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DISSERTATIONES

79

A SINGULAR VALUE DECOMPOSITION OF MATRICES IN A SPACE WITH AN INDEFINITE SCALAR PRODUCT

SEPPO HASSI

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Preface

After receiving my Phil.cand. examination in Mathematics I was employed as an assistant in the Department of Statistics of the University of Helsinki. Beside the daily work in this Department I have enjoyed the opportunity to continue my studies in Mathematics and Statistics and in 1988 to begin the work on this doctoral thesis.

The idea for this study grew from my earlier interest in the theory and applications of singular values and the related canonical representation on operators, called singular value decomposition. That earlier selection of my interest was greatly influenced by discussions with Professor Seppo Mustonen. This thesis is devoted to the study of the singular value decomposition of complex matrices in spaces with an indefinite metric.

I wish to express my sincere gratitude to Professor Ilppo Simo Louhivaara for his invaluable advice and encouragement during the completion of this work. I am grateful to Professor Hannu Niemi for reading the manuscript and making valuable comments. My thanks are due to Professor Seppo Mustonen for his suggestions with the early stages of my studies and for making it possible to type this manuscript in SURVO 84C.

Finally, I would like to thank my wife Merja for her unfailing interest and support.

Helsinki, October 1990

Seppo Hassi

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Introduction

According to a result of Eckart and Young [7] any complex matrix A can always be factorized into the form $A = UDV^*$ with (in the case of a usual "definite" scalar product) unitary matrices U and V and a diagonal matrix D with nonnegative entries, known as the singular value decomposition (s.v.d.) of the matrix A. (See also Eckart – Young [6].) This decomposition can even be found in papers of authors like J.J. Sylvester as early as the end of the 19th century. A generalization of it to infinite-dimensional spaces can be found for example in the book of Gohberg and Kreĭn [8] (p. 28 for the Schmidt expansion of compact operators and Chapter II for many important properties of the related s-numbers).

Our aim is to present this factorization for matrices, when the usual underlying scalar products are replaced by indefinite ones. It is well-known that turning to indefinite scalar products brings substantial changes to the geometry and the spectral properties of the usual operator classes on these spaces. A significant difference will also be encounted when presenting a singular value decomposition with respect to indefinite scalar products. It is no longer possible to present a s.v.d. for every matrix. The existence of this representation will be shown to depend heavily on the spectral structure of a related matrix $A^{[*]}A$. Simultaneously, for example, all the singular subspace pairs are forced to be nondegenerate with respect to the underlying scalar products.

Some basic results on finite-dimensional linear spaces with an indefinite scalar product that are relevant to our work are presented in Section 1.

In Section 2 a singular value decomposition (s.v.d.) is defined for spaces with an indefinite scalar product, and a necessary and sufficient condition for the existence of this representation is given. In Sections 3 and 4 two classes of matrices having a singular value decomposition with respect to indefinite scalar products are constructed. The first one is the class of sub-Pesonen matrices. The second one is based on semi-bounded scalar products. In Section 5 a connection of our approach to the so-called generalized singular value decomposition (g.s.v.d.) is illustrated.

Section 6 is devoted to the study of the stability of indefinite singular value decompositions of Section 2. Namely, the nonexistence of an H-s.v.d. for every matrix leads us to ask what happens to the H-s.v.d. if the elements of the matrix are perturbed. In general the answer to this question is found to be negative in the sense that the matrix under study can totally loose this decomposition property under arbitrary small perturbations. Section 6.1 identifies the class of all matrices behaving well in this respect, more formally, having a stable H-s.v.d. In Section 6.2 the behaviour of the H-singular values under analytic perturbations of the matrix elements will be considered.

In Section 7 the existence of H-s.v.d. for different kinds of plus matrices is investigated. Special attention is paid to conditions under which a specified plus matrix has a stable H-s.v.d. To illustrate the power of the class of sub-Pesonen matrices we shall prove in the end of this section approximation results which show how it is possible to approximate plus matrices with (stable) sub-Pesonen matrices. Finally, in Section 8 a connection between the singular values and the eigenvalues is generalized to the indefinite scalar product case.

Some of the results of the work have close connections to the recent investigation of Gohberg, Lancaster and Rodman (cf. e.g. [9]) on H-self-adjoint matrices.

1. Preliminaries

In this section we recall some basic tools and terminology in indefinite scalar product spaces needed later on. For the general treatments of indefinite scalar product spaces the reader is referred to Bognár [4] and Gohberg, Lancaster and Rodman [9] and to the recent work of Azizov and Iokhvidov [2].

Denote by $[.,.]$ an *indefinite scalar product* on $\mathbf{C}^n$, the vector space of ordered systems of n complex numbers. By definition, $[.,.]$ is a *Hermitean sesquilinear* form satisfying the following condition: the identity $[x, y] = 0$ for every $y \in \mathbf{C}^n$ implies $x = 0$. Hence we are dealing with *nondegenerate scalar product spaces*, or finite-dimensional *Krein spaces*. The only exceptions appearing in sequel are the semidefinite scalar products (thus automatically degenerate) related here with the study of the generalized singular value decomposition. (See also Lemma 4.1.)

For every vector $x \in (\mathbf{C}^n,[.,.])$ the *inner square* $[x, x]$ of x is a real number. As usual a vector $x \in (\mathbf{C}^n,[.,.])$ is called *positive*, *negative*, *neutral*, *nonnegative* or *nonpositive* with respect to $[.,.]$ if, respectively, $[x,x] > 0$, $[x,x] < 0$, $[x,x] = 0$, $[x,x] \geq 0$ or $[x,x] \leq 0$ holds.

A *linear subspace* $\mathcal{M} \subset (\mathbf{C}^n,[.,.])$ is said to be *positive* (*negative*) if $[x,x] > 0$ (respectively $[x,x] < 0$) for every $x \in \mathcal{M}$, $x \neq 0$. We refer to positive and negative linear subspaces with the common name of *definite linear subspaces*.

Similarly, the linear subspace $\mathcal{M}$ is called *neutral*, *nonnegative* or *nonpositive* if, respectively, $[x,x] = 0$, $[x,x] \geq 0$ or $[x,x] \leq 0$ for every $x \in \mathcal{M}$. These linear subspaces are referred to by the common term of *semidefinite linear subspaces*. The following lemma reflects a special nature of indefinite scalar product spaces.

1.1. Lemma. *Every linear subspace containing positive and negative vectors, contains nonzero neutral vectors, too.*

(For this cf. e.g. Azizov – Iokhvidov [2], Proposition I.1.9.) Let $[.,.]$ be an indefinite scalar product on $\mathbf{C}^n$. Then the *orthogonal companion* $\mathcal{M}^{[\perp]}$ of a subset $\mathcal{M} \subset \mathbf{C}^n$ with respect to $[.,.]$ is defined by

$$\mathcal{M}^{[\perp]} = \{x \in \mathbf{C}^n \mid [x, y] = 0 \text{ for all } y \in \mathcal{M}\}.$$

Clearly $\mathcal{M}^{[\perp]}$ is a linear subspace, but in general, it need not be a direct complement to $\mathcal{M}$, i.e., one can have $\mathcal{M}^{[\perp]} \cap \mathcal{M} \neq \{0\}$. However, for any given linear subspace $\mathcal{M}$ in a nondegenerate scalar product space $(\mathbf{C}^n,[.,.])$ the identity

$$\dim \mathcal{M} + \dim \mathcal{M}^{[\perp]} = n \tag{1.1}$$

is satisfied.

The linear subspace $\mathcal{M}^{\circ} = \mathcal{M} \cap \mathcal{M}^{[\perp]}$ is called the *isotropic part* of the linear subspace $\mathcal{M}$. A vector $x \in \mathcal{M}^{\circ}$, $x \neq 0$, is said to be an *isotropic vector* for the linear subspace $\mathcal{M}$.

A linear subspace $\mathcal{M}$ is *nondegenerate*, if the condition $x \in \mathcal{M}$, $[x, y] = 0$ for every $y \in \mathcal{M}$ implies $x = 0$. Obviously, $\mathcal{M}$ is nondegenerate if and only if $\mathcal{M}^{[\perp]}$ is a direct complement to $\mathcal{M}$, i.e., $\mathcal{M}^{\circ} = \mathcal{M}^{[\perp]} \cap \mathcal{M} = \{0\}$. It follows, that for nondegenerate linear subspaces $\mathcal{M}$ and only for them one can construct an $[.,.]$-*orthonormal basis*, i.e. a basis $x_1,\dots,x_s$, $s = \dim \mathcal{M}$, of $\mathcal{M}$ satisfying

$$[x_i, x_j] = \begin{cases} \pm 1 & \text{if } i=j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Furthermore, there exists a well-defined $[.,.]$-*orthogonal projection* $\mathcal{P}_{\mathcal{L}}$ onto a linear subspace $\mathcal{L}$ satisfying $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}}^2$, $\mathcal{R}(P_{\mathcal{L}}) = \mathcal{L}$ and $[\mathcal{P}_{\mathcal{L}}x, x] = [x, \mathcal{P}_{\mathcal{L}}x]$ for all $x \in \mathbf{C}^n$, if and only if $\mathcal{L}$ is nondegenerate. The following lemma is obvious.

1.2. Lemma. *The orthogonal direct sum of finitely many nondegenerate, positive, nonnegative, negative, nonpositive or neutral (respectively) linear subspaces is nondegenerate, positive, nonnegative, negative, nonpositive or neutral linear subspace.*

The correspondence $[.,.] = (H.,.)$ for an $n \times n$ *invertible Hermitean matrix* H is repeatedly used with a possible subscript. Here $(.,.)$ denotes the standard inner product of complex vectors

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i, \qquad (x, x) = \|x\|^2.$$

The *inertia of any Hermitean matrix* K is defined by $\operatorname{In} K = (p_K, q_K, n_K)$, where p_K, q_K and n_K, respectively, are the numbers of the positive, negative and zero eigenvalues of K, taking multiplicities into account.

For $[.,.] = (H.,.)$, we call the number $\kappa = \min\{p_H, q_H\}$ the *rank of the indefiniteness* of the space $(\mathbf{C}^n,[.,.])$.

1.3. Lemma. *For a Hermitean and invertible $n \times n$ matrix H the greatest possible dimension of a positive (of a nonnegative) linear subspace $\mathcal{L} \subset (\mathbf{C}^n,(H.,.))$ is equal to p_H and the greatest possible dimension of a negative (of a nonpositive) linear subspace $\mathcal{M} \subset (\mathbf{C}^n,(H.,.))$ is equal to q_H. Furthermore, for a neutral linear* *subspace* $\mathcal{N} \subset (\mathbf{C}^n,(H.,.))$ *the greatest possible dimension is equal to the rank of indefiniteness* κ.

A neutral linear subspace $\mathcal{M}$ is called *maximal neutral* if for any neutral linear subspace $\mathcal{M}' \supset \mathcal{M}$ we have $\mathcal{M}' = \mathcal{M}$. *Maximal positive*, *maximal nonnegative*, *maximal negative* and *maximal nonpositive* linear subspaces are defined similarly.

Next we recall some definitions and results concerning *linear operators* acting between indefinite scalar product spaces. We shall frequently identify operators with matrices in the usual way.

Let H_1, respectively H_2, be a Hermitean invertible $m \times m$, respectively $n \times n$, matrix. The H_1H_2*-adjoint* $A^{[*]}$ of an $m \times n$ matrix A is defined by the identity

$$(1.3) \qquad [Ax, y]_1 = [x, A^{[*]}y]_2$$

for all $x \in \mathbf{C}^n$ and $y \in \mathbf{C}^m$, where $[.,.]_j = (H_j.,.)$, $j = 1, 2$. This gives $A^{[*]} = H_2^{-1}A^*H_1$, where A^* denotes the usual *conjugate transpose* of A. A square matrix A is called H_1H_2*-self-adjoint* if $A = A^{[*]}$. Similarly other classes of matrices (e.g. H_1H_2*-unitary*) are defined. When $H_1 = H_2 = H$, term H*-self-adjoint* (instead of HH-self-adjoint) is used.

A linear subspace $\mathcal{L}$ is called *invariant under* A ($A \in \mathbf{C}^{n\times n}$), shortly A*-invariant*, if A maps the vectors in $\mathcal{L}$ into $\mathcal{L}$, i.e. if $A(\mathcal{L}) \subset \mathcal{L}$.

The *rank*, the *null-space* and the *range of* A are denoted by $r(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. For these the following connections are still valid

$$\mathcal{R}(A)^{[\perp]} = \mathcal{N}(A^{[*]}), \qquad \mathcal{N}(A)^{[\perp]} = \mathcal{R}(A^{[*]}).$$

For a square matrix A the *spectrum of* A (i.e. the set of *eigenvalues* of A) is denoted by $\sigma(A)$. The *principal subspace* of A belonging to the eigenvalue $\lambda \in \sigma(A)$ is defined by

$$\mathcal{S}_A(\lambda) = \bigcup_{i=1}^{\infty} \mathcal{N}((A - \lambda I)^i).$$

An eigenvalue λ of A is called *semisimple* if $\mathcal{S}_A(\lambda) = \mathcal{N}(A - \lambda I)$; otherwise λ is said to be a *nonsemisimple* eigenvalue of A.

1.4. Lemma. *Let A be an H-self-adjoint square matrix. If λ is an eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A. Moreover, if $\lambda, \mu \in \sigma(A)$ such that $\lambda \neq \bar{\mu}$, then $\mathcal{S}_A(\lambda)$ is H-orthogonal to $\mathcal{S}_A(\mu)$. Especially if $\lambda \in \sigma(A)$ and* $\operatorname{Im} \lambda \neq 0$ *then $\mathcal{S}_A(\lambda)$ is H-neutral.*

A similar result, the symmetry of the spectrum $\sigma(A)$ relative to the real axis R replaced by the symmetry relative to the unit circle, holds for H-unitary matrices as well.

1.5. Lemma. *Let U be an H-unitary matrix. If λ is an eigenvalue of U, then $\lambda^* = 1/\bar{\lambda}$ is also an eigenvalue of U. Moreover, if $\lambda, \mu \in \sigma(U)$ such that $\lambda \neq \mu^*$, then $\mathcal{S}_U(\lambda)$ is H-orthogonal to $\mathcal{S}_U(\mu)$. Especially if $\lambda \in \sigma(A)$ and $|\lambda| \neq 1$ then $\mathcal{S}_A(\lambda)$ is H-neutral.*

(For the proofs of the two above lemmas cf. e.g. Bognár [4], pp. 32–35.) If a complex matrix A is self-adjoint with respect to some indefinite scalar product then by Lemma 1.4 the spectrum $\sigma(A)$ of A lies symmetrically relative to the real axis in the complex plane. A more detailed way of describing the structure of an H-self-adjoint matrix A can be obtained through a *canonical form* of the pair (A, H). See e.g. Uhlig [24].

A pair (A_1, H_1) of matrices is said to be *unitary similar* to a pair (A_2, H_2) if there exists a *nonsingular* $n \times n$ matrix T such that

$$A_1 = T^{-1}A_2T \text{ and } H_1 = T^*H_2T.$$

This means that A_1 and A_2 are *similar*, H_1 and H_2 are *congruent*, and that these transformations can be obtained by a common matrix T.

Let now A be a square $(n \times n)$ H-self-adjoint matrix. We denote for the real eigenvalues λ_j of A the usual *Jordan blocks* by $J(\lambda_j)$. Further, for nonreal eigenvalues μ_k we define

$$C(\mu_k) = J(\mu_k) \oplus J(\overline{\mu}_k) = \begin{bmatrix} J(\mu_k) & 0 \\ 0 & J(\overline{\mu}_k) \end{bmatrix},$$

if $J(\mu_k)$ and $J(\overline{\mu}_k)$ are Jordan blocks of the same size for the eigenvalues μ_k and $\overline{\mu}_k$.

1.6. Lemma. *A square $n \times n$ matrix A is H-self-adjoint if and only if the pair (A, H) is unitary similar to a pair $(J, P_{e,J})$ where*

$$J = J(\lambda_1) \oplus \ldots \oplus J(\lambda_p) \oplus C(\lambda_{p+1}) \oplus \ldots \oplus C(\lambda_{p+q})$$

is a Jordan normal form of A and

$$P_{e,J} = e_1P_1 \oplus \ldots \oplus e_pP_p \oplus P_{p+1} \oplus \ldots \oplus P_{p+q}.$$

Here $0 \le p \le n$ and $0 \le q \le (n-p)/2$. Further, the vector $e = (e_1,\ldots,e_p)$ consists of an ordered set of signs $+1$ and -1 corresponding to the real Jordan blocks $J(\lambda_j)$ of J, and the matrices

$$P_i = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}, \qquad i = 1,\ldots, p+q,$$

are standard involutory permutation matrices of sizes equal to those of the blocks $J(\lambda_j)$ of J (for $j = 1,\ldots, p$) and the blocks $C(\lambda_k)$ of J (for $k = p+1,\ldots, p+q$).

