ANNALES ACADEMIÆ SCIENTIARUM FENNICÆ

MATHEMATICA

DISSERTATIONES

134

DIAMONDS ON LARGE CARDINALS

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HELSINKI 2003
SUOMALAINEN TIEDEAKATEMIA

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Academia Scientiarum Fennica
ISSN 1239-6303
ISBN 951-41-0929-5

Received 13 November 2003

2000 Mathematics Subject Classification:
Primary 03E05; Secondary 03E35, 03E55.

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HELSINKI 2003

Acknowledgements

I am grateful for having had the opportunity to work under the supervision of Professor Jouko Väänänen. He suggested to me the subject that was to be the starting point of this work that constitutes a part of my doctoral dissertation.

I would also like to thank all the other members of the Helsinki Logic Group, who have contributed to a pleasant and inspiring working environment. Especially I wish to thank Docent Tapani Hyttinen for always taking the time to discuss and giving his guidance.

Finally I wish to thank Professor Boban Veličković and Docent Kerkko Luosto for carefully reading the manuscript and suggesting many improvements.

Helsinki, November 2003 Alex Hellsten

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1. Introduction

Let κ be a weakly compact cardinal. Consider the sets of the form $\{\alpha < \kappa : \langle V_\alpha, \epsilon, U \cap V_\alpha \rangle \models \neg\phi\}$, where $U \subseteq V_\kappa$ and ϕ is a Π^1_1-sentence such that $\langle V_\kappa, \epsilon, U \rangle \models \phi$. This collection of sets generates a normal ideal over κ, a proper extension of the non-stationary ideal over κ. Sets of positive measure with respect to this ideal are called weakly compact. Thus every weakly compact subset of κ is stationary but not vice versa.

The following combinatorial principle was defined by Sun in [18] and independently by Shelah in [16]: There exists a sequence $(A_\alpha : \alpha < \kappa)$ such that

$$\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$$

is weakly compact for every set A. This strengthening of the classical diamond principle can be referred to as the weakly compact diamond. It is applied in [16] where Shelah and Väänänen show that weakly compact diamond together with the assumption $2^\kappa = \kappa^+$ implies that the logic $\mathscr{L}_{\kappa\kappa}$ has no strongest extension with certain Löwenheim-Skolem and compactness properties.

In [16] it was stated without proof that weakly compact diamond holds for every measurable cardinal, holds in the constructible universe for every weakly compact cardinal, and can be obtained through forcing. After providing proofs of these three claims we discovered that Sun had independently proved the first two claims in [18]. Indeed in both proofs of the first claim, measurability is replaced with ineffability, a considerably weaker assumption. However this work initiated a broader study of normal ideals over regular cardinals and related diamond principles with the main focus on weak compactness.

In Section 2.4 the concept of an n-club is defined and it is proved that the n-club sets generate the Π^1_n-filter, provided that it is proper. The Π^1_1-filter is the dual of the weakly compact ideal. In the light of the results presented in this paper it can be argued that the n-clubs are a rather natural and canonical generalisation of closed unbounded sets. In Section 5.3 it is shown that a weakly compact set may be killed using forcing, by shooting a 1-club through its complement. It is interesting to note that the weak compactness of the complement of the set to be killed is the only assumption necessary for this argument. In this respect the weakly compact ideal behaves like the non-stationary ideal on $\aleph_1$.

In Section 3.1 a general definition for ideals like $\mathrm{ND}_{\aleph_1}$, the "no diamond" ideal, is provided and some basic facts about these ideals are presented. Then the facts about ineffability and the axiom of constructibility implying weakly compact diamond are proved in a slightly more general form in sections 3.3 and 4.2.

Forcing arguments are developed in Section 5. Weakly compact diamond holds in many forcing extensions, but it can also be killed using a forcing notion that preserves weak compactness. The consistency of the failure of weakly compact diamond on a weakly compact cardinal follows from a result by Hauser [8].

This monograph is part of the doctoral dissertation of the author, which also includes the paper [9].

2. Ideals over regular cardinals

An *ideal* over a regular cardinal κ is a nonempty collection $I \subseteq \mathscr{P}(\kappa)$ which is closed under finite unions and subsets. The *trivial* ideal over κ is $\{\varnothing\}$ and an ideal is called *proper* if it is not $\mathscr{P}(\kappa)$. Thus an ideal I is proper if and only if $\kappa \notin I$. The trivial ideal is a special case of a *principal* ideal; an ideal of the form $\{X \subseteq \kappa : X \cap A = \varnothing\}$ where A is some fixed subset of κ. The collection $[\kappa]^{<\kappa}$ is sometimes called the *ideal of small sets* and it is proper and non-principal.

For $W \subseteq \mathscr{P}(\kappa)$ we let W^* denote the collection $\{X \subseteq \kappa : \kappa \setminus X \in W\}$. A collection $F \subseteq \mathscr{P}(\kappa)$ is a *filter* over the regular cardinal κ if F^* is an ideal over κ. If this is the case, $I = F^*$ will be called the *dual ideal* of F and $F = I^*$ is called the *dual filter* of I. All properties of ideals have their analogues for filters, which can be thought of to be defined via the dual. Thus a principal filter is the dual of a principal ideal and so forth. Also, if we find it more convenient to define some concept for filters, then it is tacitly meant to be defined on ideals too, via the dual.

