_y(\tau) - \gamma_{y'}(\tau)|.$$

$\square$

2.2. Non-smooth connections

Considering an Ehresmann connection as a section of a vector bundle allows us to work with measurable and distributional Ehresmann connections. For the rest of the text, we assume $\mathcal{M}$ to be Riemannian and complete.

If $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ is measurable, we call it a *measurable Ehresmann connection form.* For every $(x, y) \in \Omega \times \mathcal{M}$ it defines a subspace $\mathcal{H}_{(x,y)} \subset T_{(x,y)}(\Omega \times \mathcal{M}) \subset T(\Omega \times \mathcal{M})$ given by

$$\mathcal{H}_{(x,y)} := \{X + \rho_{(x,y)}(X) : X \in T_x\Omega\}.$$

Let

$$\mathcal{H} = \bigcup_{(x,y)\in\Omega\times\mathcal{M}} \mathcal{H}_{(x,y)} \subset T(\Omega \times \mathcal{M}).$$

This is the *measurable Ehresmann connection associated to ρ.*

Definition 2.16. Let $\mathcal{M}$ be a smooth Riemannian manifold. We define $\mathcal{A}(\Omega \times \mathcal{M})$ to be the set of $\rho \in L^\infty(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^1\Omega)$ such that

(1) $\|d_\Omega \rho\|_\infty < \infty$ and
(2) $\|\nabla^{\mathcal{M}}\rho\|_\infty < \infty$.

We define $\mathcal{A}_{\mathrm{loc}}(\Omega \times \mathcal{M})$ to be the set of $\rho \in L^\infty_{\mathrm{loc}}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^1\Omega)$ such that

(1) ρ is uniformly vertically Lipschitz, *i.e.* $\|\nabla^{\mathcal{M}}\rho\|_\infty < \infty$; *n.b.* this condition is not local,
(2) $d\rho \in L^\infty_{\mathrm{loc}}(\Omega \times \mathcal{M}, T\mathcal{M} \otimes \Lambda^2\Omega)$, and
(3) for every $U \subset \mathcal{M}$, $\rho|\Omega \times U \in L^\infty(\Omega \times U, TU \otimes \Lambda^1\Omega)$.

Proposition 2.17. *Let $\psi : U \subset\subset \mathcal{M} \to V \subset \mathbb{R}^m$ be a coordinate chart smooth up to the boundary and compactly contained in $\mathcal{M}$. Let $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Omega$ be a measurable Ehresmann connection form. Then the coordinate transformation $\mathrm{id}_\Omega \times \psi$ induces an Ehresmann connection form $\tilde{\rho} : \Omega \times V \to TV \otimes \Lambda^1\Omega$ satisfying*

$$\begin{array}{ccc} \Omega \times M & \xrightarrow{\rho} & T\mathcal{M} \otimes \Lambda^1\Omega \\ {\scriptstyle \mathrm{id}_\Omega \times \psi^{-1}}\uparrow & & \uparrow{\scriptstyle D\psi^{-1}\otimes \mathrm{id}_{\Lambda^1\Omega}} \\ \Omega \times V & \xrightarrow[\tilde{\rho}]{} & TV \otimes \Lambda^1\Omega \end{array} .$$

We call $\tilde{\rho}$ a *coordinate representation* of ρ on (U, ψ).

Proof. Denote by Φ the diffeomorphism $\mathrm{id}_\Omega \times \psi : \Omega \times U \to \Omega \times V$. Then $\Phi_* : T(\Omega \times U) \to T(\Omega \times V)$ is a vector bundle isomorphism. In particular, $\Phi_* : 0_\Omega \oplus TU \to 0_\Omega \oplus TV \cong \mathbb{R}^m \times (\Omega \times V)$.

Similarly $\Phi^{-1*} : \Lambda^1(\Omega \times U) \to \Lambda^1(\Omega \times V)$ is a vector bundle isomorphism taking $T^*\Omega \oplus 0_U$ to $T^*\Omega \oplus 0_V$.

We combine these two maps to yield a map $\Psi := \Phi_* \otimes \Phi^{-1*} : TU \otimes \Lambda^1\Omega \to TV \otimes \Lambda^1\Omega$. The new connection form $\tilde{\rho} : \Omega \times V \to TV \otimes \Lambda^1\Omega$ is defined as

$$\tilde{\rho} := \Psi \circ \rho \circ \Phi^{-1}.$$

□

Proposition 2.18. *Let $U \subset\subset \mathcal{M}$, $\psi : U \to V$ be a coordinate neighbourhood on $\mathcal{M}$, which is smooth up to the boundary of U. Let $\rho \in \mathcal{A}_{\mathrm{loc}}(\Omega \times \mathcal{M})$ be an Ehresmann connection form, and let $\tilde{\rho} : \Omega \times V \to TV \otimes \Lambda^1\Omega$ denote its coordinate representation* *on (U,ψ). Suppose further that $|d_\Omega\rho|$ is essentially bounded on $\Omega\times U$. Then $\tilde\rho\in\mathcal{A}(\Omega\times U)$.*

Proof. It is sufficient to verify this calculation in coordinates. First because U is compactly contained in $\mathcal{M}$, ρ is essentially bounded on U. Furthermore ψ, the coordinate chart, is smooth and hence C^1 up to the boundary. Hence $\tilde\rho$ is essentially bounded on $\Omega\times V$.

Now the vertical covariant derivative $\nabla^{\mathcal{M}}\rho$ is essentially bounded and given by

$$\nabla^{\mathcal{M}}_\beta\rho^\alpha_i=\partial_{y^\beta}\rho^\alpha_i+\Gamma^\alpha_{\mu\beta}\rho^\mu_i$$

which rearranged yields

$$\partial_{y^\beta}\rho^\alpha_i=\nabla^{\mathcal{M}}_\beta\rho^\alpha_i-\Gamma^\alpha_{\mu\beta}\rho^\mu_i.$$

where $\Gamma^\alpha_{\mu\beta}$ are the Christoffel symbols of the metric on $\mathcal{M}$. Because ψ is C^1 up to the boundary of U, the summands are continuous up to the boundary of V. Consequently $\partial_{y^\beta}\rho^\alpha_i$ is the sum of two essentially bounded functions, and hence is itself essentially bounded. The operator d_Ω does not change under the coordinate transformation, so $d_\Omega\tilde\rho$ is merely $(\Psi^{-1})^*d_\Omega\rho$. Hence $d_\Omega\tilde\rho$ is essentially bounded. □

Definition 2.19. Let $\rho\in\mathcal{A}_{\mathrm{loc}}(\Omega\times\mathcal{M})$. We define

$$F_\rho\in L^\infty_{\mathrm{loc}}(\Omega\times\mathcal{M},T\mathcal{M}\otimes\Lambda^2\Omega)$$

by

$$F_\rho(X,Y)=(\nabla^{\mathcal{M}}_{\rho(X)}\rho)(Y)-(\nabla^{\mathcal{M}}_{\rho(Y)}\rho)(X)+d_\Omega(X,Y)$$

for $X,Y\in\mathbb{R}^n$.

2.3. Holonomy bounds for smooth connections

An important notion for a connection is that of *holonomy*. Intuitively the holonomy of a connection tells us how much the start and endpoints of a lift along a closed loop differ. In this section we make more concrete the notion that "curvature is an infinitesimal measure of holonomy".

Definition 2.20. Let $\sigma:[0,1]\to\Omega$ be a closed absolutely continuous curve and $\rho\in\Gamma(TM\otimes\Lambda^1\Omega)$ a smooth Ehresmann connection form on $\Omega\times\mathcal{M}$. *The holonomy along ρ about σ starting at $y\in\mathcal{M}$* is defined by

$$\mathrm{Hol}(\rho,\sigma,y):=d(\gamma_y(0),\gamma_y(1)),$$

where γ_y is the lift of σ along ρ starting at y. Note that this is independent of a choice of parametrisation of σ. Consequently if $\sigma|(0,1)$ is a homeomorphism and $\sigma(0)=\sigma(1)=x_0$, we can define the holonomy about (S,x_0) along ρ starting at y to be

$$\mathrm{Hol}(\rho,(S,x_0),y)=\mathrm{Hol}(\rho,\sigma,y),$$

where $S=\sigma([0,1])$.

Lemma 2.21. *Let $c:\Omega\to\mathbb{R}$ be a continuous function and $\rho:\Omega\times\mathcal{M}\to T\mathcal{M}\otimes\Lambda^1\Omega$ a smooth Ehresmann connection form, vertically Lipschitz with respect to c and bounded by C. Suppose that $\sigma:[0,1]\to\Omega$ is an absolutely continuous curve, and let $R=C\ell(\sigma)$, where $\ell(\sigma)$ is the length of σ. If $y\in\mathcal{M}$ is such that $B(y,R)\subset\subset\mathcal{M}$, then a lift of σ starting at y exists.*

Proof. This is a straightforward application of the existence and uniqueness of ODEs. □

DEFINITION 2.22. We say that a continuous map $\sigma : [0,1]^2 \to \Omega$ is a *null homotopy* if

- $\sigma(0,t) = \sigma(0,0)$ for every $t \in [0,1]$; and
- $\sigma(s,0) = \sigma(s,1) = \sigma(0,0)$ for every $s \in [0,1]$.

If for every $s \in [0,1]$ the map $t \mapsto \sigma(s,t)$ is absolutely continuous, we say σ is an *AC null homotopy.*

DEFINITION 2.23. Let $\Sigma \subset \Omega$ be a closed subset. We say that $\sigma : [0,1]^2 \to \Omega$ is a *null-homotopic parametrisation of* Σ if

- σ is an AC null-homotopy,
- $\sigma([0,1]^2) = \Sigma$, and
- $\sigma|(0,1)^2 : (0,1)^2 \to \sigma((0,1)^2)$ is a homeomorphism.

Given a null-homotopic parametrisation of Σ, we define the *interior of* Σ, $\mathrm{Int}(\Sigma) = \sigma((0,1)^2)$.

DEFINITION 2.24. Let $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$ be a smooth Ehresmann connection form. If σ is an AC null homotopy, we say that $\gamma : [0,1]^2 \to \Omega \times \mathcal{M}$ is a *lift of* σ *along* ρ *starting at* $y \in \mathcal{M}$ if for every $s \in [0,1]$, the map $t \mapsto \gamma(s,t)$ is the lift of the curve $t \mapsto \sigma(s,t)$ along ρ starting at $y \in \mathcal{M}$.

DEFINITION 2.25. Let $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$ be a smooth Ehresmann connection form and $\sigma : [0,1]^2 \to \Sigma \subset \Omega$ be a null-homotopic parametrisation of Σ. Let $\gamma : [0,1]^2 \to \Sigma \times \mathcal{M}$ be a lift of σ along ρ starting at $y \in \mathcal{M}$. We say that $h : \mathrm{Int}(\Sigma) \to \mathcal{M}$ is a *height function on* Σ *for* γ if

$$\gamma(s,t) = (\sigma(s,t), h(\sigma(s,t)))$$

for every $(s,t) \in (0,1)^2$.

LEMMA 2.26. *Let* $c : \Omega \to \mathbb{R}$ *be a continuous function and let* $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ *be a smooth Ehresmann connection form, vertically Lipschitz with respect to* c *and bounded by* C. *Let* $\sigma : [0,1]^2 \to \Omega$ *be a null homotopy which is* C^2 *up to the boundary. Let* $y_0 \in \mathcal{M}$ *be a point such that* $B(y_0, C\sup_{s\in[0,1]} \ell(\sigma_s)) \subset\subset \mathcal{M}$, *where* $\sigma_s : [0,1] \to \Omega$ *is the curve* $t \mapsto \sigma(s,t)$. *Then there exists a lift* $\gamma : [0,1]^2 \to \Omega \times \mathcal{M}$ *of* σ *along* ρ *starting at* y_0 *and*

$$\mathrm{Hol}(\rho, \sigma_1, y_0) \leq \int_{[0,1]^2} \exp\left(\int_t^1 c(\sigma_s(t'))\,|\partial_t\sigma_s|\,dt'\right) |F_\rho \circ \gamma||\partial_t\sigma \wedge \partial_s\sigma|\,ds\,dt, \tag{2.7}$$

where F_ρ *is the curvature of* ρ.

Proof. The condition on y_0 and Lemma 2.21 guarantee the existence of the lift $\gamma : [0,1]^2 \to \Omega \times \mathcal{M}$. The goal is to show that the holonomy of the extremal curve in the null homotopy σ can be found by the s-derivative of the lift γ.

We embed $\mathcal{M}$ isometrically into some $\mathbb{R}^N$ using a Nash embedding [Nas56]. In this case we can embed $\Omega \times \mathcal{M}$ in $\mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^{n+N}$, so we can identify vectors and derivations.

Now $\sigma : [0,1]^2 \to \mathbb{R}^n$ and $\gamma : [0,1]^2 \to \mathbb{R}^{n+N}$. We identify vectors of $\mathbb{R}^n$ with those of $\mathbb{R}^{n+N}$ via the standard inclusion $v \mapsto (v,0) \in \mathbb{R}^n \times \mathbb{R}^N = \mathbb{R}^{n+N}$. Furthermore, the Levi-Civita connection is given by standard partial differentiation, so ∇ now denotes the gradient operator. Let $S = \partial_s\sigma$ and $T = \partial_t\sigma$. These are maps $[0,1]^2 \to \mathbb{R}^n \subset$

$\mathbb{R}^{n+N}$. Since σ and γ are C^2 we have $\partial_s\partial_t\sigma = \partial_t\partial_s\sigma$ and $\partial_s\partial_t\gamma = \partial_t\partial_s\gamma$, and so $\partial_t S = \partial_s T$. Since γ is a lift σ,

$$\partial_t\gamma_s = \partial_t\sigma_s + (\rho\circ\gamma)\cdot\partial_t\sigma_s$$

for all $s \in [0,1]$. We have denoted by $(\rho\circ\gamma)\cdot\partial_t\sigma_s$ the vector field

$$(s,t) \mapsto \rho(\gamma(s,t))(\partial_t\sigma_s(t)).$$

By the chain rule

$$\nabla(\rho\circ\gamma) = (\nabla_{\nabla\gamma}\rho)\circ\gamma.$$

In particular

$$\partial_s(\rho\circ\gamma) = (\nabla_{\partial_s\gamma}\rho)\circ\gamma$$

and

$$\begin{aligned}\partial_t(\rho\circ\gamma) &= (\nabla_{\partial_t\gamma}\rho)\circ\gamma\\ &= (\nabla_{[\partial_t\sigma+(\rho\circ\gamma)\cdot\partial_t\sigma]}\rho)\circ\gamma\\ &= (\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma.\end{aligned}$$

Let

$$E := \partial_s\gamma - (S + (\rho\circ\gamma)\cdot S).$$

We examine the t derivative of E:

$$\begin{aligned}\partial_t E &= \partial_t\partial_s\gamma - \partial_t\partial_s\sigma - (\partial_t(\rho\circ\gamma))\cdot\partial_s\sigma - (\rho\circ\gamma)\cdot\partial_t\partial_s\sigma\\ &= \partial_s\partial_t\gamma - \partial_s\partial_t\sigma - [(\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma]\cdot\partial_s\sigma - (\rho\circ\gamma)\cdot\partial_t\partial_s\sigma\\ &= \partial_s[(\rho\circ\gamma)\cdot\partial_t\sigma] - [(\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma]\cdot\partial_s\sigma - (\rho\circ\gamma)\cdot\partial_t\partial_s\sigma\\ &= [\partial_s(\rho\circ\gamma)]\cdot\partial_t\sigma + (\rho\circ\gamma)\cdot\partial_s\partial_t\sigma - [(\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma]\cdot\partial_s\sigma\\ &\quad - (\rho\circ\gamma)\cdot(\partial_t\partial_s\sigma)\\ &= [(\nabla_{\partial_s\gamma}\rho)\circ\gamma]\cdot T - [(\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma]\cdot S\\ &= [(\nabla_{\partial_s\gamma}\rho)\circ\gamma]\cdot T - [(\nabla_{[S+(\rho\circ\gamma)\cdot S]}\rho)\circ\gamma]\cdot T\\ &\quad + [(\nabla_{[S+(\rho\circ\gamma)\cdot S]}\rho)\circ\gamma]\cdot T - [(\nabla_{[T+(\rho\circ\gamma)\cdot T]}\rho)\circ\gamma]\cdot S\\ &= [(\nabla_{[\partial_s\gamma-S-(\rho\circ\gamma)\cdot S]}\rho)\circ\gamma]\cdot T + (F_\rho\circ\gamma)(S,T)\\ &= [(\nabla_E\rho)\circ\gamma](T) + (F_\rho\circ\gamma)(S,T).\end{aligned}$$

For every $s\in[0,1]$, the map $t\mapsto E(s,t)$ is absolutely continuous and hence $|\partial_t|E|| \le |\partial_t E|$ for almost every $t\in[0,1]$. Hence

$$\begin{aligned}|\partial_t|E|| &\le |\partial_t E|\\ &\le |(F_\rho\circ\gamma)(S,T)| + |((\nabla_E\rho)\circ\gamma)(T)|\\ &\le |(F_\rho\circ\gamma)(S,T)| + |E||T|(c\circ\sigma)\end{aligned}$$

almost everywhere in $[0,1]^2$. Let $f : [0,1]^2 \to \mathbb{R}$ be the function $|E|$, and let us introduce a function $g : [0,1]^2 \to (0,\infty)$ given by

$$(s,t) \mapsto \exp\left(-\int_0^t (c\circ\sigma)(s,t')|T|\,dt'\right).$$

Then $\partial_t g = -|T|(c\circ\sigma)g$ and $|\partial_t f| \le |(F_\rho\circ\gamma)(S,T)| + |T|(c\circ\sigma)\, f$, and hence

$$\begin{aligned} g(s,1)f(s,1)-&g(s,0)f(s,0) \\ &= \int_0^1 \partial_t[g(s,t)f(s,t)]\,dt \\ &\le \int_0^1 -|T|(c\circ\sigma)\,g\,f + |T|(c\circ\sigma)\,g\,f + |F_\rho\circ\gamma(S,T)|g\,dt \\ &\le \int_0^1 |F_\rho\circ\gamma(S,T)|g\,dt. \end{aligned}$$

Now $\partial_s\gamma$ and S are both zero for $t=0$, while $S(s,1)=0$. Hence $f(s,0)=0$ and $f(s,1)=|\partial_s\gamma(s,1)|$ for $s\in[0,1]$. This yields

$$g(s,1)|\partial_s\gamma(s,1)| \le \int_0^1 g(s,t)|F_\rho\circ\gamma(S,T)|\,dt$$

for every $s\in[0,1]$. Consequently

$$\begin{aligned} |\partial_s\gamma(s,1)| &\le \int_0^1 \frac{g(s,t)}{g(s,1)}|(F_\rho\circ\gamma)(S,T)|\,dt \\ &\le \int_0^1 \exp\left(\int_t^1 (c\circ\sigma)(s,t')\,|T|\,dt'\right)|(F_\rho\circ\gamma)(S,T)|\,dt \end{aligned}$$

for every $s\in[0,1]$.

Because F_ρ is antisymmetric, we have that $|F_\rho(S,T)|$ is bounded by $|F_\rho|\,|S\wedge T|$. Hence when we integrate with respect to s from 0 to 1, we get

$$\begin{aligned} d(\gamma(1,1),\gamma(0,1)) &\le \int_0^1 |\partial_s\gamma(s,1)|\,ds \\ &\le \iint_{[0,1]^2} \exp\left(\int_t^1 (c\circ\sigma)(s,t')\,|T|\,dt'\right)|F_\rho\circ\gamma|\,|S\wedge T|\,dt\,ds. \end{aligned}$$

But $\gamma(0,1)=\gamma(1,0)=\gamma(0,0)$, from which the result follows. □

Lemma 2.27. *Let $\Sigma\subset\Omega$ be a closed convex subset, contained in an affine two-plane, which is homeomorphic to a closed disk. For $0\le r\le 1$ there exists a path $\zeta_r:[0,1]\to\Sigma$, and a null homotopy $\sigma^r:[0,1]^2\to\Sigma$, satisfying properties*

(1) σ^1 is a null-homotopic parametrisation of Σ;
(2) $\zeta_1|[0,1)$ is an arc, and $\partial\Sigma=\zeta_1([0,1])$;
(3) for $0\le r\le 1$, σ^r is a smooth embedding on $(0,1)^2$;
(4) $t\mapsto\sigma^r(1,t)$ is equal to ζ_r;
(5) $\lim_{r\to 1}\|\zeta_r-\zeta_1\|_{1,1}=0$;
(6) for $r<1$, σ^r is smooth up to the boundary of $[0,1]^2$;
(7) for every $0\le s<1$

$$\lim_{r\to 1}\|\sigma^r(s,\cdot)-\sigma^1(s,\cdot)\|_{1,1}=0.$$

Proof. Without loss of generality we may assume that $\Sigma\subset\mathbb{R}^2=\mathbb{C}\subset\mathbb{R}^2\times\mathbb{R}^{n-2}$ and that $0\in\Sigma$. Suppose that $B(0,\varepsilon)\subset\Sigma\subset B(0,R)$. Since Σ is convex, $\partial\Sigma$ is Lipschitz regular [Mor08, Lemma 3.4.1]. Let $\zeta:S^1\to\partial\Sigma$ be the bi-Lipschitz projection

$$e^{2\pi it}\mapsto R(t)e^{2\pi it},$$

where $R : \mathbb{R} \to [0, \infty)$ is a 1-periodic Lipschitz map.

Let $\varphi : \mathbb{R} \to [0, \infty)$ be a smooth function compactly supported in $[-1, 1]$ satisfying

$$\varphi(x) = \varphi(-x)$$

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1.$$

Let $a > \varepsilon/\|R'\|_\infty$. Define the map $F : [0, 1] \times \mathbb{R} \to [0, R] \times \mathbb{R}$ by

$$(r, t) \mapsto \left([1 - a(1 - r)] \int_{\mathbb{R}} \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} R(t')\, dt', t \right),$$

which is smooth on $[0, 1) \times \mathbb{R}$. Let F_1 and F_2 denote the coordinate functions of F, *i.e.* $F(r, t) = (F_1(r, t), F_2(r, t))$. Then

$$\begin{aligned} \partial_t F_1(r, t) &= [1 - a(1 - r)] \int_{\mathbb{R}} \frac{\varphi'\left(\frac{t'-t}{1-r}\right)}{1 - r} \frac{-1}{1 - r} R(t')\, dt' \\ &= [1 - a(1 - r)] \int_{\mathbb{R}} \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} R'(t')\, dt'. \end{aligned}$$

So $|\partial_t F_1| \le \|R'\|_\infty$.

Then

$$\begin{aligned} \partial_r F_1(r, t) = a &\int_{\mathbb{R}} \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} R(t')\, dt' \\ &+ [1 - a(1 - r)] \int_{\mathbb{R}} R(t') \frac{\varphi'\left(\frac{t'-t}{1-r}\right)}{(1 - r)} \frac{t' - t}{(1 - r)^2}\, dt' \\ &+ [1 - a(1 - r)] \int_{\mathbb{R}} R(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{(1 - r)^2}\, dt' \\ = a &\int_{\mathbb{R}} R(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r}\, dt' - [1 - a(1 - r)] \Bigg[\int_{\mathbb{R}} R'(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} \frac{t' - t}{1 - r}\, dt' \\ &+ \int_{\mathbb{R}} R(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} \frac{1}{1 - r}\, dt' - \int_{\mathbb{R}} R(t') \frac{\varphi\left(\frac{t-t'}{1-r}\right)}{(1 - r)^2}\, dt' \Bigg] \\ = a &\int_{\mathbb{R}} R(t') \frac{\varphi\left(\frac{t-t'}{1-r}\right)}{1 - r} dt' + [1 - a(1 - r)] \int_{\mathbb{R}} R'(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} \frac{t - t'}{1 - r}\, dt'. \end{aligned}$$

It follows from this that $|\partial_r F_1(r, t)| \le aR + [1 + a]\|R'\|_\infty$. Furthermore, because $R(t) > \varepsilon$ for every $t \in \mathbb{R}$,

$$\begin{aligned} |\partial_r F(r, t)| &\ge a\varepsilon - |1 - a(1 - r)| \left| \int_{\mathbb{R}} R'(t') \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} \frac{t' - t}{1 - r}\, dt' \right| \\ &\ge a\varepsilon - |1 - a(1 - r)| \int_{t-(1-r)}^{t+(1-r)} |R'(t')| \frac{\varphi\left(\frac{t'-t}{1-r}\right)}{1 - r} \left| \frac{t' - t}{1 - r} \right| dt' \\ &\ge a\varepsilon - [1 - a(1 - r)]\|R'\|_\infty \int_{r-1}^{1-r} \frac{\varphi\left(\frac{t'}{1-r}\right)}{1 - r} \left| \frac{t'}{1 - r} \right| dt' \\ &\ge a\varepsilon - [1 - a(1 - r)]\|R'\|_\infty > 0. \end{aligned}$$

Hence there is a $\delta > 0$ such that for some $1/2 < r_0 < 1$ and every $r > r_0$, $\delta < |\partial_r F_1(r,t)|$. Furthermore $J_f > 0$ on $[r_0, 1) \times \mathbb{R}$, and so F is a diffeomorphism on $[r_0, 1) \times \mathbb{R}$.

The functions $t \mapsto F(r,t)$ converge, for every $1 \le p < \infty$, in $W^{1,p}([0,1], \mathbb{R})$ to $t \mapsto R(t)$ as $r \to 1$ by standard results on convolution [Eva98, 5.3.1].

Let $f : [0,1] \to \mathbb{R}$ be the map $t \mapsto F(r_0, t)$ and let g be the map $t \mapsto \partial_r F(r_0, t)$. Assume $F(r,t) > \varepsilon$ for all $t \in [0,1]$ and $r \in [r_0, 1]$. For every $t \in [0,1]$ define a piecewise linear function $p_t' : [1/2, r_0] \to \mathbb{R}$ by

$$p_t'(1/2) = \varepsilon, \quad p_t'(1/2 + \delta_1) = c(t), \quad p_t'(r_0 - \delta_1) = c(t), \quad \text{and} \quad p_t'(r_0) = g(t).$$

Define $p_t(r) = \varepsilon/2 + \int_{1/2}^r p_t'(r')\, dr'$. Then

$$p_t(r_0) = c(t)(r_0 - 1/2) + (\varepsilon - c(t))\delta_1/2 + (g(t) - c(t))\delta_1/2$$

Choose $0 < \delta_1 < \operatorname{ess\,inf}_t \min\{2f(t)/(g(t) + \varepsilon), (r_0 - 1/2)\}$, and set

$$c(t) = [f(t) - (\varepsilon + g(t))\frac{\delta_1}{2}](r_0 - 1/2 - \delta_1)^{-1}.$$

Then $p_t(r_0) = f(t)$, $c(t) > 0$ and $c(t)$ is smooth for all $t \in \mathbb{R}$. It follows $p_t'(r) > 0$ for all $t \in \mathbb{R}$ and $r \in [1/2, r_0]$.

The function $\hat{G} : [0,1] \times \mathbb{R} \to [0, R] \times \mathbb{R}$

$$(r,t) \mapsto \begin{cases} (\varepsilon r, t) & r < 1/2, \\ (p_t(r), t) & 1/2 \le r < r_0, \\ (F(r,t), t) & r_0 \le r \end{cases}$$

is a uniformly Lipschitz homeomorphism onto its image, and is a C^1-smooth diffeomorphism on $[0,1) \times \mathbb{R}$.

Define $\tilde{G} : [0,1] \times \mathbb{R} \to [0,R] \times \mathbb{R}$ by taking a smooth function sufficiently C^1-close to $\hat{G}$ on $[1/4, 1/2 + r_0/2]$, and equal to $\hat{G}$ on $([0, 1/4) \cup (1/2 + r_0/2, 1]) \times \mathbb{R}$. Now $\tilde{G}$ is a C^∞-smooth diffeomorphism on $[0,1) \times \mathbb{R}$.

Let $\Phi : [0,R] \times \mathbb{R} \to \mathbb{C}$ be the map $(r,t) \mapsto re^{2\pi i t}$, let $G : B(0,1) \to \mathbb{C}$ be given by $\Phi \circ \tilde{G}$ and let $g : [0,1]^2 \to B(0,1)$ be the map $(s,t) \mapsto se^{2\pi i t} + (1-s)$. For $r \in [0,1]$ we define $\zeta_r : [0,1] \to \mathbb{C}$ by $t \mapsto G(re^{2\pi i t})$ and $\sigma_r : [0,1] \times [0,1] \to \mathbb{R}^2$ by $(s,t) \mapsto G(rg(s,t))$. For $r > 1/2 + r_0/2$, $\tilde{G}(r,t) = F(r,t)$, hence, for every $1 \le p < \infty$,

$$\lim_{r \to 1} \|\zeta_r - \zeta_1\|_{1,p} = 0.$$

Fix $0 < s < 1$. Let $\varepsilon > 0$. Let $c_r : [0,1] \to B(0,1)$ denote the path $t \mapsto rg(s,t)$. There is an $\alpha > 0$ such that the length of $G(c_r([0,\alpha] \cup [1-\alpha, 1]))$ is less than ε for every $r \le 1$, and there is an $\tilde{R} < 1$ such that $G(c_r((\alpha, 1-\alpha))) \subset B(0, \tilde{R})$.