For a proof of Lemma 1.6 we refer to Gohberg–Lancaster–Rodman [9], pp. 37–43.

We shall deal especially with matrices of the form $A^{[*]}A$. These are automatically square and H_2-self-adjoint (cf. paragraph following (1.3)).

There is a natural limitation for matrices of this kind, which was obtained by Bognár and Krámli [5] in a more general context for self-adjoint operators on *J-spaces* (*Krein spaces*) of infinite dimensions. Namely, a continuous self-adjoint operator A on a J-space $\mathcal{H}$ is of the form $A = C^{[*]}C$ for some continuous linear operator C if and only if for the intrinsic dimensions k^+ and k^- of a fundamental decomposition of $\mathcal{H}$ and for the A-intrinsic dimensions k_A^+ and k_A^- of an A-fundamental decomposition of $\mathcal{H}$ the inequalities

$$\text{(a)} \quad k_A^+ \le k^+ \text{ and} \qquad \text{(b)} \quad k_A^- \le k^-$$

are satisfied (cf. Bognár – Krámli [5], Theorem 1 and Theorems 2–4).

In our case conditions (a) and (b) can be equivalently stated for Hermitean matrices in terms of the inertia of matrices, i.e. the ordered triple of the numbers of the positive, the negative and the zero eigenvalues, respectively, as follows: Let (p_1,q_1,n_1) and (p_2,q_2,n_2) denote the inertias of the Hermitean matrices H and HA, respectively. A factorization $A = C^{[*]}C$ is possible with some matrix C if and only if we have $p_2 \le p_1$ and $q_2 \le q_1$.

2. A singular value decomposition with respect to indefinite scalar products

This section gives the definition of the *singular value decomposition* (*s.v.d.*) in the case that the underlying spaces are equipped with indefinite scalar products. The main result states a necessary and sufficient condition for the existence of such a s.v.d. in terms of the related matrix $A^{[*]}A$ and the linear subspace $\mathcal{R}(A)$ determined by A.

Let $[.,.]_1 = (H_1.,.)$ and $[.,.]_2 = (H_2.,.)$ be two indefinite scalar products on $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively.

2.1. Definition. *Any factorization for an $m \times n$ matrix $A \in \mathbf{C}^{m\times n}$ of the form*

$$A = U D V^{-1},$$

where D is a real and diagonal $m \times n$ matrix with nonnegative entries and the column vectors $u_1,\dots,u_m$ and $v_1,\dots,v_n$ of $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{n\times n}$ satisfy

$$[u_i, u_j]_1 = \pm\, \delta_{ij} \ \textit{and} \quad [v_i, v_j]_2 = \pm\, \delta_{ij},$$

respectively, is called a s.v.d. of A with respect to $[.,.]_1$ and $[.,.]_2$, or, in short, an H_1H_2-s.v.d. of the matrix A.

The diagonal elements d_i of the matrix D are referred to as the *singular values of A with respect to* $[.,.]_1$ *and* $[.,.]_2$, the corresponding column vectors u_i in U (respectively v_i in V) are the *left* (respectively *right*) *singular vectors of A with respect to* $[.,.]_1$ *and* $[.,.]_2$. Using matrix notations we have

(2.1) $$U^{-1}AV = D,\ U^*H_1U = S_1 \text{ and } V^*H_2V = S_2,$$

where S_i is a diagonal square matrix with diagonal entries ±1, these appearing according to the inertia of the matrix H_i, $i = 1, 2$. Note that the orthogonality conditions in Definition 2.1 guarantee the nonsingularity of U and V. Thus, we are looking for a simple canonical form for three matrices simultaneously by using two transformations.

2.2. Remark. It is not possible to generalize the factorization of Definition 2.1 by allowing the diagonal entries d_i in D to be any complex numbers. This is seen by multiplying in such a factorization the possibly complex matrix D (and then also e.g. U) by another diagonal complex matrix having all the diagonal entries of unit moduli.

A complex square matrix A will be called *r-diagonable* if it is similar to a real diagonal matrix, i.e., if all the eigenvalues of A are real ($\sigma(A) \subset \mathrm{R}$) and semisimple ($S_A(\lambda_i) = \mathcal{N}(A - \lambda_i I),\ \lambda_i \in \sigma(A)$). Now, if A is also H-self-adjoint it follows from Lemmas 1.6 and 1.4 that all the eigenspaces of A are nondegenerate (cf. (1.2)) and that the eigenspaces corresponding to the different eigenvalues of A are H-orthogonal to each other. The result of the next lemma is well-known. We shall later prove a more general result as Lemma 6.13.

2.3. Lemma. *The range $\mathcal{R}(A)$ of A is H_1-nondegenerate if and only if $\mathcal{N}(A^{[*]}A) = \mathcal{N}(A)$ or equivalently $r(A^*H_1A) = r(A)$.*

2.4. Theorem. *An $m \times n$ matrix A ($m \geq n$) has a s.v.d. with respect to $[.,.]_1$ and $[.,.]_2$, if and only if the matrix $A^{[*]}A$ is r-diagonable and $\mathcal{R}(A)$ is a nondegenerate linear subspace of $(\mathbf{C}^m,[.,.]_1)$.*

Proof. Suppose that A satisfies the conditions of the theorem and write, by the observations just made above, $A^{[*]}A = V\Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, λ_i real, and $V^*H_2V = S_2$, where $S_2 = \mathrm{diag}(s_1^{(2)},\dots, s_n^{(2)})$, $s_i^{(2)} = +1$ or -1.

Choose r columns of V (with $r = \mathrm{rank}\ A$) corresponding to $\lambda_i \neq 0$, say $v_1,\dots,v_r$, such that the vectors

(2.2) $$\tilde{u}_i = Av_i, \quad \text{for } i = 1,\dots,r$$

become linearly independent. This is possible by the nondegeneracy of $\mathcal{R}(A)$. For $i, j = 1,\dots,r$ we have

$$\begin{aligned} \tilde{u}_i^{\,*}H_1\tilde{u}_j &= v_i^*A^*H_1Av_j \\ &= v_i^*H_2A^{[*]}Av_j \\ &= \lambda_j v_i^*H_2v_j \\ &= \begin{cases} \pm\lambda_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

Define $u_i = \tilde{u}_i/|\lambda_i|^{1/2}$ for $i = 1,\dots,r$. By the nondegeneracy of $\mathcal{R}(A)$ and hence of $\mathcal{R}(A)^{[\perp]}$, extend with $u_{r+1},\dots,u_m$ the set $\{u_1,\dots, u_r\}$ to an H_1-orthonormal basis of $\mathbf{C}^m$ (see (1.1) and (1.2)). By relation (2.2) the identity

$$A\,v_i = u_i|\lambda_i|^{1/2}, \qquad i = 1,\dots,r \tag{2.3}$$

holds. Next, note that by Lemma 2.3 $A^{[*]}Ax = 0$ implies $Ax = 0$, since $\mathcal{R}(A)$ is nondegenerate, so that the identity (2.3) holds in fact for every index $i = 1,\dots,n$. Using matrices this gives $A = UDV^{-1}$, where D is $m \times n$ and contains $D_1 = \mathrm{diag}(|\lambda_1|^{1/2},\dots,|\lambda_n|^{1/2})$ in the upper part and zeros elsewhere.

Conversely, if $A = UDV^{-1}$ is a s.v.d. of A with respect to $[.,.]_1$ and $[.,.]_2$, then

$$V^{-1}(A^{[*]}A)\,V = S_2 D^* S_1 D,$$

where S_1 and S_2 are as in (2.1). Thus, the matrix $A^{[*]}A$ is r-diagonable. The representation $A = UDV^{-1}$ readily shows, that $\mathcal{R}(A)$ has an H_1-orthonormal basis and hence is nondegenerate. This completes the proof.

There are many direct consequences of Theorem 2.4, analogous to the ordinary singular value decomposition, holding for the H_1H_2-s.v.d. of Definition 2.1. For example, the rank of A, $r(A)$, equals the number of the nonzero singular values in any s.v.d. of the matrix A.

The next example however shows, that in general not all matrices admit an H_1H_2-s.v.d., if the underlying scalar products are indefinite.

2.5. Example. Let $A = J(\alpha)$ be a Jordan block with eigenvalue $\alpha \in \mathrm{C}$. Define $H_1 = H_2 = P$, where P is a permutation matrix of the same size as A, as described in Lemma 1.6. Now one has $A^{[*]}A = PJ^*PJ = \overline{J}J$ and thus

$$A^{[*]}A = \begin{bmatrix} \overline{\alpha} & 1 & 0 & \cdots & 0 \\ & \overline{\alpha} & 1 & 0 & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & \overline{\alpha} \end{bmatrix} \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ & \alpha & 1 & 0 & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & \alpha \end{bmatrix} = \begin{bmatrix} |\alpha|^2 & 2\mathrm{Re}\,\alpha & 1 & 0 & \cdots & 0 \\ & |\alpha|^2 & 2\mathrm{Re}\,\alpha & 1 & 0 & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 1 \\ & & & & \ddots & 2\mathrm{Re}\,\alpha \\ 0 & & & & & |\alpha|^2 \end{bmatrix}.$$

This matrix is not r-diagonable, unless it is a scalar or of the order 2×2 with $i\,\alpha \in \mathrm{R}$.

The matrix $A^{[*]}A$ can be r-diagonable, i.e. $\sigma(A^{[*]}A) \subset \mathrm{R}$ and $S_{A^{[*]}A}(\lambda) = \mathcal{N}(A^{[*]}A - \lambda I)$ for all $\lambda_i \in \sigma(A^{[*]}A)$, but A need not have an H_1H_2-s.v.d.

2.6. Example. Let A be a matrix, the columns of which span a neutral subspace of $(\mathbf{C}^m,[.,.]_1)$. Then $A^{[*]}A = 0$ and, thus, it is trivially r-diagonable. However, the column space $\mathcal{R}(A)$ of A is degenerate as a neutral subspace. So A does not have a s.v.d. with respect to $[.,.]_1$, $[.,.]_2$ for any $[.,.]_2$.

Obviously, all the linear subspaces spanned by the left (respectively right) singular vectors corresponding to a fixed singular value of A, called *singular (vector) subspaces* later on, are nondegenerate subspaces of $(\mathbf{C}^m,[.,.]_1)$ (respectively of $(\mathbf{C}^n,[.,.]_2)$)

(cf. (1.2)). The following proposition is also noted and easily established.

2.7. Proposition. *The singular values and the related singular subspaces of A with respect to* $[.,.]_1$ *and* $[.,.]_2$ *are uniquely determined by A,* $[.,.]_1$ *and* $[.,.]_2$.

From Theorem 2.4 we get further

2.8. Corollary. *If there exists a s.v.d. with respect to* $[.,.]_1$ *and* $[.,.]_2$ *for A, then the same is true for* $A^{[*]}$. *Especially, in this case both of the matrices* $A^{[*]}A$ *and* $AA^{[*]}$ *are r-diagonable and the subspaces* $\mathcal{R}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A^{[*]})$ *and* $\mathcal{N}(A^{[*]})$ *are simultaneously nondegenerate.*

Proof. The first assertion follows by noting that, if $A = UDV^{-1}$ then we have $A^{[*]} = V_2 D^* U_2^{-1}$, where $U_2 = US_1$ and $V_2 = VS_2$ and S_1, S_2 are as in (2.1). The latter part of the corollary is then a direct consequence of these representations.

2.9. Corollary. *If the scalar product* $[.,.]_2$ *is definite, then every matrix* $A \in \mathbf{C}^{m\times n}$, *which satisfies one of the conditions*

(1) $AA^{[*]}$ *is r-diagonable,*

(2) $\mathcal{R}(A)$ *is nondegenerate,*

(3) $\mathcal{N}(A^{[*]})$ *is nondegenerate*

with respect to the other scalar product $[.,.]_1$, *has a s.v.d. with respect to these scalar products. If on the other hand* $[.,.]_1$ *is the definite scalar product the conditions should be correspondingly modified.*

Proof. Accordingly, suppose that $[.,.]_2$ is definite. Then automatically $\mathcal{R}(A^{[*]})$ and $\mathcal{N}(A)$ are nondegenerate as definite linear subspaces. Furthermore, $A^{[*]}A$ is r-diagonable as an H_2-self-adjoint matrix, when H_2 is positive or negative definite (cf. e.g. Lemma 6.8 in Section 6).

Now, any of the conditions (1) – (3) implies the existence of s.v.d. for A or $A^{[*]}$ by Theorem 2.4 and so for A by Corollary 2.8.

2.10. Corollary. *If both of the scalar products* $[.,.]_1$ *and* $[.,.]_2$ *are definite, then every matrix in* $\mathbf{C}^{m\times n}$ *has a singular value decomposition with respect to* $[.,.]_1$ *and* $[.,.]_2$.

2.11. Remark. As can be seen from the proof of Corollary 2.8 (see also the proof of Theorem 2.4) if A has a s.v.d. with respect to $[.,.]_1$ and $[.,.]_2$ the squares of the singular values d_i of A, equivalently of $A^{[*]}$, satisfy $d_i^2 = |\lambda_i|$, where $\lambda_i \in \sigma(A^{[*]}A)$, equivalently $\lambda_i \in \sigma(AA^{[*]})$. Furthermore, for the sign of $\lambda_i \neq 0$, $\lambda_i \in \sigma(A^{[*]}A) \subset \mathbf{R}$, (or of $\lambda_i \neq 0$, $\lambda_i \in \sigma(AA^{[*]}) \subset \mathbf{R}$) we have the identity $\operatorname{sgn} \lambda_i = [u_i, u_i]_1 [v_i, v_i]_2$, where u_i (respectively v_i) is the left (respectively the right) singular vector of A corresponding to the singular value d_i, $d_i^2 = |\lambda_i|$, of A. The singular vectors $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are the eigenvectors of $AA^{[*]}$ and $A^{[*]}A$, respectively.

2.12. Remark. In the statistical literature factorizations called *weighted singular value decompositions* (cf. e.g. Rao – Mitra [23], p. 7) are not equivalent to those of Corollary 2.10. In fact, they do not share in any direct way the common properties of the canonical singular value decomposition, which in turn are readily available for the *definite singular value decompositions* of Corollary 2.10, since these are directly based on the scalar products of the underlying spaces. Many properties, analogous to those known for the canonical s.v.d., the canonical singular values and the related singular vectors, for any other definite s.v.d. of matrices, for their singular values and vectors can still be derived. The author has presented these results in the unpublished thesis in his Statistics studies "Properties and statistical applications of a general singular value decomposition of matrices" (in Finnish).

3. A class of matrices admitting an H-singular value decomposition

For simplicity let $[.,.] = (H.,.)$ define an indefinite scalar product on $\mathbf{C}^n$. Our aim is to introduce a class of such square matrices for which a singular value decomposition with respect to $[.,.]$ always exists.

One defines a matrix B to be a *Pesonen matrix*, if it is H-self-adjoint and the relations

$$[x, x] = 0 \quad \text{and} \quad [Bx, x] = 0$$

do not hold simultaneously for any $x \neq 0$, $x \in \mathbf{C}^n$. Pesonen matrices were first investigated by Pesonen [22]. (See also Kühne [18], Bognár [4].)

A matrix $A \in \mathbf{C}^{n\times n}$ will be called a *sub-Pesonen matrix*, if $\mathcal{R}(A)$ is nondegenerate, and the matrix $B = A^{[*]}A$ is a Pesonen matrix. Define

$$\wp_H = \{A \in \mathbf{C}^{n\times n} \mid A \text{ or } A^{[*]} \text{ is sub-Pesonen}\}.$$

The scalar

$$C(A, B) = \inf_{\|x\|=1} [(x^*Ax)^2 + (x^*Bx)^2]$$

is sometimes called the *Crawford number* of the *symmetric pencil* $A - \lambda B$. So especially, if $C(H, HB) > 0$ for $B = A^{[*]}A$ and $\mathcal{R}(A)$ is nondegenerate then we have $A \in \wp_H$.

3.1. Lemma. *Every A-invariant subspace $\mathcal{K}$ of a Pesonen matrix A is nondegenerate.*

Proof. Denote by $\mathcal{K}^0 = \mathcal{K} \cap \mathcal{K}^{[\perp]}$ the isotropic part of the linear subspace $\mathcal{K}$. For $x \in \mathcal{K}^0$ one has $[x, x] = 0$ and $[Ax, x] = 0$, since $\mathcal{K}$ is A-invariant. This implies $x = 0$. Thus, $\mathcal{K}^0 = \{0\}$ and $\mathcal{K}$ is nondegenerate.