A subset of κ which is not in the ideal I is said to have *positive measure* with respect to I. Thus the complement $\mathscr{P}(\kappa) \setminus I$ is the collection of sets of positive measure, and it is denoted I^+. If F is a filter, F^+ means $(F^*)^+$. Likewise we can say that sets in the ideal have measure 0 and sets in the dual filter have measure 1. But note that this makes sense only for proper ideals, since an ideal I is proper if and only if the dual filter is contained in I^+, or equivalently, I is disjoint from its dual.

The following observations are trivial, but we want to state them explicitly since they are used tacitly, it seems, in virtually every proof that concerns itself with ideals. They can also be seen as a motivation for the measure theoretic terminology.

(a) A set has positive measure if and only if it intersects every set in the filter.

(b) A set is in the filter if and only if it intersects every set of positive measure.

(c) If E has positive measure and X is in the filter, then $E \cap X$ has positive measure.

Note that this holds regardless of whether the ideal in question is proper, which highlights our convention that there are no sets of positive measure with respect to the non-proper ideal over κ.

We shall only be interested in proper ideals that extend the ideal of small sets. These are necessarily non-principal. From now on we shall include the condition $[\kappa]^{<\kappa} \subseteq I$ in the definition of I being an *ideal* over a regular cardinal κ. We would also like to include $\kappa \notin I$ in the conditions, but we shall frequently encounter explicit definitions of ideals which may not be proper in all situations. And some simple facts can be more conveniently stated if $\mathscr{P}(\kappa)$ is considered to be an ideal. Thus we shall take the following standpoint. In some contexts ideals are taken to be proper by definition, and in some contexts not. Usually it is evident which of the two "definitions" is meant.

2.1. Completeness

Let $(X_\alpha : \alpha < \kappa)$ be a sequence of subsets of κ. The *diagonal intersection* $\triangle_{\alpha<\kappa} X_\alpha$ is the set

$$\bigcap_{\alpha<\kappa} (X_\alpha \cup [\alpha, \kappa)) = \{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} X_\alpha\}$$

and the *diagonal union* $\triangledown_{\alpha<\kappa} X_\alpha$ is the set

$$\bigcup_{\alpha<\kappa} (X_\alpha \cap [0, \alpha)) = \{\beta < \kappa : \beta \in \bigcup_{\alpha<\beta} X_\alpha\}.$$

Since $\triangledown_{\alpha<\kappa} X_\alpha = \kappa \backslash (\triangle_{\alpha<\kappa}(\kappa \backslash X_\alpha))$, a collection $W \subseteq \mathscr{P}(\kappa)$ is closed under diagonal intersections if and only if W^* is closed under diagonal unions.

Let I be an ideal over κ and let μ be another regular cardinal. The ideal I is said to be *μ-complete* if it is closed under unions of cardinality less than μ. As is customary, $\aleph_1$-completeness is referred to as *σ-completeness* or countable completeness. An ideal is *normal* if it is closed under diagonal unions.

2.1.1. Lemma. *If a collection $W \subseteq \mathscr{P}(\kappa)$ is closed under subsets, diagonal unions, and the operations $X \mapsto X \cup \alpha$ for $\alpha < \kappa$, then W is a normal κ-complete ideal.*

Proof. We only need to verify that W is closed under unions of cardinality less than κ. But this is clear since $\bigcup_{\xi<\alpha} X_\xi \subseteq (\triangledown_{\xi<\kappa} X_\xi) \cup \alpha$ for any ordinal $\alpha < \kappa$ and sequence $(X_\xi : \xi < \kappa)$ of subsets of κ. □

The canonical example of a normal ideal is the *ideal of non-stationary sets*, i.e. the collection of subsets of κ that are disjoint from some closed unbounded subset of κ. We shall denote this ideal by NS_κ. In fact NS_κ is the least normal ideal over κ. This is because any closed unbounded set C may be written as $\triangle_{\alpha<\kappa}[\min(C \cap (\alpha,\kappa)),\kappa)$, a diagonal intersection of final segments.

It is clear that an ideal over κ can not be μ-complete for any $\mu > \kappa$. By Lemma 2.1.1 it follows that every normal ideal over κ is κ-complete. Thus one can say that normality is a stronger requirement than μ-completeness for any relevant μ. The facts discussed above depend on our convention that all ideals extend the ideal of small sets by definition. This is the main motivation for having this convention.

A function f from a set of ordinals to the ordinals is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in \mathrm{dom}(f) \setminus \{0\}$. The connection between normal ideals and regressive functions is the following:

2.1.2. Lemma. *Suppose that $W \subseteq \mathscr{P}(\kappa)$ is closed under subsets. The following conditions are equivalent:*

(i) *W is closed under diagonal unions*

(ii) *For every $X \subseteq \kappa$ and $f : X \to \kappa$, if f is a regressive function such that $f^{-1}\{\alpha\} \in W$ for every $\alpha < \kappa$ then $X \in W$.*

Proof. Suppose that W is closed under diagonal unions and $f : X \to \kappa$ is regressive. If $\beta \in X$ then $\beta \in f^{-1}\{f(\beta)\}$ and thus $X \subseteq \bigtriangledown_{\alpha<\kappa} f^{-1}\{\alpha\}$. It follows that X must be in W if $f^{-1}\{\alpha\} \in W$ for every $\alpha < \kappa$.