Because G is a diffeomorphism on $\overline{B(0,\tilde{R})}$, it follows that $G \circ c_r|(\alpha, 1-\alpha)$ converges in $W^{1,1}((\alpha, 1-\alpha), \mathbb{C})$ to $G \circ c_1|(\alpha, 1-\alpha)$. Furthermore for any curve γ

$$\begin{aligned} \|\gamma\|_{1,1} &\le \|D\gamma\|_1 + \|\gamma\|_1 \\ &\le 2\|D\gamma\|_1 + |\gamma(0)| \\ &\le 2\ell(\gamma) + |\gamma(0)|. \end{aligned}$$

Hence

$$\begin{aligned}
\|\sigma^r(s,\cdot)-&\sigma^1(s,\cdot)\|_{1,1} \\
&\le \|G\circ c_r - 2G\circ c_1\|_{W^{1,1}(\alpha,1-\alpha)} \\
&\quad + \|\sigma^r(s,\cdot)-\sigma^1(s,\cdot)\|_{W^{1,1}((0,\alpha)\cup(1-\alpha,1))} \\
&\le \|G\circ c_r - 2G\circ c_1\|_{W^{1,1}(\alpha,1-\alpha)} + 2\ell(G(c_r((0,\alpha)\cup(1-\alpha,1)))) \\
&\quad + 2\ell(G(c_1((0,\alpha)\cup(1-\alpha,1)))) + (1-r) \\
&\le \|G\circ c_r - G\circ c_1\|_{W^{1,1}(\alpha,1-\alpha)} + 4\varepsilon + 1 - r.
\end{aligned}$$

Now if we take the limit as $r \to 1$ we get that

$$\limsup_{r\to 1} \|\sigma^r(s,\cdot)-\sigma^1(s,\cdot)\|_{1,1} \le 4\varepsilon.$$

But $\varepsilon > 0$ was arbitrary, and hence

$$\lim_{r\to 1} \|\sigma^r(s,\cdot)-\sigma^1(s,\cdot)\|_{1,1} = 0.$$

□

We denote by $\mathcal{H}^2$ the Hausdorff 2-measure normalised to be equal to the Lebesgue 2-measure.

Lemma 2.28. *Let $U \subset\subset \mathcal{M}$ be a coordinate neighbourhood and let $\Sigma \subset \Omega$, be a convex subset homeomorphic to a closed disk contained in an affine two-plane. Let $|\partial\Sigma|$ denote the length of the boundary. Let $\rho \in \Gamma(T\mathcal{M}\otimes\Lambda^1\Omega)$ be a smooth Ehresmann connection form for which*

$$|\rho| \le C' \quad \text{and} \quad |\nabla^{\mathcal{M}}\rho| \le C \text{ on } \Omega\times U.$$

For $0 \le r \le 1$ let $\zeta_r : [0,1] \to \Sigma$ be a path and $\sigma^r : [0,1]^2 \to \Sigma$ be a null homotopic parametrisation satisfying properties 1-7 of Lemma 2.27. If there is a $y \in U$ satisfying $\mathrm{dist}(y,\partial U) \ge C'|\partial\Sigma|$ *then for $0 \le r \le 1$ there is a lift $\gamma^r : [0,1]^2 \to \Sigma\times U$ of σ^r along ρ starting at y, such that*

$$\lim_{r\to 1} \gamma^r(x) = \gamma^1(x)$$

for every $x \in [0,1]^2$. Subsequently there exists a height function $h : \mathrm{Int}(\Sigma) \to U$ for γ^1 given by

$$h = \pi_U \circ \gamma \circ (\sigma^1|(0,1)^2)^{-1},$$

satisfying

$$\mathrm{Hol}(\rho,\zeta^1,y) \le e^{C|\partial\Sigma|} \int_\Sigma |F(x,h(x))|\, d\mathcal{H}^2(x). \tag{2.8}$$

Proof. Let $\sigma_1,\sigma_2 \in W^{1,1}([0,1],\Sigma)$. Let γ_i denote the lift along ρ of σ_i starting at y. Note that

$$|\sigma_1(t)-\sigma_2(t)| \le \|\sigma_1-\sigma_2\|_{W^{1,1}},$$

and as such $|\gamma_1(0)-\gamma_2(0)| \le |\sigma_1(0)-\sigma_2(0)| \le \|\sigma_1-\sigma_2\|_{1,1}$. Then

$$\begin{aligned}
|\partial_t|\gamma_1-\gamma_2|| &\le |\dot\sigma_1 + \rho\circ\gamma_1\cdot\dot\sigma_1 - \dot\sigma_2 - \rho\circ\gamma_2\cdot\dot\sigma_2| \\
&\le |\dot\sigma_1-\dot\sigma_2| + |\rho\circ\gamma_1\cdot\dot\sigma_1 - \rho\circ\gamma_2\cdot\dot\sigma_1| + |\rho\circ\gamma_2\cdot\dot\sigma_1 - \rho\circ\gamma_2\cdot\dot\sigma_2| \\
&\le |\dot\sigma_1-\dot\sigma_2| + \|\nabla^{\mathcal{M}}\rho\|_\infty|\dot\sigma_1||\gamma_1-\gamma_2| + \|\rho\|_\infty|\dot\sigma_1-\dot\sigma_2|
\end{aligned}$$

Recall that $\|\rho\|_\infty \le C'$ and $\|\nabla^{\mathcal{M}}\rho\| \le C$. We can once again rearrange and multiply with $e^{-C\int_0^t |\dot\sigma^1|\,dt'}$ to arrive at

$$\begin{aligned}\partial_t\left[e^{-C\int_0^t |\dot\sigma_1|\,dt'}|\gamma_1-\gamma_2|\right] &\le (1+C')e^{-C\int_0^t |\dot\sigma_1|\,dt'}|\dot\sigma_1-\dot\sigma_2|\\ &\le (1+C')|\dot\sigma_1-\dot\sigma_2|.\end{aligned}$$

When we integrate both sides from 0 to $\tau \le 1$, we arrive at

$$e^{-C\int_0^\tau |\dot\sigma_1|\,dt}|\gamma_1(\tau)-\gamma_2(\tau)| - |\gamma_1(0)-\gamma_2(0)| \le (1+C')\|\sigma_1-\sigma_2\|_{1,1}.$$

Hence

$$|\gamma_1(\tau)-\gamma_2(\tau)| \le e^{C\|\dot\sigma_1\|_1}(2+C')\|\sigma_1-\sigma_2\|_{1,1}.$$

Let $\sigma^r : [0,1]^2 \to \Sigma$ and $\zeta_r : [0,1] \to \Sigma$ be the null-homotopies and paths given by Lemma 2.27. Let $\gamma^r : [0,1]^2 \to \Sigma \times U$ be the lift of σ^r along ρ starting at y. Then

$$|\gamma^r(s,t)-\gamma^1(s,t)| \le e^{C\|\dot\sigma^1(s,\cdot)\|_1}(2+C')\|\sigma^r(s,\cdot)-\sigma^1(s,\cdot)\|_{1,1},$$

and

$$|\gamma^r(1,t)-\gamma^1(1,t)| < e^{C\|\dot\zeta_r\|_1}(2+C')\|\zeta_r-\zeta_1\|_{1,1}.$$

We can now take the limit $r \to 1$ to get $\lim_{r\to1}\gamma^r(s,t) = \gamma^1(s,t)$.

Because for $r < 1$ the null-homotopies σ^r are smooth (and hence C^2) up to the boundary, we can apply Lemma 2.26 and deduce that

$$\begin{aligned}\mathrm{Hol}(\rho,\zeta_1,y) = d(\gamma^1(1,0),\gamma^1(1,1)) &= \lim_{r\to1} d(\gamma^r(1,0),\gamma^r(1,1))\\ &= \lim_{r\to1}\mathrm{Hol}(\rho,\zeta_r,y)\\ &\le \lim_{r\to1} e^{C\ell(\zeta_r)}\int_{[0,1]^2}|F_\rho\circ\gamma^r||\partial_s\sigma^r\wedge\partial_t\sigma^r|\,dsdt\\ &\le e^{C|\partial\Sigma|}\int_{[0,1]^2}|F_\rho\circ\gamma^1||\partial_s\sigma^1\wedge\partial_t\sigma^1|\,dsdt,\end{aligned}$$

where the last limit is taken using the Dominated Convergence Theorem, which can be applied because F_ρ is bounded and continuous, and γ^r converges pointwise to γ^1. Lastly, because σ^1 is a null homotopic parametrisation of Σ, we can change variables using $\sigma^1|(0,1)^2 : (0,1)^2 \to \mathrm{Int}(\Sigma)$, and we arrive at

$$\begin{aligned}\mathrm{Hol}(\rho,(\partial\Sigma,x_0),y) &\le e^{C|\partial\Sigma|}\int_{\mathrm{Int}(\Sigma)}|F_\rho(x,h(x))|\,d\mathcal{H}^2(x)\\ &\le e^{C|\partial\Sigma|}\int_{\Sigma}|F_\rho(x,h(x))|\,d\mathcal{H}^2(x).\end{aligned}$$

□

Lemma 2.29. *Let $\rho : \Omega \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\Omega$ be a smooth Ehresmann connection form satisfying*

$$|\rho| \le C' \text{ and } |\nabla^{\mathcal{M}}\rho| \le C,$$

$\Sigma \subset \Omega$, a convex subset of an affine two-plane in Ω, which is homeomorphic to a closed disk. Let $\partial\Sigma$ denote the boundary of Σ, and $|\partial\Sigma|$ its $\mathcal{H}^1$ measure. Suppose $y \in \mathcal{M}$ is such that $B(y,|\partial\Sigma|C') \subset\subset \mathcal{M}$. Then there is a Lipschitz null homotopic parametrisation $\sigma : [0,1]^2 \to \Omega$ of Σ and a lift $\gamma : [0,1]^2 \to \mathcal{M}$ of σ along ρ starting at y satisfying

$$|\gamma(x_1)-\gamma(x_2)| \le (e^{4\|D\sigma\|_\infty\|\nabla^{\mathcal{M}}\rho\|_\infty}\|F_\rho\|_\infty\|D\sigma\|_\infty + \|\rho\|_\infty)|x_1-x_2|$$

for every $x_1, x_2 \in [0,1]^2$.

Proof. Lemmata 2.27 and 2.28 guarantee the existence of σ and γ.

We define an Ehresmann connection form $\sigma^*\rho : [0,1]^2 \times \mathcal{M} \to T\mathcal{M} \otimes \Lambda^1\mathbb{R}^2$ by

$$\sigma^*\rho(a\partial_s + b\partial_t) = \rho(a\partial_s\sigma + b\partial_t\sigma).$$

The map σ is smooth and Lipschitz up to the boundary. Consequently so is $\sigma^*\rho$. Furthermore it satisfies

$$\|\sigma^*\rho\|_\infty \leq \|D\sigma\|_\infty\|\rho\|_\infty$$

and

$$\|\nabla^{\mathcal{M}}\sigma^*\rho\|_\infty \leq \|D\sigma\|_\infty\|\nabla^{\mathcal{M}}\rho\|_\infty,$$

and as such

$$\|F_{\sigma^*\rho}\|_\infty \leq \|D\sigma\|^2_\infty\|F_\rho\|_\infty.$$

Let $x_1 = (s_1, t_1)$ and $x_2 = (s_2, t_2)$ in $(0,1)^2$. Let Q denote the quadrilateral which is the convex hull of (s_2, t_2), $(s_2, 0)$, $(s_1, 0)$, (s_1, t_1).

Let $\zeta : [0,1] \to \partial Q$ denote a closed piecewise linear path. Let $\zeta(0) = (s_2, t_2)$, $\zeta(\tau_1) = (s_2, 0)$, $\zeta(\tau_2) = (s_1, 0)$ and $\zeta(\tau_3) = (s_1, t_1)$. Let $\tilde{\gamma}$ denote the lift of ζ along ρ starting at $\gamma(x_2)$. Subsequently $\tilde{\gamma}(t_1) = y$, $\tilde{\gamma}(t_2) = y$, and $\tilde{\gamma}(t_3) = \gamma(x_1)$. Let $y' = \tilde{\gamma}(1)$. Then

$$\mathrm{Hol}(\sigma^*\rho, \zeta, \gamma(x_2)) = d(y', \gamma(x_2)).$$

Now

$$\begin{aligned}
|\gamma(x_1) - &\gamma(x_2)| \\
&\leq |\gamma(x_1) - y'| + |\gamma(x_2) - y'| \\
&\leq \mathrm{Hol}(\sigma^*\rho, \zeta, \gamma(x_2)) + \|\sigma^*\rho\|_\infty|x_2 - x_1| \\
&\leq e^{4\|\nabla^{\mathcal{M}}\sigma^*\rho\|_\infty}\|F_{\sigma^*\rho}\|_\infty|x_1 - x_2|\max\{r_1, r_2\} + \|\sigma^*\rho\|_\infty|x_2 - x_1| \\
&\leq \|D\sigma\|_\infty(e^{4\|D\sigma\|_\infty\|\nabla^{\mathcal{M}}\rho\|_\infty}\|F_\rho\|_\infty\|D\sigma\|_\infty + \|\rho\|_\infty)|x_1 - x_2|.
\end{aligned}$$

□

LEMMA 2.30. *Let $U \subset\subset \mathcal{M}$ be a coordinate neighbourhood, and let A be the intersection of an affine two-plane with Ω. Let Σ be a convex domain in A homeomorphic to a closed disk with boundary length $|\partial\Sigma|$. Let $\rho \in \Gamma(T\mathcal{M} \otimes \Lambda^1\Omega)$ be a smooth Ehresmann connection form satisfying*

$$|\nabla^{\mathcal{M}}\rho| \leq C \text{ and } |\rho| \leq C'$$

on $A \times U$. Let $R > 0$, $U' \subset B(y_0, R)$ be open and $\mathrm{dist}(y_0, \partial U) > C'|\partial\Sigma| + e^{C|\partial\Sigma|}R$. *Let $h : \Sigma \to U'$ denote a height function for a lift $\gamma : [0,1]^2 \to \Sigma \times \mathcal{M}$, of $\sigma : [0,1]^2 \to \Sigma$ a null homotopic parametrisation of Σ with $\sigma(0,0) = x_0$, and let $B_x := B(h_y(x), e^{C|\partial\Sigma|}R)$. Then*

$$\int_{U'} \mathrm{Hol}(\rho, (\partial\Sigma, x_0), y)\, dy \leq e^{(m+1)C|\partial\Sigma|}\int_\Sigma\int_{B_x}|F(x,y)|\, dy\, dx. \tag{2.9}$$

Proof. We can integrate both sides of (2.8) with respect to $y \in U'$ to yield

$$\begin{aligned}\int_{U'} \mathrm{Hol}(\rho, (\partial\Sigma, x_0), y)\, dy &\leq \int_{U'} \int_{\Sigma} e^{C|\partial\Sigma|} |F(x, h_y(x))|\, dx\, dy \\ &= \int_{\Sigma} \int_{U'} e^{C|\partial\Sigma|} |F(x, h_y(x))|\, dy\, dx.\end{aligned}$$

By Lemma 2.15 the map $f_x : U' \to U$, $y \mapsto h_y(x)$ is bi-Lipschitz with constant $e^{C|\partial\Sigma|}$. By assumption $U' \subset B(y_0, R) \subset\subset U$ and hence $f_x(U') \subset B(h(x), e^{C|\partial\Sigma|}R) = B_x \subset\subset U$. Furthermore the Jacobian determinant of f_x, $J_{f_x} : U' \to \mathbb{R}$ satisfies

$$e^{-mC|\partial\Sigma|} \leq J_{f_x}(y) \leq e^{mC|\partial\Sigma|}.$$

Consequently

$$\begin{aligned}\int_{U'} \mathrm{Hol}(\rho, (\partial\Sigma, x_0), y)\, dy &\leq \int_{\Sigma} \int_{U'} e^{C|\partial\Sigma|} |F(x, f_x(y))|\, dy\, dx \\ &= \int_{\Sigma} \int_{U'} e^{C|\partial\Sigma|} |F(x, f_x(y))| \frac{1}{J_{f_x}(y)} J_{f_x}(y)\, dy\, dx \\ &\leq e^{(m+1)C|\partial\Sigma|} \int_{\Sigma} \int_{f_x(U')} |F(x, y)|\, dy\, dx \\ &\leq e^{(m+1)C|\partial\Sigma|} \int_{\Sigma} \int_{B_x} |F(x, y)|\, dy\, dx.\end{aligned}$$

□

2.4. Smooth approximation of non-smooth connections

For this section we let U be a smooth bounded open subset of $\mathbb{R}^m$, Ω be a smooth bounded domain in $\mathbb{R}^n$ and $\rho : \Omega \times U \to TU \otimes \Lambda^1\Omega$ be a locally integrable Ehresmann connection form.

The main goal of this section is to construct a smooth approximation for $\rho \in \mathcal{A}(\Omega \times U)$ with desirable convergence properties.

Theorem 2.31. *Let $\rho \in \mathcal{A}(\Omega \times U)$. Let $\varepsilon > 0$*

$$U^\varepsilon = \{y \in U : \mathrm{dist}(y, \partial U) > \varepsilon\} \text{ and } \Omega^\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

For every $\varepsilon > 0$ there is a $\rho^\varepsilon \in \Gamma(TU^\varepsilon \otimes \Lambda^1\Omega^\varepsilon)$ such that for every $K \subset\subset U$ and $\Omega' \subset\subset \Omega$

(1) for $1 \leq p < \infty$

$$\lim_{\varepsilon\to 0} \int_{\Omega'} \sup_{y\in K} |\rho_{x,y} - \rho^\varepsilon_{x,y}|^p\, dx = 0;$$

(2) $\|\rho^\varepsilon\|_\infty \leq \|\rho\|_\infty$, $\|d_\Omega\rho^\varepsilon\|_\infty \leq \|d_\Omega\rho\|_\infty$ and $\|\nabla^U\rho^\varepsilon\|_\infty \leq \|\nabla^U\rho\|_\infty$;

(3) for $1 \leq p < \infty$

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\Omega'\times K} |\rho - \rho^\varepsilon|^p\, dxdy &= \lim_{\varepsilon\to 0} \int_{\Omega'\times K} |d_\Omega\rho - d_\Omega\rho^\varepsilon|^p\, dxdy \\ &= \lim_{\varepsilon\to 0} \int_{\Omega'\times K} |\nabla^U\rho - \nabla^U\rho^\varepsilon|^p\, dxdy = 0\end{aligned}$$

and

(4) for $1 \le p < \infty$

$$\lim_{\varepsilon \to 0} \int_{\Omega' \times K} |F_\rho - F_{\rho^\varepsilon}|^p \, dxdy = 0.$$

Properties 2 and 3 hold for generic smooth approximations. In order to get property 1 we define horizontal and vertical mollifications of ρ. Property 4 follows from property 3.

Let $\phi : [0, \infty) \to [0, \infty)$ denote a smooth non-negative decreasing function, which is constant on a neighbourhood of 0, is equal to 0 in $[1, \infty)$, and satisfies

$$\int_0^\infty \phi(t) \, dt = 1.$$

For every $k \in \mathbb{N}$ we define $c^k : (0, \infty) \to \mathbb{R}$ by

$$c^k(\varepsilon) = \left(\int_{\mathbb{R}^k} \phi(|x|/\varepsilon) \, dx \right)^{-1}$$

for $\varepsilon > 0$.

For $\varepsilon > 0$ let U^ε denote the set

$$U^\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}.$$

Let $\rho : \Omega \times U \to \mathbb{R}^m \otimes \Lambda^1\Omega$ be measurable. Then define $V^\varepsilon(\rho) : \Omega \times U^\varepsilon \to \mathbb{R}^m \otimes \Lambda^1\Omega$ by

$$(x, y) \mapsto \int_U \rho_{x,y'} \phi(|y - y'|/\varepsilon) c^m(\varepsilon) \, dy'.$$

Similarly set

$$\Omega^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\},$$

and define $H^\varepsilon(\rho) : \Omega^\varepsilon \times U \to \mathbb{R}^m \otimes \Lambda^1\Omega$ by

$$H^\varepsilon(\rho)_{x,y} := \int_\Omega \rho_{x',y} \phi(|x - x'|/\varepsilon) c^n(\varepsilon) \, dx'$$

for $(x, y) \in \Omega^\varepsilon \times U$.

We apply both operators V^ε and H^ε simultaneously, yielding

$$\rho^\varepsilon = H^\varepsilon(V^\varepsilon(\rho)) : \Omega^\varepsilon \times U^\varepsilon \to \mathbb{R}^m \otimes \Lambda^1\Omega.$$

By Fubini's theorem, $\rho^\varepsilon = V^\varepsilon(H^\varepsilon(\rho))$ for every $\varepsilon > 0$. We have the following proposition.

PROPOSITION 2.32. *Let* $\rho : \Omega \times U \to \mathbb{R}^m \otimes \Lambda^1\Omega$ *be a locally integrable function. Then* $\rho^\varepsilon : U^\varepsilon \times \Omega^\varepsilon \to \mathbb{R}^m \otimes \Lambda^1\Omega$ *is smooth.*

Proof. If we expand ρ^ε, we get

$$\rho^\varepsilon_{x,y}(\cdot) = \iint_{\Omega \times U} \phi\left(\frac{|x' - x|}{\varepsilon}\right) \phi\left(\frac{|y' - y|}{\varepsilon}\right) c^n(\varepsilon) c^m(\varepsilon) \rho_{x',y'}(\cdot) \, dy' \, dx'$$

for every $(x, y) \in \Omega^\varepsilon \times U^\varepsilon$, which is a convolution of ρ with the smooth compactly supported function

$$(x, y) \mapsto \phi(|x|/\varepsilon)\phi(|y|/\varepsilon) c^n(\varepsilon) c^m(\varepsilon).$$

Hence it is smooth [Eva98, C.4]. $\square$

Corollary 2.33. *Let $\rho \in \mathcal{A}(\Omega \times U)$. Then*

$$\|\rho^\varepsilon\|_\infty \leq \|\rho\|_\infty,\ \|d_\Omega \rho^\varepsilon\|_\infty \leq \|d_\Omega \rho\|_\infty,\ \text{and } \|\nabla^U \rho^\varepsilon\|_\infty \leq \|\nabla^U \rho\|_\infty;$$

and for any $\Omega' \subset\subset \Omega$ and $K \subset\subset U$

$$\lim_{\varepsilon\to 0} \int_{\Omega'\times K} |\rho - \rho^\varepsilon|^p \, dxdy = \lim_{\varepsilon\to 0} \int_{\Omega'\times K} |d_\Omega \rho - d_\Omega \rho^\varepsilon|^p \, dxdy$$

$$= \lim_{\varepsilon\to 0} \int_{\Omega'\times K} |\nabla^U \rho - \nabla^U \rho^\varepsilon|^p \, dxdy = 0.$$

Proof. This is a standard result of smooth approximations and linear differential operators. [Eva98, §5.3.2]. □

The most important aspect of the regularity condition $\|\nabla^U \rho\|_\infty < \infty$ is that $\rho \in \mathcal{A}(\Omega \times U)$ restricts to a Lipschitz continuous function on $\{x\} \times U$, for almost every $x \in \Omega$. This, along with the structure of our smooth approximations, allows us to prove property 1 of Theorem 2.31.

Lemma 2.34. *Let $\rho \in \mathcal{A}(\Omega \times U)$. Then for every $\Omega' \subset\subset \Omega$ and $K \subset\subset U$*

$$\int_\Omega \sup_{y\in K} |\rho^\varepsilon_{x,y} - \rho_{x,y}|^p \, dx \to 0 \tag{2.10}$$

as $\varepsilon \to 0$.

Proof. The proof of this statement mimics the proof of convergence of smooth approximations of L^p functions through continuous approximations [Eva98, §C.4]. The key step in this proof is to recognise ρ as a map from Ω to the space of continuous vector valued forms $\rho : \Omega \to C^0(U, \mathbb{R}^m \otimes \Lambda^1\Omega)$.

First we note that, for any test function $\psi \in C_0^\infty(\Omega \times U, \mathbb{R}^m \otimes \Lambda^1\Omega \otimes \mathbb{R}^m)$, we have that

$$\int_{\Omega\times U} \left\langle \rho_{x,y}, (\nabla^U)^* \psi(x,y) \right\rangle \, dx\, dy = \int_{\Omega\times U} \langle \theta_{x,y}, \psi(x,y) \rangle \, dx\, dy,$$

where $\theta_{x,y} : \Omega \times \mathcal{M} \to \mathbb{R}^m \otimes \Lambda^1\Omega \otimes \mathbb{R}^m$ is an essentially bounded measurable function.

In terms of the partial derivative with respect to the coordinates $y = (y^1, \ldots, y^m)$ in U,

$$\int_{\Omega\times U} \left\langle \rho_{x,y}, -\partial_{y^i} \phi \right\rangle \, dx\, dy = \int_{\Omega\times U} \left\langle \theta^i_{x,y}, \phi \right\rangle \, dx\, dy$$

for every $\phi \in C_0^\infty(\Omega \times U, \mathbb{R}^m \otimes \Lambda^1\Omega)$.

Let $\phi_k : U \to \mathbb{R}^m \otimes \Lambda^1\Omega$, $k \in \mathbb{N}$, be a collection of smooth compactly supported functions which are dense in $C_0^1(U, \mathbb{R}^m \otimes \Lambda^1\Omega)$. Let $\chi \in C_0^\infty(\Omega)$ be a smooth compactly supported function. Then for $k \in \mathbb{N}$ the map $(x, y) \mapsto \chi(x)\phi_k(y)$ is compactly supported in $\Omega \times U$, so

$$\int_{\Omega\times U} \left\langle \rho_{x,y}, -\chi(x)\partial_{y^i} \phi_k(y) \right\rangle \, dx\, dy = \int_{\Omega\times U} \left\langle \rho_{x,y}, -\partial_{y^i}(\chi(x)\phi_k(y)) \right\rangle \, dx\, dy$$

$$= \int_{\Omega\times U} \left\langle \theta^i_{x,y}, \chi(x)\phi_k(y) \right\rangle \, dx\, dy.$$

Then by Fubini's Theorem

$$\int_\Omega \chi(x) \int_U \left\langle \rho_{x,y}, -\partial_{y^i}\phi_k(y)\right\rangle \, dy \, dx = \int_\Omega \chi(x) \int_U \left\langle \theta^i_{x,y}, \phi_k(y)\right\rangle \, dy \, dx.$$

But because χ was arbitrary this implies that the function

$$x \mapsto \int_U \langle \rho_{x,y}, -\partial_{y^i}\phi_k\rangle - \langle \theta^i_{x,y}, \phi_k\rangle \, dy$$

is zero for almost every $x \in \Omega$. Hence there is a set $\Omega_k \subset \Omega$ of full measure, such that for every $x \in \Omega_k$

$$\int_U \langle \rho_{x,y}, -\partial_{y^i}\phi_k\rangle \, dy = \int_U \langle \theta^i_{x,y}, \phi_k\rangle \, dy.$$

Now for $x \in \Omega_\infty := \bigcap_j \Omega_j$, and for every $k \in \mathbb{N}$ we have

$$\int_U \langle \rho_{x,y}, -\partial_{y^i}\phi_k\rangle \, dy = \int_U \langle \theta^i_{x,y}, \phi_k\rangle \, dy.$$

Suppose $\phi \in C_0^\infty(U, \mathbb{R}^m \otimes \Lambda^1\Omega)$. Then by relabelling we can assume $\phi_k \to \phi$ in C^1. Hence, by the Dominated Convergence Theorem, for every $x \in \Omega_\infty$ and $1 \le i \le m$

$$\begin{aligned}
\int_U \langle \rho_{x,y}, -\partial_{y^i}\phi\rangle \, dy &= \lim_{k\to\infty} \int_U \langle \rho_{x,y}, -\partial_{y^i}\phi_k\rangle \, dy \\
&= \lim_{k\to\infty} \int_U \langle \theta^i_{x,y}, \phi_k\rangle \, dy \\
&= \int_U \langle \theta^i_{x,y}, \phi\rangle \, dy.
\end{aligned}$$

But $|\Omega \setminus \Omega_\infty| = 0$, so for almost every $x \in \Omega$ and every $\phi \in C_0^\infty(U, \mathbb{R}^m \otimes \Lambda^1\Omega \otimes \mathbb{R}^m)$

$$\int_U \langle \rho_{x,y}, (\nabla^U)^*\phi\rangle \, dy = \int_U \langle \theta_{x,y}, \phi\rangle \, dx \, dy.$$

Furthermore for almost every $x \in \Omega$ the map $y \mapsto \theta_{x,y}$ is essentially bounded. As a result, for almost every $x \in \Omega$ the map $\rho_{x,\cdot} : U \to \mathbb{R}^m \otimes \Lambda^1\Omega$ is in $W^{1,\infty}(\Omega \times U, \mathbb{R}^m \otimes \Lambda^1\Omega)$ and has a bounded Lipschitz continuous representative with Lipschitz constant $C = \|\nabla^U \rho\|_\infty$. Let $C' = \|\rho\|_\infty$.

Now we will proceed to create a simple approximation to ρ in the sense previously outlined. Let $K \subset\subset U$ be a compact subset. Consider the set of functions $K \to \mathbb{R}^m \otimes \Lambda^1\Omega$ with L^∞-norm bounded by C' which are Lipschitz with Lipschitz constant bounded by C. By the Arzelá–Ascoli Theorem this is a totally bounded subset of the space $C^0(K, \mathbb{R}^m \otimes \Lambda^k\Omega)$. Hence for any $\delta > 0$ this set has a finite δ-net. Choose $\delta > 0$, and let θ_i, $i = 1, \ldots, N$, denote the corresponding net.

For $i = 1, \ldots, N$ let $A_i := \{x \in \Omega : \sup_{y \in K} |\rho_{x,y} - \theta_i(y)| < \delta\}$. The union $\bigcup_{i=1}^N A_i$ is a set of full measure in Ω. Define inductively $B_1 = A_1$, and for $i \ge 1$, $B_{i+1} = A_{i+1} \setminus \cup_{k \le i} B_k$. Let $f : \Omega \to C^0(K, \mathbb{R}^m \otimes \Lambda^1\Omega)$ be the simple function

$$f_x(\cdot) := \sum_{i=1}^N \theta_i(\cdot)\chi_{B_i}(x),$$

where $x \in \Omega$. It follows immediately that $\operatorname{esssup}\{|\rho_{x,y} - f_x(y)| : (x,y) \in \Omega \times K\} \leq \delta$.

Our next step is to create a continuous approximation to our simple approximation. To do this, we approximate the characteristic function χ_{B_i} by continuous functions. For $i = 1, \dots, N$, we fix $\phi_i : \Omega \to [0,1]$ such that

$$\|\phi_i - \chi_{B_i}\|_p \leq \delta/(C\,N).$$

Then define $g : \Omega \to C^0(K, \mathbb{R}^m \otimes \Lambda^1\Omega)$ by

$$g_x(\cdot) = \sum_{i=1}^{N} \theta_i(\cdot)\,\phi_i(x)$$

for $x \in \Omega$. It follows immediately from the triangle inequality that

$$\left(\int_\Omega \sup_{y\in K} |f_x(y) - g_x(y)|^p\,dx\right)^{1/p} \leq \delta,$$

and hence

$$\left(\int_\Omega \sup_{y\in K} |\rho_{x,y} - g_x(y)|^p\,dx\right)^{1/p} \leq (1 + |\Omega|^{1/p})\delta.$$

The map $(x,y) \mapsto g_x(y)$ is continuous on $\Omega \times K$ and can be extended to a continuous map $\Omega \times U \to \mathbb{R}^m \otimes \Lambda^1\Omega$ by extending the function $\theta_1, \dots, \theta_N$ to compactly supported functions in U. As such we can take a smooth approximation as in Proposition 2.32. Denote this approximation by g^ε.