3.2. Proposition. *If $A \in \wp_H$ then A has a singular value decomposition with respect to $(H.,.)$.*

Proof. By Theorem 2.4 and Corollary 2.8 it is enough to verify that the eigenvalues of $A^{[*]}A$ are real and semisimple if A, for example, is sub-Pesonen.

Since principal subspaces of $A^{[*]}A$ are invariant and those of them corresponding to a nonreal eigenvalue of $A^{[*]}A$ are neutral (cf. Lemma 1.4), the spectrum $\sigma(A^{[*]}A)$ of $A^{[*]}A$ must be real by Lemma 3.1.

Suppose then that $\lambda \in \sigma(A^{[*]}A)$ is real but nonsemisimple. Then, for some vector $x \neq 0$ and some integer $r \geq 1$ we have

$$x_1 = (A^{[*]}A - \lambda I)^r x \neq 0 \text{ and } (A^{[*]}A - \lambda I)^{r+1} x = 0.$$

This implies for x_1

$$[x_1, x_1] = [(A^{[*]}A - \lambda I)^{2r} x, x] = 0.$$

Thus, x_1 is a nonzero neutral eigenvector of $A^{[*]}A$. This contradicts Lemma 3.1. Hence $\lambda \in \sigma(A^{[*]}A)$ must be semisimple. The claim is proved.

It should be noted that the nondegenerateness condition included in the definition of $\wp_H$ is essential for Proposition 3.2 in the sense that $A^{[*]}A$ can be a Pesonen matrix even in the case of a matrix A with a degenerate $\mathcal{R}(A)$ (and thus A without any H-s.v.d.).

From the proof of Proposition 3.2 one can see that the stronger claim of A itself to be a Pesonen matrix always guarantees the existence of an H-s.v.d. for A. However, all Pesonen matrices are not contained into the class $\wp_H$ introduced above.

4. Another class of matrices with an H-singular value decomposition

Another class of matrices in $\mathbf{C}^{m \times n}$ having a s.v.d. with respect to two indefinite scalar products, one of which is supposed to be "properly" indefinite (i.e. with rank of indefiniteness $\kappa > 0$), will be given.

Let $[.,.]$ and $[.,.]'$ be two scalar products on $\mathbf{C}^n$. Assume that $[.,.]$ is properly indefinite. Then the following fundamental lemma holds for the *semiboundedness* of the scalar product $[.,.]'$ with respect to $[.,.]$.

4.1. Lemma. *If* $[x, x] = 0$ *implies* $[x, x]' \geq 0$, *then there exists a real scalar* $\mu \in \mathrm{R}$ *such that*

$$[x, x]' \geq \mu [x, x]$$

holds for every $x \in \mathbf{C}^n$. *The scalar* μ *is given by*

$$\mu = \inf_{[y,y]=1} [y, y]'.$$

(For this compare Azizov – Iokhvidov [2], Corollary 1.1.36, Bognár [4], Theorem II.6.2 or Iokhvidov – Kreĭn – Langer [12], Lemma II.6.1.)

Denote by $\mathcal{N}_n^0$ (respectively by $\mathcal{N}_m^0$) the set of all neutral vectors of $(\mathbf{C}^n, [.,.]_2)$ (respectively of $(\mathbf{C}^m, [.,.]_1)$. Let $T \in \mathbf{C}^{m \times n}$ be such that

$$(4.1) \qquad Tx \in \mathcal{N}_m^0 \quad \text{for every } x \in \mathcal{N}_n^0,$$

that is $[x, x]_2 = 0$ implies $[Tx, Tx]_1 = 0$.

4.2. Proposition. *Suppose* $[.,.]_2$ *is properly indefinite. If* T *satisfies* (4.1) *and* $\mathcal{R}(T) \not\subset \mathcal{N}_m^0$, *then* T *has a s.v.d. with respect to* $[.,.]_1$ *and* $[.,.]_2$.

Proof. Applying Lemma 4.1 to scalar products $[x, y]_2$ and $[x, y]_2' = [Tx, Ty]_1$ and to $[x, y]_2$ and $-[x, y]_2'$ yields

$$[x, x]_2' \geq \mu' [x, x]_2 \text{ and } -[x, x]_2' \geq \mu'' [x, x]_2$$

with some $\mu', \mu'' \in \mathbb{R}$ and for every $x \in \mathbb{C}^n$. Thus,

$$0 \geq (\mu' + \mu'')[x, x]_2$$

holds for all x. The indefiniteness of $[.,.]_2$ implies $\mu' + \mu'' = 0$. Hence we have

$$[x, x]_2' = \mu' [x, x]_2$$

for every x and, further, by the polarization formula (cf. e.g. Bognár [4], p. 4), one has in fact

$$[Tx, Ty]_1 = \mu' [x, y]_2 \tag{4.2}$$

for every $x, y \in \mathbb{C}^n$. Assumption $\mathcal{R}(T) \not\subset \mathcal{N}_m^0$ implies $\mu' \neq 0$. Thus, the identity (4.2) shows that T satisfies the conditions of Theorem 2.4, i.e., T has a singular value decomposition with respect to $[.,.]_1$ and $[.,.]_2$.

The s.v.d. of T in Proposition 4.2 can be explicitly calculated. For example, all the nonzero H_1H_2-singular values of T are equal.

5. Connection with generalized singular value decomposition

Next we briefly consider the so-called *generalized singular value decomposition* (g.s.v.d.). This canonical form of a matrix pair (A, B) is due to Van Loan (cf. Van Loan [25]). The connection of the H_1H_2-s.v.d. in Theorem 2.4 to g.s.v.d. is illustrated. In this section degenerate scalar product spaces will also be accepted as indicated later on. We state below without proof the g.s.v.d.-theorem (cf. e.g. Golub – Van Loan [11], Theorem 8.6.4).

5.1. Theorem. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{p\times n}$. *If* $m \geq n$ *then there exist unitary matrices* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{p\times p}$ *and an invertible matrix* $X \in \mathbb{C}^{n\times n}$ *such that*

$$U^*AX = D_A = \operatorname{diag}(a_1, a_2, \ldots, a_n), \qquad a_i \geq 0,$$

and

$$V^*BX = D_B = \operatorname{diag}(b_1, b_2, \ldots, b_q), \qquad q = \min\{p, n\},$$

where $b_1 \geq b_2 \geq \ldots b_r > b_{r+1} = \ldots = b_q = 0$, $r = \operatorname{rank} B$.

We note that here the condition $m \geq n$ is essential in the sense that in the opposite case $m < n$ simple examples can be derived such that the factorization of Theorem 5.1 is not available (cf. Van Loan [25] or Paige – Saunders [21].) The relations in Theorem 5.1 can be expressed analogously in the form

$$(5.1)\qquad \begin{aligned} U^*AY &= D = \operatorname{diag}(d_1,\dots,d_n), & d_i \geq 0 \quad \text{and} \\ V^*BY &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, & r = \operatorname{rank} B, \end{aligned}$$

where $Y = X\,\Upsilon_B^{-1}$, $D = D_A \Upsilon_B^{-1}$ and $\Upsilon_B = D_{B,r} \oplus I_{n-r}$, $D_{B,r}$ being the $r \times r$ left-hand upper corner of D_B.

However, identities in (5.1) can be interpreted as a "singular value decomposition" for A with respect to $H_1 = I_m$ and $H_2 = B^*B$ satisfying

$$(5.2)\qquad A = UDY^{-1}, \quad Y^*H_2Y = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Conversely, suppose that for a given matrix A there exists a s.v.d. of the form

$$(5.3)\qquad A = UDY^{-1}, \quad Y^*H_2Y = D_{r(B)},$$

where $D_{r(B)} = \operatorname{diag}(j_1,\dots,j_n)$ with $j_i = 1$ or $= 0$. In fact, this can always be achieved with the aid of Corollary 2.10, for example, by occasionally extending $H_2 = B^*B$ to a positive definite matrix in a suitable way.

Using the relations in (5.3) one can easily prove the g.s.v.d.-theorem, stated above.

If $m \geq n$ then (and only then), by permuting the columns of Y and the first n columns of U, (5.3) can be expressed in the form (5.2). Let us define $V = [BY_1;\, V_2]$, where Y_1 contains the first r columns of Y and V_2 is any $p \times (p-r)$ submatrix with orthonormal columns and in addition satisfies $B^*V_2 = 0$. Then $V^*V = I_p$, and (5.2) yields

$$U^*AY = D = D_A, \qquad V^*BY = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = D_B,$$

i.e., a g.s.v.d. of the pair (A, B).

Note that, contrary to g.s.v.d., there are no restrictions for the numbers m, n or p in the s.v.d. of (5.3).

5.2. Remark. Van Loan introduced in his doctoral thesis (1973) the numbers a_i/b_i, $b_i \neq 0$, $i = 1,\dots,r$, and these are usually called the *generalized singular values* of the pair (A, B) (cf. Golub – Van Loan [11], p. 319). As seen above, these numbers are just the singular values of A with respect to I_m and B^*B. Thus, the previous discussion gives an equivalent description of g.s.v.d. as a s.v.d. with respect to one semidefinite (degenerate) scalar product. (For an efficient algorithm and applications of g.s.v.d. to constrained least squares, for example, the reader is referred to the book of Golub and Van Loan [11].)

6. Perturbations of singular value decompositions in a space with an indefinite metric

In this section we take $[.,.] = (H.,.)$ and study the perturbations of square matrices. In particular, the effect of perturbations to the stability of singular value decomposition in an indefinite scalar product space is considered. Some of the results in this section have connections with the stability results known for H-self-adjoint matrices (cf. e.g. Gohberg – Lancaster – Rodman [9]).

6.1. Stability of the H-s.v.d. under general perturbations

An H-s.v.d. is called stable if under small perturbations of the elements of a matrix A having a singular value decomposition with respect to $[.,.] = (H.,.)$ the perturbed matrix C also has this representation with respect to $[.,.]$. More formally, this situation is expressed by saying that A has an *H-stable H-s.v.d.* If also H is allowed to vary and in the perturbed pair (C, K) the matrix C has a K-s.v.d. for every pair (C, K), K Hermitean and nonsingular, sufficiently close to the pair (A, H), the matrix A is simply said to have a *stable H-s.v.d.*

Note that, if the scalar product $[.,.]$ under consideration is definite, the perturbation problem just posed is of no interest, since in that case every matrix trivially has an H-stable H-s.v.d. and also a stable H-s.v.d. (cf. Corollary 2.10).

We shall show that not all matrices having a singular value decomposition with respect to $[.,.]$ have a stable s.v.d. in the sense indicated above.

The aim of this section is to derive the class of all matrices having a stable (respectively H-stable) H-s.v.d. The answer to this question is roughly found to be the following one: a matrix $A \in \mathbf{C}^{n\times n}$ has a stable (H-stable) H-s.v.d. exactly if A or its H-adjoint $A^{[*]}$ as a linear mapping locally looks like a linear transformation defined on an ordinary inner product space, i.e., on a vector space equipped with a definite scalar product. This localization can be achieved with the aid of spectral subspaces.

In particular, we shall confine our attention to the behaviour of H-singular values and the corresponding singular subspaces under perturbations of the fixed unperturbed matrix A. For any set $\Omega \subset \mathbf{C}$ and any $A \in \mathbf{C}^{n\times n}$ define

$$\mathcal{S}_A(\Omega) = \operatorname*{span}_{\lambda \in \Omega \cap \sigma(A)} \mathcal{S}_A(\lambda).$$

Let us introduce the following definition.

6.1. Definition. *Let $\mathfrak{I} \subset \mathrm{R}$ be an open set of the real line R in the complex plane $\mathbf{C}$. A matrix $A \in \mathbf{C}^{n\times n}$ is said to have an H-s.v.d. with respect to $\mathfrak{I}$, shortly an $\mathfrak{I}$-s.v.d., if the preimage $f^{-1}\mathfrak{I}$ in R of the set $\mathfrak{I}$ under $f: \mathrm{R} \to \mathrm{R}$, $x \to |x|^{1/2}$, contains only semisimple eigenvalues of $A^{[*]}A$ and for the restriction $A|_{\mathcal{S}_{A^{[*]}A}(f^{-1}(\mathfrak{I}))}$ the image $\mathcal{R}(A|_{\mathcal{S}_{A^{[*]}A}(f^{-1}(\mathfrak{I}))})$ is a nondegenerate linear subspace with respect to $(H.,.)$.*

Accordingly, A is said to have an *$\mathfrak{I}$-stable ($\mathfrak{I}_H$-stable) H-s.v.d.*, if matrices C, close enough to A, have a K-s.v.d. with respect to $\mathfrak{I}$, for every Hermitean and nonsingular K sufficiently close to H (respectively have an H-s.v.d. with respect to $\mathfrak{I}$) and the number of the K-singular values (respectively the number of the H-singular values) in $\mathfrak{I}$, taking multiplicities into account, is constant.

The topology we shall use in the spaces of matrices is the norm topology defined by any of the equivalent norms on $\mathbf{C}^{m\times n}$.

The following result is an immediate consequence of the definitions introduced above.

6.2. Corollary. *The set of all matrices having a stable (respectively an H-stable) H-s.v.d. is open in the topological space of all $n \times n$ matrices.*

Before going into some further details in the study of the perturbation problem suggested above, we shall consider the *Gram operator of a subspace.* Suppose $\mathcal{L}$ is an arbitrary linear subspace of the space $(\mathbf{C}^n,[.,.])$. Obviously, by restricting the indefinite scalar product $[.,.]$ to $\mathcal{L}\times\mathcal{L}$ we get a new scalar product space $(\mathcal{L},[.,.]\,|_{\mathcal{L}\times\mathcal{L}})$. By Riesz's lemma, there exists a unique linear operator $G_{\mathcal{L}}:\mathcal{L}\to\mathcal{L}$ such that $[x,y]=(G_{\mathcal{L}}x,y)$ holds for every $x,y\in\mathcal{L}$. The operator $G_{\mathcal{L}}:\mathcal{L}\to\mathcal{L}$ is called the *Gram operator of the linear subspace $\mathcal{L}$.*

If we denote by $P_{\mathcal{L}}$ the orthogonal projection (in the usual sense) onto $\mathcal{L}$, it is obvious that we have $G_{\mathcal{L}}=P_{\mathcal{L}}H\,|_{\mathcal{L}}=P_{\mathcal{L}}HP_{\mathcal{L}}\,|_{\mathcal{L}}$, i.e., $G_{\mathcal{L}}$ is obtained by a compression of H to the linear subspace $\mathcal{L}$.

Operator $G_{\mathcal{L}}$ is Hermitean. Let $G_{\mathcal{L}}=\sum_{i=1}^{l}\lambda_i P_i$ be the spectral resolution of $G_{\mathcal{L}}$. Define $\mathcal{L}^-=P_{\mathcal{L}}^-\mathcal{L}$, $\mathcal{L}^+=P_{\mathcal{L}}^+\mathcal{L}$ and $\mathcal{L}^0=P_{\mathcal{L}}^0\mathcal{L}$, where

$$P_{\mathcal{L}}^-=\sum_{\lambda_i<0}P_i,\quad P_{\mathcal{L}}^+=\sum_{\lambda_j>0}P_j \ \text{ and } \ P_{\mathcal{L}}^0=I-P_{\mathcal{L}}^--P_{\mathcal{L}}^+.$$

Here I denotes the identity operator on $\mathcal{L}$. The following lemma is well-known (cf. e.g. Azizov – Iokhvidov [2], Theorem 1.6.4) and shows how $G_{\mathcal{L}}$ describes the structure of the subspace $\mathcal{L}$.

6.3. Lemma. *In the orthogonal direct sum $\mathcal{L}=\mathcal{L}^+\oplus\mathcal{L}^-\oplus\mathcal{L}^0$, $\mathcal{L}^+$ is H-positive, $\mathcal{L}^-$ is H-negative and $\mathcal{L}^0$ is the isotropic part of $\mathcal{L}$. These subspaces are H-orthogonal as well.*

Lemma 6.3 is characteristic, since it can be verified that in any decomposition of the linear subspace $\mathcal{L}=\mathcal{L}'_+[+]\mathcal{L}'_-[+]\mathcal{L}'_0$ into an H-orthogonal direct sum of an H-positive $\mathcal{L}'_+$, an H-negative $\mathcal{L}'_-$ and an H-neutral linear subspace $\mathcal{L}'_0$, the dimensions of these subspaces coincide with the dimensions of the subspaces $\mathcal{L}^+$, $\mathcal{L}^-$ and $\mathcal{L}^0$, respectively, appearing in Lemma 6.3. These dimensions are then given by the inertia of the corresponding Gram operator $G_{\mathcal{L}}$. The ordered triple $\operatorname{In}\mathcal{L}=(\dim\mathcal{L}^+,\dim\mathcal{L}^-,\dim\mathcal{L}^0)$ is called the *inertia index of the linear subspace $\mathcal{L}$*. The number $\kappa_{\mathcal{L}}=\min\{\dim\mathcal{L}^+,\dim\mathcal{L}^-\}$ is said to be the *rank of indefiniteness of the linear subspace $\mathcal{L}$*.