For the other direction fix a sequence $(X_\alpha : \alpha < \kappa)$ of sets in W and let $X = \bigtriangledown_{\alpha<\kappa} X_\alpha$. It is straightforward to define a regressive function $f : X \to \kappa$ such that $\beta \in X_{f(\beta)}$ for every $\beta \in X$. Now $f^{-1}\{\alpha\} \subseteq X_\alpha \in W$ for every $\alpha < \kappa$. $\square$

Condition (ii) for $\mathrm{NS}_{\aleph_1}$ in the previous lemma is the classical Fodor's lemma.

2.2. Saturated ideals

Let I be an ideal. Two sets X and Y in I^+ are said to be *almost disjoint* with respect to I if $X \cap Y \in I$. Let μ be a regular or finite cardinal. The ideal I is μ-*saturated* if every subcollection of I^+ of pairwise almost disjoint sets has cardinality less than μ. The least μ such that I is μ-saturated is denoted $\mathrm{sat}\, I$.

Consider I^+ to be ordered by inclusion. Then I is μ-saturated if and only if I^+ satisfies the μ-chain condition in the sense of the standard definition for forcing notions.

A 2-saturated ideal is a *prime ideal.* As an exception to the convention that the same terminology is used for analogous properties of filters, the dual filter of a prime ideal is an *ultrafilter*. For a prime ideal I over κ it holds that $I \cup I^* = \mathscr{P}(\kappa)$ and thus a prime ideal is a maximal ideal.

For an ideal I over κ and a set $E \subseteq \kappa$ we let $I|E$ denote the collection $\{X \subseteq \kappa : X \cap E \in I\}$. It is straightforward to see that $I|E$ is an ideal with the same closure properties that I satisfies, i.e. if I is μ-complete then $I|E$ is μ-complete and if I is normal then $I|E$ is normal. Note that $I|E$ is the ideal generated by the set $I \cup \{\kappa \setminus E\}$. A basic observation is that

(a) $I \subseteq I|E$

(b) $I|E = I$ if and only if $E \in I^*$

(c) $I|E$ is proper if and only if $E \notin I$.

For the sake of completeness we wish to present the following two lemmas that are due to Baumgartner, Taylor, and Wagon [4].

2.2.1. Lemma. *Let I and J be ideals over a regular cardinal κ such that $I \subseteq J$, and let μ be a regular cardinal. If either one of the conditions*

(i) *I is μ-saturated and J is μ-complete*

(ii) *I is κ^+-saturated and J is normal*

hold, then $J = I|E$ for some $E \subseteq \kappa$.

Proof. Assume that I is μ-saturated and suppose that J is μ-complete unless $\mu = \kappa^+$ in which case we assume that J is normal. Let $\{X_i : i < \gamma\}$ be a maximal collection of sets in $J \setminus I$ that are pairwise almost disjoint with respect to I. Since I is μ-saturated, $\gamma < \mu$. Let $S = \triangledown_{i<\kappa} X_i$ if $\gamma = \kappa$ or else let $S = \bigcup_{i<\gamma} X_i$. Let $E = \kappa \setminus S$. We shall show that $J = I|E$.

By our assumptions $S \in J$ and thus it is clear that $I|E \subseteq J$. It is straightforward to see that $E \cap X_i \subseteq i+1$ for every $i < \gamma$ (in fact $E \cap X_i = \varnothing$ in the case $S = \bigcup_{i<\gamma} X_i$.) Let $X \in J \setminus I$ be arbitrary. Since $X \cap E \cap X_i \subseteq E \cap X_i \in I$, the set $X \cap E$ is almost disjoint from X_i for every $i < \gamma$. By the maximality of $\{X_i : i < \gamma\}$ we must have $X \cap E \in I$. □

Note that the conclusion $J = I|E$ of the lemma above can be thought of as meaning that J has a maximal element with respect to I. Namely $S \in J$ is maximal in the sense that $X \setminus S \in I$ for every $X \in J$ if and only if $J = I|(\kappa \setminus S)$.

2.2.2. Lemma. *Let I be an ideal over a regular cardinal κ and let μ be another regular cardinal.*

(a) *If I is μ-complete but not μ-saturated then I can be extended to a μ-complete ideal which is not of the form $I|E$.*

(b) *If I is normal but not κ^+-saturated then I can be extended to a normal ideal which is not of the form $I|E$.*

Proof. Let $\{X_i : i < \mu\}$ be a subcollection of $\mathscr{P}(\kappa)$ and suppose that $X_i \cap X_j \in I$ whenever $i \neq j$. Let K be an ideal over μ and consider the collection J of all $X \subseteq \kappa$ such that

$$\{i < \mu : X \cap X_i \notin I\} \in K.$$

It is straightforward to check that J is an ideal and that $I \cup \{X_i : i < \mu\} \subseteq J$. Clearly J is a proper ideal if and only if $\{i < \mu : X_i \notin I\} \notin K$. If $J \subseteq I|E$ for some $E \subseteq \kappa$ then $E \in J$ directly by the definition and the fact that $\{X_i : i < \mu\} \subseteq I|E$. Thus if J is a proper ideal it can not be of the form $I|E$.