Now for $\varepsilon < \min\{\operatorname{dist}(K, \partial U), \operatorname{dist}(\Omega', \partial\Omega)\}$

$$\begin{aligned}\left(\int_{\Omega'} \sup_{y\in K} |\rho_{x,y} - \rho^\varepsilon_{x,y}|^p\,dx\right)^{1/p} \leq{}& \left(\int_{\Omega'} \sup_{y\in K} |\rho_{x,y} - g_x(y)|^p\,dx\right)^{1/p}\\ &+ \left(\int_{\Omega'} \sup_{y\in K} |g_x(y) - g^\varepsilon_x(y)|^p\,dx\right)^{1/p}\\ &+ \left(\int_{\Omega'} \sup_{y\in K} |g^\varepsilon_x(y) - \rho^\varepsilon_{x,y}|^p\,dx\right)^{1/p}.\end{aligned}$$

The first term on the right hand side was shown to be less than $\delta(1 + |\Omega|^{1/p})$. The second term is the difference of a uniformly continuous function and its smooth approximation and converges to 0 as $\varepsilon \to 0$. As for the final term, let $h = g - \rho$. The smooth approximation is a linear operation. This means that $g^\varepsilon - \rho^\varepsilon = h^\varepsilon$. Because $y \mapsto V^\varepsilon(\rho)_{x,y}$ is a smooth approximation of $y \mapsto \rho_{x,y}$ on U for almost every $x \in \Omega'$, it follows that

$$\sup_{y\in K} |H^\varepsilon(V^\varepsilon(h_{x,y}))| \leq H^\varepsilon(\sup_{y\in K} V^\varepsilon(|h_{x,y}|))$$

for almost every $x \in \Omega'$ and $0 < \varepsilon < \min\{\mathrm{dist}(K, \partial U), \mathrm{dist}(\Omega', \partial\Omega)\}$. By integrating with respect to x we obtain

$$\begin{aligned}
\left(\int_{\Omega'} \sup_{y\in K} |h^\varepsilon|^p \, dx\right)^{1/p} &\leq \left(\int_{\Omega'} H^\varepsilon (\sup_{y\in K} |h_{x,y}|)^p \, dx\right)^{1/p} \\
&\leq \|\phi * |h|\|_{L^p(\Omega')} \\
&\leq \|\phi\|_{L^1(\Omega)} \left\|\sup_{y\in K} |h_{x,y}|\right\|_{L^p(\Omega')} \\
&\leq \left\|\sup_{y\in K} |h_{x,y}|\right\|_{L^p(\Omega'} = \left(\int_{\Omega'} \sup_{y\in K} |h_{x,y}|^p \, dx\right)^{1/p}.
\end{aligned}$$

But

$$\left(\int_{\Omega'} \sup_{y\in K} |h_{x,y}|^p \, dx\right)^{1/p} = \left(\int_{\Omega'} \left(\sup_{y\in K} |\rho_{x,y} - g_x(y)|\right)^p \, dx\right)^{1/p} \leq (1 + |\Omega|^{1/p})\delta.$$

Thus,

$$\limsup_{\varepsilon\to 0} \left(\int_{\Omega'} \sup_{y\in K} |\rho_{x,y} - \rho^\varepsilon_{x,y}|^p \, dx\right)^{1/p} \leq 2(1 + |\Omega|^{1/p})\delta.$$

Since $\delta > 0$ was arbitrary,

$$\lim_{\varepsilon\to 0} \left(\int_{\Omega^\varepsilon} \sup_{y\in K} |\rho_{x,y} - \rho^\varepsilon_{x,y}|^p \, dx\right)^{1/p} = 0.$$

□

Proof of Theorem 2.31. Let $\Omega \subset\subset \Omega$ and $K \subset\subset U$. We prove

$$\iint_{\Omega'\times K} |F_{\rho^\varepsilon} - F_\rho|^p \, dx \, dy \to 0, \tag{2.11}$$

We apply equation (2.6) denoting ρ^ε by ϱ. In this case

$$\begin{aligned}
|F_\varrho - F_\rho| \leq &\left| \left(\varrho^a_i \frac{\partial \varrho^b_k}{\partial y^a} - \varrho^a_k \frac{\partial \varrho^b_i}{\partial y^a} - \rho^a_i \frac{\partial \rho^b_k}{\partial y^a} + \rho^a_k \frac{\partial \rho^b_i}{\partial y^a}\right) \partial y^b \otimes dx^i \wedge dx^k \right| \\
&+ |d_\Omega \varrho - d_\Omega \rho|. \\
\leq &\left| \left(\varrho^a_i \frac{\partial \varrho^b_k}{\partial y^a} - \varrho^a_k \frac{\partial \varrho^b_i}{\partial y^a} - \rho^a_i \frac{\partial \varrho^b_k}{\partial y^a} + \rho^a_k \frac{\partial \varrho^b_i}{\partial y^a}\right.\right. \\
&\left.\left. + \rho^a_i \frac{\partial \varrho^b_k}{\partial y^a} - \rho^a_k \frac{\partial \varrho^b_i}{\partial y^a} - \rho^a_i \frac{\partial \rho^b_k}{\partial y^a} + \rho^a_k \frac{\partial \rho^b_i}{\partial y^a}\right) \partial y^b \otimes dx^i \wedge dx^k \right| \\
&+ |d_\Omega \varrho - d_\Omega \rho| \\
\leq &\, 2|\varrho - \rho| \, |\nabla^U \varrho| + 2|\rho| \, |\nabla^U \varrho - \nabla^U \rho| + |d_\Omega \varrho - d_\Omega \rho|
\end{aligned}$$

almost everywhere in $\Omega\times U$. We can then apply Minkowski's inequality, and replacing ϱ with ρ^ε yields

$$\|F_\rho - F_{\rho^\varepsilon}\|_p \leq 2\|\nabla^U \rho\|_\infty \|\rho^\varepsilon - \rho\|_p + 2\|\rho\|_\infty \|\nabla^U \rho - \nabla^U \rho^\varepsilon\|_p + \|d\rho - d\rho^\varepsilon\|_p.$$

By Corollary 2.33, the right hand side converges to 0 as ε tends to 0, yielding the desired result.

This completes the proof of Theorem 2.31. $\square$

REMARK 2.35. The operator $\rho \mapsto F_\rho$ is nonlinear, so we cannot expect

$$\|F_{\rho^\varepsilon}\|_\infty \leq \|F_\rho\|_\infty,$$

but we can still bound $\|F_{\rho^\varepsilon}\|_\infty$, because

$$\begin{aligned}\|F_{\rho^\varepsilon}\|_\infty &\leq 2\|\rho^\varepsilon\|_\infty\|\nabla^U\rho^\varepsilon\|_\infty + \|d_\Omega\rho^\varepsilon\|_\infty \\ &\leq 2\|\rho\|_\infty\|\nabla^U\rho\|_\infty + \|d_\Omega\rho\|_\infty.\end{aligned}$$

2.5. HOLONOMY BOUNDS FOR NON-SMOOTH CONNECTIONS

In this section we consider $\rho \in \mathcal{A}(\Omega \times \mathcal{M})$ restricted to a ball $B(x_0, r_0) \subset\subset \Omega$ and define a radial lift along ρ of paths $t \mapsto x_0 + tv$ for $t \in [0, r_0]$ and $v \in S^{n-1}$.

DEFINITION 2.36. Let ρ be a map $\Omega \times U \to \mathbb{R}^m \otimes \Lambda^1\Omega$. Let A be a k-dimensional affine subspace in $\mathbb{R}^n$. We define the restriction of ρ to A, denoted $\rho|A$, to be a map $\rho|A : (\Omega \cap A) \times U \to \mathbb{R}^m \otimes \Lambda^1 A$, induced by the inclusion map on forms:

$$\mathrm{Id}_{\mathbb{R}^m} \otimes i^* : \mathbb{R}^m \otimes \Lambda^1\Omega \to \mathbb{R}^m \otimes \Lambda^1 A,$$

taking $v \otimes \alpha$ to $v \otimes i^*(\alpha)$. The map $\rho|A$ is defined to be $\mathrm{Id}_{\mathbb{R}^m} \otimes i^*(\rho)$.

In this sense, we discuss the restriction of ρ to lines. Fix a point $y \in U$ and a point $x_0 \in \Omega$. Then we can consider the restriction of ρ to all of the lines containing x_0. When we talk about a property "almost everywhere", we will specify a normalised Hausdorff k-measure $\mathcal{H}^k$ for which said property holds.

LEMMA 2.37. *Let $\rho \in \mathcal{A}(\Omega \times U)$. Denote by $l_v = \{x + tv : t \in [0, \infty)\}$ the ray through $x_0 \in \mathbb{R}^n$ in the direction v. Then for $\mathcal{H}^{n-1}$-almost every point $v \in S^{n-1}$,*

$$\|\rho|l_v\|_\infty \leq \|\rho\|_\infty \text{ and } \|\nabla^U\rho|l_v\|_\infty \leq \|\nabla^U\rho\|_\infty.$$

Furthermore,

$$d_{\Omega\cap l_v}\rho|l = 0,$$

and hence $\rho|l_v \in \mathcal{A}(\Omega \cap l_v \times U)$.

Proof. Let $E \subset \Omega$ denote the set of points $y \in \Omega$ satisfying

$$|\text{ess sup}_{\,y}\rho| > \|\rho\|_\infty \text{ or } |\text{ess sup}_{\,y}\nabla^U\rho| > \|\nabla^U\rho\|_\infty.$$

Then

$$0 = |E| = \int_\Omega \chi_E \, dx \geq \int_{S^{n-1}} \int_0^{r_0} \chi_E(x_0 + tv)t^{n-1} \, dt \, dv$$

for every $r_0 > 0$. Thus, for $\mathcal{H}^{n-1}$-almost every $v \in S^{n-1}$, the map $(0, r_0) \to \mathbb{R}$, $t \mapsto \chi_E(x_0 + tv)t^{n-1}$, is equal to 0 for almost every $t \in [0, r_0]$. Hence, for $\mathcal{H}^{n-1}$-almost every $v \in S^{n-1}$, $\chi_E(c_0 + tv)$ is equal to 0 for almost every $t \in [0, r_0]$.

To see that $d_{\Omega\cap l_v}\rho = 0$, it is enough to note that $d_A\rho = 0$ for any one-dimensional space A, as there are no non-zero two-forms on the space. $\square$

Let $\rho \in \mathcal{A}(\Omega \times U)$, let y be a point in U, and $x_0 \in \Omega$. Let $r_0 > 0$ and

$$r_0 < \min\{\mathrm{dist}(y, \partial U)/\|\rho\|_\infty, \mathrm{dist}(x_0, \partial\Omega)\}.$$

Lemma 2.37 tells us that for almost every $v \in S^{n-1}$, ρ has a well defined restriction to the ray from x_0 in the direction v. Furthermore the initial value problem

$$f'(t) = \rho_{x_0+tv,f(t)}v, \quad f(0) = y$$

has a unique solution $f_v^y : [0, r_0] \to U$.

Definition 2.38. Let $\rho \in \mathcal{A}(\Omega \times U)$, let y be a point U, and $x_0 \in \Omega$. Let $0 < r_0 < \min\{\mathrm{dist}(y, \partial U)/\|\rho\|_\infty, \mathrm{dist}(x_0, \partial\Omega)\}$. Define $\gamma_y : B(x_0, r_0) \subset \Omega \to U$

$$\gamma_y(x) = f^y_{\frac{x-x_0}{|x-x_0|}}(|x - x_0|) \tag{2.12}$$

for $x \in B(x_0, r_0)$. We call γ_y the *radial lift along ρ centered at x_0 starting at y.*

Corollary 2.39. *The radial lift is well defined for all $x \in l_v \cap B(x_0, r_0)$ for $\mathcal{H}^{n-1}$-almost every $v \in S^{n-1}$ and hence is defined for $\mathcal{H}^n$ almost every $x \in B(x_0, r_0)$. Furthermore the radial lift is uniquely defined by its starting point and*

$$\exp(-\|\nabla^U\rho\|_\infty|x - x_0|)|y - y'| \le |\gamma_y(x) - \gamma_{y'}(x)| \le \exp(\|\nabla^U\rho\|_\infty|x - x_0|)|y - y'|.$$

Proof. By Lemma 2.37, a well defined restriction of ρ is defined along the ray $x_0 + tv$ for $\mathcal{H}^{n-1}$-almost every $v \in S^{n-1}$, and by the theory of existence and uniqueness of ODEs [Lev55], γ is well defined and unique. The claim follows when we apply Lemma 2.15. $\square$

Theorem 2.40. *Let y be any point in U, x_0 any point in Ω, and $r_0 > 0$ such that*

$$r_0 < \min\{\mathrm{dist}(y, \partial U)/(4\|\rho\|_\infty), \mathrm{dist}(x_0, \partial\Omega)\}.$$

Let $\rho \in \mathcal{A}(\Omega \times U)$ and let γ be the radial lift along ρ centered at x_0 starting at y. Then γ has a Lipschitz representative satisfying

$$|\gamma(x_1) - \gamma(x_2)| \le (\|\rho\|_\infty + C(\rho, r_0)\|F_\rho\|_\infty \max\{|x_1 - x_0|, |x_2 - x_0|\})|x_1 - x_2|$$

for every x_1 and x_2 in $B(x_0, r_0)$.

To prove the theorem we define *typical planes* in Ω and show that γ is Lipschitz almost everywhere on a typical plane. Then we show that almost every plane is typical. Let $\rho \in \mathcal{A}(\Omega \times U)$ and $\rho^\varepsilon \in \Gamma(TU \otimes \Lambda^1\Omega)$ satisfying

(1) for $1 \le p < \infty$

$$\|\rho^\varepsilon - \rho\|_p,\ \|d_\Omega\rho^\varepsilon - d_\Omega\rho\|_p \text{ and } \|\nabla^U\rho^\varepsilon - \nabla^U\rho\|_p$$

converge to 0 as $\varepsilon \to 0$;

(2) and

$$\|\rho^\varepsilon\|_\infty \le \|\rho\|_\infty,\ \|d_\Omega\rho^\varepsilon\|_\infty \le \|d_\Omega\rho\|_\infty \text{ and } \|\nabla^U\rho^\varepsilon\|_\infty \le \|\nabla^U\rho\|_\infty.$$

A plane $P \subset \Omega$ containing x_0 is *typical for $\rho^\varepsilon \to \rho$* if

(1) $\rho|P \in \mathcal{A}(P \times U)$ and has norm bounds

$$\|\rho|P\|_\infty \le \|\rho\|_\infty,\ \|(d_\Omega\rho)|P\|_\infty \le \|d_\Omega\rho\|_\infty, \|(\nabla^U\rho)|P\|_\infty \le \|\nabla^U\rho\|_\infty,$$

and

$$\|(F_\rho)|P\|_\infty \le \|F_\rho\|_\infty;$$

(2) and there is a sequence $\varepsilon_k \to 0$ such that

$$\|(\rho^{\varepsilon_k} - \rho)|P\|_1,\ \|(d_\Omega\rho^{\varepsilon_k} - d_\Omega\rho)|P\|_1, \text{ and } \|(\nabla^U\rho^{\varepsilon_k} - \nabla^U\rho)|P\|_1$$

converge to 0 as $k \to \infty$.

Lemma 2.41. *Let $\rho \in \mathcal{A}(\Omega \times U^\circ)$, $y \in U' \subset\subset U \subset\subset U^\circ$ $x_0 \in \Omega' \subset\subset \Omega$,*

$$0 < r_0 < \inf_{y\in U'}(\operatorname{dist}(y, \partial U))\,(4\,\|\rho\|_\infty)^{-1},$$

and for $\varepsilon > 0$ let $\rho^\varepsilon \in \Gamma(TU \otimes \Lambda^1\Omega')$ be smooth Ehresmann connection forms satisfying

$$\|\rho^\varepsilon\|_\infty \leq \|\rho\|_\infty, \quad \|d_\Omega \rho^\varepsilon\|_\infty \leq \|d_\Omega \rho\|_\infty \quad \text{and} \quad \|\nabla^U \rho^\varepsilon\|_\infty \leq \|\nabla^U \rho\|_\infty.$$

Let γ_y and γ_y^ε be the radial lifts centered at x_0 starting at y along ρ and ρ^ε, respectively. Suppose that x_1 and x_2 are two points in Ω' not colinear with x_0, and as such defining a plane $P \subset \Omega'$. If P is typical for $\rho^\varepsilon \to \rho$ and

(1) for almost every $x \in P$, $\gamma_y^{\varepsilon_k}(x) \to \gamma_y(x)$,
(2) and $\gamma_y^{\varepsilon_k}(x_i)$ converges uniformly in y to $\gamma_y(x_i)$ for $i = 1, 2$, that is, for every $\delta > 0$ there exists an $k_0 > 0$ such that for every $y \in U'$, $i = 1, 2$ and $k > k_0$

$$|\gamma_y^{\varepsilon_k}(x_i) - \gamma_y(x_i)| < \delta,$$

where $\varepsilon_k \to 0$ is the sequence for which P is typical for $\rho^\varepsilon \to \rho$;

then there is a constant $C = C(\rho, r_0)$ such that

$$|\gamma_y(x_1) - \gamma_y(x_2)| \leq (\|\rho\|_\infty + C(\rho, r_0)\|F_\rho\|_\infty \min\{|x_0 - x_1|, |x_0 - x_2|\})|x_1 - x_2|. \tag{2.13}$$

Proof. Let $\rho^k = \rho^{\varepsilon_k}$ and $\gamma_y^k = \gamma_y^{\varepsilon_k}$. By the triangle inequality, for any $y, y' \in U'$

$$\begin{aligned} |\gamma_y(x_1) - \gamma_y(x_2)| \leq{}& |\gamma_y(x_1) - \gamma_y^k(x_1)| + |\gamma_y(x_2) - \gamma_y^k(x_2)| + |\gamma_{y'}^k(x_1) - \gamma_y^k(x_1)| \\ &+ |\gamma_{y'}^k(x_2) - \gamma_y^k(x_2)| + |\gamma_{y'}^k(x_1) - \gamma_{y'}^k(x_2)|. \end{aligned}$$

By the convergence $\gamma_y^k(x_i) \to \gamma_y(x_i)$ as $k \to \infty$, we have that the terms $|\gamma_y^k(x_i) - \gamma_y(x_i)|$ tend to zero as $k \to \infty$. By Corollary 2.39, we have that

$$\begin{aligned} |\gamma_y^k(x_i) - \gamma_{y'}^k(x_i)| &\leq \exp(\|\nabla^U \rho^k\|_\infty |x_0 - x_i|)|y - y'| \\ &\leq \exp(\|\nabla^U \rho^k\|_\infty r_0)|y - y'|, \end{aligned}$$

for all $k \in \mathbb{N}$ and $i = 1, 2$. Let $\eta_y^k : B(x_2, r_0) \to U$ be the radial lift centered at x_2 starting at $\gamma_y^k(x_2)$. That is, η_y^k is the solution of the initial value problem

$$\begin{gathered} \frac{d}{dt}\eta_y^k\,(x_2 + t(x - x_2)/|x - x_2|) = \rho^k_{x_2+t(x-x_2)/|x-x_2|,\eta_y^k} \cdot \left(\frac{x - x_2}{|x - x_2|}\right) \\ \eta_y^k(x_2) = \gamma_y^k(x_2) \end{gathered}$$

at $t = |x - x_2|$. Then we use the triangle inequality to yield

$$|\gamma_{y'}^k(x_1) - \gamma_{y'}^k(x_2)| \leq |\gamma_{y'}^k(x_1) - \eta_{y'}^k(x_1)| + |\eta_{y'}^k(x_1) - \gamma_{y'}^k(x_2)|,$$

for all $k \in \mathbb{N}$. Let Δ be the triangle $[x_0, x_1, x_2]$. We note that $|\gamma_{y'}^k(x_1) - \eta_{y'}^k(x_1)| = \operatorname{Hol}(\rho^k, (\partial\Delta, x_0), \gamma_{y'}^k(x_1))$. Because $\eta_{y'}^k$ is the solution to an initial value problem in the direction $x - x_2$, we have that

$$|\eta_{y'}^k(x_1) - \gamma_{y'}^k(x_2)| = |\eta_{y'}^k(x_1) - \eta_{y'}^k(x_2)| \leq \|\rho^k\|_\infty |x_2 - x_1|.$$

Combining these yields

$$|\gamma_{y'}^k(x_1) - \gamma_{y'}^k(x_2)| \leq \operatorname{Hol}(\rho^k, (\partial\Delta, x_1), \gamma_{y'}^k(x_1)) + \|\rho^k\|_\infty |x_2 - x_1|.$$

We combine these to obtain

$$\begin{aligned}|\gamma_y(x_1)-\gamma_y(x_2)| \le\ & |\gamma_y(x_1)-\gamma_y^k(x_1)|+|\gamma_y(x_2)-\gamma_y^k(x_2)|\\ & + 2\exp(r_0\|\nabla^U\rho^k\|_\infty)|y'-y| + \|\rho^k\|_\infty|x_2-x_1|\\ & + \mathrm{Hol}(\rho^k,(\partial\Delta,x_1),\gamma_{y'}^k(x_1)),\end{aligned}$$

for every $k\in\mathbb{N}$. If we integrate with respect to $y'\in B := B(y,r)\subset U$, and let k tend to zero, noting that the norms $\|\rho^k\|_\infty$ and $\|\nabla^U\rho^k\|_\infty$ are bounded respectively by $\|\rho\|_\infty$ and $\|\nabla^U\rho\|_\infty$, we arrive at

$$\begin{aligned}|\gamma_y(x_1)-\gamma_y(x_2)|\,|B| &\le 2\exp(r_0\|\nabla^U\rho\|_\infty)\int_B |y'-y|\,dy' + \|\rho\|_\infty|x_2-x_1|\,|B|\\ &\quad + \limsup_{k\to 0}\int_B \mathrm{Hol}(\rho^k,(\partial\Delta,x_1),\gamma_{y'}^k(x_1))\,dy'\\ &\le 2\exp(r_0\|\nabla^U\rho\|_\infty)|B|\,r\ + \|\rho\|_\infty|x_2-x_1|\,|B|\\ &\quad + \limsup_{k\to 0}\int_B \mathrm{Hol}(\rho^k,(\partial\Delta,x_1),\gamma_{y'}^k(x_1))\,dy'.\end{aligned}$$

We first apply Lemma 2.28 to get a height function $h^k : \mathrm{Int}(\Delta)\to U$ for a lift $\gamma^k : [0,1]^2\to\Delta\times U$ of a null homotopic parametrisation $\sigma : [0,1]^2\to\Delta$ along ρ^k starting at y. By Lemma 2.29 the functions γ^k are uniformly Lipschitz, and their image is bounded. Hence by the Arzelà–Ascoli theorem there is a subsequence of $k\to\infty$ such that $\gamma^k\to\gamma$, where $\gamma : [0,1]^2\to\Delta\times U$ is Lipschitz. Consequently $h^k(x)$ converges for every $x\in\mathrm{Int}(\Delta)$ to $h(x)=\pi_U\circ\gamma\circ\sigma^{-1}(x)$. Let $C=\|\nabla^U\rho\|_\infty$ and $B_x^k = B(h^k(x), e^{C|\partial\Delta|}r)$ and $B_x = B(h(x), e^{C|\partial\Delta|}r)$ for every $x\in\Delta$. Then $\chi(B_x^k)$ converges to $\chi(B_x)$ almost everywhrere. Then we can apply Lemma 2.30 noting that $|\partial\Delta|<4r_0$ and choosing $r>0$ such that $\inf_{k,x}\mathrm{dist}(B_x^\varepsilon,\partial U)>\|\rho\|_\infty 4r_0$. We arrive at

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{B_{x_0}}\mathrm{Hol}(\rho^\varepsilon,(\partial\Delta,x_1),\gamma_{y'}^\varepsilon(x_1))\,dy'\\ &\qquad\le e^{C(m+1)4r_0}\limsup_{\varepsilon\to 0}\int_\Delta\int_{B_x^\varepsilon}|F^\varepsilon|\,dy'\,dx\\ &\qquad\le e^{C(m+1)4r_0}\limsup_{\varepsilon\to 0}\Bigg[\int_\Delta\int_{B_x}|F|\,dy'\,dx\\ &\qquad\quad+\int_\Delta\int_U|\chi(B_x)(y')-\chi(B_x^\varepsilon)(y')||F| + \int_\Delta\int_U||F^\varepsilon|-|F||\,dy\,dx\Bigg]\\ &\qquad\le e^{C(m+1)4r_0}\int_\Delta\int_{B_x}|F|\,dy\,dx\\ &\qquad\le e^{C(m+1)4r_0}e^{Cm4r_0}|B|\|F\|_\infty\min\{|x_0-x_1|,|x_0-x_2|\}|x_1-x_2|\end{aligned}$$

by the Dominated Convergence Theorem and the convergence of F^ε in Theorem 2.31. Let $C(\rho,r_0)=e^{4(2m+1)\|\nabla^U\rho\|_\infty r_0}$. Then

$$\begin{aligned}&|\gamma_y(x_1)-\gamma_y(x_2)|\,|B|\\ &\qquad\le 2\exp(r_0\|\nabla^U\rho\|_\infty)r|B| + \|\rho\|_\infty|x_2-x_1|\,|B|\\ &\qquad\quad + C(\rho,r_0)|B|\|F\|_\infty\min\{|x_0-x_1|,|x_0-x_2|\}|x_1-x_2|.\end{aligned}$$

Because $r \in (0, r_1)$ was arbitrary, we have that

$$|\gamma_y(x_1) - \gamma_y(x_2)| \leq (\|\rho\|_\infty + C(\rho, r_0)\|F\|_\infty \min\{|x_2 - x_0|, |x_1 - x_0|\})|x_2 - x_1|,$$

which proves the result. □

2.5.1. **Almost every plane is typical.** Now that we have demonstrated how we can get Lipschitz behaviour, we must demonstrate that the points satisfying the criteria of Lemma 2.42 form a set of full measure in $B(x_0, r_0) \times B(x_0, r_0)$.

LEMMA 2.42. *There is a subsequence $\varepsilon_i \to 0$ such that for $\mathcal{H}^{n-1}$-almost every $v \in S^1$, such that for every $\delta > 0$ there is an $i_0 \geq 0$ such that for every $i > i_0$ and every $y \in U'$*

$$\sup_{t \in [0, r_0)} |\gamma_y^{\varepsilon_i}(x_0 + tv) - \gamma_y(x_0 + tv)| < \delta.$$

Proof. For $v \in S^1$ and $t \in [0, 1]$ denote $x_t(v) = (x_0 + tv)$, and when it is unambiguous by just x_t. Consider $\gamma^\varepsilon(tv + x_0) - \gamma(tv + x_0)$. By Lemma 2.34

$$\begin{aligned}
\left|\frac{d}{dt}|\gamma_y^\varepsilon(x_t) - \gamma_y(x_t)|\right| &\leq \left|\frac{d}{dt}\gamma_y^\varepsilon(x_t) - \frac{d}{dt}\gamma_y(x_t)\right| \\
&\leq |\rho^\varepsilon_{x_t, \gamma_y^\varepsilon(x_t)} v - \rho_{x_t, \gamma_y(x_t)} v| \\
&\leq |\rho^\varepsilon_{x_t, \gamma_y^\varepsilon(x_t)} - \rho^\varepsilon_{x_t, \gamma_y(x_t)}| + |\rho^\varepsilon_{x_t, \gamma_y(x_t)} - \rho_{x_t, \gamma_y(x_t)}| \\
&\leq \|\nabla^U \rho^\varepsilon\|_\infty |\gamma_y^\varepsilon(x_t) - \gamma_y(x_t)| + \sup_{y \in U'} |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}|.
\end{aligned} \tag{2.14}$$

for almost every $t \in [0, 1]$.