6.4. Remark. It is easily checked that $\operatorname{In}\mathcal{L}=\operatorname{In}G_{\mathcal{L}}$ equals to the difference $\operatorname{In}(A^*HA)-(0,0,\operatorname{codim}\mathcal{L})$ where A is any square matrix satisfying $\mathcal{R}(A)=\mathcal{L}$.

6.5. Lemma. *Suppose* $\mathcal{L}, \mathcal{M} \subset (\mathbf{C}^n,[.,.])$ *are linear subspaces such that* $\mathcal{L} \cap \mathcal{M} = \{0\}$ *and* $\mathcal{L}\,[\perp]\,\mathcal{M}$. *Then we have* $\mathrm{In}(\mathcal{L} \oplus \mathcal{M}) = \mathrm{In}\,\mathcal{L} + \mathrm{In}\,\mathcal{M}$ *and* $\mathrm{In}\,G_{\mathcal{L}\oplus\mathcal{M}} = \mathrm{In}\,G_{\mathcal{L}} + \mathrm{In}\,G_{\mathcal{M}}$.

Proof. Since $\mathcal{L}\,[\perp]\,\mathcal{M}$ the first statement $\mathrm{In}(\mathcal{L} \oplus \mathcal{M}) = \mathrm{In}\,\mathcal{L} + \mathrm{In}\,\mathcal{M}$ follows directly from Lemma 1.2 by the condition $\mathcal{L} \cap \mathcal{M} = \{0\}$. For the second statement $\mathrm{In}\,G_{\mathcal{L}\oplus\mathcal{M}} = \mathrm{In}\,G_{\mathcal{L}} + \mathrm{In}\,G_{\mathcal{M}}$ we use the identity $\mathrm{In}\,G_{\mathcal{K}} = \mathrm{In}\,\mathcal{K}$.

Lemma 6.5 can be extended in a straightforward manner into the direct H-orthogonal sum of finitely many linear subspaces.

We shall next consider the stability of nondegenerateness of the range space $\mathcal{R}(A)$ of a matrix A under perturbations of A and H, respectively. Note that, if H is nonsingular then also matrices close enough to H are nonsingular. This can be seen, for example, by the continuity of the ordinary singular values with respect to the matrix elements.

6.6. Theorem. *Let the range* $\mathcal{R}(A)$ *be a nondegenerate subspace of* $(\mathbf{C}^n,(H.,.))$. *Then under sufficiently small perturbations of A and H, respectively, the range* $\mathcal{R}(B)$ *of B, the perturbed A, remains nondegenerate with respect to* $(K.,.)$, *the perturbed* $(H.,.)$, *K Hermitean and nonsingular, if and only if the orthogonal companion* $\mathcal{R}(A)^{[\perp]}$ *of* $\mathcal{R}(A)$ *is a definite linear subspace with respect to* $(H.,.)$. *The same condition is necessary and sufficient for* $\mathcal{R}(B)$, *B close enough to A, to be nondegenerate in the case where* $[.,.]$ *is not allowed to be perturbed at all.*

Proof. If $\mathcal{R}(A)^{[\perp]}$ were not a definite linear subspace of $(\mathbf{C}^n,(H.,.))$ then we could find a nonzero neutral vector $z_0 \in \mathcal{R}(A)^{[\perp]}$ (cf. Lemma 1.1) and a nonzero vector $y_0 \in \mathcal{N}(A)$. Define

$$A(\xi) = A + \xi\, z_0 y_0^*,$$

$\xi \in \mathbf{C}$. Then we have $r(A(\xi)) = r(A) + 1$ for any $\xi \neq 0$ and $A(\xi) \to A$ as $\xi \to 0$. However, $A(\xi)^* H A(\xi) = A^* H A$ and thus, $\mathcal{R}(A(\xi))$ is a degenerate subspace of $(\mathbf{C}^n,(H.,.))$ for every $\xi \neq 0$. This shows, in both cases, the necessity of the condition $\mathcal{R}(A)^{[\perp]}$ to be definite with respect to $(H.,.)$.

Suppose conversely, that $\mathcal{R}(A)^{[\perp]}$ is a definite linear subspace with respect to $(H.,.)$. Let us write $B = A + E$. We understand E to be a perturbation of A. Denote by P_{A*} the orthogonal projection (in the usual sense) onto $\mathcal{R}(A^*)$ and write

$$A + E = (A + E P_{A*}) + E(I - P_{A*}) = A' + E'.$$

Then $\mathcal{R}(A'^*) \perp \mathcal{R}(E'^*)$ and hence, by writing full rank decompositions (say singular value decompositions) for A' and E', for example, one sees that $r(A' + E') = r[A'; E']$.

One must show $\mathcal{R}(B) = \mathcal{R}[A'; E']$ to be nondegenerate with respect to $(K.,.)$ for every matrix $B = A + E$ and K, K Hermitean and nonsingular, sufficiently close to A and H, respectively. For this, it is enough to prove that $\mathcal{R}(A')$ is nondegenerate and its K-orthogonal companion $\mathcal{R}(A')^{[\perp]}$ is a definite subspace with respect to

$(K.,.)$, whenever $B \in \mathcal{U}_A$ and $K \in \mathcal{U}_H$, K Hermitean and nonsingular, for some neighbourhoods $\mathcal{U}_A$ and $\mathcal{U}_H$ of A and H, respectively. Indeed, denote by $\mathcal{P}_K$ the K-orthogonal projection onto $\mathcal{R}(A')$. (Such a $\mathcal{P}_K$ exists, if $\mathcal{R}(A')$ is K-nondegenerate). By writing $E' = \mathcal{P}_K E' + (I - \mathcal{P}_K)E'$, we find out that $\mathcal{R}(B)$ equals to the direct K-orthogonal sum of $\mathcal{R}(A')$ and a subspace belonging to $\mathcal{R}(A')^{[\perp]}$. But, if $\mathcal{R}(A')^{[\perp]}$ is definite, then any $\mathcal{M} \subset \mathcal{R}(A')^{[\perp]}$ is definite and hence nondegenerate, too. Thus, it only remains to refer to Lemma 1.2 to establish the nondegenerateness of the range space $\mathcal{R}(B)$ of B. We are now going to prove that $\mathcal{R}(A')$ is nondegenerate and its K-orthogonal companion $\mathcal{R}(A')^{[\perp]}$ is definite with respect to $(K.,.)$.

By representation $A' = A + E\,P_{A*}$ one has $r(A') \le r(A)$ so that

$$r(A'^{*}KA') \le r(A') \le r(A) = r(A^{*}HA), \tag{6.1}$$

since $\mathcal{R}(A)$ is nondegenerate. Let $\mathcal{U}_A$ and $\mathcal{U}_H$ be so small neighbourhoods of A and H, respectively, so that the numbers of the positive and the numbers of the negative eigenvalues of the matrix $K \in \mathcal{U}_H$ and the matrix $A'^{*}KA'$, $B \in \mathcal{U}_A$, are at least the same as are the corresponding numbers for H and $A^{*}HA$, respectively. Such neighbourhoods $\mathcal{U}_A$ and $\mathcal{U}_H$ exist by the continuity of the eigenvalues with respect to the matrix elements. (We have $A' \to A$ as $E \to 0$.)

By (6.1) this implies that $\mathcal{R}(A')$ is nondegenerate with respect to $(K.,.)$. In fact, we have verified that $\mathrm{In}(A'^{*}KA') = \mathrm{In}(A^{*}HA)$ and $\mathrm{In}\,K = \mathrm{In}\,H$ for every $B \in \mathcal{U}_A$ and $K \in \mathcal{U}_H$, K Hermitean and nonsingular.

To prove the definiteness of $\mathcal{R}(A')^{[\perp]}$, first note that, as $\mathcal{R}(A')$ is nondegenerate, the identity

$$\mathrm{In}\,\mathcal{R}(A') + \mathrm{In}\,\mathcal{R}(A')^{[\perp]} = \mathrm{In}\,(\mathcal{R}(A') \oplus \mathcal{R}(A')^{[\perp]}) = \mathrm{In}\,K \tag{6.2}$$

holds by Lemma 6.5. In (6.2) the first and the last terms are constant in the neighbourhoods $\mathcal{U}_A$ of A and $\mathcal{U}_H$ of H, since $\mathrm{In}(A'^{*}KA')$ and $\dim \mathcal{R}(A')$ are constant there (cf. Remark 6.4). Hence, the same is true for the second term on the left of (6.2). Thus, the subspace $\mathcal{R}(A')^{[\perp]}$ is definite with respect to $(K.,.)$ for every $B \in \mathcal{U}_A$ and $K \in \mathcal{U}_H$, K Hermitean and nonsingular. This completes the proof of the theorem.

6.7. Corollary. *If $r(A) < \kappa$, where κ = rank of indefiniteness of the whole space $(\mathbf{C}^n,[.,.])$, then the H-s.v.d. of A, if exists, is neither stable nor H-stable.*

We are now going into some further details in the study of the stability of H-s.v.d. under perturbations.

We give the next result, Lemma 6.8, in the case of an infinite-dimensional *Pontryagin space Π_k*, i.e. in a Krein space of finite rank k of positivity. It gives bounds for the numbers of the different kinds of eigenvalues of symmetric operators acting on a Pontryagin space Π_k. (See also the proof of Corollary 2.9.)

6.8. Lemma. *For any Π_k-symmetric operator A on a Pontryagin space Π_k the following statements hold:*

(a) *A has at most k eigenvalues in the half-plane* $\operatorname{Im} z > 0$ *and at most k eigenvalues in the half-plane* $\operatorname{Im} z < 0$ *(taking multiplicities into account).*

(b) *The number of the nonsemisimple eigenvalues of A will not exceed k.*

(For this compare e.g. Bognár [4], Theorem IX.4.6 and Theorem IX.4.8.) Obviously, in our case we take $\Pi_k = (\mathbf{C}^n,(H.,.))$ with $k = \min\{p, q\}$ denoting the *index*, ind H, of the matrix H, i.e., the minimum of the numbers p of the positive and q of the negative eigenvalues of H (taking multiplicities into account). In this case we could prove Lemma 6.8 with the aid of the canonical form of Lemma 1.6 by using Lemma 1.3.

For a fixed H-self-adjoint matrix A the sums of the multiplicities of the real and the semisimple eigenvalues of H-self-adjoint matrices B from some neighbourhood $\mathcal{U}_A$ of A can vary quite freely, in the limits of Lemma 6.8. However, for fixed H and A (A an H-self-adjoint matrix) the difference of the sums of the multiplicities of the positive and the negative eigenvalues of K-self-adjoint matrices B, where K (Hermitean and non-singular) is taken from some neighbourhood $\mathcal{U}_H$ of H and B from some neighbourhood $\mathcal{U}_A$ of A, obeys more "regular" rules.

Recall that the *signature*, $\operatorname{sig} H$, of a Hermitean matrix H denotes the difference of the sums of the multiplicities of the positive and the negative eigenvalues of H.

Let $\mathcal{L} \subset (\mathbf{C}^n,(H.,.))$ be a linear subspace. Suppose $\mathcal{L} = \mathcal{L}^+[+]\mathcal{L}^-[+]\mathcal{L}^0$ is an H-orthogonal direct sum of an H-positive subspace $\mathcal{L}^+$, an H-negative subspace $\mathcal{L}^-$ and an H-neutral subspace $\mathcal{L}^0$. (Such an H-orthogonal decomposition of $\mathcal{L}$ exists by Lemma 6.3.)

6.9. Definition. *We call the difference* $\operatorname{sig} \mathcal{L} = \dim \mathcal{L}^+ - \dim \mathcal{L}^-$ *the signature of the linear subspace* $\mathcal{L}$.

By the discussion following Lemma 6.3, Definition 6.9 is adequately formulated and, clearly, if $G_{\mathcal{L}}$ is the Gram operator of $\mathcal{L}$ then $\operatorname{sig} \mathcal{L} = \operatorname{sig} G_{\mathcal{L}}$. Furthermore, by Remark 6.4 we have

$$\operatorname{sig} \mathcal{L} = \operatorname{sig}(A^*HA) \text{ for any matrix } A \in \mathbf{C}^{n\times n} \text{ such that } \mathcal{R}(A) = \mathcal{L}. \tag{6.3}$$

Let $\mathcal{S}_A(\lambda)$ and $G_A(\lambda)$, respectively, be the principal subspace and the corresponding Gram operator of the subspace $\mathcal{S}_A(\lambda)$ belonging to the eigenvalue λ of $A \in \mathbf{C}^{n\times n}$. We call a linear subspace $\mathcal{M}$ *a spectral subspace of* A if $\mathcal{M}$ can be obtained as a linear span of a collection of the principal subspaces $\mathcal{S}_A(\lambda)$ of A. Especially, for any set $\Omega \subset \mathbf{C}$ the Gram operator of $\mathcal{S}_A(\Omega) = \operatorname{span}\{\mathcal{S}_A(\lambda) \mid \lambda \in \Omega \cap \sigma(A)\}$ is denoted by $G_A(\Omega)$.

6.10. Lemma. *Let* $A \in \mathbf{C}^{n\times n}$ *be H-self-adjoint. For any open interval* $(\mu_1, \mu_2) \subset \mathbf{R}$ *such that* $\mu_1, \mu_2 \notin \sigma(A)$ *we have*

$$\operatorname{sig} \mathcal{S}_A(\mu_1, \mu_2) = \operatorname{sig} G_A(\mu_1, \mu_2) = \tfrac{1}{2}[\operatorname{sig}(\mu_2 H - HA) - \operatorname{sig}(\mu_1 H - HA)]. \tag{6.4}$$

Proof. By Lemma 1.4 the principal subspaces $\mathcal{S}_A(\lambda_i)$, $\lambda_i \in (\mu_1, \mu_2)$, satisfy the conditions of Lemma 6.5. Thus,

$$\operatorname{sig} G_A(\mu_1, \mu_2) = \operatorname{sig} \mathcal{S}_A(\mu_1, \mu_2) = \sum_{\lambda_i \in (\mu_1, \mu_2) \cap \sigma(A)} \operatorname{sig} \mathcal{S}_A(\lambda_i).$$

Since the right-hand side of (6.4) is clearly additive on acceptable subdivisions of the interval (μ_1, μ_2), we can suppose that $(\mu_1, \mu_2) \cap \sigma(A) = \{\lambda_i\}$. We make use of the canonical form of Lemma 1.6. By (6.3) we have

$$\operatorname{sig} \mathcal{S}_A(\lambda_i) = \operatorname{sig}(e_i P(\lambda_i)), \qquad \lambda_i \in \sigma(A), \tag{6.5}$$

where $e_i P(\lambda_i)$ refers to the block of H corresponding to the Jordan block $J(\lambda_i)$ of A in the canonical form of the pair (A, H).

Next note that, for any $\lambda_j \in \mathrm{R}$, $\lambda_j \neq 0$, we have $\operatorname{sig}(P(\lambda_j)J(\lambda_j)) = \operatorname{sgn} \lambda_j \cdot \operatorname{sig} P(\lambda_j)$. Hence, with $(\mu_1, \mu_2) \cap \sigma(A) = \{\lambda_i\}$ the last term of (6.4) becomes

$$\begin{aligned}\operatorname{sig}(\mu_2 H - HA) - \operatorname{sig}(\mu_1 H - HA) &= \operatorname{sig}(\mu_2 P - PJ) - \operatorname{sig}(\mu_1 P - PJ)\\ &= \operatorname{sig}(e_i P(\lambda_i)J(\mu_2 - \lambda_i)) - \operatorname{sig}(e_i P(\lambda_i)J(\mu_1 - \lambda_i))\\ &= 2\operatorname{sig}(e_i P(\lambda_i)).\end{aligned}$$

Combining this identity with (6.5) proves the lemma.

The two matrices on the right-hand side of (6.4) are Hermitean and invertible and thus the inertias and signatures of them are not changed under small perturbations of A and H, respectively. These observations lead us to the following lemma.