For (a) let $K = [\mu]^{<\mu}$. Then since both I and K are μ-complete, also J must be μ-complete. Using the fact that I is not μ-saturated we can assume that $X_i \notin I$ for every $i < \mu$. Collecting the facts stated above, we have a proof of (a). For (b) we still use $K = [\mu]^{<\mu}$ but with $\mu = \kappa^+$. Again it is not hard to verify that J is normal because I is normal and K is κ^+-complete. □

2.3. Indescribability

We shall be dealing with higher order formulae in an extended language of set theory that in addition to the relation symbol $\in$ may include a finite number of unary predicate symbols and binary relation symbols. The higher order variables are always unary.

We shall often neglect to explicitly state the language we are using, but instead a statement like $\langle V_\kappa, \in, U\rangle \models \phi$ is tacitly expressing that ϕ is in the language consisting of $\in$ and one unary predicate symbol U. In fact any finite number of predicates and relations may be coded into one unary predicate using a first order definable coding. Thus we shall always use only one unary predicate unless some other language is motivated by notational convenience.

A formula is always equivalent with a formula in prenex normal form in which the quantifiers of the highest order are all collected in the beginning of the formula. Furthermore adjacent quantifiers of the same kind and order may be contracted into one, by coding the two variables into one by a first order definable coding.

Suppose that quantifiers of the highest order appearing in a formula ϕ have order $p+1$. Let us assume that the quantifiers of order $p+1$ are all collected in the beginning of ϕ and that they alternate so that no two existential nor two universal quantifiers are next to each other. Let n be the number of quantifiers of order $p+1$ in ϕ. Then if the first quantifier is existential ϕ is said to be a Σ^p_n-formula and if the first quantifier is universal then ϕ is a Π^p_n-formula.

Also formulae that are obviously equivalent to a Π^p_n-formula or a Σ^p_n-formula are said to be Π^p_n or Σ^p_n respectively. This hierarchy of formulae and definable concepts is often referred to as the *Levy hierarchy* because a study of it was initiated by Levy in [13]. We shall mainly be interested in Π^1_n-formulae since they provide a generalisation of the non-stationary ideal that seems to be fruitful in many ways. By Π^1_0-formulae we mean first order formulae in a language including at least one unary predicate.

Let $\phi(m)$ be a Π^1_n-formula where the free variable m is of first order. We say that $\phi(m)$ is *universal* if the following holds: For every Π^1_n-sentence σ there exists a number $m < \omega$ such that $\langle V_\kappa, \in, U\rangle \models \sigma$ if and only if $\langle V_\kappa, \in, U\rangle \models \phi[m]$ for any regular uncountable cardinal κ and predicate $U \subseteq V_\kappa$. We extend the definition to other formulae in the Levy hierarchy in the obvious way.

We shall now define an universal Π^1_1-formula $\phi(m)$. Fix some Gödel numbering of formulae with one free second order variable X and with all other variables bound and of first order. Let $\phi_m(X)$ denote the formula numbered with m. Simply by formalising the truth definition we can find a formula $\theta(T, X)$ with the following properties: Apart from the free second order variables T and X displayed, $\theta(T, X)$ contains only bounded first order variables. Furthermore for every $X \subseteq V_\kappa$,

$$\langle V_\kappa, \in, U\rangle \models \theta[T, X]$$

if and only if T is the set of pairs (m, a) such that $m < \omega$, a is an assignment function $\omega \to V_\kappa$ for the first order variables, and $\langle V_\kappa, \in, U\rangle \models_a \phi_m[X]$. Since every Π^1_1-sentence σ is equivalent to the sentence $\forall X \phi_m(X)$ for some $m < \omega$,

$$\forall X \forall T(\theta(T, X) \to \forall a \in {}^\omega V_\kappa((m, a) \in T))$$

is an universal formula. Note that the building blocks defined above can also be put together as

$$\exists X \exists T(\theta(T, X) \wedge \forall a \in {}^\omega V_\kappa((m, a) \in T))$$

which is an universal Σ^1_1-formula.

The following results about universal formulae have been well known since early developments of the subject.

2.3.1. Lemma. *There exists an universal Π^1_n-formula for every positive integer n.*

Proof. We argued above that universal Π^1_1-formulae exist. The only change required to that argument is that the second order variable X in $\phi_m(X)$ and $\theta(T, X)$ must be replaced by a string of n second order variables. Then

$$\forall X \exists Y \exists T(\theta(T, X, Y) \wedge \forall a \in {}^\omega V_\kappa((m, a) \in T))$$

is an universal Π^1_2-formula and

$$\forall X \exists Y \forall Z \forall T(\theta(T, X, Y, Z) \to \forall a \in {}^\omega V_\kappa((m, a) \in T))$$

is an universal Π^1_3-formula and so forth, where the two forms alternate depending on whether n is even or odd. Note that θ is a different formula for each n. □

Of course universal Σ^1_n-formulae exist too, and in fact the result generalises to all higher order formulae in the Levy hierarchy, but we shall only be needing the result of Lemma 2.3.1.