Now define $f : [0, 1] \to \Omega$ by $f(t) := |\gamma_y^\varepsilon(x_t) - \gamma_y(x_t)|$. It follows that f is absolutely continuous, for almost every v and by (2.14)

$$|f'(t)| \leq \|\nabla^U \rho\|_\infty f(t) + \sup_{y \in U'} |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}|.$$

Denoting $C = \|\nabla^U \rho\|_\infty$ we have

$$f'(t) \leq C f(t) + \sup |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}|,$$

for almost every $t \in [0, 1]$. We can multiply both sides by e^{-Ct} to yield

$$e^{-Ct} f'(t) \leq C e^{-Ct} f(t) + e^{-Ct} \sup_{y \in U'} |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}|,$$

for almost every $t \in [0, 1]$. Now we can combine the terms involving f to one side and bound e^{-Ct} by 1 to arrive at

$$e^{-Ct} f'(t) - C e^{-Ct} f(t) \leq \sup_{y \in U'} |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}|,$$

for almost every $t \in [0, 1]$. Since $f(0) = 0$, we arrive at

$$e^{-C\tau} f(\tau) \leq \int_0^\tau \sup_{y \in U'} |\rho^\varepsilon_{x_t, y} - \rho_{x_t, y}| \, dt.$$

Thus

$$\begin{aligned} f(\tau) &\leq e^{C\tau} \int_0^\tau \sup_{y \in U'} |\rho^\varepsilon_{x_t,y} - \rho_{x_t,y}| \, dt \\ &\leq e^C \int_0^{r_0} \sup_{y \in U'} |\rho^\varepsilon_{x_t,y} - \rho_{x_t,y}| \, dt. \end{aligned}$$

We integrate with respect to $v \in S^{n-1}$ to get

$$\begin{aligned} &\int_{S^{n-1}} \sup_{\substack{\tau \in [0,r_0) \\ y \in U'}} |\gamma^\varepsilon_y(x_\tau(v)) - \gamma_y(x_\tau(v))| \, dv \\ &\qquad \leq e^C \int_{S^{n-1}} \int_0^{r_0} \sup_{y \in U'} |\rho^\varepsilon_{x_t(v),y} - \rho_{x_t(v),y}| \, dt \, dv \\ &\qquad = e^C \int_{S^{n-1}} \int_0^{r_0} \sup_{y \in U'} |\rho^\varepsilon_{x_t(v),y} - \rho_{x_t(v),y}| \frac{1}{t^{n-1}} t^{n-1} \, dt \, dv \\ &\qquad \leq e^C \int_{B(x_0,r_0)} \sup_{y \in U'} |\rho^\varepsilon_{x,y} - \rho_{x,y}| \frac{1}{|x - x_0|^{n-1}} \, dx \\ &\qquad \leq e^C \left(\int_{B(0,r_0)} \left(\frac{1}{|x|^{n-1}} \right)^{p'} dx \right)^{1/p'} \left\| \sup_{y \in U'} |\rho^\varepsilon_{x,y} - \rho_{x,y}| \right\|_{L^p(B(x_0,r_0))}, \end{aligned} \tag{2.15}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. We know that $\|r^{1-n}\|_{L^{p'}(B(0,r_0))}$ is finite for $p' < n/(n-1)$, so by the convergence of $\rho^\varepsilon \to \rho$ in Theorem 2.31 we have that the right hand side of (2.15) converges to 0 as ε converges to 0. Hence there is a sequence $\varepsilon_i \to 0$ such that for almost every $v \in S^{n-1}$

$$\sup_{\substack{\tau \in [0,r_0) \\ y \in U'}} |\gamma^{\varepsilon_i}_y(x_0 + \tau v) - \gamma_y(x_0 + \tau v)| \to 0$$

as $i \to \infty$. □

Corollary 2.43. *Let ε_i be a sequence as in Lemma 2.42. The set $\tilde{F}$ of points $x \in B(x_0, r_0)$ for which $\sup_{y \in U'} |\gamma^{\varepsilon_i}_y(x) - \gamma_y(x)| \to 0$ is of full measure;*

$$|B(x_0, r_0) \setminus \tilde{F}| = 0.$$

Proof. The set $\tilde{S} \subset S^{n-1}$ of points $v \in S^{n-1}$ for which

$$\sup_{\substack{\tau \in [0,r_0) \\ y \in U'}} |\gamma^{\varepsilon_i}_y(x_0 + \tau v) - \gamma_y(x_0 + \tau v)| \to 0 \tag{2.16}$$

is of full measure. Let $\tilde{E}$ be the set of points $x_0 + \tau v$, where $\tau \in [0, r_0]$ and $v \in \tilde{S}$. Then

$$\sup_{y \in U'} |\gamma^{\varepsilon_i}_y(x) - \gamma_y(x)| \to 0$$

as $i \to \infty$ for every $x \in \tilde{E}$ and $\tilde{E}$ has full measure in $B(x_0, r_0)$.

□

Let G be the set of point–pairs that define a plane through x_0:

$$G := \{(x_1, x_2) \in B(x_0, r_0) \times B(x_0, r_0) : \dim(\operatorname{span}\{x_1 - x_0, x_2 - x_0\}) = 2\}.$$

With this defined, we can denote for $(x_1, x_2) \in G$ the unique plane through x_0 containing x_1 and x_2 by P_{x_1,x_2}.

Define now the following "good" set

$$\hat{F}(\rho, \rho^\varepsilon) := \{(x_1, x_2) \in G : P_{x_1,x_2} \text{ is typical for } \rho^\varepsilon \to \rho\}.$$

For every $x_1 \in B(x_0, r_0)$, we define

$$\hat{F}_{x_1}(\rho, \rho^\varepsilon) := \hat{F}(\rho, \rho^\varepsilon) \cap (\{x_1\} \times B(x_0, r_0)).$$

LEMMA 2.44. *Let $\delta > 0$, $\varrho \in \mathcal{A}(B(0, 1 + \delta) \times U)$ where $B(0, 1) \subset \mathbb{R}^n$, and $\varrho^\varepsilon = H^\varepsilon V^\varepsilon(\varrho)$ for $\varepsilon > 0$. Let e_n denote the n^{th} standard basis vector of $\mathbb{R}^n$, and let $U' \subset\subset U$. Then*

$$\mathcal{H}^{n-2}(S^{n-2} \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon)) = 0,$$

where $\mathcal{H}^{n-2}$ is the Hausdorff $(n-2)$-measure in $\mathbb{R}^n$, and S^{n-2} is the unit sphere in $\mathbb{R}^{n-1}$ embedded into the first $n-1$ coordinates of $\mathbb{R}^n$.

Proof. Let $p > n/2$ and $f^\varepsilon \in L^p(B(0,1)\times U')$ converge in norm to $f \in L^p(B(0,1)\times U')$ as $\varepsilon \to 0$. Then

$$\begin{aligned}
&\int_{S^{n-2}} \int_{P_{e_n,\theta}} \int_{U'} |f^\varepsilon - f| \, dy \, d\mathcal{H}^2(x) \, d\mathcal{H}^{n-2}(\theta) \\
&\qquad \le \int_{S^{n-2}} \int_{P_{e_n,\theta}} \int_{U'} |f^\varepsilon - f||x|^{n-2}|x|^{2-n} \, d\mathcal{H}^2(x) \, dy \, d\mathcal{H}^{n-2}(\theta) \\
&\qquad \le \||x|^{2-n}\|_{L^{p'}(B(0,1))} |U'|^{1/p'} \|f^\varepsilon - f\|_{L^p(B(0,1)\times U')},
\end{aligned}$$

where p and p' are Hölder conjugates. By assumption $\|f^\varepsilon - f\|_p \to 0$ as $\varepsilon \to 0$ and $\||x|^{2-n}\|_{p'} < \infty$, consequently there is a sequence $\varepsilon_i \to 0$ such that for almost every $\theta \in S^{n-2}$

$$\lim_{i\to\infty} \int_{P_{e_n,\theta}} \int_{U'} |f^{\varepsilon_i} - f| \, dy \, d\mathcal{H}^2(x) = 0.$$

By Theorem 2.31, for $n/2 < p < \infty$, ρ^ε, $d_\Omega \rho^\varepsilon$ and $\nabla^U \rho^\varepsilon$ converge in their respective L^p spaces to ρ, $d_\Omega \rho$ and $\nabla^U \rho$ respectively, consequently their is a sequence $\varepsilon_i \to 0$ such that for almost every $\theta \in S^{n-2}$

$$\|(\rho^{\varepsilon_i} - \rho)|P_{e_n,\theta}\|_1, \ \|(d_\Omega \rho^{\varepsilon_i} - d_\Omega \rho)|P_{e_n,\theta}\|_1, \text{ and } \|(\nabla^U \rho^{\varepsilon_i} - \nabla^U \rho)|P_{e_n,\theta}\|_1$$

converge to 0 as $i \to \infty$.

Now assume $f \in L^\infty(B(0,1) \times U')$ is non-negative and let E denote the set of points $(x, y) \in B(0, 1) \times U'$ such that $f(x, y) > \|f\|_\infty$. Then

$$\begin{aligned}
0 = |E| &= \int_{B(0,1)\times U'} \chi_E(x, y) \, dx \, dy \\
&= \int_{S_{n-2}} \int_{P_{e_n,\theta}} \int_{U'} \chi_E(x, y)|x|^{n-2} \, dy \, d\mathcal{H}^2(x) \, d\mathcal{H}^{n-2}(\theta).
\end{aligned}$$

Hence, for almost every $\theta \in S^{n-2}$ and almost every $x \in P_{e_n,\theta}$, $\chi_E(x, y) = 0$. Consequently for almost every $\theta \in S^{n-2}$

$$\operatorname{ess\,sup}\{f(x, y) : x \in P_{e_n,\theta} \times U'\} \le \|f\|_\infty.$$

By letting f be $|\varrho|$, $|d_\Omega \varrho|$, or $|\nabla^U \rho|$ as appropriate, we get that for almost every $\theta \in S^{n-2}$, $P_{e_n,\theta}$ is typical for $\varrho^\varepsilon \to \varrho$. $\square$

Corollary 2.45. *Let U, U', ϱ and ϱ^ε be as in Lemma 2.44. Then*

$$\mathcal{H}^n(\mathbb{R}^n \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon)) = 0.$$

Proof. Let $\Psi : S^{n-2} \times (0,\infty) \times \mathbb{R} \to \mathbb{R}^n \setminus \mathbb{R}e_n$ be the map $(\theta, r, z) \mapsto r\theta + ze_n$. Now $P_{e_n, r\theta+ze_n} = P_{e_n, r'\theta+z'e_n}$ for $r, r' \in (0,\infty)$ and $z, z' \in \mathbb{R}$. Consequently

$$\mathbb{R}^n \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon) = \Psi((S^{n-2} \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon)) \times (0,\infty) \times \mathbb{R}).$$

By Lemma 2.44, $\mathcal{H}^{n-2}(S^{n-2} \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon) = 0$, and so

$$\mathcal{H}^n[(S^{n-2} \setminus \hat{F}_{e_n}(\varrho, \varrho^\varepsilon)) \times (0,\infty) \times \mathbb{R})] = 0.$$

The function Ψ preserves sets of measure 0, which proves the claim. $\square$

Proposition 2.46. *Let $\rho \in \mathcal{A}(B(x_0, r_0) \times U))$ and $\rho^\varepsilon = H^\varepsilon(V^\varepsilon(\rho))$. For every $x_1 \in B(x_0, r_0) \setminus \{x_0\}$*

$$\mathcal{H}^n((\{x_1\} \times B(x_0, r_0)) \setminus \hat{F}_{x_1}) = 0.$$

Proof. Define $\phi_{x_1} : B(x_0, r_0) \setminus \{x_0\} \to S^{n-1}$ by

$$\phi_{x_1}(x) = A\frac{x - x_0}{|x_1 - x_0|},$$

where A is a rotation satisfying

$$A\frac{x_1 - x_0}{|x_1 - x_0|} = e_n.$$

Then

$$\begin{aligned}\mathcal{H}^n(\{x_1\} \times B(x_0, r_0) \setminus \hat{F}_{x_1}(\rho, \rho^\varepsilon)) &\le \mathcal{H}^n(\mathbb{R}^n \setminus \phi_{x_1}(\hat{F}_{x_1}(\rho, \rho^\varepsilon))) \\ &\le \mathcal{H}^n(\mathbb{R}^n \setminus \hat{F}'_{e_n}(\tilde{\rho}, \tilde{\rho}^\varepsilon)),\end{aligned}$$

where $\tilde{\rho}_{x,y} = (\phi_{x_1}^{-1})^* \rho_{\phi_{x_1}^{-1}(x), y}$. By Corollary 2.45

$$\mathcal{H}^n(\mathbb{R}^n \setminus \hat{F}'_{e_n}(\tilde{\rho}, \tilde{\rho}^\varepsilon)) = 0.$$

$\square$

Corollary 2.47. *The set $\hat{F}(\rho, \rho^\varepsilon)$ has full measure, i.e.*

$$\mathcal{H}^{2n}(B(x_0, r_0) \times B(x_0, r_0) \setminus \hat{F}(\rho, \rho^\varepsilon)) = 0.$$

Proof. By Fubini's theorem

$$|B(x_0, r_0) \times B(x_0, r_0) \setminus \hat{F}(\rho, \rho^\varepsilon)| = |G \setminus \hat{F}| = \int_{x_1} |G \setminus \hat{F}_{x_1}| \, d\mathcal{H}^n(x_1) = 0.$$

$\square$

Proof of Theorem 2.40. Let $\rho^\varepsilon \in \Gamma(TU \otimes \Lambda^1\Omega)$ be as in Theorem 2.31. Now we need merely show that the set of point pairs which satisfy the conditions of Lemma 2.41 is of full measure.

The set $\hat{F}(\rho, \rho^\varepsilon)$ of point pairs which span a typical plane is of full measure by Corollary 2.47.

Conditions 1 and 2 in Lemma 2.41 hold for point pairs in the set $\tilde{F} \times \tilde{F}$, where $\tilde{F}$ is given by Corollary 2.43

The set $\tilde{F}$ is of full measure in $B(x_0, r_0)$ and hence the set $\tilde{F} \times \tilde{F}$ is of full measure in $B(x_0, r_0) \times B(x_0, r_0)$. Lastly the set of point pairs

$$\hat{F}(\rho, \rho^\varepsilon) \cap \tilde{F} \times \tilde{F},$$

has full measure, and every pair in it satisfies the conditions of Lemma 2.41. □

2.5.2. **The proof of Theorem 1.1.** In section 2.3 we defined the holonomy of a smooth connection about an absolutely continuous curve. If $\rho \in \mathcal{A}(\Omega \times U)$, it is not possible to define *a priori* a lift of an absolutely continuous curve. We have, however, shown that radial lifts along ρ are Lipschitz functions. Consequently we can define the holonomy about a triangle by taking successive radial lifts along the vertices.

Let $\rho \in \mathcal{A}(\Omega \times U)$, $U' \subset\subset U$ be a domain and $y \in U'$. Let

$$0 < r_0 < \min\{\operatorname{dist}(U', \partial U)/(4\|\rho\|_\infty), \operatorname{dist}(y, \partial U')/(12\|\rho\|_\infty)\}.$$

By Theorem 2.40 we have that $\gamma_{x,y} : B(x, r_0) \times U' \to U$, the radial lift along ρ centered at $x_0 \in \Omega$ and starting at y, has a Lipschitz representative. Let $x_0, x_1, x_2 \in \Omega$ be points with pairwise-distance less than r_0. For $i = 0, 1, 2$ define $f_i : B(x_i, r_0) \times U' \to U$

$$f_i(x, y) = \gamma_{x_i,y}(x) \tag{2.17}$$

for $x \in B(x_i, r_0)$ and $y \in U'$). Note that $(x_i, y, x) \mapsto \gamma_{x_i y}(x)$ is jointly continuous in all variables. For any $y' \in U'$ satisfying $\operatorname{dist}(y', \partial U') > 4r_0\|\rho\|_\infty$, we have that $f_i(x_j, y') \in U'$ for $i = 0, 1, 2$. In particular, if $\operatorname{dist}(y, \partial U') > 12r_0\|\rho\|_\infty$, then

$$f_{i_2}(x_{i_0}, f_{i_1}(x_{i_2}, f_{i_0}(x_{i_1}, y))) \in U'$$

and is well defined for any permutation of (i_0, i_1, i_2) of $\{0, 1, 2\}$.

Definition 2.48. The $\mathcal{A}$-holonomy of ρ about the triangle $\Delta = [x_0, x_1, x_2]$ based at x_0 starting at y is

$$\operatorname{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) = |y - f_2(x_0, f_1(x_2, f_0(x_1, y)))|.$$

Remark 2.49. A smooth connection form $\rho \in \Gamma(\Omega \times U)$ is also in $\mathcal{A}(\Omega \times U)$. Consequently we can define both its holonomy and $\mathcal{A}$-holonomy about a triangle $\Delta = [x_0, x_1, x_2]$. But

$$\operatorname{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) = \operatorname{Hol}(\rho, (\partial\Delta, x_0), y),$$

as the radial lift along ρ starting at x_i evaluated at x_j is just the lift along ρ of the straight line $t \mapsto (x_i + t(x_j - x_i))$.

Theorem (1.1). *Let Ω and U be smooth bounded domains. Let $\rho \in \mathcal{A}(\Omega \times U)$ be an Ehresmann connection form and $U' \subset\subset U$ be a domain. Let $r_0 < d(U', \partial U)/(4\|\rho\|_\infty)$. There is a constant $C = C(\rho, r_0)$ such that for every $y \in U'$, and $x_0, x_1, x_2 \in \Omega$, if* $\operatorname{dist}(y, U') > 12r_0\|\rho\|_\infty$ *and $|x_i - x_j| < r_0$ for every $i, j = 0, 1, 2$, then*

$$\operatorname{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) \leq C(\rho, r_0)\|F_\rho\||\Delta| \tag{2.18}$$

where $\Delta = [x_0, x_1, x_2]$.

Proof. Let f_i be as in (2.17) and let $\rho^\varepsilon \in \Gamma(\Omega \times U)$ be a smooth approximations as in Theorem 2.31. We show that

$$\lim_{\varepsilon\to 0} \mathrm{Hol}(\rho^\varepsilon, (\partial\Delta, x_0), y) = \mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y)$$

for $y \in U'$.

Let $y_1 = f_0(x_1, y)$, $y_2 = f_1(x_2, y_1)$, $\hat{y}_1^\varepsilon = \gamma^\varepsilon_{x_0,y}(x_1)$, $\hat{y}_2^\varepsilon = \gamma^\varepsilon_{x_1,y_1}(x_2)$, and $\tilde{y}_2^\varepsilon = \gamma^\varepsilon_{x_1,\hat{y}_1^\varepsilon}(x_2)$. Let $C = \|\nabla^U \rho\|_\infty$. Then

$$\begin{aligned}
|\mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta,&x_0), y) - \mathrm{Hol}(\rho^\varepsilon, (\partial\Delta, x_0), y)| \\
&= |y - f_2(x_0, y_2)| - |y - \gamma^\varepsilon_{x_2,\tilde{y}_2}(x_0)|| \\
&\le |f_2(x_0, y_2) - \gamma^\varepsilon_{x_2,\tilde{y}_2}(x_0)| \\
&\le |f_2(x_0, y_2) - \gamma^\varepsilon_{x_2,y_2}(x_0)| + |\gamma^\varepsilon_{x_2,y_2}(x_0) - \gamma^\varepsilon_{x_2,\hat{y}_2}(x_0| \\
&\quad + |\gamma^\varepsilon_{x_2,\hat{y}_2}(x_0) - \gamma^\varepsilon_{x_2,\tilde{y}_2}(x_0)| \\
&\le |f_2(x_0, y_2) - \gamma^\varepsilon_{x_2,y_2}(x_0)| + e^{C|x_2-x_0|}(|y_2 - \hat{y}_2^\varepsilon| + |\hat{y}_2^\varepsilon - \tilde{y}_2^\varepsilon|)
\end{aligned}$$

by Lemma 2.15. First we examine $\gamma^\varepsilon_{x_2,y_2} : B(x_2, r_0) \to U$. By Theorem 2.40, for $\varepsilon > 0$ this is a family of uniformly Lipschitz maps whose image is contained in a compact set. By Lemma 2.42, there is a sequence $\varepsilon_i \to 0$ for which these functions converge pointwise almost everywhere to γ_{x_2,y_y}, where $\gamma_{x_2,y_2} : B(x_2, r_0) \to U$ is the radial lift along ρ centered at x_2 starting at y_2. By the Arzelá–Ascoli theorem there is a uniformly converging subsequence. But $\gamma_{x_2,y_2}(x) = f_2(x, y_2)$ and hence

$$\lim_{i\to\infty} |y^{\varepsilon_i} - \tilde{y}_0^{\varepsilon_i}| = \lim_{\varepsilon_i\to 0} |f_2(x_0) - \gamma^{\varepsilon_i}_{x_2,y_2}(x_0)| = 0.$$

Now

$$|y_2 - \hat{y}_2^\varepsilon| = |\gamma_{x_1,y_1}(x_2) - \gamma^\varepsilon_{x_1,y_1}|.$$

We similarly deduce that there is a sequence $\varepsilon_i \to 0$ such that $\gamma^{\varepsilon_i}_{x_1,y_1} : B(x_1, r_0) \to U$ converges uniformly to γ_{x_1,y_1}, and so

$$\lim_{i\to\infty} |y_2 - \hat{y}_2^\varepsilon| = \lim_{\varepsilon\to 0} |\gamma_{x_1,y_1}(x_2) - \gamma^\varepsilon_{x_1,y_1}(x_2)| = 0.$$

By Lemma 2.15

$$\begin{aligned}
|\hat{y}_2^\varepsilon - \tilde{y}_2^\varepsilon| &= |\gamma^\varepsilon_{x_1,y_1}(x_2) - \gamma^\varepsilon_{x_1,\hat{y}_1}(x_2)| \\
&\le e^{C|x_2-x_1|}|\hat{y}_1^\varepsilon - y_1| \\
&= e^{C|x_2-x_1|}|\gamma^\varepsilon_{x_0,y}(x_1) - \gamma_{x_0,y}(x_1)|.
\end{aligned}$$

Once again we deduce that for some sequence $\varepsilon_i \to 0$ $\gamma^\varepsilon_{x_0,y}$ converges uniformly to $\gamma_{x_0,y}$, and hence

$$\lim_{i\to\infty} |\hat{y}_2^\varepsilon - \tilde{y}_2^\varepsilon| = 0.$$

Now we may conclude that

$$\mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) = \lim_{\varepsilon\to 0} \mathrm{Hol}(\rho^\varepsilon, (\partial\Delta, x_0), y).$$

We note that the functions $y \mapsto f_i(x, y)$ are uniformly Lipschitz in U' (with Lipschitz constant $\exp(C \max\{|x_0-x_1|, |x_1-x_2|, |x_2-x_0|\})$ and hence $y \mapsto \mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y)$ is Lipschitz in U'. Similarly $y \mapsto \mathrm{Hol}(\rho^\varepsilon, (\Delta, x_0), y)$ is uniformly (in ε) Lipschitz in U'.

By Lemmata 2.27 and 2.28 there is a null homotopic parametrisation of Δ, $\sigma : [0,1]^2 \to \Delta$, a lift $\gamma^\varepsilon : [0,1]^2 \to \Delta \times U$ of σ along ρ^ε starting at y, and a height function $h^\varepsilon : \Delta \to U$ for γ^ε given by

$$h^\varepsilon = \pi_U \circ \gamma^\varepsilon \circ \sigma^{-1}.$$

Let $r > 0$ be sufficiently small that for every $y' \in B(y, r)$, $\mathrm{Hol}_{\mathcal{A}}(\rho, \partial\Delta, y')$ is defined. For $x \in \Delta$ let $B^\varepsilon_x = B(h^\varepsilon(x), e^{C4r_0}r)$. As in the proof of Lemma 2.41, h^ε converges for almost every $x \in \mathrm{Int}(\Delta)$ to some $h : \mathrm{Int}(\Delta) \to U$.

Assume that x_0, x_1 and x_2 span a typical plane for $\rho^\varepsilon \to \rho$. Then there is a sequence $\varepsilon_i \to 0$ such that $F_{\rho^{\varepsilon_i}}|P_{x_1,x_2}$ converges almost everywhere to $F_\rho|P_{x_1,x_2}$ and $\|F_{\rho|P_{x_1,x_2}}\|_\infty \leq \|F_\rho\|_\infty$. Then $|F_{\rho^{\varepsilon_i}}(x,y)|\chi(B^{\varepsilon_i}_x)(y)$ converges for almost every $(x, y) \in \Delta \times U$ to $|F_\rho(x,y)|\chi(B_x)(y)$. Let $\beta(m) = |B(0,1)|^{-1}$ in $\mathbb{R}^m$ and let $C = \|\nabla^U \rho\|_\infty$. Consequently

$$\begin{aligned}
\mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y) &= \lim_{r\to 0} \beta(m) r^{-m} \int_{B(y,r)} \mathrm{Hol}_{\mathcal{A}}(\rho, (\partial\Delta, x_0), y')\, dy' \\
&= \lim_{r\to 0} r^{-m}\beta(m) \int_{B(y,r)} \lim_{\varepsilon\to 0} \mathrm{Hol}(\rho^\varepsilon, (\partial\Delta, x_0), y')\, dy \\
&= \lim_{r\to 0}\lim_{\varepsilon\to 0} r^{-m}\beta(m) \int_{B(y,r)} \mathrm{Hol}(\rho^\varepsilon, (\partial\Delta, x_0), y'\, dy' \\
&\leq \lim_{r\to 0}\lim_{\varepsilon\to 0} \beta(m) r^{-m} e^{C(m+1)4r_0} \int_\Delta \int_{B^\varepsilon_x} |F^\varepsilon|\, dy'\, dx \\
&\leq \lim_{r\to 0} \beta(m) r^{-m} e^{C(m+1)4r_0} \int_\Delta \int_{B_x} |F|\, dy'\, dx \\
&\leq \lim_{r\to 0} \beta(m) e^{C(2m+1)4r_0} |\Delta| \|F\|_\infty.
\end{aligned}$$

Hence (2.18) holds for every $x_1, x_2 \in B(x_0, r_0)$ for which P_{x_1,x_2} is typical for $\rho^\varepsilon \to \rho$. However, for every $x_0 \in \Omega$ and $y \in U'$

$$(x_1, x_2) \mapsto \mathrm{Hol}_{\mathcal{A}}(\rho, (\partial[x_0, x_1, x_2], x_0), y)$$

is continuous on $B(x_0, r_0) \times B(x_0, r_0)$. But the set of vertices for which (2.18) holds is of full measure and hence dense. Consequently it holds for all $x_1, x_2 \in B(x_0, r_0)$. Thus (2.18) holds with $C(\rho, r_0) = e^{4C(2m+1)r_0}$. □

2.6. A Frobenius theorem for non-smooth connections

Theorem 2.50. *Let $\rho \in \mathcal{A}(\Omega \times U)$ be an Ehresmann connection form with curvature*

$$F_\rho = 0.$$

Let $U' \subset U$, $y \in U'$ and r_0 satisfy

$$r_0 < \min\{d(U', \partial U)/4\|\rho\|_\infty, \mathrm{dist}(y, \partial U')/(16\|\rho\|_\infty)\}. \tag{2.19}$$

Then there exists a radial lift $\gamma_y : B(x_0, r_0) \to U$ along ρ centered at x_0 starting at y satisfying

$$D_x\gamma_y = \rho_{x,\gamma_y(x)} \text{ a.e., and } \gamma_y(x_0) = y.$$

Furthermore, suppose $\eta : B(x_0, r_0) \to U$ is Lipschitz, such that $\eta(x_0) = y$ and $D_x\eta = \rho_{x,\eta(x)}$. Then $\eta = \gamma_y$.

Proof. By Theorem 1.1, if the curvature of ρ is zero, we know that the $\mathcal{A}$-holonomy about any triangle is zero. Since $r_0 > 0$ satisfies (2.19) there exists a radial lift $\gamma : B(x_0, r_0) \to U$ starting at y which is Lipschitz continuous by Theorem 2.40.

There exists an $\varepsilon > 0$ such that for any $x' \in B(x_0, \varepsilon)$ there is a radial lift $\gamma_{x'} : B(x', r_0 + \varepsilon) \to U$ along ρ starting at $\gamma(x')$ and (2.19) holds for $r_0 + \varepsilon$. For any $x \in B(x_0, r_0)$ we can apply Theorem 1.1 to get that the $\mathcal{A}$-holonomy about the triangle $[x, x_0, x']$ starting at $\gamma(x)$ is 0. Consequently $\gamma_{x'}(x) = \gamma(x)$ for every $x' \in B(x_0, \varepsilon)$ and every $x \in B(x_0, r_0)$. Furthermore,

$$D\gamma(x) \cdot \frac{x - x'}{|x - x'|} = D\gamma^{x'}(x) \cdot \frac{x - x'}{|x - x'|} = \rho_{x,\gamma^{x'}(x)} \cdot \frac{x - x'}{|x - x'|}$$

for almost every $x' \in B(x_0, \varepsilon)$. Hence

$$D\gamma(x) = \rho_{x,\gamma(x)}.$$

Uniqueness follows from uniqueness for the radial lift in Corollary 2.39. □

Definition 2.51. Let ρ be a connection one-form on $\Omega \times \mathcal{M}$ of class $\mathcal{A}_{\text{loc}}(\Omega \times \mathcal{M})$ with 0 curvature. For every $(x_0, y_0) \in \Omega \times \mathcal{M}$ there is a coordinate neighbourhood of y_0, $\Psi : U \to \mathbb{R}^m$, an r_0 such that

$$0 < r_0 < d(y_0, \partial U)/(4\|\Psi_*\rho\|_\infty)$$

and a Lipschitz map $\eta_{x_0,y_0} : B(x_0, r_0) \to \mathcal{M}$ given by the radial lift along $\Psi_*\rho$ post-composed with Ψ. As a consequence, η_{x_0,y_0} satisfies

$$D_x\eta_{x_0,y_0} = \rho_{x,\eta_{x_0,y_0}(x)} \quad \text{and} \quad \eta_{x_0,y_0}(x_0) = y_0.$$

We call this a *lift along ρ centered at x_0, y_0*.

Let (Y, d) be a compact metric space and $f : Y \to \Omega$ a Lipschitz map. We say that a Lipschitz map $\tilde{f} : Y \to \mathcal{M}$ is a *lift* of f along ρ if there are numbers $r, r_0 > 0$ such that for every $z \in Y$ and $z' \in B(z, r)$

$$\tilde{f}(z') = \eta_{f(z),\tilde{f}(z)}(f(z')),$$

where $\eta : B(f(z), r_0) \to \mathcal{M}$ is the lift along ρ centered at $f(z), \tilde{f}(z)$.

Lemma 2.52. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold and (Y, d_0) be a compact metric space. Let $y_0 \in \mathcal{M}$ and $R > 0$ and let ρ be a connection one-form of class $\mathcal{A}_{\text{loc}}(\Omega \times \mathcal{M})$ with zero curvature. Let $H : Y \times [0, 1] \to \Omega$ be a Lipschitz map. There is an $R' > 0$ such that for any Lipschitz map $f : Y \times \{0\} \to B(y_0, R)$ which is a lift of $H|Y \times \{0\}$ along ρ, there is a unique Lipschitz map $\tilde{H} : Y \times [0, 1] \to B(y_0, R')$ which is a lift of H along ρ and satisfies $\tilde{H}(z, 0) = f(z, 0)$ for every $z \in Y$.*

Proof. The proof follows that of the homotopy lifting property for covering spaces *cf.* [Hat02, Proposition 1.30]. Let c denote the Lipschitz constant of H, $C = \|\nabla^{\mathcal{M}}\rho\|_\infty$ and D the diameter of Y. Suppose $t \mapsto \tilde{H}(z, t)$ is the solution of the initial value problem

$$\frac{d}{dt}\tilde{H}(z, t) = \rho_{H(z,t),\tilde{H}(z,t)} \cdot \frac{d}{dt}H(z, t)$$

$$\tilde{H}(z, 0) = f(z, 0).$$

Consequently

$$d(y_0, \tilde{H}(z,t)) \leq d(y_0, \tilde{H}(z,0)) + d(\tilde{H}(z,0), \tilde{H}(z,t))$$
$$\leq R + \|\rho\|_\infty e^{cC},$$

where the norm of ρ is taken over the domain $\Omega \times B(y_0, r)$, which is finite because $\rho \in \mathcal{A}_{\mathrm{loc}}(\Omega \times \mathcal{M})$. We set $R' = R + \|\rho\|_\infty e^{cC}$.