6.11. Lemma. *Let $A \in \mathrm{C}^{n \times n}$ be H-self-adjoint and let $\Phi \subset \mathrm{R}$ be an open set of the real line R in the complex plane C such that the boundary $\partial\Phi$ of Φ relative to R does not intersect $\sigma(A)$.*

(i) *Then for every $K \in \mathcal{U}_H$, K Hermitean and nonsingular, and for every K-self-adjoint $B \in \mathcal{U}_A$, the identity*

$$\operatorname{sig} \mathcal{S}_B(\Phi) = \operatorname{sig} \mathcal{S}_A(\Phi), \quad \textit{equivalently} \quad \operatorname{sig} G_B(\Phi) = \operatorname{sig} G_A(\Phi),$$

holds for sufficiently small neighbourhoods $\mathcal{U}_H$ of H and $\mathcal{U}_A$ of A, respectively.

(ii) *For $K \in \mathcal{U}_H$ (K Hermitean and nonsingular) and $B \in \mathcal{U}_A$ (B K-self-adjoint), with $\mathcal{U}_A$ and $\mathcal{U}_H$ small enough, the number $\rho_\Phi(B)$ of the eigenvalues of B in Φ (taking multiplicities into account) satisfies the inequality*

$$\rho_\Phi(B) \geq \sum_{\lambda \in \Phi \cap \sigma(A)} |\operatorname{sig} \mathcal{S}_A(\lambda)| = \sum_{\lambda \in \Phi \cap \sigma(A)} |\operatorname{sig} G_A(\lambda)|,$$

(iii) *the number $\eta_\Phi(B)$ of the nonsemisimple eigenvalues of B in Φ satisfies the inequality*

$$\eta_\Phi(B) \leq \tfrac{1}{2}\Big(n_\Phi(A) - \sum_{\lambda \in \Phi \cap \sigma(A)} |\operatorname{sig} \mathcal{S}_A(\lambda)|\Big) = \tfrac{1}{2}\Big(n_\Phi(A) - \sum_{\lambda \in \Phi \cap \sigma(A)} |\operatorname{sig} G_A(\lambda)|\Big),$$

where $n_\Phi(A)$ denotes the number of all the eigenvalues (taking multiplicities into account) of the matrix A in Φ.

Proof. For the first part of the lemma suppose Φ_j are the connected components of the open set Φ. Since there is only a finite number of connected components containing eigenvalues of the matrix B, we can apply Lemma 6.10 to each Φ_j and thus find neighbourhoods $\mathcal{U}_A$ of A and $\mathcal{U}_H$ of H such that $\operatorname{sig} G_B(\Phi_j) = \operatorname{sig} G_A(\Phi_j)$ is simultaneously satisfied for all Φ_j. The equality $\operatorname{sig} \mathcal{S}_B(\Phi) = \operatorname{sig} \mathcal{S}_A(\Phi)$ (equivalently $\operatorname{sig} G_B(\Phi) = \operatorname{sig} G_A(\Phi)$) then follows from Lemma 6.5.

To prove (ii) and (iii), we first apply part (i) to small separate intervals I_λ, $\lambda \in I_\lambda$, of every distinct $\lambda \in \Phi \cap \sigma(A)$ to get the identity

$$(6.6) \qquad \sum_{\lambda \in \Phi \cap \sigma(B)} |\operatorname{sig} \mathcal{S}_B(\lambda)| = \sum_{\lambda \in \Phi \cap \sigma(A)} |\operatorname{sig} \mathcal{S}_A(\lambda)|,$$

for any acceptable K and B close enough to H and A, respectively. Then for (iii) part (b) of Lemma 6.8 is applied to the restricted $B\,|_{\mathcal{S}_B(I_\lambda)}$ in the space $(\mathcal{S}_B(I_\lambda), (G_B(I_\lambda)\cdot,\cdot))$ with every $\lambda \in \Phi \cap \sigma(A)$ again. It is then enough to take into account that the estimation $\rho_\Phi(B) \le n_\Phi(A)$ holds for every B sufficiently close to A by the assumption made on Φ. Similarly, part (ii) can be proved with the aid of Lemma 6.8 (a) but, in fact, it follows directly from (6.6).

6.12. Corollary. *For any open set $\Phi \subset \mathbb{R}$ such that $\partial\Phi \cap \sigma(A) = \emptyset$ the signature* sig $\mathcal{L}$ *of any spectral subspace $\mathcal{L} = \mathcal{S}_A(\Phi)$, related to an arbitrary H-self-adjoint matrix A, is invariant under small perturbations of A and H.*

However, the dimension of such a spectral subspace $\mathcal{S}_A(\Phi)$ (equivalently the rank of the Gram operator $G_A(\Phi)$ of $\mathcal{S}_A(\Phi)$) need not, in general, be constant in any neighbourhood of (A, H).

6.13. Lemma. *Let $A \in \mathbf{C}^{m\times n}$ and let $\mathcal{K}$ be an $A^{[*]}A$-invariant $[.,.]_2$-nondegenerate linear subspace. Then the image $A(\mathcal{K}) \subset \mathcal{R}(A)$ of $\mathcal{K}$ under A is nondegenerate with respect to $[.,.]_1$ if and only if the inclusion $\mathcal{N}(A^{[*]}A) \cap \mathcal{K} \subset \mathcal{N}(A)$ holds.*

Proof. If the relation $\mathcal{N}(A^{[*]}A) \cap \mathcal{K} \subset \mathcal{N}(A)$ is not valid, there exists a vector $y \in \mathcal{N}(A^{[*]}A) \cap \mathcal{K}$ such that $y \notin \mathcal{N}(A)$. But, then $z = Ay \neq 0$ and $[z, Ax]_1 = [A^{[*]}Ay, x]_2 = 0$ for every $x \in \mathcal{K}$ and z is isotropic for $A(\mathcal{K})$. Thus, if $A(\mathcal{K})$ is nondegenerate then we have $\mathcal{N}(A^{[*]}A) \cap \mathcal{K} \subset \mathcal{N}(A)$.

Suppose conversely that $z \neq 0$, $z = Ay$, $y \in \mathcal{K}$, is isotropic for $A(\mathcal{K})$. Since $\mathcal{K}$ is $A^{[*]}A$-invariant, $A^{[*]}Ay \in \mathcal{K}$ and for all $x \in \mathcal{K}$ we have $[A^{[*]}Ay, x]_2 = [Ay, Ax]_1 = 0$, as $z = Ay$ is isotropic for $A(\mathcal{K})$. Thus, $A^{[*]}Ay$ is H_2-orthogonal to $\mathcal{K}$. But, $\mathcal{K}$ was nondegenerate and hence $A^{[*]}Ay = 0$ showing that $y \in \mathcal{N}(A^{[*]}A) \cap \mathcal{K}$ although $y \notin \mathcal{N}(A)$. This gives the converse statement that the inclusion $\mathcal{N}(A^{[*]}A) \cap \mathcal{K} \subset \mathcal{N}(A)$ implies the nondegenerateness of $A(\mathcal{K})$. The lemma is proved.

Note that the $A^{[*]}A$-invariance and nondegenerateness of $\mathcal{K}$ were used only to prove the sufficiency part of the lemma.

We are ready to give a complete characterization of the classes of matrices having a stable or H-stable H-s.v.d., or more generally, having an $\mathfrak{I}$-stable or an $\mathfrak{I}_H$-

stable H-s.v.d., respectively.

6.14. Theorem. *Let $A \in \mathbf{C}^{n\times n}$. Further let $\mathfrak{I} \subset \mathbf{R}$ be an open set of the real line $\mathbf{R}$ in the complex plane $\mathbf{C}$ such that A has an H-s.v.d. with respect to $\mathfrak{I}$ and that the boundary $\partial\mathfrak{I}$ of $\mathfrak{I}$ relative to $\mathbf{R}$ contains no H-singular values of A so that $f^{-1}(\partial\mathfrak{I}) \cap \sigma(A^{[*]}A) = \emptyset$. The H-s.v.d. of A with respect to $\mathfrak{I}$ is $\mathfrak{I}$-stable if and only if either the left or the right singular subspace corresponding to each nonzero H-singular value d_i of A in $\mathfrak{I}$ is definite with respect to $(H.,.)$ and both are definite for the zero H-singular value of A, if $0 \in \mathfrak{I}$. This same condition is necessary and sufficient for the $\mathfrak{I}$-s.v.d. of A to be $\mathfrak{I}_H$-stable, as well.*

Proof. We first prove the sufficiency part of the theorem. Let $K \in \mathcal{U}_H$, K Hermitean and invertible, and $C \in \mathcal{U}_A$, and let $\mathcal{U}_A$ and $\mathcal{U}_H$ be such small neighbourhoods of A and H that the conclusions of Lemma 6.11 are valid for K, $C^{[*]}C$ and Φ, where $\Phi = f^{-1}(\mathfrak{I})$ denotes the preimage of $\mathfrak{I}$ under f as in Definition 6.1, containing the numbers $\lambda_i = d_i^2$ and $\lambda_i = -d_i^2$ corresponding to the H-singular values $d_i \in \mathfrak{I}$ of A. Such an $\mathcal{U}_A$ and $\mathcal{U}_H$ exist, since $C^{[*]}C$ is K-self-adjoint and $C^{[*]}C \to A^{[*]}A$ as $C \to A$ and $K \to H$.

By Remark 2.11 the sign of any nonzero eigenvalue $\lambda_i \in \sigma(A^{[*]}A)$ is equal to

$$\text{sgn}\,\lambda_i = (Hu_i, u_i)\,(Hv_i, v_i), \tag{6.7}$$

where u_i and v_i are the left and the right singular vectors of A corresponding to d_i. Hence, the definiteness assumptions on the H-singular values $d_i \in \mathfrak{I}$ imply that the linear subspaces $\mathcal{S}_{A^{[*]}A}(\lambda_i)$, $|\lambda_i| = d_i^2$, are definite and, thus the equality

$$|\,\text{sig}\,\mathcal{S}_{A^{[*]}A}(\lambda_i)\,| = \dim \mathcal{S}_{A^{[*]}A}(\lambda_i)$$

holds for every eigenvalue $\lambda_i \in \sigma(A^{[*]}A) \cap \Phi$. (For $\lambda_i = 0$, if $0 \in \mathfrak{I}$, this follows from $\mathcal{N}(A^{[*]}A) = \mathcal{N}(A^{[*]}A) \cap \mathcal{S}_{A^{[*]}A}(f^{-1}(\mathfrak{I})) \subset \mathcal{N}(A)$; see Lemma 6.13.) Hence, parts (ii) and (iii) of Lemma 6.11 give us the inequalities

$$\rho_\Phi(C^{[*]}C) \geq \sum_{\lambda \in \Phi \cap \sigma(A^{[*]}A)} \dim \mathcal{S}_{A^{[*]}A}(\lambda_i) =: k_0 \tag{6.8}$$

and

$$\begin{aligned} \eta_\Phi(C^{[*]}C) &\leq \tfrac{1}{2}\Big(n_\Phi(A^{[*]}A) - \sum_{\lambda \in \Phi \cap \sigma(A^{[*]}A)} \dim \mathcal{S}_{A^{[*]}A}(\lambda)\Big) \\ &= \tfrac{1}{2}\big(n_\Phi(A^{[*]}A) - k_0\big) \end{aligned} \tag{6.9}$$

for every $C \in \mathcal{U}_A$ and $K \in \mathcal{U}_H$, K Hermitean and invertible. But $n_\Phi(A^{[*]}A) = k_0$ and thus (6.9) implies $\eta_\Phi(C^{[*]}C) = 0$, i.e., all the eigenvalues of $C^{[*]}C$ in Φ are semisimple.

On the other hand, the continuity of the eigenvalues λ_i and the assumptions on $\mathfrak{I}$ guarantee that there exist some neighbourhoods of A and H such that the the number of the eigenvalues of $C^{[*]}C$ in Φ is at most k_0. So making $\mathcal{U}_A$ and $\mathcal{U}_H$ smaller, if necessary, we can find neighbourhoods $\mathcal{V}_A$ of A and $\mathcal{V}_H$ of H, respectively, such that the equality holds in (6.8), i.e., the number of the eigenvalues of $C^{[*]}C$ in Φ

is the same for every $C \in \mathcal{V}_A$ and $K \in \mathcal{V}_H$.

It remains to consider the nondegeneracy of the space $\mathcal{R}_{\mathfrak{I}}(C) := \mathcal{R}(C\,|_{S_{C^{[*]}C}(f^{-1}(\mathfrak{I}))})$. Suppose $C \in \mathcal{V}_A$ and $K \in \mathcal{V}_H$ with K Hermitean and nonsingular. If $0 \notin \mathfrak{I}$ then we have $\mathcal{N}(C^{[*]}C) \cap S_{C^{[*]}C}(f^{-1}(\mathfrak{I})) = \{0\}$ and hence $\mathcal{R}_{\mathfrak{I}}(C)$ is nondegenerate by Lemma 6.13. If $0 \in \mathfrak{I}$, then $\mathcal{N}(C^{[*]}C) \cap S_{C^{[*]}C}(f^{-1}(\mathfrak{I})) = \mathcal{N}(C^{[*]}C)$. Hence, for the proof of the statement that $\mathcal{R}_{\mathfrak{I}}(C)$ is K-nondegenerate for any $C \in \mathcal{V}_A$ and $K \in \mathcal{V}_H$ we need, again by Lemma 6.13, only to verify $\mathcal{N}(C^{[*]}C) \subset \mathcal{N}(C)$, i.e., that the whole range $\mathcal{R}(C)$ of C is K-nondegenerate. But this can be achieved by Theorem 6.6, as the left singular subspace, i.e. the linear subspace $\mathcal{N}(A^{[*]}) = \mathcal{R}(A)^{[\perp]}$, was assumed to be definite. Indeed, by making $\mathcal{V}_A$ and $\mathcal{V}_H$ smaller, if necessary, we can find neighbourhoods $\mathcal{W}_A$ of A and $\mathcal{W}_H$ of H, respectively, such that for every $C \in \mathcal{W}_A$ and $K \in \mathcal{W}_H$, K Hermitean and nonsingular, $\mathcal{R}(C)$ and, thus $\mathcal{R}_{\mathfrak{I}}(C)$ also, is K-nondegenerate.

We have shown that for $K \in \mathcal{W}_H$, K Hermitean and nonsingular, and for $C \in \mathcal{W}_A$, C has a K-s.v.d. with respect to $\mathfrak{I}$ and the number of the K-singular values of C in Φ is constant. In the terminology introduced above, A has an $\mathfrak{I}$-stable and, thus, also an $\mathfrak{I}_H$-stable H-s.v.d.

To prove the converse statement, suppose first that A has an H-s.v.d. with respect to $\mathfrak{I}$ but that for some nonzero H-singular value d_i of A contained in $\mathfrak{I}$ neither the left nor the right singular subspace is definite. By restricting A to the linear subspace $S_{A^{[*]}A}(f^{-1}(\mathfrak{I}))$, we essentially can assume that A has an H-s.v.d. $A = UDV^{-1}$.

It is easily seen that (after a permutation) there is a 2×2 block $D_i = d_i I_2$ in D such that for the corresponding left and right singular vectors, without loosing generality, the related block in $S_1 = U^*HU$ is $S_1{}^i = \operatorname{diag}(+1,-1)$ and the block in $S_2 = V^*HV$ is $S_2{}^i = \pm\operatorname{diag}(+1,-1)$. Denote

$$D_i(\xi) = \begin{bmatrix} d_i & \xi \\ -\xi & d_i \end{bmatrix} = d_i I_2 + \xi(e_1 e_2{}^* - e_2 e_1{}^*), \quad \xi \in \mathrm{C},$$

and let $D(\xi) = \operatorname{diag}(d_{j_1},\ldots,D_i(\xi),\ldots,d_{j_p})$. Then we have $D(\xi) \to D$ and $C(\xi) = UD(\xi)V^{-1} \to A$ as $\xi \to 0$.

We have $C(\xi)^{[*]}C(\xi) = VS_2D^*(\xi)S_1D(\xi)V^{-1}$. Here $S_2D^*(\xi)S_1D(\xi)$, similar to $C(\xi)^{[*]}C(\xi)$, is block diagonal and contains the block

$$E = S_2^iD^*(\xi)S_1^iD(\xi) = \pm\begin{bmatrix} d_i^2 - |\xi|^2 & 2d_i\mathrm{Re}\ \xi \\ -2d_i\mathrm{Re}\ \xi & d_i^2 - |\xi|^2 \end{bmatrix}.$$

Hence, E is a sum of the matrix $\pm(d_i^2 - |\xi|^2)I_2$ and of a skew-adjoint matrix. Thus, whenever ξ satisfies $\mathrm{Re}\ \xi \neq 0$, the eigenvalues of E are not real. So the number of the H-singular values of C in $\mathfrak{I}$ is not constant. This shows that the H-s.v.d. of the matrix A is not stable with respect to $\mathfrak{I}$, if the definiteness condition on $d_i \in \mathfrak{I}$, $d_i \neq 0$, is not satisfied.