In the case of order $p+1$ the variables X and T are of order $p+1$ and all other variables in $\phi_m(X)$ and $\theta(T,X)$ are bounded and of order at most p. The assignment functions a have to assign values to all variables of order at most p. Thus the pairs $(m,a) \in T$ and the functions a must be coded using a flat pairing function in order to have $T \in V_{\kappa+p}$. After these changes the proof is the same as in the second order case.

A subset X of a regular cardinal κ is Π^1_n-*indescribable* if for every Π^1_n-sentence ϕ and every unary predicate $U \subseteq V_\kappa$ such that $\langle V_\kappa, \in, U\rangle \models \phi$, there exists an ordinal $\alpha \in X$ such that $\langle V_\alpha, \in, U \cap V_\alpha\rangle \models \phi$. A subset of κ which is not Π^1_n-indescribable is said to be Π^1_n-*describable.*

The following two lemmas also constitute well known observations (see e.g. Jech [10, Lemma 32.3]).

2.3.2. Lemma. *A set $X \subseteq \alpha$ is first order indescribable if and only if α is inaccessible and X is stationary.*

2.3.3. Lemma. *A set is Σ^1_{n+1}-indescribable if and only it is Π^1_n-indescribable.*

The following result is due to Levy [13].

2.3.4. Theorem. *For every natural number n and cardinal κ the collection of Π^1_n-describable subsets of κ is a normal ideal.*

Proof. Let W be the collection of Π^1_n-describable subsets of κ. It is evident that W is closed under subsets and rather easy to see that $X \cup \alpha \in W$ for every $X \in W$ and $\alpha < \kappa$. By Lemma 2.1.1 it then suffices to check that W is closed under diagonal unions.

Let $f : X \to \kappa$ be regressive where $X \subseteq \kappa$ and suppose that $f^{-1}\{i\} \in W$ for every $i < \kappa$. Thus for each $i < \kappa$ there is a Π^1_n-sentence ϕ_i and a predicate $U_i \subseteq V_\kappa$ such that $\langle V_\kappa, \in, U_i\rangle \models \phi_i$ and

$$\langle V_\alpha, \in, U_i \cap V_\alpha\rangle \models \neg\phi_i \tag{1}$$

whenever $f(\alpha) = i$.

By Lemma 2.3.1 there exists an universal Π^1_n-formula. Let $\phi(U_i, m)$ be the formula obtained from the universal formula by replacing occurrences of the unary predicate symbol with the second order variable U_i. Let g be a function $\kappa \to \omega$ such that ϕ_i is equivalent to $\phi(U_i, g(i))$. Put $U = \{(\xi, i) : \xi \in U_i\}$. Now

$$\langle V_\kappa, \in, U, g\rangle \models \forall i(\phi(U_i, g(i))) \tag{2}$$

and the right hand side is a Π^1_n-sentence if formalised properly. Of course U_i and $g(i)$ are expressed in the formalisation using the predicate symbols for U and g.

Now suppose that $\alpha \in X$ and fix $i = f(\alpha)$. Since (1) holds and $i < \alpha$,

$$\langle V_\alpha, \in, U \cap V_\alpha, g|\alpha\rangle \not\models \forall i(\phi(U_i \cap V_\alpha, g(i)))$$

where the right hand side is the same as in (2) when rendered formally. Thus $X \in W$ and by Lemma 2.1.2 it follows that W is closed under diagonal unions. □

We shall simply talk about the Π^1_n-*ideal* over κ when we mean the ideal of Π^1_n-describable subsets of κ. In some connections where the cardinal κ is clear from the context we let the symbol Π^1_n denote this ideal. By Lemma 2.3.2 the Π^1_0-ideal over κ is proper if and only if κ is inaccessible, and then it coincides with NS_κ. The Π^1_1-indescribable sets are also called *weakly compact* and the Π^1_1-ideal over κ will be denoted WC_κ. Sometimes we refer to WC_κ as the *weakly compact ideal.*

2.4. n-closed sets

Lemma 2.3.1 has the following well known application.

2.4.1. Lemma. *For every $n < \omega$ there exists a Π^1_{n+1}-sentence ϕ such that a set $X \subseteq \kappa$ is Π^1_n-indescribable if and only if $\langle V_\kappa, \in, X\rangle \models \phi$.*

Proof. We can find a Π^1_1-sentence that in $\langle V_\kappa, \in, X\rangle$ expresses that κ is inaccessible and X is stationary. Thus the case $n = 0$ is handled by Lemma 2.3.2. Assume that n is positive for the rest of the proof.