Because $\mathcal{M}$ is complete, $\overline{B}(y_0, R')$ is compact. Hence there is an $r_1 > 0$ such that for any $(x, y) \in H(Y \times [0,1]) \times B(y_0, R')$ there exists $\eta_{x,y} : B(x, r_1) \to \mathcal{M}$ a lift along ρ centered at x, y. Because H is Lipschitz, there is an $r > 0$ such that for any $z \in Y$ and $t \in [0,1]$, $H(B(z,r) \times (t-r, t+r)) \subset B(H(z,t), r_1)$. Let $\{z_i : i = 1, \ldots, N\}$ be an r-net for Y, and let $0 = t_0 < t_1 < \cdots < t_k = 1$ satisfy $t_{i+1} - t_i < r$ for $i = 0, \ldots, k$.

We proceed by induction. Suppose $\tilde{H}_{i,j} : B(z_i, r) \times [0, t_j] \to \mathcal{M}$ is a lift of $H|B(z_i, r) \times ([0, t_j + r) \cap [0,1])$ along ρ. Define $H_{i,j+1} : B(z_i, r) \times [0, t_{j+1} + r) \to \mathcal{M}$ by

$$H_{i,j+1}(z,t) = \begin{cases} H_{i,j}(z,t), & t \in [0, t_j] \\ \eta_{H(z_i, t_{j+1}), \tilde{H}_{i,j}(z_i, t_j)}(H(z,t)), & t_j < t < t_{j+1} + r. \end{cases}$$

Define $\tilde{H}_{i,0} : B(y_i, r) \times [0, r)$ by

$$\tilde{H}_{i,0}(z,t) = \eta_{H(z_i,0), f(z_i,0)}(H(z,t)).$$

The uniqueness of the lift $\eta_{H(z_i,t_j), \tilde{H}_{i,j}(z_i,t_j)}$ guarantees that $\tilde{H}_{i,j+1}$ is continuous and is a lift of $H|B(z_i, r) \times ([0, t_{j+1} + r) \cap [0,1])$. By induction we have constructed a map $\tilde{H}_i : B(z_i, r) \times [0,1] \to \mathcal{M}$. Suppose $B(z_i, r) \cap B(z_j, r) \neq \varnothing$. Once again by the uniqueness of radial lifts the maps $\tilde{H}_{i,0}$ and $\tilde{H}_{j,0}$ are equal on $(B(z_i, r) \cap B(z_j, r)) \times [0, r)$. By induction $\tilde{H}_i$ is equal to $\tilde{H}_j$ on $(B(z_i, r) \cap B(z_i, r)) \times [0,1]$. Similarly any other lift G equal to f on $Y \times \{0\}$ is equal to $\tilde{H}$.

□

Corollary 2.53 (Theorem 1.2). *Let $(\mathcal{M}, g)$ be a smooth complete Riemannian manifold, and $\Omega \subset \mathbb{R}^n$ a connected and simply connected domain, and $\rho \in \mathcal{A}_{\mathrm{loc}}(\Omega \times \mathcal{M})$ an Ehresmann connection one-form with zero curvature, that is*

$$F_\rho = 0 \text{ almost everywhere.}$$

Then for every $y \in \mathcal{M}$ and $x_0 \in \Omega$, there is a unique Lipschitz map $\gamma_y : \Omega \to \mathcal{M}$ such that

$$D_x \gamma_y = \rho_{x, \gamma_y(x)},$$
$$\gamma_y(x_0) = y.$$

Proof. Let $\sigma_1, \sigma_2 : [0,1] \to \Omega$ be two Lipschitz paths satisfying $\sigma_i(0) = x_0$ $\sigma_i(1) = x_2$. Let $\gamma_i : [0,1] \to \mathcal{M}$ be the lift along ρ of σ_i starting at y. Let $H : [0,1] \times [0,1] \to \Omega$ be a Lipschitz map such that $H(0,t) = \sigma_1(t)$, $H(1,t) = \sigma_2(t)$, $H(s,0) = x_0$ and $H(s,1) = x_1$ for $s, t \in [0,1]$. We define a lift $f : [0,1] \times 0 \to \mathcal{M}$ of $H|[0,1] \times \{0\}$ by $f(s,0) = y$. We apply Lemma 2.52 to construct a lift $\tilde{H} : [0,1] \times [0,1] \to \mathcal{M}$ of H along ρ equal to f on $[0,1] \times \{0\}$. Consequently $\gamma_1(t) = \tilde{H}(0,t)$ and $\gamma_2(t) = \tilde{H}(1,t)$. Because $\tilde{H}$ is a lift of H, $s \mapsto \tilde{H}(s,1)$ is constant for $s \in [0,1]$. Consequently $\gamma_1(1) = \gamma_2(1)$. We define $\gamma(x)$ to be the value of the lift of any Lipschitz path $\sigma : [0,1] \to \Omega$ from x_0 to x starting at y. By the preceding reasoning, this value is independent of the choice of path.

Lastly the condition

$$D_x\gamma_y = \rho_{x,\gamma_y}$$

follows from the fact that $\gamma(x') = \eta_{x,\gamma(x)}(x')$ wherever the latter is defined. □

2.6.1. **Frobenius' theorem for Lipschitz distributions.** In this section we show that Frobenius' integrability condition for Lipschitz distributions follows from Theorem 1.2.

THEOREM 2.54 (cf. Theorem A1 in [Sim96]). *Let $k \leq n$ be positive integers, let $\mathcal{M}$ be a C^2 n-manifold and let $\mathcal{H}$ be a rank k Lipschitz hyperplane distribution in $\mathcal{M}$. If $\mathcal{H}$ is involutive almost everywhere then it is integrable. Furthermore the integral submanifolds are of class $C^{1,1}$.*

We say a rank k hyperplane distribution, $\mathcal{H} \subset T\mathcal{M}$ is *continuous* if it is everywhere locally given by the span of continuous vector fields. If it is given locally by the span of Lipschitz vector fields, we say it is *Lipschitz.* If $X, Y : \mathcal{M} \to T\mathcal{M}$ are Lipschitz vector fields we can define their commutator in coordinates, that is if

$$X = X^i\partial_i, \quad Y = Y^i\partial_i$$

where $\partial_i = \frac{\partial}{\partial x^i}$, then

$$[X,Y] = [X,Y]^i\partial_i \quad [X,Y]^i = X^j\frac{\partial Y^i}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j},$$

which is defined almost everywhere and is locally essentially bounded.

A Lipschitz Hyperplane distribution $\mathcal{H}$ is said to be *involutive almost everywhere* if for every pair of Lipschitz vector fields $X, Y : \mathcal{M} \to \mathcal{H}$, and for almost every $x \in \mathcal{M}$

$$[X,Y](x) \in \mathcal{H}.$$

Let U and V be subsets of $\mathbb{R}^n$. Define the set

$$U + V = \{x + y : x \in U,\, y \in V\}.$$

LEMMA 2.55. *Let $\Omega \subset \mathbb{R}^n$ be a connected domain. Let $\mathcal{H} \subset T\Omega$ be a continuous rank k-distribution. Then for every $x_0 \in \Omega$ there is a k-dimensional subspace $P_0 \subset \mathbb{R}^n$, relatively open $U_0 \subset P_0$ and $V_0 \subset P_0^\perp$ and a connection one-form $\rho : U_0 \times V_0 \to P_0^\perp \otimes \Lambda^1 P_0$ such that $x_0 \in U_0 + V_0$, and for every $x \in U \times V$*

$$\mathcal{H}_x = \{X + \rho(x)\cdot X : X \in P_0\} \times \{y\}.$$

Furthermore $\mathcal{H}$ is Lipschitz, if and only if ρ is a Lipschitz continuous function.

Proof. Without loss of generality we may assume that $x_0 = 0$ and $\mathcal{H}_0 = \mathbb{R}^k \subset \mathbb{R}^n$. Let $\varepsilon > 0$ be such that there are continuous vector fields $X_1, X_2, \ldots, X_k : B(0,\varepsilon) \to \mathbb{R}^n$ spanning $\mathcal{H}|B(0,\varepsilon)$. Suppose further that there exist continuous vector fields $X_{k+1}, \ldots, X_n : B(0,\varepsilon) \to \mathbb{R}^n$ which, for every $x \in B(0,\varepsilon)$, extend $X_1, \ldots, X_k$ to a basis of $\mathbb{R}^n$. We apply Gram-Schmidt orthogonalisation to get continuous orthonormal vector fields $\hat{X}_1, \ldots, \hat{X}_n : B(0,\varepsilon) \to \mathbb{R}^n$, such that $\hat{X}_1, \ldots, \hat{X}_k$ span $\mathcal{H}$, and $\hat{X}_{k+1}, \ldots, \hat{X}_m$ span $\mathcal{H}^\perp$, the orthogonal complement of $\mathcal{H}$. In particular if the X_i are Lipschitz continuous then so are the $\hat{X}_i$. Set e_i to be the standard Cartesian basis vectors in $\mathbb{R}^n$, and without loss of generality assume that $e_i = \hat{X}_i(0)$ for $i = 1, \ldots, n$. Then $\{e_1, \ldots, e_k\}$ is an orthonormal basis of $\mathbb{R}^k$ and $\{e_{k+1}, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^{n-k}$.

Define the matrix field

$$\Pi_{\mathcal{H}} : B(0,\varepsilon) \to \mathcal{M}_{n\times n}, \quad x \mapsto \sum_{i=1}^{k} \hat{X}_i(x) \otimes \langle \hat{X}_i(x), \cdot\rangle.$$

Then $\Pi_{\mathcal{H}}(x) \cdot \Pi_{\mathcal{H}}(x) = \Pi_{\mathcal{H}}(x)$ and $\Pi_{\mathcal{H}}(x)^t = \Pi_{\mathcal{H}}(x)$ for every $x \in B(0,\varepsilon)$. And for every $x \in B(0,\varepsilon)$ and $X \in \mathbb{R}^n$, $\Pi_{\mathcal{H}}(x) \cdot X \in \mathcal{H}_x$.

Let Π^k and Π_{n-k} denote the matrices of the projections onto, respectively, the first k-coordinates and last $n-k$ coordinates of $\mathbb{R}^n$.

$\Pi_{\mathcal{H}}$ is continuous because it is the sum of products of continuous functions and $\Pi_{\mathcal{H}}(0) = \Pi^k$. If the X_i are Lipschitz, then so is $\Pi_{\mathcal{H}}$.

Consider for every $x \in B(0,\varepsilon)$ the matrix $\Pi^k \cdot \Pi_{\mathcal{H}}(x)$. For any vector $X \in \mathbb{R}^k$,

$$\begin{aligned}|(\Pi^k \cdot \Pi_{\mathcal{H}}(x)) \cdot X - X| &= |\Pi^k \cdot (\Pi_{\mathcal{H}}(x) - \Pi^k)X| \\ &\le \|\Pi^k\|\|\Pi_{\mathcal{H}}(x) - \Pi_{\mathcal{H}}(0)\||X| \\ &\le \|\Pi_{\mathcal{H}}(x) - \Pi_{\mathcal{H}}(0)\||X|.\end{aligned}$$

By the continuity of $\Pi_{\mathcal{H}}$ we have that there is a $\delta > 0$ such that for all $x \in B(0,\delta)$ $\|\Pi_{\mathcal{H}}(x) - \Pi_{\mathcal{H}}(0)\| < 1/2$. For every $x \in B(0,\delta)$ let $(\Pi^k \cdot \Pi_{\mathcal{H}}(x))|\mathbb{R}^k : \mathbb{R}^k \to \mathbb{R}^k$ denote the linear map

$$X \mapsto \Pi^k \cdot \Pi_{\mathcal{H}}(x) \cdot X.$$

Then

$$\|(\Pi^k \cdot \Pi_{\mathcal{H}}(x))|\mathbb{R}^k - \mathrm{Id}_{\mathbb{R}^k}\| < 1/2$$

for every $x \in B(0,\delta)$. Hence there is a $(k \times k)$ inverse matrix $R(x)$. Furthermore for every $x \in B(0,\delta)$, matrix multiplication by $\Pi_{\mathcal{H}}(x)$ is a linear isomorphism from $\mathbb{R}^k \to \mathcal{H}_x$ and matrix multiplication by Π^k is a linear isomorphism from $\mathcal{H}_x$ to $\mathbb{R}^k$. The function $R : B(x_0,\delta) \to GL(k)$ is continuous because $\Pi_{\mathcal{H}} : B(0,\delta) \to \mathcal{M}_{n\times n}$ is continuous. In particular, if $\Pi_{\mathcal{H}}$ is Lipschitz then so is R.

Let $i_{n,k}$ denote the $(n \times k)$-matrix which takes $\mathbb{R}^k$ to the first k components of $\mathbb{R}^n$. Define $Q : B(0,\delta) \to \mathcal{M}_{n\times k}$ by $Q(x) = \Pi_{\mathcal{H}}(x) \cdot i_{n,k} \cdot R(x) \in \mathcal{M}_{n\times k}$ for $x \in B(0,\delta)$. Then $\Pi^k \cdot Q(x) = \mathrm{Id}_{\mathbb{R}^k}$. Define the map $\rho : B(0,\delta) \to \mathcal{M}_{(n-k)\times k}$ by $x \mapsto \Pi_{n-k} \cdot Q(x)$. If Q is continuous, then so is ρ. If Q is Lipschitz then so is ρ.

For any $Z \in \mathcal{H}_x$ $Z = \Pi^k \cdot Z + \Pi_{n-k} \cdot Z$. Let $X = \Pi^k \cdot Z$, then $\Pi^k \cdot Q(x)X = X$. The vector Z is the unique vector in $\mathcal{H}$ with P-component X, so $Z = Q(x) \cdot X$, and so

$$Z = \Pi^k \cdot Q(x) \cdot X + \Pi_{n-k} \cdot Q(x) \cdot X = X + \rho(x) \cdot X.$$

Let $U = B(0,\delta/\sqrt{2}) \subset \mathbb{R}^k$ and $V = B(0,\delta/\sqrt{2}) \subset \mathbb{R}^{n-k}$, then $U + V \subset B(0,\delta) \subset \mathbb{R}^n$. This gives us a map $\rho : U + V \to \mathcal{M}_{(n-k)\times k}$ but $\mathcal{M}_{(n-k)\times k}$ is isomorphic to $TV \otimes \Lambda^1 U$.

Suppose ρ is Lipschitz continuous. Then the vector fields given by $x \mapsto e_i + \rho(x) \cdot e_i$ for $i = 1,\dots,k$ are Lipschitz and span $\mathcal{H}$. Hence $\mathcal{H}$ is Lipschitz.

□

LEMMA 2.56. *Let $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^{m-k}$ be bounded domains and $\rho : U \times V \to \mathbb{R}^{m-k} \otimes \Lambda^1 U$ be a Lipschitz continuous connection one-form. Then $\rho \in \mathcal{A}(U \times V)$. Furthermore the distribution*

$$\mathcal{H} = \{(X \oplus \rho(y) \cdot X, y) \in \mathbb{R}^m \times (U \times V) : \ y \in U \times V, \, X \in \mathbb{R}^k$$

is involutive almost everywhere if and only if $F_\rho = 0$ almost everywhere.

Proof. Let $x_1, \ldots, x_k$ denote the k coordinates of $\mathbb{R}^k$ and $y_1, \ldots, y_{m-k}$ the coordinates of $\mathbb{R}^{m-k}$. Because ρ is Lipschitz, it is in $W^{1,\infty}(U \times V, \mathbb{R}^n \otimes \Lambda^1 U)$. Hence ρ is of class $\mathcal{A}(U \times V)$.

First we note that the proof of Lemma 2.8 holds for X and Y Lipschitz vector fields. Hence for Lipschitz vector fields $Z_1 = X + \rho \cdot X$ and $Z_2 = Y + \rho \cdot Y$ the following holds almost everywhere

$$\begin{aligned}[Z_1, Z_2] &= [X + \rho \cdot X, Y + \rho \cdot Y] \\ &= \nabla_{X+\rho\cdot X} Y - \nabla_{Y+\rho\cdot Y} X + \rho \cdot (\nabla_{X+\rho\cdot X} Y + \nabla_{Y+\rho\cdot Y} X) + F_\rho(X, Y).\end{aligned}$$

By construction $\mathcal{H} \oplus \mathbb{R}^{n-k} = \mathbb{R}^n$ so

$$[Z_1, Z_2] \in \mathcal{H}$$

almost everywhere if and only $F_\rho = 0$ almost everywhere. □

Proof of Theorem 2.54. Because $\mathcal{H}$ is Lipschitz, around every point $z_0 \in \mathcal{M}$ we can choose a chart (φ, Ω) such that $\varphi(\Omega) = U \times V$, $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^{m-k}$ and $D\varphi(\mathcal{H})$ is given by $\{X + \rho \cdot X : X \in \mathbb{R}^k\}$ where $\rho \in \mathcal{A}(U \times V)$ by Lemma 2.56. Suppose without loss of generality that $\varphi(z_0) = (0, 0)$. Because $\mathcal{H}$ is involutive almost everywhere, it follows that $F_\rho = 0$ almost everywhere. Hence by Lemma 2.55, there is a $V' \subset V$ containing 0 and a unique $\gamma : U \times V' \to U \times V$ satisfying

$$\partial_i \gamma(x, y) = e_i + \rho \cdot e_i \text{ and } \gamma(x_0, y) = (x_0, y),$$

where e_i is the i^{th} standard basis vector of $\mathbb{R}^k$ and $i = 1, \ldots, k$. Then we can define a homeomorphism $\gamma_{z_0} : U \times V' \to \mathcal{M}$ by $\gamma_{z_0} = \varphi^{-1} \circ \gamma$, which satisfies $\partial_i \gamma_{z_0} \in \mathcal{H}$ for $i = 1, \ldots k$. In particular the map $x \mapsto \gamma_{z_0}(x, y)$ has a Lipschitz continuous derivative and hence is $C^{1,1}$-smooth.

Given two points z_1 and z_2 in $\mathcal{M}$, and corresponding charts (φ_1, Ω_1), (φ_2, Ω_2), $\varphi_i(z_i) = (x_i, y_i)$, define $\gamma_{z_1} : U_1 \times V_1 \to \mathcal{M}$ and $\gamma_{z_2} : U_2 \times V_2 \to \mathcal{M}$.

We can define sets $O_i = \gamma_{z_i}(U_i \times \{y_i\})$ $i = 1, 2$, and maps $\psi_i = \pi_{U_i \times \{y_i\}} \circ \varphi_i : O_i \to U_i \subset \mathbb{R}^k$, where $\pi_{U_i \times \{y_i\}}$ is the Cartesian projection $U_i \times V_i \to U_i \times \{y_i\}$. Then $\gamma_{z_i} | U_i \times \{y_i\}$ is the inverse map of ψ_i.

By uniqueness of the lifts

$$\gamma_{z_1} \circ \varphi_1 \circ \varphi_2^{-1} = \gamma_{z_2}.$$

on $\psi_2(O_1 \cap O_2)$, and hence the intersection $O_1 \cap O_2$ is locally homeomorphic to an open subset of $\mathbb{R}^k$. Define the transition map

$$\psi_{1,2} : \psi_1(O_1 \cap O_2) \to \psi_2(O_1 \cap O_2), \quad x \mapsto \psi_2(\gamma_{z_1}(x, y_1)).$$

The transition maps are compositions of C^2- and $C^{1,1}$-smooth maps, and hence $C^{1,1}$-smooth. □

CHAPTER 3

Quasiconformal co-frames and p-harmonic maps to $SO(n)$

In this chapter we investigate the connections between $\mathbb{R}^n$-valued one-forms minimising the norm of their exterior derivative over a fixed conformal class and p-harmonic maps to $SO(n)$.

The orthogonal group $O(n)$ is the space of $(n \times n)$-matrices R satisfying

$$R^t R = I.$$

The special orthogonal group $SO(n)$ is the subset of $O(n)$ of matrices with determinant 1. The group $O(n)$ is homeomorphic to two disjoint copies of $SO(n)$, and $SO(n)$ is the connected component of the identity of $O(n)$.

These groups are both Lie groups. Because $SO(n)$ is the connected component of $I \in O(n)$ we have that the Lie algebras of $SO(n)$ and $O(n)$ are the same. We denote this space by $\mathfrak{so}_n$ and it is the space of antisymmetric $(n \times n)$-matrices with Lie bracket given by the commutator of two elements:

$$[A, B] = AB - BA$$

for every $A, B \in \mathfrak{so}_n$. The adjoint action is given by conjugation with elements of $O(n)$: for every $R \in O(n)$

$$\mathrm{Ad}_R : \mathfrak{so}_n \to \mathfrak{so}_n, \quad \mathrm{Ad}_R(u) = R^{-1} u R.$$

The conformal group $CO(n)$ is the group of $(n \times n)$-matrices λR where $\lambda \in \mathbb{R} \setminus \{0\}$ and $R \in O(n)$. The group $CO^+(n)$ is the set of *positively oriented conformal matrices*, that is conformal matrices with positive determinant. We denote

$$CO_0^+(n) = CO^+(n) \cup \{0\} \quad \text{and} \quad CO_0(n) = CO(n) \cup \{0\}.$$

The sets $CO(n)$ and $CO^+(n)$ are (non-compact) Lie groups.

Once again $CO^+(n)$ is the connected component of the identity in $CO(n)$, so $CO^+(n)$ and $CO(n)$ have the same Lie algebra, namely $(\mathbb{R}I) \oplus \mathfrak{so}_n$ where $\mathbb{R}I$ is the space of matrices equal to a constant times the identity matrix. The Lie-bracket is once again given by the commutator.

Conformal matrices have a rather nice property *cf.* [IM01, (9.39)]

Proposition 3.1. *Let A be a matrix, then*

$$|A|^n \geq n^{n/2} |\det A|,$$

and $A \in CO_0^+$ *if and only if*

$$|A|^n = n^{n/2} \det A.$$

Here $|A|$ is the non-normalised Hilbert-Schmidt norm $\sqrt{\sum_{i,j=1}^n a_{ij}^2}$ of $A = (a_{ij})$

For an invertible matrix A we define the *outer distortion*

$$K^O(A) = |A|^n \det A^{-1} \tag{3.1}$$

and the *inner distortion*

$$K^I(A) = |A^{-1}|^n \det A. \tag{3.2}$$

For a matrix field $A : \Omega \to \mathcal{M}_{n\times n}$ which is invertible almost everywhere, we define $K^O(A) = \operatorname{esssup}_x K^O(A(x))$ and $K^I(A) = \operatorname{esssup}_x K^I(A(x))$.

If $\rho : \Omega \to \mathbb{R}^n \otimes \Lambda^1\Omega$ is a measurable co-frame, and there is a $K > 0$ for which

$$|\rho(x)|^n \leq K \det \rho(x)$$

for almost every $x \in \Omega$, we say that ρ is a K-*quasiconformal* co-frame; by Proposition 3.1 $K \geq n^{n/2}$.

Let $\Sigma \subset \Omega$ be a measurable set and let $\rho : \Omega \to \mathbb{R}^n \otimes \Lambda^1\Omega$ given by $\rho^i = P^i_j dx^j$ be a K-quasiconformal co-frame which vanishes almost nowhere on Σ. It has a *dual frame* $R : \Sigma \to \mathbb{R}^n \otimes T\Omega$ given by

$$R_i = R_i^j \partial_j.$$

where R_i^j is the inverse matrix field to P^i_j, *i.e.* $R^i_j P^j_k = \delta^i_k$. Trivially

$$R_i \otimes \rho^i(x) = \mathrm{Id}_{T_x\Omega} \quad \text{and} \quad \rho^i(R_j)(x) = \delta^i_j$$

for almost every $x \in \Sigma$.

Lemma 3.2. *Let $\rho : \Omega \to \mathbb{R}^n \otimes \Lambda^1\Omega$ be a K-quasiconformal co-frame. Suppose that for every $x \in \Sigma \subset \Omega$, $\rho(x) \neq 0$. Then for almost every $x \in \Sigma$ the map*

$$\rho(x)_\wedge : \mathfrak{so}_n \otimes \Lambda^1\Omega \to \mathbb{R}^n \otimes \Lambda^2\Omega \qquad \alpha \mapsto \alpha \wedge \rho(x)$$

is invertible and has inverse given by

$$\rho(x)_\wedge^{-1} : \mathbb{R}^n \otimes \Lambda^2\Omega \to \mathfrak{so}_n \otimes \Lambda^1\Omega$$

$$\beta \mapsto \frac{1}{2}\left(R_i(x) \llcorner \beta^j - R_j(x) \llcorner \beta^i + \beta^k(R_i(x), R_j(x))\rho^k(x)\right)$$

where $R \in \mathbb{R}^n \otimes T\Omega$ is the dual frame to ρ. The norm of this map is bounded in the following way

$$|\rho(x)_\wedge^{-1}| \leq C(n)(K^I(P))^{1/n} \det \rho^{-1/n}(x), \tag{3.3}$$

where $K^I(P)$ is the inner distortion of the coefficient matrix P of ρ.

Proof. The norm bound follows from the fact that

$$|R_j(x)| \leq |P^{-1}| \leq (K^I)^{1/n} \det \rho^{-1/n},$$

by virtue of being the dual frame to $\rho^j(x)$. Let $\alpha \in \mathfrak{so}_n \otimes \Lambda^1\Omega$. Then

$$\begin{aligned}
&2\rho(x)_\wedge^{-1}(\alpha \wedge \rho(x))^{ij} \\
&\qquad = R_i(x)\llcorner(\alpha^{jk} \wedge \rho^k(x)) - R_j(x)\llcorner(\alpha^{ik} \wedge \rho^k(x)) \\
&\qquad\quad + (\alpha^{kl} \wedge \rho^l(x))(R_i(x), R_j(x))\rho^k(x) \\
&\qquad = \alpha^{jk}(R_i(x))\rho^k(x) - \rho^k(x)(R_i(x))\alpha^{jk} - \alpha^{ik}(R_j(x))\rho^k(x) \\
&\qquad\quad + \rho^k(x)(R_j(x))\alpha^{ik} + \alpha^{kl}(R_i(x))\rho^l(x)(R_j(x))\rho^k(x) \\
&\qquad\quad - \alpha^{kl}(R_j(x))\rho^l(x)(R_i(x))\rho^k(x) \\
&\qquad = \alpha^{jk}(R_i(x))\rho^k(x) - \delta_i^k\alpha^{jk} - \alpha^{ik}(R_j(x))\rho^k(x) + \delta_j^k\alpha^{ik} \\
&\qquad\quad + \alpha^{kl}(R_i(x))\delta_j^l\rho^k(x) - \alpha^{kl}(R_j(x))\delta_i^l\rho^k(x) \\
&\qquad = \alpha^{ij} - \alpha^{ji} \\
&\qquad = 2\alpha^{ij}.
\end{aligned}$$

□

For $1 \le p < \infty$ and $\rho \in W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^k\Omega)$ define the *exterior energy* $\mathcal{E}_p$ of ρ to be

$$\mathcal{E}_p(\rho) := \int_\Omega |d\rho|^p \, dx. \tag{3.4}$$

3.1. p-HARMONIC MAPS AND $SO(n)$

Throughout this chapter, we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded Euclidean domain with smooth boundary. Of special interest in this study is the space $W^{1,p}(\Omega, SO(n))$ of Sobolev maps to the Lie group $SO(n)$. For $1 \le p \le \infty$ the space of *p-Sobolev maps to SO(n)* denoted $W^{1,p}(\Omega, SO(n))$ is defined to be the subset of elements of $W^{1,p}(\Omega, \mathcal{M}_{n\times n})$ with values almost everywhere in $SO(n)$. Of special interest in this study is the so called *p-energy* of a Sobolev map:

$$\mathcal{I}_p : W^{1,p}(\Omega, SO(n)) \to \mathbb{R}, \qquad \sigma \mapsto \int_\Omega |d\sigma|^p. \tag{3.5}$$

The *left (resp. right) Darboux derivative* of $\sigma \in W^{1,p}(\Omega, SO(n))$ is $D_L\sigma := \sigma^{-1}d\sigma$ (resp. $(D_R\sigma)\sigma^{-1}$). *A priori* we would expect $\sigma^{-1}d\sigma \in L^p(\Omega, \mathcal{M}_{n\times n} \otimes \Lambda^1\Omega)$, as $d\sigma \in L^p(\Omega, \mathcal{M}_{n\times n} \otimes \Lambda^1\Omega)$ and $\sigma^{-1} \in L^\infty(\Omega, SO(n))$. In fact we have the following

LEMMA 3.3. *Let $1 \le p \le \infty$ and $\sigma \in W^{1,p}(\Omega, SO(n))$. Then $D_L\sigma$ is $\mathfrak{so}_n$-valued,* i.e. $D_L\sigma \in L^p(\Omega, \mathfrak{so}_n \otimes \Lambda^1\Omega)$. *(Similarly for $D_R\sigma$).*

Proof. Consider first an absolutely continuous function $\sigma : [0,1] \to SO(n)$. We can extend this to a map $\sigma : \mathbb{R} \to SO(n)$ by letting it be constant outside the endpoints. Let $\sigma^\varepsilon : \mathbb{R} \to \mathcal{M}_{n\times n}$ be a standard smooth approximation satisfying the properties

(1) $\sigma^\varepsilon \to \sigma$ uniformly, *i.e.* $\|\sigma^\varepsilon - \sigma\|_\infty \to 0$ as $\varepsilon \to 0$, and $\|\sigma^\varepsilon\|_\infty \le \|\sigma\|_\infty$;
(2) $d\sigma^\varepsilon \to d\sigma$ in the L^1 norm, *i.e.*

$$\int_\mathbb{R} |d\sigma^\varepsilon(x) - d\sigma(x)| \, dx \to 0$$

as $\varepsilon \to 0$, and $\|d\sigma^\varepsilon\|_1 \le \|d\sigma\|_1$;
(3) $\sigma^\varepsilon \in C^\infty(\mathbb{R}, \mathcal{M}_{n\times n})$.