Suppose then $0 \in \mathfrak{I}$ and that the $\mathfrak{I}$-s.v.d. of A is $\mathfrak{I}_H$-stable. Then in some neighbourhood $\mathcal{U}_A$ of A the linear subspace $\mathcal{R}(C\,|_{S_{C^{[*]}C}(f^{-1}(\mathfrak{I}))})$ is H-nondegenerate for $C \in \mathcal{U}_A$. It follows from Lemma 6.13 that the whole range $\mathcal{R}(C)$ of every $C \in \mathcal{U}_A$ must be H-nondegenerate, too. By Theorem 6.6 $\mathcal{R}(A)^{[\perp]} = \mathcal{N}(A^{[*]})$, equivalently the left singular subspace of A corresponding to $d_i = 0$, must be H-definite. It remains to prove that the right singular subspace of A corresponding to $d_i = 0$, i.e. the space $\mathcal{N}(A) = \mathcal{R}(A^{[*]})^{[\perp]}$, is H-definite. If $\mathcal{N}(A)$ is not H-definite, then as in the proof of Theorem 6.6 we can find in every neighbourhood of A a matrix C such that $\mathcal{N}(C) = \mathcal{N}(C^{[*]}C)$ is H-degenerate. (Define $C(\xi) = A + \xi y_0[.,z_0]$, where $0 \neq z_0 \in \mathcal{N}(A)$ is H-neutral and $0 \neq y_0 \in \mathcal{R}(A)^{[\perp]}$ is H-positive or H-negative.) For such a C, the number $\lambda = 0$ is not a semisimple eigenvalue of $C^{[*]}C$. This contradicts the $\mathfrak{I}_H$-stability of A and completes the proof of the theorem.

For sub-Pesonen matrices, introduced in Section 3, we have the following result.

6.15. Proposition. *A sub-Pesonen matrix $A \in \wp_H$ has a stable H-s.v.d. and an H-stable H-s.v.d. simultaneously, and these exist if and only if the subspace $\mathcal{N}(A^{[*]})$ is definite.*

Proof. We first show that the left or the right singular subspace of $A \in \wp_H$ for any nonzero H-singular value d_i is definite. For this it is enough to verify that the eigenspaces $\mathcal{N}(A^{[*]}A - \lambda_i I)$, $\lambda_i \in \sigma(A^{[*]}A)$, of $A^{[*]}A$ are definite (cf. identity (6.7) above).

But, by Lemma 3.1 $A^{[*]}A$ cannot have any nonzero neutral eigenvectors, which readily implies the definiteness of every eigenspace $\mathcal{N}(A^{[*]}A - \lambda_i I) = \mathcal{S}_{A^{[*]}A}(\lambda_i)$, $\lambda_i \in \sigma(A^{[*]}A)$, of $A^{[*]}A$. (If $0 \in \sigma(A^{[*]}A)$, the statement holds also for this eigenvalue).

Hence, A fulfills the requirements of Theorem 6.14 exactly if the linear subspace $\mathcal{N}(A^{[*]})$ is definite, since $\mathcal{N}(A) = \mathcal{N}(A^{[*]}A)$.

6.2. Analytic perturbations and H-singular values

Let $A(t) = \sum_{i=0}^{\infty} t^i A_i$ be some *analytic $n \times n$ matrix-valued function* defined in some neighbourhood of $t = 0$. Suppose that $A(0) = A$ is the unperturbed matrix under investigation. It can be asked whether under such "regular" perturbations as above, the H-singular values $d_i(t)$ of $A(t)$, if they exist, behave in some regular way, too. More precisely: can they always be chosen to be analytic functions in t? To make use of the results known for analytic matrix functions and especially for their eigenvalues we shall restrict our interest to a *real parameter* $t \in \mathbb{R}$. (Note that $d_i(t) \in \mathbb{R}^+$ by definition. Further, if $A(z)$ is both Hermitean and analytic matrix function of a complex argument z then it follows from Cauchy–Riemann equations that $A(z)$ must be constant.)

6.16. Theorem. *Let A be nonsingular and let $A(t)$, $A(0) = A$, be any analytic $n \times n$ matrix-valued function on a real interval $\mathcal{U}$ around $t = 0$. Suppose that A has an H-s.v.d. with respect to $[.,.] = (H.,.)$. Then there exists a real interval $\mathcal{V}$ around $t = 0$ such that $A(t)$ has an H-s.v.d. and the H-singular values of $A(t)$ can be taken to be analytic functions on $\mathcal{V}$ if and only if for every H-singular value d_i of A the left or the right singular subspace is definite, i.e., if and only if the H-s.v.d. of A is H-stable.*

Proof. First suppose that $A = A(0)$ has an H-stable H-s.v.d. By Theorem 6.14 there exists a neighbourhood $\mathcal{U}_0$ of $t = 0$ such that every matrix $A(t)$, $t \in \mathcal{U}_0$, has an H-s.v.d. Since A is nonsingular, also $A(t)$ is nonsingular in some neighbourhood $\mathcal{V}_0$ of

$t = 0$. Define $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_0 \cap \mathcal{V}_0$ and

$$A_1(t) = \begin{bmatrix} 0 & A(t) \\ A(t)^{[*]} & 0 \end{bmatrix}, \qquad t \in \mathcal{V}.$$

Then $A_1(t)$ is analytic on $\mathcal{V}$ and $(H \oplus H)$-self-adjoint. It can be easily verified that the positive eigenvalues $\lambda_i(t) \in \sigma(A_1(t))$ of $A_1(t)$ for any $t \in \mathcal{V}$ are equal to the H-singular values $d_i(t)$ of $A(t)$ originating from the positive eigenvalues $d_i(t)^2$ of $A(t)^{[*]}A(t)$. Note that, by the continuity of eigenvalues and the nonsingularity of $A(t)$ and hence of $A_1(t)$ on $\mathcal{V}$ the signs of the branches of the real eigenvalues $\lambda_i(t) \in \sigma(A_1(t))$ cannot change on $\mathcal{V}$, since we have $\sigma(A_1(t)) \subset \mathrm{R} \cup i\mathrm{R}$ for every $t \in \mathcal{V}$ by the H-stability of the H-s.v.d. of A. One must show that the branches $\lambda_i = d_i$ with $\lambda_i(t) = d_i(t) > 0$ are analytic functions on $\mathcal{V}$.

From the theory of algebraic functions it is known that for any polynomial $P(z, \lambda) = \sum_{j=0}^{n} b_j(z)\lambda^j$, having as its coefficients analytic functions b_j ($b_n(z) \equiv 1$) defined in some neighbourhood $|z-z_0| < \varepsilon$ of z_0, the zeros $\lambda_i(z)$ of $P(z, \lambda)$ can be represented as functions of z in the form of Puiseux series, $\lambda_i(z) = \sum_{k=0}^{\infty} a_k[\zeta(z-z_0)^{1/g}]^k$, where $g \geq 1$ is an integer less than or equal to the multiplicity of $\lambda_i(z_0)$ as a root of $P(z_0, \lambda) = 0$, ζ denotes a gth root of the unit and $(z-z_0)^{1/g}$ is a fixed branch of $(z-z_0)^{1/g}$. (For detailed information see Baumgärtel [3], especially pp. 370–371.) In particular, this result is applicable to the eigenvalues of an analytic matrix function $C(z)$, as the coefficients of $\det(C(z) - \lambda I)$ are then analytic functions in z.

It follows that for an eigenvalue $d(t)$ of $A_1(t)$, $t \in \mathcal{V}$, where $d(t) \to d(0) > 0$ as $t \to 0$, we can write a fractional power series expansion in $t^{1/g}$, say

$$\begin{aligned} (6.10) \qquad d(t) &= d(0) + \sum_{k=1}^{\infty} d_{(k)}[t^{1/g}]^k \\ &= d(0) + \sum_{k=1}^{\infty} d_{(k)}(|t|^{1/g})^k[\cos(k(\arg t)/g) + i\sin(k(\arg t)/g)], \end{aligned}$$

where g is a positive integer, less than or equal to the algebraic multiplicity of $d(0)$ as an eigenvalue of $A_1(0)$, i.e., $g \leq \dim S_{A_1}(d(0))$. Since $d(t)$ is real on $\mathcal{V}$, a limit argument for $t > 0$, $t \in \mathcal{V}$, shows that the coefficients $d_{(k)}$ in the expansion (6.10) are real numbers. On the other hand for $t < 0$, $t \in \mathcal{V}$, the imaginary parts $d_{(k)}(|t|^{1/g})^k \cdot \sin(k\pi/g)$ of $d_{(k)}[t^{1/g}]^k$ must all vanish. That is, in (6.10) $d_{(k)} = 0$ unless k is of the form $k = p\,g$, $p \in \mathbf{N}$, i.e., $d(t)$ is analytic on $\mathcal{V}$.

To prove that the H-singular values $d(t) \to d(0)$ obtained from the negative eigenvalues of $A(t)^{[*]}A(t)$ are analytic on $\mathcal{V}$, we make use of the matrix function

$$A_2(t) = \begin{bmatrix} 0 & i\,A(t) \\ i\,A(t)^{[*]} & 0 \end{bmatrix}, \qquad t \in \mathcal{V}.$$

The function $A_2(t)$ is self-adjoint with respect to $H \oplus (-H)$ and the positive eigenvalues $\mu_i(t)$ of $A_2(t)$ are just the H-singular values $d_i(t)$ of $A(t)$ obtained from the negative eigenvalues of $A(t)^{[*]}A(t)$. A similar argument as above shows that also these H-singular values define analytic functions on $\mathcal{V}$.

The converse statement is obvious. The latter part of the proof of Theorem 6.14 gives an analytic matrix function which has no H-s.v.d. for any $t \in \mathrm{R} \setminus \{0\}$. (See also Example 6.19 below.)

6.17. Remark. The result of Theorem 6.16 is applicable also for the case where the matrix H has been replaced by an analytic matrix function $H(t)$, $t \in \mathrm{R}$, $H(t)$ Hermitean and nonsingular. The same proof will apply with some obvious changes.

6.18. Remark. We can leave out the nonsingularity assumption made on A in Theorem 6.16, if we allow such H-singular values for which $d(t) \to 0$ as $t \to 0$ to attain also negative values. To verify this, we first recall that for $\gamma_i(t) \in \sigma(A(t)^{[*]}A(t))$, $|\gamma_i(t)| = d_i(t)^2$, the identity $\operatorname{sgn} \gamma_i(t) = [u_i(t), u_i(t)]\,[v_i(t), v_i(t)]$ holds, where $u_i(t)$ and $v_i(t)$ are the left and the right singular vectors of $A(t)$ ($t \in \mathcal{V}$) corresponding to $d_i(t)$ (cf. Remark 2.11). Then by the H-stability of the H-s.v.d. of $A = A(0)$, the linear subspaces $\mathcal{N}(A(0)^{[*]}A(0))$ and $\mathcal{N}(A(0)A(0)^{[*]})$ are definite, and furthermore, taking the invariance of the signature stated in Corollary 6.12 into account, it is possible to find a neighbourhood $\mathcal{V}' \subset \mathcal{V}$ of $t = 0$ such that $[u_i(t), u_i(t)]$, $[v_i(t), v_i(t)]$ and, thus also $\operatorname{sgn} \gamma_i(t)$ is constant for $t \in \mathcal{V}'$. Moreover, this sign, $\operatorname{sgn} \gamma_i(t)$, is the same for every $\gamma_i(t)$, $|\gamma_i(t)| = d_i(t)^2$, having the property $\gamma_i(t) \to 0$ as $t \to 0$. It follows that we can still use $A_1(t)$ (if $\operatorname{sgn} \gamma_i(t) > 0$) or $A_2(t)$ (if $\operatorname{sgn} \gamma_i(t) < 0$) to prove that also the signed H-singular values $d_i(t) \in \mathrm{R}$ of $A(t)$, with $d_i(t) \to 0$ for $t \to 0$, define analytic functions d_i on $\mathcal{V}'$.

If A does not have a stable or an H-stable H-s.v.d. a more detailed result is valid. If the definiteness condition on the singular subspace pairs is violated the analytic nature of the H-singular values $d(t)$ can be lost totally. In the next example an analytic matrix function is given for which even a squared H-singular value $d(t)^2$ need not be analytic on any real neighbourhood of the origin.

6.19. Example. Without loss of generality it is enough to consider the 2×2 case using $H = \operatorname{diag}(1,-1)$. Define

$$A(\varepsilon) = \begin{bmatrix} d-\varepsilon-\varepsilon^2 & \varepsilon^2-\varepsilon \\ \varepsilon-\varepsilon^2 & d+\varepsilon+\varepsilon^2 \end{bmatrix}, \qquad \varepsilon \in \mathrm{R},$$

where $d \neq 0$ refers to a singular value having both singular subspaces nondefinite. Then $A(\varepsilon)$ is analytic and $A(\varepsilon) \to \operatorname{diag}(d, d)$ as $\varepsilon \to 0$. However, for

$$A(\varepsilon)^{[*]}A(\varepsilon) = \begin{bmatrix} d^2-2d(\varepsilon+\varepsilon^2)+4\varepsilon^3 & 2d(\varepsilon^2-\varepsilon) \\ 2d(\varepsilon-\varepsilon^2) & d^2+2d(\varepsilon+\varepsilon^2)+4\varepsilon^3 \end{bmatrix}$$

the eigenvalues and also the squares of H-singular values (for $\varepsilon \geq 0$) are nonanalytic at $\varepsilon = 0$. Indeed, one has

$$A(\varepsilon)^{[*]}A(\varepsilon) = (d^2+4\varepsilon^3)I_2 + 2d \begin{bmatrix} -\varepsilon-\varepsilon^2 & \varepsilon^2-\varepsilon \\ \varepsilon-\varepsilon^2 & \varepsilon+\varepsilon^2 \end{bmatrix}.$$

This gives the eigenvalues $\lambda_\varepsilon = d^2 + 4\varepsilon^3 \pm 2d(4\varepsilon^3)^{1/2}$ which are not analytic in any real neighbourhood of $\varepsilon = 0$.

7. The H-s.v.d. of plus matrices

Let $(\mathbf{C}^n, [.,.])$ be a properly indefinite scalar product space (i.e. $\kappa > 0$). We denote by $\mathcal{B}_n^+$ (respectively $\mathcal{B}_n^-$) the set of all nonnegative (respectively nonpositive) vectors in $(\mathbf{C}^n, [.,.])$. In this section we will study the H-s.v.d. of matrices in some important subclasses of matrices operating in $(\mathbf{C}^n, [.,.])$. The first step in this direction was taken up already in Section 4.

7.1. Definition. *A matrix $A \in \mathbf{C}^{n\times n}$ is said to be H-noncontractive if $[Ax, Ax] \geq [x, x]$ for all $x \in \mathbf{C}^n$. $A \in \mathbf{C}^{n\times n}$ is said to be a plus matrix if $A\mathcal{B}_n^+ \subset \mathcal{B}_n^+$, i.e. if A carries nonnegative vectors onto nonnegative vectors.*

Clearly, every scalar multiple of an H-noncontractive matrix is a plus matrix. Moreover, any matrix A satisfying $\mathcal{R}(A) \subset \mathcal{B}_n^+$ is a plus matrix too. The following lemma is fundamental (cf. e.g. Kreĭn – Shmul'yan [15]) and gives a converse statement.

7.2. Lemma. *Let A be a plus matrix. Then there exists a scalar $\mu \geq 0$ such that*

$$[Ax, Ax] \geq \mu\, [x, x],$$

holds for every $x \in \mathbf{C}^n$.

Proof. The assertion follows by applying Lemma 4.1 to scalar products $[x, x]$ and $[Ax, Ax]$ to get the desired estimation with

$$\mu = \mu_+(A) = \inf_{[x,x]=1} [Ax, Ax] \geq 0.$$

This proves the assertion.

By Lemma 7.2 for any plus matrix A the matrix $A^{[*]}A - \mu_+(A)I$ is *H-nonnegative*, $A^{[*]}A - \mu_+(A)I \geq_H 0$, by which we understand that $A^*HA - \mu_+(A)H \geq 0$.

If for a plus matrix A the scalar $\mu_+(A) > 0$ then A is called a *strict plus matrix*. In case $\mu_+(A) = 0$ A is said to be a *nonstrict plus matrix*. Thus, a strict plus matrix is collinear with an H-noncontractive matrix and for a nonstrict plus matrix A we have $\mathcal{R}(A) \subset \mathcal{B}_n^+$.