Let $\phi(U, m)$ be obtained from an universal Π^1_n-formula in the same way as in the proof of Theorem 2.3.4. A proper formalisation of

$$\forall U \forall m(\phi(U, m) \to \exists \alpha \in X(\langle V_\alpha, \in\rangle \models \phi(m, U \cap V_\alpha)))$$

is the required Π^1_{n+1}-sentence involving the unary predicate X. The formalisation of $\langle V_\alpha, \in\rangle \models \phi(m, U \cap V_\alpha)$ can be assumed to imply that α is a regular uncountable cardinal. □

A subset X of κ is $(n+1)$-*closed* if $\alpha \in X$ whenever $\alpha < \kappa$ and $X \cap \alpha$ is Π^1_n-indescribable as a subset of α. If X is both $(n+1)$-closed and Π^1_n-indescribable then X is said to be a $(n+1)$-*club* subset of κ. We let 0-*club* stand for closed unbounded.

In [18] Sun introduced a notion of 1-club as follows. A subset X of κ is 1-club if it is stationary and closed in the sense that for every regular $\alpha < \kappa$, if $X \cap \alpha$ is stationary then $\alpha \in X$. For this notion Sun also proved Theorem 2.4.2 in the case $n = 1$. For our purposes the difference between our definition of 1-club using Π^1_0-indescribability from Sun's notion is merely technical. In fact Sun's notion can be seen as the analogue using *weak* indescribability.

2.4.2. Theorem. *If a cardinal κ is Π^1_n-indescribable then a set $X \subseteq \kappa$ is in the Π^1_n-filter if and only it contains an n-club.*

Proof. For $n = 0$ the theorem follows from Lemma 2.3.2. Suppose now that $n \geq 1$ and X is n-club. We shall show that X is in the Π^1_n-filter. Let E be Π^1_n-indescribable. By Lemma 2.4.1 there is a Π^1_n-sentence ϕ that expresses Π^1_{n-1}-indescribability. Thus $\langle V_\kappa, \in, X\rangle \models \phi$ and therefore there exists an ordinal $\alpha \in E$ such that $\langle V_\alpha, \in, X \cap \alpha\rangle \models \phi$. It follows that $\alpha \in X \cap E$.

For the other direction suppose that $X = \{\alpha < \kappa : \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \phi\}$ where $U \subseteq V_\kappa$ and ϕ is a Π^1_n-sentence such that $\langle V_\kappa, \in, U\rangle \models \phi$. We shall show that X is n-club. Clearly X is Π^1_{n-1}-indescribable. Towards contradiction assume that $\alpha \notin X$ although $\alpha < \kappa$ and $X \cap \alpha$ is Π^1_{n-1}-indescribable. By Lemma 2.3.3 the set $X \cap \alpha$ is also Σ^1_n-indescribable and since $\langle V_\alpha, \in, U \cap V_\alpha\rangle \models \neg\phi$ there exists an ordinal $\beta \in X \cap \alpha$ such that $\langle V_\beta, \in, U \cap V_\beta\rangle \models \neg\phi$. But this is a contradiction by the definition of X. $\square$

The *Mahlo operation* $M(X)$ on subsets of κ has traditionally been defined by putting

$$M(X) = \{\alpha < \kappa : \mathrm{cf}\,\alpha > \omega \text{ and } X \cap \alpha \text{ is stationary}\}.$$

We define the operations $M^n : [\kappa]^\kappa \to \mathscr{P}(\kappa)$ for $n < \omega$ as follows:

$$M^n(X) = \{\alpha < \kappa : X \cap \alpha \text{ is } \Pi^1_n\text{-indescribable}\}.$$

We conclude this section by a simple fact that for $n = 0$ is analogous to a result proved by Baumgartner, Taylor, and Wagon [4] for the Mahlo operation. Their formulation is slightly different and related to the concept of an M-ideal defined in [4]. Note that the conclusion $M^n(E) \in (\Pi^1_{n+1})^*$ can be seen a straightforward generalisation of the fact that the set of limit points of an unbounded set is closed unbounded.

2.4.3. Lemma. *If E is a Π^1_n-indescribable subset of κ then the set $M^n(E)$ is in the Π^1_{n+1}-filter but not in the filter $(\Pi^1_n|E)^*$, whereby $\Pi^1_n|E \neq \Pi^1_{n+1}$.*

Proof. That $M^n(E)$ is in the Π^1_{n+1}-filter is an almost immediate corollary of Lemma 2.4.1. Suppose towards contradiction that $M^n(E) \in (\Pi^1_n|E)^*$. It means that there exists a set X in the Π^1_n-filter such that $X \cap E \subseteq M^n(E)$. We can even assume that $X = \{\alpha < \kappa : \langle V_\alpha, \in, U \cap V_\alpha\rangle \models \phi\}$ for some $U \subseteq V_\kappa$ and Π^1_n-sentence ϕ. Let $\alpha = \min(X \cap E)$. Since $\alpha \in M^n(E)$ there exists an ordinal $\beta \in X \cap E \cap \alpha$, a contradiction. $\square$

An immediate corollary of Theorem 2.3.4 and Lemmas 2.2.1 and 2.4.3 is that the Π^1_n-ideal over κ is not κ^+-saturated whenever the Π^1_{n+1}-ideal is proper. (If Π^1_{n+1} is not proper then trivially $\Pi^1_n|E = \Pi^1_{n+1}$ for any $E \in \Pi^1_n$.) In [4] it is proved that it suffices that κ is *greatly Mahlo* for NS_κ not to be κ^+-saturated. The version of Lemma 2.4.3 for the Mahlo operation is a central ingredient in Solovay's [17] classical result that any stationary subset of a regular cardinal κ can be split into κ pairwise disjoint stationary sets.