The existence of such an approximation is standard and can be found in, for instance [Eva98, §5.3.1,§C.4].

Then

$$
\begin{aligned}
\int_0^1 &|(D\sigma^t)\sigma + \sigma^t D\sigma - (D(\sigma^\varepsilon)^t\sigma^\varepsilon + (\sigma^\varepsilon)^t D\sigma^\varepsilon)|\,dx \\
&\le \int_{\mathbb{R}} |D(\sigma^\varepsilon)^t - D\sigma^t||\sigma^\varepsilon|\,dx + \int_{\mathbb{R}} |D(\sigma^t)||\sigma^\varepsilon - \sigma|\,dx \\
&\quad + \int_{\mathbb{R}} |(\sigma^\varepsilon)^t - \sigma^t||D\sigma^\varepsilon|\,dx + \int_{\mathbb{R}} |\sigma^t||D\sigma^\varepsilon - D\sigma|\,dx \\
&\le (\|D(\sigma^\varepsilon) - D\sigma\|_1(\|\sigma\|_\infty + \|\sigma^\varepsilon\|_\infty) \\
&\quad + (\|D\sigma\|_1 + \|D\sigma^\varepsilon\|_1)\|\sigma^\varepsilon - \sigma\|_\infty \\
&\le 2\|\sigma\|_\infty\|D\sigma - D\sigma^\varepsilon\|_1 + 2\|D\sigma\|_1(\|\sigma^\varepsilon - \sigma\|_\infty).
\end{aligned}
$$

Now $\|\sigma - \sigma^\varepsilon\|_\infty \to 0$ and $\|D\sigma - D\sigma^\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. It follows that

$$\int_{\mathbb{R}} |D\sigma^t\sigma + \sigma^t D\sigma - D((\sigma^\varepsilon)^t\sigma^\varepsilon)|\,dx \to 0$$

as $\varepsilon \to 0$. As such $(\sigma^\varepsilon)^t(\sigma^\varepsilon)$ converges in $W^{1,1}(\mathbb{R}, \mathcal{M}_{n\times n})$. Since $(\sigma^\varepsilon)^t\sigma^\varepsilon$ converges uniformly to I, and $d((\sigma^\varepsilon)^t\sigma^\varepsilon)$ converges in $L^1(\mathbb{R}, \mathcal{M}_{n\times n})$, $d[(\sigma^\varepsilon)^t\sigma^\varepsilon]$ must converge to the weak derivative of I, which is zero [Eva98, 5.2.1]. Hence

$$(D\sigma^t)\sigma + \sigma^t D\sigma = \lim_{\varepsilon\to 0}(D\sigma^\varepsilon)^t\sigma^\varepsilon + (\sigma^\varepsilon)^t D\sigma^\varepsilon = DI = 0, \text{ a.e.}$$

Consequently $\sigma^t D\sigma = \sigma^{-1}D\sigma$ is an antisymmetric matrix.

Now if $\sigma \in W^{1,p}(\Omega, SO(n))$ then it is absolutely continuous on almost every line. As such for almost every line ℓ in Ω, the restriction of σ to this line is absolutely continuous. It follows that the partial derivatives of $\sigma : \Omega \to SO(n)$, when multiplied by σ^{-1} on the left, are antisymmetric matrices. And hence $D_L\sigma = \sigma^{-1}d\sigma$ is an antisymmetric matrix valued one-form almost everywhere.

Lastly $d\sigma \in L^p(\Omega, \mathcal{M}_{n\times n} \otimes \Lambda^1\Omega)$ so $\sigma^{-1}d\sigma \in L^p(\Omega, \mathcal{M}_{n\times n} \otimes \Lambda^1\Omega)$. But by the preceding reasoning $\sigma^{-1}d\sigma(x) \in \mathfrak{so}_n \otimes \Lambda^1\Omega$ for almost every $x \in \Omega$, and so $\sigma^{-1}d\sigma \in L^p(\Omega, \mathfrak{so}_n \otimes \Lambda^1\Omega)$. □

Remark 3.4. Let $1 \le p < \infty$ and $\sigma \in W^{1,p}(\Omega, SO(n))$. Then

$$\mathcal{I}^p(\sigma) = \int_\Omega |D_L\sigma|^p\,dx = \int_\Omega |D_R\sigma|^p\,dx.$$

This is a direct consequence of Proposition 1.13, as $d\sigma$ is a matrix valued one-form, and left and right multiplication by an orthogonal matrix preserves the Hilbert–Schmidt norm.

Lemma 3.5. *Let $p' \ge n$, $1 < p \le p'$ and $1 < q \le \infty$. Suppose $\sigma \in W^{1,p'/(p'-1)}(\Omega, SO(n))$ and $\rho \in L^{p'}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. Suppose further that $\sigma\rho \in \mathcal{CO}^p_\rho(\Omega)$ and $(\det\rho)^{-1/n} \in L^q(\Omega)$. Then $\sigma \in W^{1,q'}(\Omega, SO(n))$ where $q' = pq/(p+q)$.*

Proof. Because $\sigma \in W^{1,p'/(p'-1)}(\Omega, SO(n))$, we know by Lemma 3.3 that $D_L\sigma \in L^{p'/(p'-1)}(\Omega, \mathfrak{so}_n \otimes \Lambda^1\Omega)$. The condition $(\det\rho)^{-1/n} \in L^q(\Omega)$ implies that ρ is essentially non-vanishing. Now by Lemma 3.2 and quasiconformality of the frame ρ,

$$\begin{aligned}\int_\Omega |D_L\sigma|^{q'}\,dx &= \int_\Omega |D_L\sigma|^{q'} \det\rho^{q'/n} \det\rho^{-q'/n}\,dx\\ &\le C(n)\int_\Omega |D_L\sigma|^{q'}|\rho|^{q'}\det\rho^{-q'/n}\,dx\\ &\le C(n,K)\int_\Omega |D_L\sigma\wedge\rho|^{q'}\det\rho^{-q'/n}\,dx\\ &\le C(n,K)\|D_L\sigma\wedge\rho\|_p^{q'}\|\det\rho^{-1/n}\|_q^{q'}.\end{aligned}$$

Thus

$$\|D_L\sigma\|_{q'} \le C(n,K)\|D_L\sigma\wedge\rho\|_p\|\det\rho^{-1/n}\|_q.$$

Now by virtue of Lemma 1.14, we know that $d(\sigma\rho) = d\sigma\wedge\rho + \sigma d\rho$, and hence we know that

$$D_L\sigma\wedge\rho = \sigma^{-1}d\sigma\wedge\rho = \sigma^{-1}d(\sigma\rho) - d\rho.$$

Consequently

$$\begin{aligned}\|D_L\sigma\|_{q'} &\le C(n,K)\|\det\rho^{-1/n}\|_q\|D_L\sigma\wedge\rho\|_p\\ &\le C(n,K)\|\det\rho^{-1/n}\|_q\left(\|d(\sigma\rho)\|_p + \|d\rho\|_p\right)\end{aligned}$$

and hence $D_L\sigma \in L^{q'}(\Omega, \mathfrak{so}_n\otimes\Lambda^1\Omega)$. Consequently $d\sigma \in L^{q'}(\Omega, \mathcal{M}_{n\times n}\otimes\Lambda^1\Omega)$. Because $\sigma : \Omega \to SO(n)$ is an $\mathcal{M}_{n\times n}$-valued 0-form, and $d\sigma \in L^{q'}(\Omega, \mathcal{M}_{n\times n}\otimes\Lambda^1\Omega)$ the claim follows. □

3.2. The Euler–Lagrange equations

THEOREM (1.5). *Let $1 < p < \infty$. If $\rho \in L^p(\Omega, \mathbb{R}^n\otimes\Lambda^1\Omega)$ is a local minimiser of $\mathcal{E}_p : \mathcal{CO}^p_{\rho_0}(\Omega) \to \mathbb{R}$, then it satisfies the Euler-Lagrange equations*

$$\int_\Omega \langle |d\rho|^{p-2}d\rho, d(\lambda\rho)\rangle\,dx = 0 \tag{3.6}$$

and

$$\int_\Omega \langle |d\rho|^{p-2}d\rho, du\wedge\rho\rangle\,dx = 0, \tag{3.7}$$

where $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$ and $\lambda \in C_0^\infty(\Omega)$.

Proof. Let $\delta > 0$ and $\alpha_t \in C^\infty(\Omega, CO^+(n))$ $t \in (-\delta, \delta)$ be a smooth one-parameter family of test functions equal to the identity on a neighbourhood of $\partial\Omega$, with $\alpha_0 \equiv I$. Then for $|t|$ small enough, there exists a smooth one-parameter family $A_t \in C_0^\infty(\Omega, \mathfrak{co}_n)$ satisfying $\alpha_t = \exp(A_t)$.

Because $\mathfrak{co}_n = \mathbb{R}\oplus\mathfrak{so}_n$ we consider $e^{t(\lambda I+u)}$ where $\lambda \in C_0^\infty(\Omega)$ and $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$. Set $\tau_t = t^{-1}(e^{t(\lambda I+u)} - I)\rho$ for $t > 0$ and $\tau_0 = (\lambda I+u)\rho$. Furthermore we note that

$\lim_{t\to 0}\tau_t(x) = \tau_0(x)$ for every $x \in \Omega$. Then

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_p(e^{t(\lambda+u)}\rho)|_{t=0} &= \lim_{t\to 0}\frac{1}{t}\int_\Omega |d(e^{t(\lambda+u)}\rho)|^p - |d\rho|^p\, dx \\
&= \lim_{t\to 0}\frac{1}{t}\int_\Omega \langle d(e^{t(\lambda+u)}\rho), d(e^{t(\lambda+u)}\rho)\rangle^{p/2} - |d\rho|^p\, dx \\
&= \lim_{t\to 0}\frac{1}{t}\int_\Omega \langle d\rho + td\tau_t, d\rho + td\tau_t\rangle^{p/2} - |d\rho|^p\, dx \\
&= \lim_{t\to 0}\frac{1}{t}\int_\Omega \Big(|d\rho|^2 + t^2|d\tau_t|^2 + 2t\,\langle d\rho, d\tau_t\rangle\Big)^{p/2} - |d\rho|^p\, dx \\
&= \lim_{t\to 0}\int_\Omega I_t\, dx,
\end{aligned}$$

where

$$I_t = \frac{1}{t}\left[(|d\rho|^2 + t^2|d\tau_t|^2 + 2\langle d\tau, d\rho\rangle)^{p/2} - |d\rho|^p\right].$$

By the monotonicity of $|\cdot|^{p/2}$, we have the estimate

$$\begin{aligned}
|I_t| &\le \frac{1}{t}\big((|d\rho|^2 + t^2|d\tau_t|^2 + 2t|\langle d\rho, d\tau_t\rangle|)^{p/2} - |d\rho|^p\big) \\
&\le \frac{1}{t}\big((|d\rho|^2 + t^2|d\tau_t|^2 + 2t|d\rho||d\tau_t|)^{p/2} - |d\rho|^p\big) \\
&\le \frac{1}{t}\left((|d\rho| + t|d\tau_t|)^p - |d\rho|^p\right) \\
&\le p(|d\rho| + t|d\tau_t|)^{p-1}|d\tau_t| \\
&\le p(|d\rho| + |d\tau_t|)^{p-1}|d\tau_t| \\
&\le C(p)(|d\rho|^{p-1}|d\tau_t| + |d\tau_t|^p).
\end{aligned}$$

for every $t > 0$ On the other hand,

$$\begin{aligned}
d\tau_t &= d\left(\sum_{k=1}^\infty (\lambda I + u)^k \frac{t^{k-1}}{k!}\rho\right) \\
&= \left(\sum_{k=1}^\infty \sum_{j=0}^{k-1} (\lambda I + u)^j (d\lambda I + du)(\lambda I + u)^{k-j-1}\frac{t^{k-1}}{k!}\right)\wedge\rho \\
&\quad + \left(\sum_{k=1}^\infty (\lambda I + u)^k \frac{t^{k-1}}{k!}\right) d\rho.
\end{aligned}$$

Thus

$$\begin{aligned}
|d\tau_t| &\le \sum_{k=1}^\infty k(|\lambda| + |u|)^{k-1}(|d\lambda| + |du|)\frac{t^{k-1}}{k!}|\rho| \\
&\quad + \sum_{k=1}^\infty (|u| + |\lambda|)^k \frac{t^{k-1}}{k!}|d\rho| \\
&\le (\|du\|_\infty + \|d\lambda\|_\infty)e^{t(\|u\|_\infty + \|\lambda\|_\infty)}|\rho| \\
&\quad + (\|u\|_\infty + \|\lambda\|_\infty)F(t(\|u\|_\infty + \|\lambda\|_\infty))|d\rho|,
\end{aligned}$$

where $F(x) = (e^x - 1)/x$. By the monotonicity of F and the exponential function, for every $0 < t \le 1$

$$|d\tau_t| \le (\|du\|_\infty + \|d\lambda\|_\infty)e^{\|u\|_\infty + \|\lambda\|_\infty}|\rho| + (\|u\|_\infty + \|\lambda\|_\infty)F(\|u\|_\infty + \|\lambda\|_\infty)|d\rho|.$$

Thus

$$|d\tau_t| \le C(|\rho| + |d\rho|), \tag{3.8}$$

where

$$C = (\|du\|_\infty + \|d\lambda\|_\infty)e^{\|u\|_\infty + \|\lambda\|} + (\|u\|_\infty + \|\lambda\|_\infty)F(\|u\|_\infty + \|\lambda\|_\infty).$$

Hence

$$|I_t| \le C\left[|d\rho|^p + |\rho|^p + |d\rho|^{p-1}|\rho|\right],$$

where $C = C(p, \|u\|_{1,\infty}, \|\lambda\|_\infty)$ Thus we get a uniform (for $0 < t \le 1$) integrable bound. Hence we can apply the Dominated Convergence Theorem and obtain:

$$\begin{aligned}
0 = \frac{d}{dt}\,\mathcal{E}_p\left(e^{t(\lambda+u)}\rho\right)\Big|_{t=0} &= \lim_{t\to 0}\int_\Omega I_t\,dx \\
&= \int_\Omega \lim_{t\to 0}\frac{1}{t}\left[(|d\rho|^2 + t^2|d\tau_t|^2 + 2t\langle d\tau_t, d\rho\rangle)^{p/2} - |d\rho|^p\right]\,dx \\
&= p\int_\Omega \langle |d\rho|^{p-2}d\rho, d\tau_0\rangle\,dx = p\int_\Omega \langle |d\rho|^{p-2}d\rho, d[(\lambda I + u)\rho]\rangle\,dx.
\end{aligned}$$

Setting $u = 0$ and then $\lambda = 0$ yields, respectively, the equations

$$\int_\Omega \langle |d\rho|^{p-2}d\rho, d(\lambda\rho)\rangle = 0, \quad \text{and} \quad \int_\Omega \langle |d\rho|^{p-2}d\rho, d(u\rho)\rangle\,dx = 0.$$

Since $\langle d\rho, u\,d\rho\rangle = 0$ almost everywhere by Lemma 1.12, the second equation can be written

$$\begin{aligned}
0 &= \int_\Omega \langle |d\rho|^{p-2}d\rho, du\wedge\rho + u\,d\rho\rangle\,dx \\
&= \int_\Omega \langle |d\rho|^{p-2}d\rho, du\wedge\rho\rangle + |d\rho|^{p-2}\langle d\rho, u\,d\rho\rangle\,dx \\
&= \int_\Omega \langle |d\rho|^{p-2}d\rho, du\wedge\rho\rangle\,dx.
\end{aligned}$$

This completes the proof. □

Equation (3.6) corresponds to Equation (7.4) in [PR11]. As a direct corollary of Theorem 1.5 we obtain a weak reverse Hölder inequality for the local minimisers of the energy functional $\mathcal{E}_p$.

Corollary 3.6 (Corollary 7.8 in [PR11]). *Let $p > n/2$. Suppose $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ is a local minimiser of $\mathcal{E}_p$. Then there exist constants $q = q(n, K) > n$ and $C = C(n, K, q) > 0$ such that for every ball B with $2B \subset\subset \Omega$*

$$\left(\int_B |\rho|^q\,dx\right)^{1/q} \le C\left(\int_{2B} |\rho|^n\,dx\right)^{1/n}.$$

The following Lemma is needed to test agains arbitrary Sobolev functions, and not just smooth functions.

Lemma 3.7. *Let $n/2 < p \leq n$. Suppose $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ satisfies the Euler-Lagrange equations* (3.6) *(respectively* (3.7)*) for any smooth test function $\lambda \in C_0^\infty(\Omega)$ (resp. $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$). Then ρ satisfies the equations for any function $\lambda \in W_0^{1,p^*}(\Omega)$ (resp. $u \in W_0^{1,p^*}(\Omega, \mathfrak{so}_n)$) where $p^* = np/(n-p)$. Furthermore, if $p = n/2$ and ρ satisfies* (3.7) *for any smooth test function $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$ then it satisfies* (3.7) *for any test function $u \in W_0^{1,n}(\Omega, \mathfrak{so}_n)$.*

Proof. We first prove the claim for (3.7). Let $u^\varepsilon \to u$ be a sequence of smooth compactly supported functions in Ω converging in $W_0^{1,p^*}(\Omega, \mathfrak{so}_n)$.

Then

$$\begin{aligned}
&\left| \int_\Omega \langle |d\rho|^{p-2} d\rho, du \wedge \rho \rangle \, dx \right| \\
&\qquad = \left| \int_\Omega \langle |d\rho|^{p-2} d\rho, du^\varepsilon \wedge \rho + (du - du^\varepsilon) \wedge \rho \rangle \, dx \right| \\
&\qquad = \left| \int_\Omega \langle |d\rho|^{p-2} d\rho, du^\varepsilon \wedge \rho \rangle \, dx \right. \\
&\qquad\qquad \left. + \int_\Omega \langle |d\rho|^{p-2} d\rho, (du - du^\varepsilon) \wedge \rho \rangle \, dx \right| \\
&\qquad = \left| \int_\Omega \langle |d\rho|^{p-2} d\rho, (du - du^\varepsilon) \wedge \rho \rangle \, dx \right| \\
&\qquad \leq \int_\Omega |d\rho|^{p-1} |du - du^\varepsilon| |\rho| \, dx \\
&\qquad \leq \|d\rho\|_p^{p-1} \left(\int_\Omega |du - du^\varepsilon|^p |\rho|^p \, dx \right)^{1/p} \\
&\qquad \leq \|d\rho\|_p^{p-1} \|du - du^\varepsilon\|_{p^*} \|\rho\|_n.
\end{aligned}$$

This is true for any $1 < p \leq n$ and every $\varepsilon > 0$, and hence

$$\int_\Omega \langle |d\rho|^{p-2} d\rho, du \wedge \rho \rangle \, dx = 0.$$

Now we prove the claim for (3.6). Let $\lambda^\varepsilon \to \lambda$ be a sequence of smooth compactly supported functions converging in $W_0^{1,p^*}(\Omega)$. Then consider

$$\begin{aligned}
&\left| \int_\Omega \langle |d\rho|^{p-2} d\rho, d(\lambda\rho) \rangle \, dx \right| \\
&\qquad = \left| \int_\Omega \langle |d\rho|^{p-2} d\rho, d(\lambda^\varepsilon \rho) \rangle \, dx + \int_\Omega \langle |d\rho|^{p-2} d\rho, d(\lambda - \lambda^\varepsilon) \wedge \rho \right. \\
&\qquad\qquad \left. + (\lambda - \lambda^\varepsilon) d\rho \rangle \, dx \right| \\
&\qquad = \left| \int_\Omega \langle |d\rho|^{p-2} d\rho, d(\lambda - \lambda^\varepsilon) \wedge \rho + (\lambda - \lambda^\varepsilon) d\rho \rangle \, dx \right| \\
&\qquad \leq \int_\Omega |d\rho|^{p-1} |d\lambda - d\lambda^\varepsilon| |\rho| + |d\rho|^p |\lambda - \lambda^\varepsilon| \, dx
\end{aligned}$$

$$\leq \|d\rho\|_p^{p-1}\|d\lambda - d\lambda^\varepsilon\|_{p^*}\|\rho\|_n + \|d\rho\|_p^p\|\lambda - \lambda^\varepsilon\|_\infty$$
$$\leq C\|d\rho\|_p^{p-1}(\|d\rho\|_p + \|\rho\|_n)\|d\lambda - d\lambda^\varepsilon\|_{p^*}.$$

The last inequality is possible because if $p > n/2$ then $p^* > n$, so we can use Morrey's inequality. Once again we allow $\varepsilon \to 0$, which yields

$$\int_\Omega \langle |d\rho|^{p-2}d\rho, d(\lambda\rho)\rangle = 0.$$

□

3.2.1. **$\mathcal{A}$-harmonic maps to $SO(n)$.** Let V be a finite dimensional inner-product space. Consider a measurable bundle map

$$\mathcal{A} : \Omega \times (V \otimes \Lambda^1\mathbb{R}^n) \to (V \otimes \Lambda^1\mathbb{R}^n).$$

We say $\mathcal{A}$ is *monotone of growth* $p \geq 1$ if

- for almost every $x \in \Omega$ the map $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous;
- there is a number $C \geq 1$ such that
$$C^{-1}|\xi|^p \leq \langle \mathcal{A}(x,\xi), \xi\rangle \leq C|\xi|^p$$
for almost every $x \in \Omega$ and $\xi \in V \otimes \Lambda^1\mathbb{R}^n$;
- for almost every $x \in \Omega$ and every $\xi, \zeta \in V \otimes \Lambda^1\mathbb{R}^n$
$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta), \xi - \zeta\rangle \geq 0$$
with equality if and only if $\xi = \zeta$; and
- for every $\lambda \in \mathbb{R}$ $\mathcal{A}(x, \lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}(x,\xi)$.

A function $f : \Omega \to V$ is said to be $\mathcal{A}$-harmonic if it satisfies

$$\int_\Omega \langle \mathcal{A}(df), du\rangle \, dx = 0$$

for all $u \in C_0^\infty(\Omega, V)$. The concept of an $\mathcal{A}$-harmonic function $f : \Omega \to V$ generalises the notion of a p-harmonic function. In general they arise as minimisers of functionals of the form

$$\mathcal{I}_\mathcal{A}(f) = \int_\Omega F(x, Df)\, dx$$

under suitable assumptions on F; see *e.g.* [HKM06, Chapter 5].

We consider maps from Ω to $SO(n)$. Let $\mathcal{A} : \Omega \times (\mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n) \to (\mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n)$ be monotone of growth $p \geq 1$.

A map $\sigma \in W^{1,p}(\Omega, SO(n))$ is said to be $\mathcal{A}$-*harmonic* if

$$\int_\Omega \langle \mathcal{A}(D_L\sigma), \mathrm{Ad}_\sigma(du)\rangle \, dx = 0$$

for all $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$.

Lemma 3.8. *Let $1 < p < \infty$. Assume that $G : \Omega \to \mathrm{End}(\mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n)$ is an essentially bounded measurable map satisfying*

(1) for every $\xi \in \mathfrak{so}_n \otimes \Lambda^1\Omega$ and almost every $x \in \Omega$

$$C^{-1}|\xi|^2 \leq \langle G(x)\xi, \xi\rangle \leq |\xi|^2$$

(2) for every $\xi, \zeta \in \mathfrak{so}_n \otimes \Lambda^1\Omega$ and almost every $x \in \Omega$

$$\langle G(x)\xi, \zeta\rangle = \langle \xi, G(x)\zeta\rangle.$$

Then

$$\mathcal{A}_G(x,\xi) = \langle G(x)\xi, \xi\rangle^{p/2-1} G(x)\xi,$$

is monotone of growth p, and a local minimiser $\sigma \in W^{1,p}(\Omega, SO(n))$ of the energy

$$\mathcal{I}_G^p(\sigma) = \int_\Omega \langle G(x) D_L\sigma(x), D_L\sigma(x)\rangle^{p/2}\, dx$$

satisfies the $\mathcal{A}$-harmonic equation

$$\int_\Omega \langle \mathcal{A}_G(x, D_L\sigma(x)), \mathrm{Ad}_{\sigma(x)}(du(x))\rangle\, dx = 0$$

for every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$.

Proof. We consider the variation $\sigma \mapsto e^{tu}\sigma$ where $t \in \mathbb{R}$ and $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$. Then

$$\begin{aligned} D_L(e^{tu}\sigma) &= \sigma^{-1}e^{-tu}de^{tu}\sigma + \sigma^{-1}d\sigma \\ &= \mathrm{Ad}_\sigma(D_L e^{tu}) + D_L\sigma. \end{aligned}$$

We get

$$\left.\frac{d}{dt}\right|_{t=0} D_L(e^{tu}\sigma) = \mathrm{Ad}_\sigma(du).$$

We differentiate the integrand of $\mathcal{I}_G^p(\sigma)$, and by Property 2

$$\begin{aligned} &\left.\frac{d}{dt}\right|_{t=0} \langle GD_L(e^{tu}\sigma), D_L(e^{tu}\sigma)\rangle^{p/2} \\ &\qquad = (p/2)\langle G\, D_L\sigma, D_L\sigma\rangle^{p/2-1}(\langle G\,\mathrm{Ad}_\sigma(du), D_L\sigma\rangle \\ &\qquad\quad + \langle G\, D_L\sigma, \mathrm{Ad}_\sigma(du)\rangle) \\ &\qquad = p\langle G\, D_L\sigma, D_L\sigma\rangle^{p/2-1}\langle G\, D_L\sigma, \mathrm{Ad}_\sigma(du)\rangle \end{aligned}$$

a.e. in Ω.

Consequently, since σ is a local minimiser of $\mathcal{I}_G^p$, we get

$$\begin{aligned} 0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{I}_G^p(e^{tu}\sigma) &= \int_\Omega \left.\frac{d}{dt}\right|_{t=0} \langle G(x)D_L(e^{tu(x)}\sigma(x)), D_L(e^{tu(x)}\sigma(x))\rangle^{p/2}\, dx \\ = p\int_\Omega \langle G(x)D_L\sigma(x), &D_L\sigma(x)\rangle^{p/2-1}\langle G(x)D_L\sigma(x), \mathrm{Ad}_{\sigma(x)}du(x)\rangle\, dx. \end{aligned}$$

Here we have brought the differentiation through the integration without an explicit justification *vis-à-vis* the proof of Theorem 1.5, because this class of functionals is more standard in the literature [HKM06, HL87]. □

When we look at the energy minimiser problem $\sigma \mapsto \mathcal{E}_p(\sigma\, d\mathbf{x})$ in this context, we arrive at the following Euler-Lagrange equation:

Corollary 3.9. *Let $\sigma \in W^{1,p}(\Omega, SO(n))$ be a local minimiser of $\sigma \mapsto \mathcal{E}_p(\sigma\, d\mathbf{x})$. Then σ satisfies the $\mathcal{A}$-harmonic equation*

$$\int_\Omega \langle \mathcal{A}(D_L\sigma), \mathrm{Ad}_\sigma(du)\rangle\, dx = 0 \tag{3.9}$$

for every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$*, where*

$$\mathcal{A} : \mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n \to \mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n, \quad \mathcal{A}(\xi) = \langle G\xi, \xi\rangle^{p/2-1} G\xi,$$

and

$$G = (\cdot \wedge d\mathbf{x})^*(\cdot \wedge d\mathbf{x}).$$

We remind the reader that $G \in \mathrm{End}(\mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n)$ is defined by

$$\langle G(\xi), \zeta\rangle = \langle \xi \wedge d\mathbf{x}, \zeta \wedge d\mathbf{x}\rangle.$$

Proof of Corollary 3.9. By Theorem 1.5, for every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$

$$\int_\Omega \langle |d\sigma \wedge d\mathbf{x}|^{p-2} d\sigma \wedge d\mathbf{x}, du \wedge \sigma d\mathbf{x}\rangle \, dx = 0.$$

Since σ takes values in $SO(n)$, we have

$$\begin{aligned} 0 &= \int_\Omega \langle |\sigma^{-1} d\sigma \wedge d\mathbf{x}|^{p-2} \sigma^{-1} d\sigma \wedge d\mathbf{x}, \sigma^{-1} du \sigma \wedge d\mathbf{x}\rangle \, dx \\ &= \int_\Omega \langle |D_L\sigma \wedge d\mathbf{x}|^{p-2} D_L\sigma \wedge d\mathbf{x}, \mathrm{Ad}_\sigma(du) \wedge d\mathbf{x}\rangle \, dx \\ &= \int_\Omega \langle D_L\sigma \wedge d\mathbf{x}, D_L\sigma \wedge d\mathbf{x}\rangle^{p/2-1} \langle D_L\sigma, \mathrm{Ad}_\sigma(du) \wedge d\mathbf{x}\rangle \, dx. \end{aligned}$$

Thus

$$\int_\Omega \langle G\, D_L\sigma, D_L\sigma\rangle^{p/2-1} \langle G\, D_L\sigma, \mathrm{Ad}_\sigma(du)\rangle \, dx = 0.$$

By Lemma 3.2 G satisfies the conditions of Lemma 3.8, and so $\mathcal{A}$ is monotone of growth p and

$$\int_\Omega \langle \mathcal{A}(D_L\sigma), \mathrm{Ad}_\sigma(du)\rangle \, dx = 0$$

for every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$. □

The interesting thing is that the linear map G in Corollary 3.9 is independent of $x \in \Omega$ and σ. In particular $\mathcal{A}$ is C^1, and we can apply existing regularity theory to yield higher regularity for this minimiser. We refer to [HL87] for a more detailed discussion.