Before we turn to analyse the H-s.v.d. of plus matrices we prove the following lemma for H-nonnegative matrices $A \geq_H 0$.

7.3. Lemma. *For $A \geq_H 0$ all the nonzero eigenvalues λ_i of A are real and semisimple. The Jordan blocks corresponding to the eigenvalue $\lambda_i = 0$ have a maximum size of 2×2. If $\lambda_i \in \sigma(A)$ and $\lambda_i > 0$ then the eigenspace $\mathcal{N}(A - \lambda_i I)$ is positive. If $\lambda_i \in \sigma(A)$ and $\lambda_i < 0$ then the eigenspace $\mathcal{N}(A - \lambda_i I)$ is negative.*

Proof. First note that $[Ax, x] = 0$ implies $Ax = 0$, since A is H-nonnegative. It follows that for $p \in \mathbf{N}^+$ the identity $A^p x = 0$ implies $A^2 x = 0$. For $\lambda_i \in \sigma(A)$ and

$x \in \mathcal{N}(A - \lambda_i I)$ we have $\lambda_i[x,x] = [Ax,x] \geq 0$. Thus $\sigma(A) \subset \mathrm{R}$, and further $\lambda_i > 0$ implies $[x,x] > 0$ and $\lambda_i < 0$ implies $[x,x] < 0$, since from $[x,x] = 0$ it follows $[Ax,x] = 0$ and hence $0 = Ax = \lambda_i x$. The definiteness of $\mathcal{N}(A - \lambda_i I)$ for $\lambda_i \neq 0$ shows also that $\lambda_i \neq 0$ is semisimple. The proof is completed.

7.4. Corollary. *Let $A \in \mathbf{C}^{n\times n}$ be H-nonnegative. Then A has an H-s.v.d. if and only if the subspace $\mathcal{N}(A)$ is nondegenerate. The H-s.v.d. of A is stable (or H-stable) if and only if $\mathcal{N}(A)$ is definite and $\sigma(A)$ does not contain points lying symmetric relative to the origin.*

Proof. By Lemma 7.3 $A^{[*]}A = A^2$ is r-diagonable (cf. p. 11). The first assertion follows then from Theorem 2.4 by noting that $\mathcal{N}(A) = \mathcal{R}(A)^{[\perp]}$.

By Remark 2.11 and Lemma 7.3 again the singular subspaces corresponding to the singular values $d_i \neq 0$ of A are definite if and only if $\sigma(A)$ does not contain points with opposite signs. It remains to refer to Theorem 6.14 to complete the proof of the corollary.

Note that if $A \geq_H 0$ and $\mathfrak{I} \subset \mathrm{R}$ is an open set for which $0 \notin \mathfrak{I} \cap \sigma(A)$ then A has an $\mathfrak{I}$-s.v.d. If further $\partial\mathfrak{I}$ does not contain points $|\lambda_i|$, $\lambda_i \in \sigma(A)$, the $\mathfrak{I}$-s.v.d. for such an A and $\mathfrak{I}$ is $\mathfrak{I}$-stable ($\mathfrak{I}_H$-stable) if again there are no points $\pm\lambda_i \in \sigma(A)$ lying symmetric relative to the origin with $|\lambda_i| \in \mathfrak{I}$.

7.5. Theorem. *Let $A \in \mathbf{C}^{n\times n}$ be a nonstrict plus matrix. Then A has an H-s.v.d. if and only if $\mathcal{R}(A)$ is positive and $\mathcal{N}(A)$ is nondegenerate. The H-s.v.d. of such an A is stable (or H-stable) if and only if $\mathcal{N}(A)$ is positive and $\mathcal{R}(A)$ is a maximal positive linear subspace of* $(\mathbf{C}^n,[.,.])$.

Proof. Since A is nonstrict plus, $\mathcal{R}(A)$ as a nonnegative linear subspace is nondegenerate if and only if it is positive. By Lemma 7.2 we have $A^{[*]}A \geq_H 0$ and hence by Lemma 7.3 $A^{[*]}A$ is r-diagonable exactly if the eigenvalue $\lambda_i = 0$ of $A^{[*]}A$ is semisimple. This last condition is equivalent to the nondegenerateness of the subspace $\mathcal{N}(A^{[*]}A)$. But for A with a nondegenerate $\mathcal{R}(A)$ we have $\mathcal{N}(A) = \mathcal{N}(A^{[*]}A)$ (cf. Lemma 2.3). This proves the first part of the theorem.

To establish the latter part, note that all the left singular subspaces corresponding to the nonzero singular values of A are positive as linear subspaces of $\mathcal{R}(A)$. Next, it is easily verified that the definiteness of $\mathcal{R}(A)^{[\perp]}$ is equivalent to the claim for the subspace $\mathcal{R}(A)$ to be a maximal positive linear subspace. By Theorem 6.14 it remains to consider the definiteness of $\mathcal{N}(A)$. Since A is nonstrict plus, $\mathcal{N}(A)$ contains a positive vector $x \neq 0$. (This is easily seen with the aid of the H-s.v.d. of A.) Hence, for a stable (or H-stable) H-s.v.d. of A the space $\mathcal{N}(A)$ must be positive.

7.6. Corollary. *Let $(H.,.)$ be an indefinite scalar product on $\mathbf{C}^n$ and let* $\mathrm{In}\, H =$ *$(p,q,0)$ be the inertia of H with $\kappa = \{p,q\} > 0$. A necessary condition for any nonstrict plus matrix A operating on $(\mathbf{C}^n,(H.,.))$ to have a stable (or H-stable) H-s.v.d. is that the inequality $p \geq q$ is satisfied.*

7.7. Remark. If A is some nonstrict plus matrix having an H-s.v.d. then also $A^{[*]}$ has an H-s.v.d. However, $A^{[*]}$ need not be a plus matrix as simple examples show. In fact, if a nonstrict plus matrix A has a stable (or an H-stable) H-s.v.d. then $A^{[*]}$ cannot be a plus matrix. Note also that if $\mathfrak{I} \subset \mathrm{R}$ is an open subset such that $0 \notin \overline{\mathfrak{I}}$ then every nonstrict plus matrix A, with H-singular values not lying on the boundary $\partial\mathfrak{I}$, has an $\mathfrak{I}$-stable (and $\mathfrak{I}_H$-stable) H-s.v.d.

The following result, concerning the H-s.v.d. of strict plus matrices, is also easily established with the aid of Lemma 7.3 by considering the matrix $A^{[*]}A - \mu_+(A)I \geq_H 0$, $\mu_+(A) > 0$, and by noting that the linear subspace $\mathcal{R}(A)^{[\perp]}$ for such an A is negative.

7.8. Theorem. *Let $A \in \mathbf{C}^{n\times n}$ be a strict plus matrix. Then A has an H-s.v.d. if and only if the linear subspace $\mathcal{N}(A^{[*]}A - \mu_+(A)I)$ is nondegenerate. Further, for A to have a stable (or H-stable) H-s.v.d. it is necessary and sufficient that $\mathcal{N}(A^{[*]}A - \mu_+(A)I)$ is definite.*

If $A \in \mathbf{C}^{n\times n}$ is a strict plus matrix and $\mathfrak{I} \subset \mathrm{R}$ is an open subset such that $\mu_+(A) \notin \overline{\mathfrak{I}}$ then A, with no H-singular values on the boundary $\partial\mathfrak{I}$, has an $\mathfrak{I}$-stable (and $\mathfrak{I}_H$-stable) H-s.v.d.

Let $A \in \mathbf{C}^{n\times n}$ and suppose that for some $\mu \in \mathrm{R}$ we have $A^{[*]}A - \mu I \geq_H 0$ and further denote

$$\mu_-(A) = \sup_{[x,x]=-1} -[Ax, Ax].$$

Then $\mu_-(A)$ is well-defined and finite ($\mu_-(A) \leq \mu$). Especially, for any plus matrix A we have $\mu_-(A) \leq \mu_+(A)$ and it is easily verified that $A^{[*]}A - \mu I \geq_H 0$ holds exactly if $\mu \in [\mu_-(A), \mu_+(A)]$. By Lemma 7.3 the linear subspace $\mathcal{N}(A^{[*]}A - \lambda I)$ is negative for any $\lambda < \mu_+(A)$ and positive for any $\lambda > \mu_-(A)$. This implies that $A^{[*]}A - \mu I$ is nonsingular for every $\mu \in (\mu_-(A), \mu_+(A))$ (if $\mu_-(A) < \mu_+(A)$).

7.9. Lemma. *Let $A \in \mathbf{C}^{n\times n}$ be a plus matrix. Then we have $\mu_-(A)$, $\mu_+(A) \in \sigma(A^{[*]}A)$. If A is a strict plus matrix then we have $\sigma(A^{[*]}A) \subset \mathrm{R}^+$ and especially $\mu_-(A) \geq 0$.*

Proof. If for example $A^{[*]}A - \mu_+(A)I \geq_H 0$ were nonsingular then for any μ sufficiently close to $\mu_+(A)$ the inequality $A^{[*]}A - \mu I \geq_H 0$ would still be valid, a contradiction to the fact that μ should lie in $[\mu_-(A), \mu_+(A)]$. Thus we have $\mu_+(A) \in \sigma(A^{[*]}A)$. The relation $\mu_-(A) \in \sigma(A^{[*]}A)$ is verified analogously.

To prove the second statement suppose that $\lambda \in \sigma(A^{[*]}A)$ and $\lambda < 0$. Then for $x \neq 0$ with $A^{[*]}Ax = \lambda x$ we have $[x, x] < 0$ since $\lambda < 0 < \mu_+(A)$ (cf. Lemma 7.3). Hence $0 < \lambda[x, x] = [Ax, Ax]$. On the other hand let $\mathcal{M}$ be some maximal positive linear subspace of $(\mathbf{C}^n, [.,.])$ such that $x\,[\perp]\,\mathcal{M}$. (Such an $\mathcal{M}$ exists, cf. Lemma 1.3 and Lemma 6.5.) Then $A\mathcal{M}$ is also maximal positive, since A is strict plus, and further for every $y \in \mathcal{M}$ we have $0 = \lambda[x, y] = [\lambda x, y] = [A^{[*]}Ax, y] = [Ax, Ay]$. Thus $Ax\,[\perp]\,A\mathcal{M}$ and $[Ax, Ax] < 0$, which gives us a contradiction. Hence, for a strict plus matrix A we must have $\sigma(A^{[*]}A) \subset \mathrm{R}^+$.

We shall next investigate some important subclasses of strict plus matrices and their H-s.v.d.

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be a *B-plus matrix* if for all $x \neq 0$ with $[x, x] \geq 0$ we have $[Ax, Ax] > 0$. $A \in \mathbb{C}^{n\times n}$ is said to be a *focusing plus matrix* if for some $\eta > 0$ the inequality $[Ax, Ax] \geq \eta\|Ax\|^2$ holds for every x with $[x, x] \geq 0$. Moreover, we call a matrix $V \in \mathbb{C}^{n\times n}$ *(uniformly) H-expansive* if for some $\delta > 0$ the inequality $[Vx, Vx] \geq [x, x] + \delta\|x\|^2$ is satisfied for any $x \in \mathbb{C}^n$. Here $\|\cdot\|$ stands for some norm on $\mathbb{C}^n$.

Clearly, H-expansive matrices are H-noncontractive and B-plus matrices and focusing plus matrices are, indeed, plus matrices.

Next we note that by making use of the H-s.v.d. (provided that it exists) of a plus matrix A we see that the following characterizations of $\mu_+(A)$ and $\mu_-(A)$ are valid

$$(7.1)\qquad \begin{aligned} \mu_+(A) &= \min\{\lambda_i \in \sigma(A^{[*]}A) \mid |\lambda_i| = d_i^2 \text{ with } [v_i, v_i] = +1\},\\ \mu_-(A) &= \max\{\lambda_i \in \sigma(A^{[*]}A) \mid |\lambda_i| = d_i^2 \text{ with } [v_i, v_i] = -1\}. \end{aligned}$$

Here v_i denotes a right singular vector corresponding to the H-singular value d_i of A.

7.10. Theorem. *For a plus matrix $A \in \mathbb{C}^{n\times n}$ the following conditions are equivalent:*

(i) *A is strict plus and has a stable H-s.v.d.,*
(ii) $0 \leq \mu_-(A) < \mu_+(A)$,
(iii) *A is nonzero and collinear with an H-expansive matrix V,*
(iv) *A is a focusing strict plus,*
(v) *A is B-plus,*
(iv) *A is strict plus and $\mathcal{N}(A^{[*]}A - \mu_+(A)I)$ is (positive) definite.*

Moreover, if A satisfies any of these conditions the same is true for $A^{[*]}$ and further $\mu_\pm(A^{[*]}) = \mu_\pm(A)$.

Proof. (i) $\Rightarrow$ (ii). Suppose first that (i) holds. By Lemma 7.9 $\mu_-(A) \geq 0$. Further $\sigma(A^{[*]}A) \subset \mathbb{R}^+$ and thus $[u_i, u_i][v_i, v_i] = +1$. It then follows from the identities (7.1) and the stability of the H-s.v.d. of A that $\mu_-(A) \neq \mu_+(A)$, i.e. $\mu_-(A) < \mu_+(A)$ holds.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds, and let $\mu \in \mathbb{R}$ be such that $\mu_-(A) < \mu < \mu_+(A)$. Then $A^{[*]}A - \mu I$ is invertible and for any $x \in \mathbb{C}^n$, $x \neq 0$, we have

$$[(A^{[*]}A - \mu I)x, x] > 0.$$

It follows that for any norm $\|\cdot\|$ on $\mathbb{C}^n$ there exists $\delta(\mu) > 0$ such that

$$((A^*HA - \mu H)x, x) \geq \delta(\mu)\|x\|$$

is satisfied, i.e. $\mu^{-1/2}A$ is H-expansive.

(iii) $\Rightarrow$ (iv). Let $A = cV$, $c \neq 0$, where V is H-expansive and let $\|\cdot\|$ be some norm on $\mathbb{C}^n$. Denote by $\|A\| = \sup \|Ax\|/\|x\|$, $x \neq 0$, the corresponding norm on $\mathbb{C}^{n\times n}$. Then for any $x \in \mathcal{B}_n^+$ we have

$$[Ax, Ax] \geq |c|^2([x, x] \ + \delta\|x\|^2) \ \geq \ |c|^2\delta\|x\|^2 \ \ \geq \ \frac{|c|^2\delta}{\|A\|^2} \ \|Ax\|^2.$$

Thus A is a focusing and strict plus matrix.

(iv) $\Rightarrow$ (v). Let A be focusing strict plus. To show that A is a B-plus matrix we verify that if $[x, x] \geq 0$ and $[Ax, Ax] = 0$ then one has $x = 0$. Indeed, for any x satisfying these two conditions we have $0 = [Ax, Ax] \geq \eta\,\|Ax\|$ with $\eta > 0$ and thus $x \in \mathcal{N}(A)$. Since $\mathcal{N}(A^{[*]}A) \supset \mathcal{N}(A)$ is negative for a strict plus matrix A (cf. Lemma 7.3) it follows that $x = 0$.

(v) $\Rightarrow$ (i). If A is a B-plus matrix then the matrix $A^{[*]}A$ is a Pesonen matrix and to show that A has a stable H-s.v.d. it is enough to verify that the linear subspace $\mathcal{R}(A)^{[\perp]}$ is definite (cf. Proposition 6.15). But a B-plus matrix A clearly maps every positive subspace onto a positive subspace in a one-to-one manner. Hence $\mathcal{R}(A)$ contains a maximal positive subspace and thus the subspace $\mathcal{R}(A)^{[\perp]}$ is contained in a negative subspace, i.e. $\mathcal{R}(A)^{[\perp]}$ is definite. That A is strict plus (and not only a plus matrix) follows then from the existence of an H-s.v.d. for A and the identities (7.1) above.

That (i) $\Leftrightarrow$ (vi) was stated already in Theorem 7.8. To complete the proof of the theorem it is enough to use the H-s.v.d. of A and again the characterizations of the numbers $\mu_{\pm}(A)$ given in (7.1).

7.11. Corollary. *If A satisfies any of the conditions (i)–(vi) of Theorem 7.10 then A belongs to the class* $\wp_H$ *of sub-Pesonen matrices.*

If $A \in \wp_H$ and the H-s.v.d. of A is stable (or H-stable) we will shortly call A a *stable sub-Pesonen* matrix. By Theorem 7.10 all (uniformly) H-expansive matrices, focusing strict plus matrices or B-plus matrices are examples of stable sub-Pesonen matrices. In the following theorem we shall establish various approximation properties of stable sub-Pesonen matrices.