3. Operations on ideals

3.1. Diamond principles

Let I be a normal ideal over a regular cardinal κ. By $\diamondsuit(I)$ we shall denote the following statement: There exists a sequence $(A_\alpha : \alpha < \kappa)$ such that

$$\{\alpha < \kappa : A \cap \alpha = A_\alpha\} \notin I$$

for every set A. The classical diamond principle $\diamondsuit_\kappa$ is $\diamondsuit(\mathrm{NS}_\kappa)$. A sequence witnessing that $\diamondsuit(I)$ holds, as $(A_\alpha : \alpha < \kappa)$ above, is said to be a *$\diamondsuit(I)$-sequence*.

The next result is well known in many settings (see e.g. [11] or [6]) but it is seldom, if ever, noted that it applies to any normal ideal whatsoever. This is why we give a complete proof. However it seems that the case $I = \mathrm{NS}_\kappa$ is in some sense the most interesting since the proof relies on closed unbounded sets.

3.1.1. Lemma. *For any normal ideal I over κ, $\diamondsuit(I)$ holds if and only if there exists a sequence $(W_\alpha : \alpha < \kappa)$ such that $|W_\alpha| = |\alpha|$ for every $\alpha < \kappa$ and*

$$\{\alpha < \kappa : A \cap \alpha \in W_\alpha\} \notin I$$

for every set A.

Proof. Clearly only one direction of the equivalence requires an argument. Assume that $(W_\alpha : \alpha < \kappa)$ is as above. Let $f : \kappa \times \kappa \to \kappa$ be a bijection. There is a closed unbounded set C such that $f[\alpha \times \alpha] = \alpha$ for every $\alpha \in C$. Because $\kappa \setminus C \in \mathrm{NS}_\kappa$ and $\mathrm{NS}_\kappa \subseteq I$ it is straightforward to use f to construct an indexed family $(B^i_\alpha : i < \alpha < \kappa)$ such that

$$\{\alpha < \kappa : B \cap (\alpha \times \alpha) = B^i_\alpha \text{ for some } i < \alpha\} \notin I \tag{3}$$

for every set B. To be more precise this is achieved by picking the sets B^i_α in such a way that $\{f[B^i_\alpha] : i < \alpha\} = W_\alpha \cap \mathscr{P}(\alpha)$ whenever $\alpha \in C$.

Let $A^i_\alpha = \{\xi < \alpha : (\xi, i) \in B^i_\alpha\}$ when $i < \alpha < \kappa$ and let A^i_α be arbitrary when $\alpha \le i < \kappa$. Consider the sequences $(A^i_\alpha : \alpha < \kappa)$ for $i < \kappa$. We shall derive a contradiction from the assumption that none of these sequences is a $\diamondsuit(I)$-sequence.

So we assume that for every $i < \kappa$ there exists a set A_i such that $X_i = \{\alpha < \kappa : A_i \cap \alpha \ne A^i_\alpha\}$ is in the dual filter F of I. By normality $X = \Delta_{i<\kappa} X_i \in F$. Let $B = \bigcup_{i<\kappa}(A_i \times \{i\})$. If $\alpha \in X$ then for all $i < \alpha$ we have $\alpha \in X_i$ and thus $A_i \cap \alpha \ne A^i_\alpha$. But if α is in the left hand side of (3) then $A_i \cap \alpha = A^i_\alpha$ for some $i < \alpha$ because for $\xi, i < \alpha$ we have $\xi \in A_i \cap \alpha$ iff $(\xi, i) \in B$ and $(\xi, i) \in B^i_\alpha$ iff $\xi \in A^i_\alpha$. By (3) we have arrived at a contradiction. □

There is an immediate simple connection between diamond principles and saturation of ideals. Let A and B be subsets of κ and let $\gamma < \kappa$ be the least ordinal such that $A \cap \{\gamma\} \neq B \cap \{\gamma\}$ i.e. γ is the least ordinal where the sets A and B differ. Then for any sequence $(A_\alpha : \alpha < \kappa)$ we have $\{\alpha < \kappa : A \cap \alpha = A_\alpha\} \cap \{\alpha < \kappa : B \cap \alpha = A_\alpha\} \subseteq \gamma + 1$. Thus $\diamondsuit(I)$ implies that I is not 2^κ-saturated.

For a normal ideal I over κ, let $\diamondsuit^*(I)$ denote the following statement: There exists a sequence $(W_\alpha : \alpha < \kappa)$ such that $|W_\alpha| = |\alpha|$ for every $\alpha < \kappa$ and

$$\{\alpha < \kappa : A \cap \alpha \in W_\alpha\} \in I^*$$

for every set A. Of course an analogous change to the initial formulation of $\diamondsuit(I)$ would result in a provably false statement. We shall also talk about $\diamondsuit^*(I)$-*sequences* in a similar fashion as with $\diamondsuit(I)$.