Corollary 3.10. *Let* $1 < p < \infty$ *and let* $\sigma \in W^{1,p}(\Omega, SO(n))$ *be a local minimiser of* $\sigma \mapsto \mathcal{E}_p(\sigma d\mathbf{x})$ *then there is an* α, $0 < \alpha \le 1$, *and a set* $\Sigma \subset\subset \Omega$ *of Hausdorff dimension less than or equal to* $n - [p] - 1$ *such that* $\sigma \in C^{1,\alpha}_{\mathrm{loc}}(\Omega \setminus \Sigma, SO(n))$*. Furthermore if* $\partial\Omega$ *is* C^2*, and* $\sigma|_{\partial\Omega}$ *is* C^1*, then* σ *is Hölder continuous up to the boundary.*

3.2.2. **Minimisers in the class of an exact frame.** In what follows we examine a K-quasiconformal map $f : \Omega \xrightarrow{\cong} \Omega' \subset \mathbb{R}^n$. We denote by $h : \Omega' \to \Omega$ the inverse f^{-1} of f. In this case the frame df is a quasiconformal frame, and $J_f = \det df$ is its Jacobian determinant. Let $d\mathbf{y}$ denote the standard Cartesian co-frame in Ω'. For almost every $y \in \Omega'$ we define the measurable map $A : \Omega' \times \mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n \to \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n$

$$A(y, \xi) = \begin{cases} (B(y))^\#(\xi \wedge d\mathbf{y}), & \text{if } dh(y) \neq 0 \\ \xi \wedge d\mathbf{y}, & \text{otherwise,} \end{cases} \tag{3.10}$$

where $B(y) = J_f^{-1/n}(h(y))Df(h(y))$. Note that $B(y)^{-1} = J_h^{-1/n}(y)(Dh(y))$. Subsequently by (3.1) and (3.2) $|B| = (K_f^O)^{1/n}$ and $|B^{-1}| = (K_f^I)^{1/n}$, where K_f^I and K_f^O are respectively the inner and outer distortions of f.

Apply Lemma 3.2 to deduce that $|d\mathbf{y}_\wedge^{-1}| \le C(n)$. Then

$$\begin{aligned} |\xi| &\le |d\mathbf{y}_\wedge^{-1}(B^{-1})^\# B^\#(\xi \wedge d\mathbf{y})| \\ &\le C(n)(K_f^I)^{2/n}|B^\#(\xi \wedge d\mathbf{y})|. \end{aligned}$$

On the other hand

$$\begin{aligned} |B^\#(\xi \wedge d\mathbf{y})| &\le (K_f^O)^{2/n}|\xi \wedge d\mathbf{y}| \\ &\le C(n)(K_f^O)^{2/n}|\xi|. \end{aligned}$$

We set

$$G(y) = A^*(y)A(y) : \mathfrak{so}_n \otimes \Lambda^1\Omega' \to \mathfrak{so}_n \otimes \Lambda^1\Omega',$$

where

$$A^*(y) : \mathbb{R}^n \otimes \Lambda^2\Omega' \to \mathfrak{so}_n \otimes \Lambda^1\Omega'$$

is the adjoint of $A(y)$.

We can then define a measurable monotone function $\mathcal{A} : \Omega' \times (\mathfrak{so}_n \otimes \Lambda^1\mathbb{R}^n) \to \mathfrak{so}_n \otimes \Lambda^1\Omega'$ of growth $n/2$ by

$$\mathcal{A}(y,\xi) = \langle G(y)\xi, \xi\rangle^{(n-4)/4} G(y)\xi \tag{3.11}$$

for every $y \in \Omega'$ and $\xi\mathfrak{so}_n \in \Lambda^1\mathbb{R}^n$

The reason for using this potential theoretic terminology is that in this context the Orthogonal Euler–Lagrange Equations (3.7) take the following form.

THEOREM (1.7). *Let $f : \Omega \to \Omega'$ be a quasi-conformal map with inverse $h : \Omega' \to \Omega$, and $\sigma : \Omega \to SO(n)$ be a measurable map such that $\tilde\sigma = \sigma \circ h \in W^{1,1}(\Omega', SO(n))$. If $d(\sigma df) \in L^{n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$ then $\tilde\sigma \in W^{1,n/2}(\Omega, SO(n)$. If σdf satisfies the Orthogonal Euler–Lagrange equations* (1.6) *then $\tilde\sigma$ satisfies an $\mathcal{A}$-harmonic equation, where $\mathcal{A}$ is given by* (3.11). *That is, for every $u \in C_0^\infty(\Omega', \mathfrak{so}_n)$*

$$\int_{\Omega'} \langle \mathcal{A}(D_L\tilde\sigma), \mathrm{Ad}_{\tilde\sigma}(\, du)\rangle = 0. \tag{3.12}$$

Proof. By quasiconformality of h, $d(h^*\sigma\, df) = h^*d(\sigma df) \in L^{n/2}(\Omega', \mathbb{R}^n \otimes \Lambda^2\Omega)$, *cf.* [GT10, Thm 6.6]. But $h^*(\sigma df) = (\sigma \circ h)d\mathbf{y} = \tilde\sigma d\mathbf{y}$. By Lemma 3.5,

$$D_L(\tilde\sigma) = (d\mathbf{y}_\wedge)^{-1}(\sigma \circ h)^{-1} d((\sigma \circ h)d\mathbf{y}) \in L^{n/2}(\Omega', \mathfrak{so}_n \otimes \Lambda^1\Omega').$$

Consequently $\tilde\sigma \in W^{1,n/2}(\Omega', SO(n)))$.

Now because σdf is a solution to (1.6) and by Proposition 1.13 left-multiplication by σ is an orthogonal operator $\mathbb{R}^n \otimes \Lambda^2\Omega \to \mathbb{R}^n \otimes \Lambda^2\Omega$, we have

$$\begin{aligned} 0 &= \int_\Omega \langle |d(\sigma df)|^{n/2-2} d(\sigma df), du \wedge \sigma df\rangle\, dx \\ &= \int_\Omega \langle |\sigma^{-1}d(\sigma df)|^{n/2-2}\sigma^{-1}d(\sigma df), \sigma^{-1}(du)\sigma \wedge df\rangle\, dx. \end{aligned}$$

We apply the change of variables $x = h(y)$ to yield

$$\begin{aligned}0 &= \int_{\Omega'} \langle |\sigma^{-1}d(\sigma df)|^{(n-4)/2}\sigma^{-1}d(\sigma df), \mathrm{Ad}_\sigma(du)\wedge df\rangle_{h(y)} J_h(y)\, dy\\ &= \int_{\Omega'} \langle |Df^\#(Dh\circ f)^\#\sigma^{-1}d(\sigma df)|^{(n-4)/2} Df^\#(Dh\circ f)^\#\sigma^{-1}d(\sigma df),\\ &\qquad Df^\#(Dh\circ f)^\#(\mathrm{Ad}_\sigma(du)\wedge df)\rangle_{h(y)} J_f(h(y))^{-1}\, dy.\end{aligned}$$

We set $B = J_f^{-1/n} Df\circ h$, $A = B^\#(\cdot\wedge d\mathbf{y})$ and $G = A^*A$. Then

$$\begin{aligned}0 &= \int_{\Omega'} \langle \left|B^\# h^*(\sigma^{-1}d(\sigma df))\right|^{(n-4)/2} B^\# h^*(\sigma^{-1}d(\sigma\, df)),\\ &\qquad\qquad B^\# h^*(\mathrm{Ad}_\sigma(du)\wedge df)\rangle_y\, dy\\ &= \int_{\Omega'} \langle |B^\#(\tilde\sigma^{-1}(d\tilde\sigma)\wedge d\mathbf{y})|^{(n-4)/2} B^\#(\tilde\sigma^{-1}(d\tilde\sigma)\wedge d\mathbf{y}),\\ &\qquad\qquad B^\#(\mathrm{Ad}_{\tilde\sigma}(d(u\circ h))\wedge d\mathbf{y})\rangle_y\, dy\\ &= \int_{\Omega'} \langle |A(D_L\tilde\sigma)|^{(n-4)/2} A(D_L\tilde\sigma), A(\mathrm{Ad}_{\tilde\sigma}(d(u\circ h)))\rangle_y\, dy\\ &= \int_{\Omega'} \langle G(D_L\tilde\sigma), D_L\tilde\sigma\rangle^{(n-4)/4} G(D_L\tilde\sigma), \mathrm{Ad}_{\tilde\sigma}(d(u\circ h))\rangle_y\, dy\\ &= \int_{\Omega'} \langle \mathcal{A}(D_L\tilde\sigma), \mathrm{Ad}_{\tilde\sigma}(d(u\circ h))\rangle_y\, dy, \qquad (3.13)\end{aligned}$$

where $\mathcal{A}$ is as in (3.11). By Lemma 3.7 the Euler–Lagrange equations (1.6) are true for any function $u \in W_0^{1,n}(\Omega, \mathfrak{so}_n)$. But h^* is a Banach space isomorphism of $W_0^{1,n}(\Omega, \mathfrak{so}_n)$ to $W_0^{1,n}(\Omega', \mathfrak{so}_n)$. Hence we may re-express (3.13) as

$$\int_{\Omega'} \langle \mathcal{A}(D_L\tilde\sigma), \mathrm{Ad}_{\tilde\sigma}(du)\rangle\, dy = 0 \qquad (3.14)$$

for any function $u \in W_0^{1,n}(\Omega', \mathfrak{so}_n)$. But $C_0^\infty(\Omega', \mathfrak{so}_n) \subset W_0^{1,n}(\Omega', \mathfrak{so}_n)$ so, in particular $\tilde\sigma$ satisfies (3.14) for any $u \in C_0^\infty(\Omega', \mathfrak{so}_n)$. □

3.3. Minimisers of exterior energy

In this section we prove the existence of minimisers for the exterior energy.

Proposition 3.11. *Let $n/2 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain for which there are no harmonic 1-forms with vanishing tangential component,* i.e. *$\mathcal{H}_T(\Omega, \Lambda^1\Omega) = \{0\}$. Let $\rho_0 \in L^n(\Omega, \mathbb{R}^n\otimes\Lambda^1\Omega)$ be a K-quasiconformal co-frame such that $d\rho_0 \in L^p(\Omega, \mathbb{R}^n\otimes\Lambda^2\Omega)$. There is a number $C = C(K,p,\Omega)$ such that for every $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$*

$$\|\rho\|_n \le C(\|\rho_0\|_n + \|d(\rho-\rho_0)\|_p). \qquad (3.15)$$

Nota bene The condition $\mathcal{H}_T(\Omega, \Lambda^1\Omega) = 0$ is equivalent to the cohomological condition $H^1(\Omega, \partial\Omega) = 0$ [DS52, Theorem 3].

Proof. The following proof is along the lines of a similar proof in [PR11]. Denote $\eta = \rho - \rho_0$. Using the Hodge decomposition [ISS]

$$\begin{aligned}L^n(\Omega,&\mathbb{R}^n \otimes \Lambda^1\Omega)\\ &= dW_0^{1,n}(\Omega,\mathbb{R}^n) \oplus d^*W_N^{1,n}(\Omega,\mathbb{R}^n \otimes \Lambda^2\Omega) \oplus \mathcal{H}^p(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)\end{aligned}$$

we can express

$$\eta = df + d^*\beta + \zeta,$$

where $f \in W_0^{1,n}(\Omega,\mathbb{R}^n)$, $\beta \in W_N^{1,n}(\Omega,\mathbb{R}^n \otimes \Lambda^2\Omega)$ and $\zeta \in \mathcal{H}^n(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$. Furthermore η is in $W_T^{d,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$. Consequently so are ζ and $d^*\beta$.

Because $\zeta \in W_T^{d,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$, it follows that $\zeta \in \mathcal{H}_T(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$, and hence $\zeta = 0$.

The condition $(d^*\beta) \in W_T^{d,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$ along with

$$dd^*\beta = d\eta = d(\rho - \rho_0) \in L^p(\Omega,\mathbb{R}^n \otimes \Lambda^2\Omega)$$

guarantees that $d^*\beta \in W_T^{d,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega) \cap W_N^{d^*,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$ hence by [ISS, (6.4)], $d^*\beta \in W_0^{1,p}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$, and by Gaffney's inequality for compactly supported forms *cf.* [ISS, Proposition 4.1]

$$\|\nabla d^*\beta\|_p \leq C(p,\Omega)\|dd^*\beta\|_p + \|d^*d^*\beta\|_p = C(p,\Omega)\|d(\rho - \rho_0)\|_p.$$

Because Ω is bounded, and $n/2 \leq p$, we have $d^*\beta \in W_0^{1,n/2}(\Omega,\mathbb{R}^n \otimes \Lambda^1\Omega)$, and

$$\|\nabla d^*\beta\|_{n/2} \leq C(p,\Omega)\|d(\rho - \rho_0)\|_p.$$

Then by the Sobolev–Poincaré inequality

$$\|d^*\beta\|_n \leq C(\Omega)\|\nabla\beta\|_{n/2} \leq C(p,\Omega)\|d(\rho - \rho_0)\|_p.$$

Let $\tau = \rho_0 + d^*\beta$, so that $\rho - \tau = df$. Then

$$\begin{aligned}\|\tau\|_n &\leq \|\rho_0\|_n + \|d^*\beta\|_n\\ &\leq \|\rho_0\|_n + C(p,\Omega)\|d(\rho - \rho_0)\|_p,\end{aligned}$$

Consider $J_f = \star(df^1 \wedge \cdots \wedge df^n)$. Then

$$\begin{aligned}J_f &= \star\left[(\rho^1 - \tau^1) \wedge \cdots \wedge (\rho^n - \tau^n)\right]\\ &= \det\rho + \star\sum_{i=1}^{n}\sum_{\substack{I \subset N\\ |I|=i}} \operatorname{sign}(I)\rho^{N\setminus I} \wedge \tau^I\\ &\geq \frac{1}{K}|\rho|^n - C(n)\sum_{i=1}^{n-i}|\rho|^{n-i}|\tau|^i\end{aligned}$$

Because $f \in W_0^{1,n}(\Omega,\mathbb{R}^n)$, when we integrate J_f over Ω it yields 0, and hence

$$\begin{aligned}0 &\geq \frac{1}{K}\int_\Omega |\rho|^n - C(n)\sum_{i=1}^{n}\int_\Omega |\rho|^{n-i}|\tau|^i\\ &\geq \frac{1}{K}\|\rho\|_n^n - C(n)\sum_{i=1}^{n}\|\rho\|_n^{n-i}\|\tau\|_n^i\end{aligned}$$

by applying Hölder's inequality.

Assume $\|\rho\|_n \leq \|\tau\|_n$. Then

$$\|\rho\|_n \leq \|\rho_0\|_n + C(p,\Omega)\|d(\rho-\rho_0)\|_p.$$

If $\|\rho\|_n > \|\tau\|_n$, then

$$0 \geq 1 - C(n)K\sum_{i=1}^{n}\frac{\|\tau\|_n^i}{\|\rho\|_n^i} \geq 1 - C(n)K\frac{\|\tau\|_n}{\|\rho\|_n - \|\tau\|_n}.$$

After rearranging,

$$\|\rho\|_n \leq (C(n)K+1)\|\tau\|_n \leq (C(n)K+1)(\|\rho_0\|_n + C(n,K,p,\Omega)\|d(\rho-\rho_0)\|_p).$$

This completes the proof. □

PROPOSITION 3.12. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Suppose that for every non-negative $\varphi \in C_0^\infty(\Omega)$*

$$\int_\Omega f\varphi\,dx \geq 0.$$

Then $f(x) \geq 0$ for almost every x.

Proof. Let $\varphi^\varepsilon = \varepsilon^{-n}\varphi(x/\varepsilon)$ be a smooth mollifier function satisfying

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \varphi \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^n}\varphi(x)\,dx = 1.$$

Then

$$0 \leq \lim_{\varepsilon\to\infty}\int_\Omega \varphi^\varepsilon(x-y)f(x)\,dx = f(y),$$

for almost every $y \in \Omega$. □

LEMMA 3.13. *Let (A_ν) be a sequence in $L^n(\Omega, CO_0^+(n))$ which converges weakly to $A \in L^n(\Omega, \mathcal{M}_{n\times n})$ as $\nu \to \infty$, and suppose* $\det A_\nu \rightharpoonup \det A$ *in the sense of distributions. Then $A \in L^n(\Omega, CO_0^+(n))$.*

Proof. By conformality, $n^{-n/2}|A_\nu|^n = \det A_\nu$. Let $\eta \in L^\infty(\Omega)$ be a non-negative function. As such $\eta|A|^{n-2}A \in L^{n/(n-1)}$ and thus

$$\begin{aligned}\int_\Omega \eta|A|^n\,dx = \int_\Omega \langle A, \eta|A|^{n-2}A\rangle\,dx &= \lim_{\nu\to\infty}\int_\Omega \langle A_\nu, \eta|A|^{n-2}A\rangle\,dx \\ &\leq \liminf_{\nu\to\infty}\|\eta^{1/n}A_\nu\|_n\|\eta^{1/n}A\|_n^{n-1}.\end{aligned}$$

Consequently

$$\int_\Omega \eta|A|^n\,dx \leq \liminf_{\nu\to\infty}\int_\Omega \eta|A_\nu|^n\,dx.$$

Hence, for $\eta \in C_0^\infty(\Omega)$

$$\int_\Omega \eta(|A|^n - n^{n/2}\det A) \leq \liminf_{\nu\to\infty}\int_\Omega \eta(|A_\nu|^n - n^{n/2}\det A_\nu)\,dx = 0.$$

Hence, by Proposition 3.12, $|A|^n = n^{n/2}\det A$ almost everywhere, and by Proposition 3.1, A is a conformal matrix almost everywhere. □

Lemma 3.13 allows us to apply the following result of Iwaniec and Lutoborski.

THEOREM 3.14 (Compensated Compactness Theorem, 5.1 in [IL93]). *Assume* $\alpha_\nu^1 \in L^{p_1}(\Omega, \Lambda^{l_1}(\Omega)), \dots, \alpha_\nu^k \in L^{p_k}(\Omega, \Lambda^{l_k}(\Omega))$ *converge weakly in their respective spaces to* $\alpha^1, \dots, \alpha^k$ *as* $\nu \to \infty$, *where* $\sum_{j=1}^k \frac{1}{p_j} = 1$, *and assume that* $\|d\alpha_\nu^j\|_{q_j} \leq C < \infty$ *for all* $j = 1, \dots, k$ *and* $\nu \geq 1$, *where* $q_j > np_j/(n+p_j)$. *Then* $\alpha_\nu^1 \wedge \cdots \wedge \alpha_\nu^k \rightharpoonup \alpha^1 \wedge \cdots \wedge \alpha^k$ *as* $\nu \to \infty$ *in the sense of distributions.*

We use this theorem to show that $\mathcal{CO}_{\rho_0}^p(\Omega)$ is closed under the weak topology.

LEMMA 3.15. *Let* $\rho_\nu \in \mathcal{CO}_{\rho_0}^p(\Omega)$, $\liminf_{\nu\to\infty} \mathcal{E}_p(\rho_\nu) < \infty$ *and* ρ_ν *converge weakly to* $\varrho \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1(\Omega))$, *then* $\mathcal{E}_p(\varrho) \leq \liminf_{\nu\to\infty} \mathcal{E}_p(\rho_\nu)$ *and* $\varrho \in \mathcal{CO}_{\rho_0}^p(\Omega)$.

Proof. The proof of weak lower-semicontinuity is classical as in [PR11]. Similarly it is straightforward to see $\varrho - \rho_0 \in W_T^{d,p}(\Omega, \mathbb{R}^n \otimes \Omega)$. The real difficulty lies in showing that ϱ is a conformal multiple of ρ_0. To do so we use the Compensated Compactness Theorem and Lemma 3.13

From the compensated compactness theorem it follows that $\det \rho_\nu \rightharpoonup \det \varrho$ in the sense of distributions. Then consider the conformal matrix fields A_ν defined uniquely by $\rho_\nu = A_\nu \rho_0 \det \rho_0^{-1/n}$, for $\det \rho_0 \neq 0$ and $A_\nu = 0$ otherwise. Hence $\det \rho_\nu = \det A_\nu$. Likewise there is a uniquely defined matrix field $A : \Omega \to \mathcal{M}_{n\times n}$ such that $\varrho(x) = A(x)\rho_0(x) \det \rho_0^{-1/n}(x)$ when $\det \rho_0(x) \neq 0$, and $A(x) = 0$ otherwise. Hence $\det \varrho = \det A$. Now

$$|A_\nu|^n = n^{n/2} \det A_\nu = n^{n/2} \det \rho_\nu,$$

so A_ν is a bounded sequence in $L^n(\Omega, \mathcal{M}_{n\times n})$. Hence by passing to a subsequence A_ν converges weakly to A in $L^n(\Omega, \mathcal{M}_{n\times n})$, and by Theorem 3.14 $\det A_\nu$ converges weakly in the sense of distributions to $\det A$. As such, by Lemma 3.13, A is a conformal matrix field, so $\varrho \in \mathcal{CO}_{\rho_0}^p(\Omega)$.

□

THEOREM (1.3). *Let* $p > n/2$, *let* Ω *be a smooth bounded domain in* $\mathbb{R}^n$ *with* $\mathcal{H}_T(\Omega, \Lambda^1\Omega) = \{0\}$ *and let* ρ_0 *be a quasiconformal co-frame in* $W^{d,p}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega) \cap L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. *Then there is a minimiser of* $\mathcal{E}_p$ *in the space* $\mathcal{CO}_{\rho_0}^p(\Omega)$.

Proof. Assume we have a minimising sequence ρ_ν for which

$$\lim_{\nu\to\infty} \mathcal{E}_p(\rho_\nu) = \inf_{\rho \in \mathcal{CO}_{\rho_0}^p(\Omega)} \mathcal{E}_p(\rho).$$

Then by Proposition 3.11

$$\begin{aligned}\|\rho_\nu\|_n &< C(p, K, \Omega)(\|\rho_0\|_n + \|d\rho_\nu - d\rho_0\|_p) \\ &\leq C(p, K, \Omega)(\|\rho_0\|_n + \mathcal{E}_p(\rho_\nu)^{1/p} + \mathcal{E}_p(\rho_0)^{1/p}).\end{aligned}$$

Hence ρ_ν is a bounded sequence in $L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. By passing to a subsequence we may assume that (ρ_ν) converges weakly to a frame $\rho \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. By Lemma 3.15, $\rho \in \mathcal{CO}_{\rho_0}^p(\Omega)$. Furthermore, by weak lower-semicontinuity of $\mathcal{E}_p$,

$$\mathcal{E}_p(\rho) \leq \lim_{\nu\to\infty} \mathcal{E}_p(\rho_\nu) = \inf_{\varrho \in \mathcal{CO}^p(\Omega)} \mathcal{E}_p(\varrho_0).$$

Since $\rho \in \mathcal{CO}_{\rho_0}^p$,

$$\mathcal{E}_p(\rho) \geq \inf_{\varrho \in \mathcal{CO}_{\rho_0}^p(\Omega)} \mathcal{E}_p(\varrho).$$

This completes the proof. □

The condition $p > n/2$ in the proof of Theorem 1.3 follows from Theorem 3.14. If we assume a higher exponent of integrability for ρ, then we can examine the situation where $d\rho$ is only $n/2$-integrable, using a modification of Theorem 3.14.

We introduce some new notation. Let $m \geq 1$ and let

$$p = (p_1, \dots, p_m) \text{ and } q = (q_1, \dots, q_m)$$

be multi-exponents, where $p_i > 1$ and $q_i > 1$. Define

$$\hat{p}_i := \left(1 - \sum_{j \neq i} \frac{1}{p_j}\right)^{-1}$$

for every $1 \leq i \leq m$ and

$$p_{ik} := \left(1 - \frac{1}{q_k} - \sum_{j \neq i,k} \frac{1}{p_i}\right)^{-1}$$

for every $1 \leq i \leq m$ and $1 \leq k \leq m$, $i \neq k$.

We will use the *Averaged Poincaré Homotopy Operator*. Let $B \subset \mathbb{R}^n$ be a ball and let $\phi \in C_0^\infty(B)$ be a non-negative function satisfying

$$\int_B \phi(y)\, dy = 1.$$

We define $T_\varphi : C^\infty(B, \Lambda^k B) \to C^\infty(B, \Lambda^{k-1} B)$ by

$$T_\varphi \alpha = \int_B \phi(y) K_y \alpha \, dy,$$

where

$$K_y(\alpha)(x) = \int_0^1 (x - y) \lrcorner \alpha(y + t(x - y))\, t^{k-1} dt.$$

for $y \in B$ and $\alpha \in C^\infty(B, \Lambda^k B)$.

The operator T_φ is well defined by the convexity of B, and it satisfies the following properties:

(1) for $1 < p < \infty$, T_φ extends to a bounded linear operator $L^p(B, \Lambda^k) \to W^{1,p}(B, \Lambda^{k-1} B)$;
(2) for $1 < p < \infty$ and $1 \leq q < p^*$, T_φ is a compact operator $L^p(B, \Lambda^k B) \to L^q(B, \Lambda^{k-1} B)$;
(3) if $\alpha \in L^1_{\text{loc}}(B, \Lambda^k B)$ and $d\alpha \in L^1_{\text{loc}}(B, \Lambda^{k+1} B)$, then $T_\varphi d\alpha$ and $dT\alpha$ belong to $L^1_{\text{loc}}(B, \Lambda^k B)$ and $\alpha = Td\alpha + dT\alpha$.

For proofs of these facts see [IL93, §4].

Lemma 3.16. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $m \geq 1$, and let $\varphi_\nu = (\varphi_\nu^1, \dots, \varphi_\nu^m)$ be an m-tuple of differential forms $\varphi_\nu^i \in L^{p_i}_{\text{loc}}(\Omega, \Lambda^{l_i}\Omega)$, $d\varphi_\nu^i \in L^{q_i}_{\text{loc}}(\Omega, \Lambda^{l_i+1}\Omega)$ where $q_i^* > \hat{p}_i$ and*

$$\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p_0} \leq 1.$$

Suppose φ_ν and $d\varphi_\nu$ converge weakly to φ and $d\varphi$ in their respective spaces as $\nu \to \infty$. Then, provided $p_i^* > \hat{p}_i$ for every $1 \leq i \leq m$,

$$\int_\Omega \eta \wedge \varphi_\nu^1 \wedge \dots \wedge \varphi_\nu^m \to \int_\Omega \eta \wedge \varphi^1 \wedge \dots \wedge \varphi^m \quad \textit{as} \quad \nu \to \infty$$

for every $\eta \in C_0^\infty(\Omega, \Lambda^{n-\sum_i l_i}\Omega)$.

Proof. This is modest generalisation of [IL93, Theorem 5.1] to the case where $\sum_{i=1}^m p_i^{-1} < 1$, and follows the original proof almost exactly. The key here is that by increasing the integrability of the forms themselves, we can decrease the integrability of the exterior derivatives. We follow the proof of Theorem 5.1 in [IL93]. We first assume that $\eta \in C_0^\infty(\Omega, \Lambda^{n-\sum_i l_i}\Omega)$ is compactly supported in a ball $B = B(x_0, r) \subset\subset \Omega$. Then consider

$$\int_\Omega \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^m - \int_\Omega \eta \wedge \varphi_\nu^1 \wedge \cdots \wedge \varphi_\nu^m = \int_\Omega \sum_{i=1}^m \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (\varphi^i - \varphi_\nu^i) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m. \tag{3.16}$$

Let T_ψ be the averaged Poincaré homotopy operator where $\psi \in C_0^\infty(B)$ is constant on the support of η. Then

$$\begin{aligned}\sum_{i=1}^m \int_\Omega \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (\varphi^i - \varphi_\nu^i) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m \\ = \sum_{i=1}^m \int_B \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (T_\psi d(\varphi^i - \varphi_\nu^i)) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m \\ + \sum_{i=1}^m \int_B \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (dT_\psi(\varphi^i - \varphi_\nu^i)) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m \\ = \sum_{i=1}^m A_i + B_i,\end{aligned}$$

where

$$A_i = \int_B \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (T_\psi d(\varphi^i - \varphi_\nu^i)) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m$$

and

$$B_i = \int_B \eta \wedge \varphi^1 \wedge \cdots \wedge \varphi^{i-1} \wedge (dT_\psi(\varphi^i - \varphi_\nu^i)) \wedge \varphi_\nu^{i+1} \wedge \cdots \wedge \varphi_\nu^m$$

for every $i = 1, \ldots, m$. To estimate A_i, we apply Hölder's inequality and arrive at

$$|A_i| \le \|\eta\|_\infty \|\varphi^1\|_{p_1} \cdots \|T_\psi(d\varphi^i - d\varphi_\nu^i)\|_{\hat{p}_i} \cdots \|\varphi_\nu^m\|_{p_m}.$$

Since $q_i^* > \hat{p}_i$, T_ψ is a compact operator. By passing to a subsequence, if necessary, $\|T_\psi(d\varphi^i - d\varphi_\nu^i)\|_{\hat{p}_i}$ converges to zero as $\nu \to \infty$.

To estimate B_i we first integrate by parts and then estimate:

$$\begin{aligned}|B_i| \le \left|\int_\Omega d\eta \wedge \varphi^1 \cdots \wedge T_\psi(\varphi^i - \varphi_\nu^i) \wedge \cdots \wedge \varphi_\nu^m\right| \\ + \sum_{k<i} \left|\int_B \eta \wedge \varphi^1 \wedge \cdots \wedge d\varphi^k \wedge \cdots \wedge T_\psi(\varphi^i - \varphi_\nu^i) \wedge \cdots \wedge \varphi_\nu^m\right| \\ + \sum_{k>i} \left|\int_B \eta \wedge \varphi^1 \wedge \cdots \wedge T_\psi(\varphi^i - \varphi_\nu^i) \wedge \cdots \wedge d\varphi_\nu^k \wedge \cdots \wedge \varphi_\nu^m\right|\end{aligned}$$

From this it follows that

$$\begin{aligned}|B_i| \leq \|d\eta\|_\infty \|\varphi^1\|_{p_j} \cdots \|T_\psi(\varphi^i - \varphi^i_\nu)\|_{\hat{p}_i} \cdots \|\varphi^m_\nu\|_{p_m} \\ + \sum_{k<i} \|\eta\|_\infty \|\varphi^1\|_{p_1} \cdots \|d\varphi^k\|_{q_k} \cdots \|T_\psi(\varphi^i - \varphi^i_\nu)\|_{p_{ik}} \cdots \|\varphi^m_\nu\|_{p_m} \\ + \sum_{k>i} \|\eta\|_\infty \|\varphi^1\|_{p_1} \cdots \|T_\psi(\varphi^i - \varphi^i_\nu)\|_{p_{ik}} \cdots \|d\varphi^k_\nu\|_{q_k} \cdots \|\varphi^m_\nu\|_{p_m}\end{aligned}$$

for every $i = 1, \ldots, m$. Because $\hat{p}_i < p^*_i$, $T_\psi : L^{p_i}(B, \Lambda^{l_i} B) \to L^{\hat{p}_i}(B, \Lambda^{l_i - 1} B)$ is compact and, by passing to a subsequence if necessary, $\|T_\psi(\varphi^i - \varphi^i_\nu)\|_{\hat{p}_i}$ converges to zero as $\nu \to \infty$.