7.12. Theorem. *The closure of the set of stable sub-Pesonen matrices in the norm topology of the linear space* $\mathbf{C}^{n\times n}$ *contains all strict plus matrices and nonstrict plus matrices A with the property* $\mu_-(A) < 0$ $(\mu_+(A) = 0)$ *as well as nonstrict plus matrices A such that* $A^{[*]}$ *also is a plus matrix.*

Proof. To construct the desired approximations using stable sub-Pesonen matrices we first denote by P^+ an orthogonal projection onto some maximal positive subspace $\mathcal{M}$ such that P^+ commutes with H, i.e. $P^+H = HP^+$. (Such a P^+ is found by taking $\mathcal{M}$ as the spectral subspace of H determined by the positive eigenvalues of H.) Then $P^+|_{\mathcal{K}}$ is injective for any nonnegative linear subspace $\mathcal{K}$: if $x \in \mathcal{K}$ and $P^+x = 0$ then we have $x = (I - P^+)x \in \mathcal{M}^{[\perp]}$ and since $\mathcal{M}^{[\perp]}$ is negative it follows that $x = 0$. Denote $P^- = I - P^+$. Then $(P^+ - P^-) \geq_H 0$ and since $(P^+ - P^-)^2 = I$ we can define a norm on $\mathbf{C}^n$ by $\|x\|^2 = [(P^+ - P^-)x, x]$. Further we denote $I_\varepsilon = I + \varepsilon(P^+ - P^-)$, $\varepsilon > 0$. Then the identity $[I_\varepsilon x, I_\varepsilon x] = (1 + \varepsilon^2)[x, x] + 2\varepsilon\|x\|^2$ holds. Thus for a strict plus matrix A we get an estimation

$$[AI_\varepsilon x, AI_\varepsilon x] \ \geq \ \mu_+(A)[(1 + \varepsilon^2)[x, x] + 2\varepsilon\|x\|^2]$$

from which it can immediately be deduced that AI_ε, $\varepsilon > 0$, is a B-plus matrix with $\mu_+(AI_\varepsilon) \geq (1+\varepsilon^2)\mu_+(A)$ and hence a stable sub-Pesonen matrix too. Since $AI_\varepsilon \to A$ as $\varepsilon \to 0$, the first assertion is proven. (The matrices $I_\varepsilon A$ would be applicable here as well.)

Next suppose that A is nonstrict plus with $\mu_-(A) < 0$. For any $\varepsilon > 0$ we define $A_\varepsilon = (I + \varepsilon P^+)A$. Then the range $\mathcal{R}(A_\varepsilon)$ of A_ε is positive, in fact

$$[A_\varepsilon x, A_\varepsilon x] = [Ax, Ax] + (\varepsilon^2 + 2\varepsilon)[P^+Ax, P^+Ax] \geq (\varepsilon^2 + 2\varepsilon)[P^+Ax, P^+Ax] > 0$$

for any x with $A_\varepsilon x \neq 0$, since if $A_\varepsilon x \neq 0$ then $Ax \neq 0$ and thus $P^+Ax \neq 0$. Especially A_ε is a plus matrix. For $\mu = \mu_-(A)/2 < 0$ the matrix $A^{[*]}A - \mu I \geq_H 0$ is invertible and thus by making $\varepsilon > 0$ small enough the same holds for A_ε. Hence we have $\mu_-(A_\varepsilon) \leq \mu_-(A)/2 < 0$. Thus, by Lemma 7.3 $\mathcal{N}(A_\varepsilon{}^{[*]}A_\varepsilon)$ is positive (and $\mu_+(A_\varepsilon) = 0$, see Lemma 7.9). This implies that $A_\varepsilon{}^{[*]}A_\varepsilon$ is a Pesonen matrix, and hence the matrix A_ε is sub-Pesonen for $\varepsilon > 0$ small enough. To find a desired approximation by stable sub-Pesonen matrices we make use of the H-s.v.d. of A_ε. Indeed, if $\mathcal{R}(A)$ is not yet maximal positive (i.e. $\mathcal{N}(A^{[*]})$ is not negative) then $A_\varepsilon' = A_\varepsilon + \delta[.,v]u$ ($\delta > 0$), where $u \in \mathcal{R}(A_\varepsilon)^{[\perp]}$ is positive and $v \in \mathcal{N}(A_\varepsilon)$, is again a sub-Pesonen matrix. By repeating this process finitely many times we can assume that $\mathcal{R}(A_\varepsilon)$ is maximal positive, i.e. that A_ε is a stable sub-Pesonen matrix (cf. Proposition 6.15).

Suppose then that A is a nonstrict plus matrix such that also $A^{[*]}$ is a plus matrix. Define $A_\rho = (I + \rho P^+)A(I + \rho P^+)$, $\rho > 0$. As above it is verified that $\mathcal{R}(A_\rho)$ is positive and thus $\mathcal{R}(A_\rho{}^{[*]}) = \mathcal{R}(A_\rho{}^{[*]}A_\rho)$ (cf. Lemma 2.3). Since $A^{[*]}$ and $I + \rho P^+$ are plus matrices, this identity shows that also $\mathcal{R}(A_\rho{}^{[*]})$ is positive. Thus $\mathcal{N}(A_\rho{}^{[*]}A_\rho) = \mathcal{N}(A_\rho) = \mathcal{R}(A_\rho{}^{[*]})^{[\perp]}$ is nondegenerate and $\lambda_i = 0$ must be a semisimple eigenvalue of $A_\rho{}^{[*]}A_\rho \geq_H 0$. Hence A_ρ has an H-s.v.d. (cf. Lemma 7.3), and by a similar extension as made above for A_ε we can find in every neighbourhood of A a matrix A_ρ' such that $\mathcal{R}(A_\rho')$ and $\mathcal{R}(A_\rho'^{[*]})$ are in fact maximal positive linear subspaces. Then $A_\rho'^{[*]}A_\rho' \geq_H 0$ and the linear subspaces $\mathcal{N}(A_\rho'^{[*]}A_\rho')$ and $\mathcal{N}(A_\rho'^{[*]})$ are negative and thus A_ρ' is a stable sub-Pesonen matrix approximating A. This proves the last part of the theorem.

A matrix $A \in \mathbf{C}^{n\times n}$ is called a *minus matrix* if $A\mathcal{B}_n^- \subset \mathcal{B}_n^-$, i.e. if A maps nonpositive vectors onto nonpositive vectors. Equivalently, one can interpret a minus matrix as a plus matrix operating on the *antispace* $(\mathbf{C}^n, (-H.,.))$ of the space $(\mathbf{C}^n, (H.,.))$. It is obvious that results analogous to those proved above for plus matrices can be established to minus matrices and to the corresponding subclasses of them as well.

8. On a connection between H-singular values and eigenvalues

It is a known fact that the ordinary (definite) singular values of a *normal matrix* coincide with the moduli of the eigenvalues of A, i.e. $d_i(A) = |\lambda_i(A)|$ if A is normal. The next theorem gives a generalization of this result into the indefinite scalar product case. For this, one needs a closer insight into the spectral structure of an H-normal matrix. As usual, any matrix commuting with its H-adjoint will be called an H-*normal matrix*. It is noted, that any complex matrix can be regarded as an H-normal matrix with respect to some indefinite scalar product. However, the existence of an

H-s.v.d. for such an A guarantees a somewhat specialized structure for the matrix A.

Let H be a Hermitean and nonsingular matrix. By $\mathcal{N}_H$ we denote the *class of square matrices* U for which $U^{[*]}U = -I$.

Since any $U \in \mathcal{N}_H$ is clearly invertible, the class $\mathcal{N}_H$ is nonempty only under specific conditions. In particular, one must have $\mathrm{In}\, H = \mathrm{In}\,(-H)$, so that the sums of the multiplicities of the positive and the negative eigenvalues of H must be equal.

Before proving our next lemma, we recall that two linear subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ are said to be *skewly linked* (or to form a *dual pair* in the sense of Bognár [4]) if $\mathcal{K}_1 \cap \mathcal{K}_2^{[\perp]} = \mathcal{K}_2 \cap \mathcal{K}_1^{[\perp]} = \{0\}$.

8.1. Lemma. *Let $U \in \mathcal{N}_H$. If ν is an eigenvalue of U then the number $\nu^* = -1/\bar{\nu}$ is also an eigenvalue of U. All the principal subspaces $\mathcal{S}_\nu$ of U are neutral. Moreover, the principal subspaces $\mathcal{S}_\nu$ and $\mathcal{S}_{\nu^*}$ are skewly linked and their sum $\mathcal{S}_\nu + \mathcal{S}_{\nu^*}$ is nondegenerate.*

Proof. It is enough to show that for $\nu, \mu \in \sigma(U)$ satisfying $\nu \neq \mu^*$ the principal subspaces $\mathcal{S}_\nu$ and $\mathcal{S}_\mu$ are H-orthogonal to each other. After this the rest can be proved as follows. First, since always $\nu \neq \nu^*$, all $\mathcal{S}_\nu$ are neutral. For $\nu \in \sigma(U)$ the orthogonal companion of $\mathcal{S}_\nu$ contains $\mathcal{S}_\nu$ and also $\mathcal{S}_\mu$ for every $\mu \neq \nu^*$. By equation (1.1) of Section 1, this gives an estimation

$$n - \dim \mathcal{S}_\nu = \dim (\mathcal{S}_\nu^{[\perp]}) \geq n - \dim \mathcal{S}_{\nu^*}.$$

Hence we have $\dim \mathcal{S}_{\nu^*} \geq \dim \mathcal{S}_\nu$, which implies $\nu^* \in \sigma(U)$. By symmetry, one has $\dim \mathcal{S}_\nu = \dim \mathcal{S}_{\nu^{**}} \geq \dim \mathcal{S}_{\nu^*}$. Thus

$$\mathcal{S}_\nu^{[\perp]} \cap \mathcal{S}_{\nu^*} = \mathcal{S}_{\nu^*}^{[\perp]} \cap \mathcal{S}_\nu = \{0\},$$

i.e., $\mathcal{S}_\nu$ and $\mathcal{S}_{\nu^*}$ are skewly linked.

The subspace $\mathcal{S}_\nu + \mathcal{S}_{\nu^*}$ is nondegenerate, since again by equation (1.1) its isotropic part must be $\{0\}$.

Next we prove the H-orthogonality of the subspaces $\mathcal{S}_\nu$ and $\mathcal{S}_\mu$ for every $\nu, \mu \in \sigma(U)$ with $\nu \neq \mu^*$. For every pair $x \in \mathcal{S}_\nu$ and $y \in \mathcal{S}_\mu$ there are $s, t \in \mathbb{N}$ such that $(U - \nu I)^s x = 0$ and $(U - \mu I)^t y = 0$. We shall prove the orthogonality of x and y.

In the case $s = t = 1$ we get

$$[x, y] = -[Ux, Uy] = -\nu\bar{\mu}\, [x, y].$$

Since $\nu \neq \mu^*$, i.e. $-\nu\bar{\mu} \neq 1$, this implies $[x, y] = 0$.

We proceed by induction on $s + t$. Denote $x_1 = (U - \nu I)x$ and $y_1 = (U - \mu I)y$. Then the H-orthogonality condition on all exponents s and t such that $s + t < n$ gives for $s + t = n$

$$\begin{aligned}[x, y] &= -[Ux, Uy] &&= -[x_1 + \nu x, y_1 + \mu y] \\ &= -[\nu x, \mu y] &&= -\nu\bar{\mu}\, [x, y].\end{aligned}$$

Hence again $[x, y] = 0$ and the lemma is proved.

In fact, the Jordan blocks of U corresponding to the eigenvalues ν and ν^* of U are of equal size. The converse statement is also true; if the spectrum of the matrix U satisfies the above symmetry relation and the sizes of the Jordan blocks of ν and ν^* are identical, then $U^{[*]}U = -I$ with respect to some indefinite scalar product. In other words, if U is similar to $-(U^*)^{-1}$ this similarity can be achieved with some Hermitean H.

Let us use for a while some of the terminology of *lattices of invariant subspaces*. First recall, that a *hyperinvariant subspace* of the matrix A is a linear subspace that is invariant under any matrix B commuting with A. It is well-known and easily verified that the collection of all hyperinvariant subspaces of the matrix A forms a lattice, the lattice operations being the sum and the intersection of the subspaces.

The lattice of the hyperinvariant subspaces for A can be explicitly derived. It contains all the principal subspaces of A, as shown by direct calculation. Accordingly, all the spectral subspaces (i.e. the sums of the principal subspaces) of A belong to it.

Let A be H-normal and suppose that A has an H-s.v.d. with respect to $[.,.] = (H.,.)$, H Hermitean and invertible. Then the left and the right singular subspaces corresponding to any H-singular value of A coincide. Furthermore, the above discussion applied to $A^{[*]}A$ tells us that every eigenspace corresponding to a fixed eigenvalue of $A^{[*]}A$ is a reducing subspace for A itself (i.e. a linear subspace such that it itself and some of its direct complements are A-invariant). This means that we can find a basis in $\mathbf{C}^n$ with respect to which A has a block diagonal structure, equivalently, A as a matrix is similar to a block diagonal matrix. Every block, say T_i, corresponds to a certain eigenvalue $\lambda_i \in \sigma(A^{[*]}A)$, hence to some H-singular value d_i of A, too. The eigenvalues of the block T_i are below referred to as the *eigenvalues of A corresponding to an H-singular value d_i of A*. The structure of A is completely characterized by the proof of the next theorem, which answers to the problem posed at the beginning of this section.

8.2. Theorem. *Let H be a Hermitean and nonsingular matrix. Suppose that A, $|A| \neq 0$, is an H-normal matrix and has an H-s.v.d. Let d_i be an H-singular value of A.*

(i) *Then the eigenvalue μ_i of A corresponding to d_i, originating from a positive eigenvalue of $A^{[*]}A$, satisfies $|\mu_i| = d_i$ if and only if the principal subspace $S_A(\mu_i)$ is nondegenerate. Furthermore, if $|\mu_i| \neq d_i$ then the principal subspace $S_A(\mu_i)$ is neutral.*

(ii) *For any singular value d_i, originating from a negative eigenvalue of $A^{[*]}A$, all the principal subspaces $S_A(\mu_i)$ for the eigenvalues μ_i of A corresponding to d_i are neutral. Moreover, the eigenvalues μ_i such that $|\mu_i| = d_i$ appear in pairs in such a way that with μ_i also $-\mu_i$ is an eigenvalue of A.*

Proof. Using the eigenvectors of $A^{[*]}A$ as a basis for $\mathbf{C}^n$, equivalently after a similarity transformation, we can assume that the matrices A, $A^{[*]}A$ and H are of the following form

$$A = T_1 \oplus \ldots \oplus T_p\,, \quad A^{[*]}A = \Lambda = \lambda_1 I_1 \oplus \ldots \oplus \lambda_p I_p$$

and

$$H = S = \text{diag}(s_1, \dots, s_n)$$

with $s_i = +1$ or -1, so that A commutes with Λ. Define $\text{sgn}\,\Lambda = \Lambda D^{-2}$, where D stands for the diagonal matrix of the H-singular values of A (cf. Remark 2.11). Then the identities

$$\Lambda = A^{[*]}A = S^{-1}A^{*}SA = SA^{*}SA$$

give for $K = K_1 \oplus \dots \oplus K_p$, $K_i = T_i/|\lambda_i|^{1/2}$, $i = 1, \dots, p$,

$$SK^{*}SK = \text{sgn}\,\Lambda .$$

Hence, K_i is either S_i-unitary or belongs to the class $\mathcal{N}_H$ determined by $[.,.] = (S_i.,.)$ according to the sign of λ_i. Here S_i denotes the diagonal block of S corresponding to K_i.

We have shown that each block T_i of A is just d_i times a matrix that is unitary with respect to an indefinite scalar product, if the corresponding eigenvalue $\lambda_i \in \sigma(A^{[*]}A)$ is positive, and d_i times a matrix that belongs to the class $\mathcal{N}_H$, if $\lambda_i \in \sigma(A^{[*]}A)$ is negative. Equivalently, this holds for the restrictions of A to the reducing subspaces determined by the principal subspaces of $A^{[*]}A$. Part (i) of the theorem follows then directly from the properties of H-unitary matrices operating on an indefinite scalar product space (cf. Lemma 1.5). Further, one gets part (ii) from Lemma 8.1 above.

Our proof gives also the following result.

8.3. Corollary. *Under the conditions of Theorem 8.2, the sum of the multiplicities of the negative eigenvalues in $\sigma(A^{[*]}A)$ is even.*

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University of Helsinki
Department of Statistics
Aleksanterinkatu 7
SF-00100 Helsinki
Finland