Since $\hat{p}_k < q^*_k$,

$$\frac{1}{q_k} - \frac{1}{n} < 1 - \sum_{j \neq k} \frac{1}{p_j}$$

and

$$\frac{1}{p_i} - \frac{1}{n} < 1 - \frac{1}{q_k} - \sum_{j \neq i,k} \frac{1}{p_j}.$$

Thus $p^*_i > p_{ik}$. Hence $T_\psi : L^{p_i}(B, \Lambda^{l_i} B) \to L^{p_{ik}}(B, \Lambda^{l_i - 1} B)$ is compact and, by passing to a subsequence if necessary, $\|T_\psi(\varphi^i - \varphi^i_\nu)\|_{p_{ik}}$ converges to 0 as $\nu \to \infty$.

We have obtained

$$\int_\Omega \eta \wedge \varphi^1_\nu \wedge \cdots \wedge \varphi^m_\nu \to \int_\Omega \eta \wedge \varsigma^1 \wedge \cdots \wedge \varphi^m$$

as $\nu \to \infty$ for any η supported in a ball $B \subset\subset \Omega$.

Assume $\eta \in C^\infty_0(\Omega, \Lambda^{n - \sum l_i} \Omega)$. We can cover the support of η by balls compactly contained in Ω. If we take a smooth finite partition of unity $\{\psi_i\}_{i=1}^N$ covering the support of η, such that each ψ_i is supported in a ball $B_i \subset\subset \Omega$, then we can calculate

$$\begin{aligned}&\lim_{\nu\to\infty} \int_\Omega \eta \wedge (\varphi^1 \wedge \cdots \wedge \varphi^m - \varphi^1_\nu \wedge \cdots \wedge \varphi^m_\nu) \\ &\qquad = \lim_{\nu\to\infty} \int_\Omega \left(\sum_{i=1}^N \psi_i \eta \right) \wedge (\varphi^1 \wedge \cdots \wedge \varphi^m - \varphi^1_\nu \wedge \cdots \wedge \varphi^m_\nu) \\ &\qquad = \sum_{i=1}^N \lim_{\nu\to\infty} \int_{B_i} \psi_i \eta \wedge (\varphi^1 \wedge \cdots \wedge \varphi^m - \varphi^1_\nu \wedge \cdots \wedge \varphi^m_\nu) \\ &\qquad = 0.\end{aligned}$$

This completes the proof. □

Corollary 3.17. *Let Ω be a domain, $p \geq n$, $q > np/((n+1)p - n(n-1))$ and let $\rho_\nu \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^1 \Omega)$ be a bounded sequence with $d\rho_\nu$ bounded in $L^q(\Omega, \mathbb{R}^n \otimes \Lambda^2 \Omega)$. Then there exists a weakly convergent subsequence (ρ_{ν_k}) tending to ρ as $k \to \infty$ such that* $\det \rho_{\nu_k} \rightharpoonup \det \rho$ *as $k \to \infty$.*

Proof. Now we apply Lemma 3.16 with $\varphi^i_\nu = \rho^i_\nu$, $p_i = p$, and $q_i = q$. Let us check that the conditions are met:

$$(q_i^*)^{-1} = (q^*)^{-1} = \frac{1}{q} - \frac{1}{n} < \frac{n+1}{n} - \frac{n-1}{p} - \frac{1}{n} = 1 - \frac{n-1}{p} = (\hat{p}_i)^{-1}.$$

Consequently

$$(q_i^*)^{-1} < (\hat{p}_i)^{-1}.$$

Furthermore $p^* = \infty > \hat{p}_i$ and hence the conditions of Lemma 3.16 are satisfied. $\square$

Remark 3.18. Because $p > n$,

$$1 - \frac{n-1}{p} > 1 - \frac{n-1}{n} = n = \left(\frac{n}{2}\right)^*,$$

and hence $q = n/2$ is admissible in Corollary 3.17.

Theorem (1.4). *Let $p > n$ and $q > np/((n+1)p - n(n-1))$. Let Ω be a smooth bounded domain and let $\rho_0 \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be a quasiconformal co-frame. Suppose $d\rho_0 \in L^q(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$. Then there exists a minimiser of $\mathcal{E}_p$ in $\mathcal{SO}^q_{\rho_0}$.*

It's worth noting that for every $\rho \in \mathcal{SO}^q_{\rho_0}(\Omega)$, it follows automatically that $\|\rho\|_p = \|\rho_0\|_p$ and hence we do not need the condition on harmonic fields as in Theorem 1.3.

Proof. Our theorem assumes that $\rho \in L^p_{\mathrm{loc}}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$, whereas Corollary 3.17 assumes that $\rho \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$. To circumvent this problem, let $U \subset\subset \Omega$.

Suppose (σ_ν) is a sequence of measurable mappings : $\Omega \to SO(n)$ so that $\rho_\nu = \sigma_\nu\rho_0$ is a minimising sequence for $\mathcal{E}_q$. Then (ρ_ν) is a bounded sequence in $L^p(U, \mathbb{R}^n \otimes \Lambda^1 U)$ and $(d(\sigma_\nu\rho_0))$ is bounded in $L^q(U, \mathbb{R}^n \otimes \Lambda^2 U)$.

Consequently we can choose a subsequence (ρ_{ν_k}), weakly converging to some $\rho \in L^n(U, \mathbb{R}^n \otimes \Lambda^1 U)$. We can apply Corollary 3.17 to yield that $\det \sigma_{\nu_k}\rho_0 \rightharpoonup \det \rho$. But $\det \sigma_{\nu_k}\rho_0 = \det \rho_0$ almost everywhere for every $k \in \mathbb{N}$, so $\det \rho = \det \rho_0$ almost everywhere. Then by Proposition 3.13, $\rho = A\rho_0$ for some conformal matrix field A. But then $\det A = 1$ almost everywhere, so A is a measurable map $\Omega \to SO(n)$. $\square$

Theorem (1.6). *Let $p > n/2$ and let Ω be a bounded smooth domain with $\mathcal{H}_T(\Omega, \Lambda^1\Omega) = 0$. Suppose $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ is a K-quasiconformal co-frame and $d\rho_0 \in L^p(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$. Then there exists a $q_0 = q_0(n, K) < n/2$ such that for every $q > q_0$ there is a $\rho \in \mathcal{CO}^q_{\rho_0}(\Omega)$ satisfying* (3.7) *with exponent q.*

Proof. First we minimise the p-energy of ρ_0 with Theorem 1.3 and get a minimiser $\varrho \in \mathcal{CO}^p_{\rho_0}(\Omega)$. The co-frame ϱ is in $L^{p'}_{\mathrm{loc}}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ by Corollary 3.6 where $p' = p'(n, K) > n$. Then, by Theorem 1.4, for any

$$q > q_0 = \frac{n\,p'}{(n+1)p' - n(n-1)}$$

there is a minimiser of $\mathcal{E}_q$ in $\mathcal{SO}^q_\varrho(\Omega)$, which is then a solution of (3.7) with exponent q. But $\varrho \in \mathcal{CO}^p_{\rho_0}(\Omega) \subset \mathcal{CO}^q_{\rho_0}(\Omega)$ and so $\mathcal{SO}^q_\varrho(\Omega) \subset \mathcal{CO}^q_{\rho_0}(\Omega)$. $\square$

Remark 3.19. It follows that for all $k \leq n$, any $\sigma_0 \in W^{1,q}(\Omega, SO(k))$, and $q > 1$ there exists a minimiser of the Dirichlet p-energy, $\sigma : \Omega \to SO(k)$, with $\sigma|_{\partial\Omega} = \sigma_0|_{\partial\Omega}$ in the Sobolev trace sense. Indeed, we may set $\rho_0 = \sigma_0 d\mathbf{x} \in L^\infty(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ and apply Theorem 1.4. This is, however, a rather roundabout way of proving the existence of a p-harmonic map $\Omega \to SO(k)$; compare with [Whi88].

3.4. Another exterior energy

In this section we define an operator $\mathfrak{A}_\rho$ and use this to define an alternate exterior energy. Let $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be a fixed essentially non-vanishing quasiconformal co-frame, and let $\rho \in \mathcal{SO}^p_{\rho_0}(\Omega)$ or $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$. The determinant of our quasiconformal co-frame plays the role of a weight for our Euler-Lagrange equations, and hence the non-vanishing assumption is a natural one to make. It arises, for instance in the case when the determinant is an A_p weight, which is the case when our co-frame is exact.

Denote by P_0 and P the matrices of coefficients of ρ_0 and ρ, respectively. Let $S_0 = \det P_0^{1/n} P_0^{-1}$, let $S_\rho = \det P^{1/n} P^{-1}$ wherever $\det \rho \neq 0$ and let $S_\rho = S_0$ otherwise. Define $\mathfrak{A}_\rho : \mathbb{R}^n \otimes \Lambda^2\Omega \to \mathbb{R}^n \otimes \Lambda^2\Omega$ by the formula

$$\mathfrak{A}_\rho(\beta) = S_\rho^\# \beta. \tag{3.17}$$

We minimise the following energy:

$$\mathcal{E}'_p(\rho) = \int_\Omega |\mathfrak{A}_\rho(d\rho)|^p. \tag{3.18}$$

The reason for adding a seemingly additional layer of complexity is that it peels away when we examine a situation analogous to Theorem 1.7, that is frames $\rho = \sigma df$, where $f : \Omega \to \Omega'$ is a quasiconformal map and $\sigma \circ f^{-1} \in W^{1,1}(\Omega', SO(n))$.

We first show that minimisers of (3.18) exist and then derive their Euler-Lagrange equations. The end result is that in certain cases the minimiser corresponds to an $\mathcal{A}$-harmonic map $\sigma : \Omega \to SO(n)$ with a C^1-smooth bundle map $\mathcal{A}$. Subsequently we can apply existing results on the higher regularity of such maps.

Lemma 3.20. *Let $1 \leq p < \infty$ and let $\rho \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be an essentially non-vanishing K-quasiconformal frame. Then*

$$C(n)^{-1}(K^O)^{-2p/n}\mathcal{E}_p(\rho) \leq \mathcal{E}'_p(\rho) \leq C(n)(K^I)^{2p/n}\mathcal{E}_p(\rho)$$

where K^O and K^I are respectively the outer and inner distortions of the coefficient matrix-field P of ρ.

Proof. This lemma is really a corollary of the fact that given a matrix-field $A : \Omega \to \mathcal{M}_{n\times n}$, the pointwise operator $A^\# : \Lambda^k\Omega \to \Lambda^k\Omega$ is bounded by the function $|A|^k$, where $|\cdot|$ is the Hilbert–Schmidt norm.

Let $A = S_\rho$. Then

$$|A| \leq \det P^{1/n}|P^{-1}| \leq (K^I)^{1/n}$$

Hence, for a 2-form $\alpha : \Omega \to \Lambda^2\Omega$

$$|A^\#\alpha|^p \leq |A|^{2p}|\alpha|^p \leq (K^I)^{2p/n}|\alpha|^p.$$

Now for the first part of the inequality. Let $B = \det P^{-1/n}P$. then $AB = I$. It is clear that $|B| \leq (K^O)^{1/n}$, and so

$$|\alpha|^p = |B^\# A^\# \alpha|^p \leq |B^\#|^{2p}|A^\#\alpha|^p \leq (K^O)^{2p/n}|A^\#\alpha|^p.$$

Thus

$$|K^O|^{-2p/n}|\alpha|^p \leq |A^\#\alpha|^p.$$

□

It is not immediate that the functional $\mathcal{E}'_p$ is lower semi-continuous. But in fact we have the following property.

Lemma 3.21. *Let $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be an essentially non-vanishing quasiconformal co-frame. Let $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$. Then for every $\alpha : \Omega \to \mathbb{R}^n \otimes \Lambda^2\Omega$*

$$|\mathfrak{A}_\rho(\alpha)| = |\mathfrak{A}_{\rho_0}(\alpha)|$$

almost everywhere, and consequently

$$\int_\Omega |S^\#_\rho d\rho|^p \, dx = \int_\Omega |S^\#_0 d\rho|^p \, dx.$$

Proof. Let $P : \Omega \to \mathcal{M}_{n\times n}$ denote the coefficient matrix-field of ρ and $P_0 : \Omega \to \mathcal{M}_{n\times n}$ denote the coefficient matrix-field of ρ_0.

Let E denote the set where $\det\rho \neq 0$ and $\det\rho_0 \neq 0$. It follows that $\rho = A\rho_0$ almost everywhere for some conformal matrix-field $A : \Omega \to CO^+_0(n)$, where $A = I$ on $\Omega \setminus E$. Then $P^{-1} = P_0^{-1} \det A^{-2/n} A^t$. When acting on $\Lambda^k T_x\Omega$ for $x \in E$

$$\begin{aligned}
(P^{-1}\det P^{1/n})^\# &= \det P^{k/n}(P_0^{-1}\det A^{-2/n}A^t)^\# \\
&= \det P_0^{k/n}\det A^{k/n}\det A^{-2k/n}(A^t)^\#(P_0^{-1})^\# \\
&= \det P_0^{k/n}\det A^{-k/n}(A^t)^\#(P_0^{-1})^\# \\
&= (\det A^{-1/n}A^t)^\#(\det P_0^{1/n}P_0^{-1})^\#.
\end{aligned}$$

Now $R := \det A^{-1/n}A$ is a measurable orthogonal matrix field, so

$$|R^\#\beta| = |\beta|,$$

for every form $\beta : \Omega \to \Lambda^k\Omega$, and hence for $\alpha : \Omega \to \Lambda^k\Omega$

$$|(P^{-1}\det P^{1/n})^\#\alpha)| = |(R^\#(P_0^{-1}\det P_0^{1/n})^\#\alpha)| = |(P_0^{-1}\det P_0^{1/n})^\#\alpha|.$$

If $\rho = 0$ then $\mathfrak{A}_\rho = \mathfrak{A}_{\rho_0}$ by construction. □

Lemma 3.22. *Let $1 < p < \infty$, and let $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ be essentially non-vanishing. Suppose (ρ_ν) is a sequence in $\mathcal{CO}^p_{\rho_0}(\Omega)$ with $\mathcal{E}'_p(\rho_\nu) \leq C$. If (ρ_ν) converges weakly to $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ as $\nu \to \infty$, then*

$$\mathcal{E}'_p(\rho) \leq \liminf_{\nu\to\infty} \mathcal{E}'_p(\rho_\nu).$$

Proof. By Lemma 3.21, we can express

$$\mathcal{E}'_p(\rho_\nu) = \int_\Omega |S^\#_0 d\rho_\nu|^p \, dx$$

and

$$\mathcal{E}'_p(\rho) = \int_\Omega |S^\#_0 d\rho|^p \, dx.$$

We introduce the norm $\|\cdot\|_{\rho_0} : L^p(\Omega, \mathbb{R}^n \otimes \Lambda^k\Omega) \to \mathbb{R}$ by defining

$$\|\alpha\|_{\rho_0} = \left(\int_\Omega |S^\#_0 \alpha|^p \, dx\right)^{1/p}.$$

It is equivalent to the usual L^p norm, *i.e.*

$$C^{-1}(n,p,K)\|\alpha\|_p \leq \|\alpha\|_{\rho_0} \leq C(n,p,K)\|\alpha\|_p.$$

By the weak lower semicontinuity of norms on Banach spaces,

$$\mathcal{E}'_p(\rho) = \|d\rho\|^p_{\rho_0} \leq \liminf_{\nu\to\infty} \|d\rho_\nu\|^p_{\rho_0} = \liminf \mathcal{E}'_p(\rho_\nu).$$

□

THEOREM 3.23. *Let $1 < q < \infty$ and $n \le p < \infty$. Suppose $\rho_0 \in L^p(\mathbb{R}^n \otimes \Lambda^1\Omega)$ is an essentially non-vanishing K-quasiconformal co-frame with finite exterior q energy,* i.e. *$\mathcal{E}'_q(\rho_0) < \infty$. If $p = n$ and $n/2 < q \le n$ then there exists a minimiser for $\mathcal{E}'_q : \mathcal{CO}^p_{\rho_0}(\Omega) \to \mathbb{R}$. If $p > n$ and $q > np/((n+1)p - n(n-1))$ then there exists a minimiser for $\mathcal{E}'_p : \mathcal{SO}^{n/2}_{\rho_0}(\Omega) \to \mathbb{R}$.*

Proof. Once again we start with a minimising sequence (ρ_ν) for $\mathcal{E}'_p$. Then, by Lemmata 3.20 and 3.11, the sequence (ρ_ν) is bounded in $\mathcal{CO}^p_{\rho_0}(\Omega)$. Hence we can choose a weakly converging subsequence, which by Lemma 3.15, converges in $\mathcal{CO}^p_{\rho_0}(\Omega)$. Lastly by Lemma 3.22, this weak limit minimises the energy. □

We now derive the Euler–Lagrange equations for $\mathcal{E}'_p : \mathcal{CO}^p_{\rho_0}(\Omega) \to \mathbb{R}$.

THEOREM 3.24. *Let $1 < p < \infty$ and let $\rho_0 \in L^n(\Omega, \mathbb{R}^n \otimes \Lambda^1(\Omega))$ be an essentially non-vanishing K-quasiconformal co-frame such that $\mathcal{E}'_p(\rho_0) < \infty$. Let $P_0 : \Omega \to \mathcal{M}_{n\times n}$ denote the coefficient matrix-field of ρ_0 with respect to $d\mathbf{x}$. Suppose $\rho \in \mathcal{CO}^p_{\rho_0}(\Omega)$ is a local minimiser of $\mathcal{E}'_p$. Then ρ satisfies the Euler-Lagrange equations*

$$\int_\Omega \langle |(P_0^{-1})^\# d\rho|^{p-2}(P_0^{-1})^\# d\rho, ((P_0^{-1})^\# du) \wedge d\mathbf{x}\rangle \det P_0^{2p/n}\, dx = 0 \tag{3.19}$$

$$\int_\Omega \langle |(P_0^{-1})^\# d\rho|^{p-2}(P_0^{-1})^\# d\rho, ((P_0^{-1})^\# (d(\lambda\rho))\rangle \det P_0^{2p/n}\, dx = 0 \tag{3.20}$$

for every $\lambda \in C_0^\infty(\Omega)$ and every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$.

If $\rho \in \mathcal{SO}^p_{\rho_0}(\Omega)$ is a local minimiser of $\mathcal{E}'_p : \mathcal{SO}^p_{\rho_0}(\Omega) \to \mathbb{R}$, then ρ satisfies (3.19) *for every $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$.*

Proof. Let $u \in C_0^\infty(\Omega, \mathfrak{so}_n)$ and $\lambda \in C_0^\infty(\Omega)$. We can again apply Lemma 3.21 to obtain for every $t \in \mathbb{R}$

$$\mathcal{E}'_p(e^{t(\lambda I+u)}\rho) = \int_\Omega \left|(\det P_0^{1/n} P_0^{-1})^\# d(e^{t(\lambda I+u)}\rho)\right|^p\, dx.$$

Denote $\det P_0^{1/n} P_0^{-1}$ by B, and note that $|B| \le C(n,K)$.

Once again we let $\tau_t = t^{-1}(e^{t(\lambda I+u)} - I)\rho$ for $0 < t \le 1$. Recall

$$|d\tau_t| \le C(|\rho| + |d\rho|)$$

pointwise almost everywhere, where $C = C(\|u\|_{1,\infty}, \|\lambda\|_{1,\infty})$; see (3.8). Then

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}'_p(e^{t(\lambda I+u)}\rho)\Big|_{t=0} \\ &\quad = \lim_{t\to 0} \frac{1}{t}\int_\Omega |B^\#(de^{t(\lambda I+u)}\rho)|^p - |B^\# d\rho|^p\, dx \\ &\quad = \lim_{t\to 0}\int_\Omega I_t\, dx, \end{aligned}$$

where

$$I_t = \frac{1}{t}[(|B^\# d\rho|^2 + t^2|B^\# d\tau_t|^2 + 2t\langle B^\# d\rho, B^\# d\tau_t\rangle)^{p/2} - |B^\# d\rho|^p]$$

For every $0 < t \leq 1$

$$
\begin{aligned}
|I_t| &\leq \frac{1}{t}\left((|B^{\#}d\rho|^2 + t^2|B^{\#}d\tau_t|^2 + 2t|B^{\#}d\tau_t||B^{\#}d\rho|)^{p/2} - |B^{\#}d\rho|^p\right)\\
&\leq \frac{1}{t}\left((|B^{\#}d\rho| + t|B^{\#}d\tau_t|)^p - |d\rho|^p\right)\\
&\leq p(|B^{\#}d\rho| + |B^{\#}d\tau_t|)^{p-1}|d\tau|\\
&\leq C(n,p,K)(|d\rho|^p + |d\rho|^{p-1}|\rho|).
\end{aligned}
$$

Once again the bound for I_t is integrable and independent of $0 < t \leq 1$. Consequently we can use the Dominated Convergence Theorem to bring the limit inside the integral

$$
\lim_{t\to 0}\int_\Omega I_t\, dx = \int_\Omega \lim_{t\to 0} I_t\, dx.
$$

Let us calculate

$$
\begin{aligned}
\lim_{t\to 0} I_t &= \lim_{t\to 0}\frac{1}{t}\left(|B^{\#}d\rho|^2 + t^2|B^{\#}d\tau_t|^2 + 2t\langle B^{\#}d\tau_t, B^{\#}d\rho\rangle\right)^{p/2} - |B^{\#}d\rho|^p\\
&= p|B^{\#}d\rho|^{p-2}\langle B^{\#}d\rho, d((\lambda I + u)\rho)\rangle
\end{aligned}
$$

For

$$
\left.\frac{d}{dt}\mathcal{E}'_p(\rho)\right|_{t=0} = 0,
$$

and $u = 0$ we arrive at (3.20).

To derive (3.19), set $\lambda = 0$ and consider

$$
\begin{aligned}
\langle B^{\#}d\rho, B^{\#}d(u\rho)\rangle &= \langle B^{\#}d\rho, B^{\#}((du)\wedge\rho)\rangle + \langle B^{\#}d\rho, B^{\#}(ud\rho)\rangle\\
&= \langle B^{\#}d\rho, B^{\#}((du)\wedge\rho)\rangle + \langle B^{\#}d\rho, uB^{\#}d\rho\rangle\\
&= \langle B^{\#}d\rho, B^{\#}((du)\wedge\rho)\rangle.
\end{aligned}
$$

This follows because $B^{\#}(A\alpha) = AB^{\#}\alpha$, for any $\alpha : \Omega \to \mathbb{R}^n \otimes \Lambda^k\Omega$ and any matrix A, and u is antisymmetric, so $\langle u\alpha, \alpha\rangle = 0$ for any $\alpha : \Omega \to \mathbb{R}^n \otimes \Lambda^k\Omega$ *cf.* Proposition 1.12. Furthermore

$$
B^{\#}(du\wedge\rho) = B^{\#}(du)\wedge B^{\#}(\rho) = B^{\#}(du)\wedge d\mathbf{x}(\det\rho)^{1/n}.
$$

Replacing B with $\det P_0^{1/n}P_0^{-1}$ and pulling out all factors of the determinant yields the desired result. $\square$

Theorem (1.8). *Let $f : \Omega \to \Omega'$ be a quasiconformal map with inverse $h : \Omega' \to \Omega$. Let $\sigma : \Omega \to SO(n)$ be a measurable map satisfying $\tilde\sigma := \sigma\circ h \in W^{1,1}(\Omega', SO(n))$. If $d(\sigma df) \in L^{n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$ then $\tilde\sigma$ is in $W^{1,n/2}(\Omega', SO(n))$. Furthermore if σdf is a solution to* (3.19) *for $p = n/2$ then $\tilde\sigma$ satisfies*

$$
\int_{\Omega'}\langle |D_L\tilde\sigma\wedge d\mathbf{y}|^{n/2-2}D_L\tilde\sigma\wedge d\mathbf{y}, \mathrm{Ad}_{\tilde\sigma}(du)\wedge d\mathbf{y}\rangle\, dx = 0
$$

for every $u \in W_0^{1,n/2}(\Omega', SO(n))$, where $d\mathbf{y}$ is the standard Cartesian co-frame on Ω'.

Proof. This is just a simple application of Theorem 3.24. In this case $\rho_0 = df$ so $P_0 = Df$, and $\rho = \sigma df$. Once again h is the inverse map to f, so $Df(h(y))^{-1} = Dh(y)$ for almost every $y \in \Omega'$.

Since $d(\sigma df) \in L^{n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^2\Omega)$, it follows that $h^*d(\sigma df) \in L^{n/2}(\Omega', \mathbb{R}^n \otimes \Lambda^2\Omega')$. But $h^*d(\sigma df) = dh^*(\sigma df) = d(\tilde{\sigma}d\mathbf{y})$. Thus, by Lemma 3.5, $\tilde{\sigma} \in W^{1,n/2}(\Omega, SO(n))$ and

$$h^*d(\sigma df) = d(\tilde{\sigma}d\mathbf{y}) = d\tilde{\sigma} \wedge d\mathbf{y}.$$

For $p = n/2$ the Euler–Lagrange equations (3.19) yield

$$\int_\Omega \langle |(Df^{-1})^\# d(\sigma df)|^{n/2-2}(Df^{-1})^\# d(\sigma df), (Df^{-1})^\#(du \wedge df)\rangle_x J_f(x)\, dx = 0.$$

We apply a change of variables $x = h(y)$ to yield

$$\begin{aligned}
0 &= \int_{\Omega'} \langle |(Df^{-1})^\# d(\sigma df)|^{n/2-2}(Df^{-1})^\# d(\sigma df), (Df^{-1})^\#(du \wedge \sigma df)\rangle_{h(y)}\, dy \\
&= \int_{\Omega'} \langle |h^*(d(\sigma df))|^{n/2-2} h^*(d(\sigma df)), h^*(du \wedge \sigma df)\rangle_y\, dy \\
&= \int_{\Omega'} \langle |d\tilde{\sigma} \wedge d\mathbf{y}|^{n/2-2} d\tilde{\sigma} \wedge d\mathbf{y}, h^*(du) \wedge \tilde{\sigma}d\mathbf{y}\rangle_y\, dy \\
&= \int_{\Omega'} \langle |\tilde{\sigma}^{-1}d\tilde{\sigma} \wedge d\mathbf{y}|^{n/2-2} \tilde{\sigma}^{-1}d\tilde{\sigma} \wedge d\mathbf{y}, \tilde{\sigma}^{-1}h^*(du) \wedge \tilde{\sigma}d\mathbf{y}\rangle_y\, dy \\
&= \int_{\Omega'} \langle |D_L\tilde{\sigma} \wedge d\mathbf{y}|^{n/2-2} D_L\tilde{\sigma} \wedge d\mathbf{y}, \mathrm{Ad}_{\tilde{\sigma}}(h^*(du)) \wedge d\mathbf{y}\rangle_y\, dy.
\end{aligned}$$

We can use similar reasoning to Lemma 3.7 to test against any $u \in W^{1,n}_0(\Omega, \mathfrak{so}_n)$. Then noting that composition with h is a linear isomorphism

$$W^{1,n}_0(\Omega, \mathfrak{so}_n) \to W^{1,n}_0(\Omega', \mathfrak{so}_n)$$

we obtain

$$\int_{\Omega'} \langle |D_L\tilde{\sigma} \wedge d\mathbf{y}|^{n/2-2} D_L\tilde{\sigma} \wedge d\mathbf{y}, \mathrm{Ad}_{\tilde{\sigma}}(dv) \wedge d\mathbf{y}\rangle\, dy = 0$$

for every $v \in W^{1,n}_0(\Omega, \mathfrak{so}_n)$. □

Theorem 1.8 somewhat unnaturally assumes that $\sigma \circ f^{-1} \in W^{1,1}(\Omega')$. This could be omitted if the following conjecture were to hold

Conjecture. *Let $\sigma : \Omega' \to SO(n)$ be a measurable map, and $\sigma d\mathbf{x} \in W^{d,n/2}(\Omega', \mathbb{R}^n \otimes \Lambda^1\Omega)$. Then $\sigma \in W^{1,n/2}(\Omega', SO(n))$.*

Assuming the conjecture, suppose $\sigma df \in W^{d,n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ is a solution to the Euler–Lagrange equations (3.19) for $p = n/2$, where $f : \Omega \to \Omega'$ is quasiconformal and $\sigma : \Omega \to SO(n)$ is measurable.

It follows that $\sigma \circ f^{-1}$ is in $W^{1,n/2}(\Omega', SO(n))$. So if σdf is a solution, then we could apply the higher regularity of $\sigma \circ f^{-1}$ to get that σ is Hölder continuous excepting a set of zero Hausdorff β-measure. The number $\beta = \beta(n, K)$ is given by the Hausdorff dimension distortion of Sobolev functions *cf.* [Kau00],

$$\beta = \frac{p(\lceil n/2\rceil - 1)}{p - \lfloor n/2\rfloor - 1}$$

and $p = p(n, K)$ is the higher integrability exponent of a K-quasiregular map in n-dimensional space. In particular β is always strictly less than n.

Furthermore if this conjecture were to hold, then it would trivialise the existence of frames minimising the exterior energy in $\mathcal{SO}^{n/2}_{df}(\Omega)$, as we could apply a classical proof of existence: any minimising sequence $\sigma_\nu df$ weakly convergent to $\rho \in W^{d,n/2}(\Omega, \mathbb{R}^n \otimes \Lambda^1\Omega)$ would give rise to a sequence $(\sigma_\nu \circ f^{-1})$ which would be bounded in $W^{1,n/2}(\Omega', SO(n))$. Hence, by passing to a subsequence (if necessary), $(\sigma_{\nu_k} \circ f^{-1})$ would strongly converge in $L^p(\Omega', \mathbb{R}^n)$ for $p < n = (n/2)^*$ and would converge pointwise almost everywhere to $\sigma \circ f^{-1} : \Omega' \to SO(n)$. Consequently ρ would be equal to $\sigma df \in \mathcal{SO}^{n/2}_{df}(\Omega)$.

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