By Lemma 3.1.1 it is clear that $\diamondsuit^*(I) \to \diamondsuit(I)$ for any normal ideal I. Let I and J be normal ideals such that $I \subseteq J$. Then $\diamondsuit(J)$ implies $\diamondsuit(I)$ but $\diamondsuit^*(I)$ implies $\diamondsuit^*(J)$. Thus the principle $\diamondsuit^*(\mathrm{NS}_\kappa)$, traditionally denoted by $\diamondsuit^*_\kappa$, is the strongest of these statements and implies both $\diamondsuit(I)$ and $\diamondsuit^*(I)$ for every normal ideal I over κ.

Let I be a normal ideal, let $E \in I^+$, and consider the following statements:

(a) There exists a sequence $(A_\alpha : \alpha \in E)$ such that $\{\alpha \in E : A \cap \alpha = A_\alpha\} \in I^+$ for every set A.

(b) There exists a sequence $(W_\alpha : \alpha \in E)$ such that $|W_\alpha| = |\alpha|$ for every $\alpha \in E$ and for every set A there exists a set $X \in I^*$ such that $A \cap \alpha \in W_\alpha$ for every $\alpha \in X \cap E$.

Traditionally (a) for $I = \mathrm{NS}_\kappa$ has been denoted $\diamondsuit_E$ or $\diamondsuit_\kappa(E)$ and (b) similarly $\diamondsuit^*_E$ or $\diamondsuit^*_\kappa(E)$. But it is not hard to see that in fact (a) and (b) are equivalent to $\diamondsuit(I|E)$ and $\diamondsuit^*(I|E)$ respectively. Therefore there is no need for us to introduce special notation for this kind of principles. But we may still use e.g. $\diamondsuit_\kappa(E)$ as a shorthand for $\diamondsuit(\mathrm{NS}_\kappa|E)$.

For a normal ideal I over κ we define the collections $\mathrm{ND}(I) \subseteq \mathscr{P}(\kappa)$ and $\mathrm{SD}(I) \subseteq \mathscr{P}(\kappa)$ as follows:

$$\mathrm{ND}(I) = \{X \subseteq \kappa : \diamondsuit(I|X) \text{ fails}\}$$

$$\mathrm{SD}(I) = \{X \subseteq \kappa : \diamondsuit^*(I|X) \text{ holds}\}.$$

The letters ND and SD refer to “no diamond” and “strong diamond” respectively.

It turns out that these collections are ideals. By ND_κ and SD_κ we shall denote the ideals $\mathrm{ND}(\mathrm{NS}_\kappa)$ and $\mathrm{SD}(\mathrm{NS}_\kappa)$ respectively. $\mathrm{ND}\,I$ is proper iff $\diamondsuit(I)$ holds, and $\mathrm{SD}\,I$ is proper iff $\diamondsuit^*(I)$ fails.

The ideal $\mathrm{ND}_{\aleph_1}$ has been studied in literature. The result that $\mathrm{ND}_{\aleph_1}$ is normal is due to Saharon Shelah and was announced in [5] where Devlin proves that $\mathrm{ND}_{\aleph_1}$ is countably complete.

3.1.2. Theorem. *If I is a normal ideal over κ then* $\mathrm{ND}(I)$ *and* $\mathrm{SD}(I)$ *are both normal ideals over κ and* $\mathrm{ND}(I) \cap \mathrm{SD}(I) = I$.

Proof. It is easy to see that $\mathrm{ND}(I)$ is closed under subsets and that $X \in \mathrm{ND}(I)$ and $\alpha < \kappa$ implies $X \cup \alpha \in \mathrm{ND}(I)$. The same holds for $\mathrm{SD}(I)$. Thus, to see that $\mathrm{ND}(I)$ and $\mathrm{SD}(I)$ are normal ideals it suffices, by Lemma 2.1.1, to check that the collections in question are closed under diagonal unions.

Let $X \subseteq \kappa$ and let $f : X \to \kappa$ be a regressive function such that $f^{-1}\{i\} \in \mathrm{ND}(I)$ for every $i < \kappa$. We shall show that $X \in \mathrm{ND}(I)$ by proving that an arbitrary sequence $(A_\alpha : \alpha < \kappa)$ can not be a $\diamondsuit(I|X)$-sequence. It follows by Lemma 2.1.2 that $\mathrm{ND}(I)$ is closed under diagonal unions.

Let $g : \kappa \times \kappa \to \kappa$ be a bijection and let C be a closed unbounded set such that $g[\alpha \times \alpha] = \alpha$ for every $\alpha \in C$. For ordinals i and α in κ we put

$$A^i_\alpha = \{\xi : (\xi, i) \in g^{-1}[A_\alpha]\}. \tag{4}$$

Since $(A^i_\alpha : \alpha < \kappa)$ can not be a $\diamondsuit(I|f^{-1}\{i\})$-sequence there exists a set $A^i \subseteq \kappa$ and a set $Y_i \in I^*$ such that

$$\{\alpha \in f^{-1}\{i\} : A^i \cap \alpha = A^i_\alpha\} \cap Y_i = \varnothing. \tag{5}$$

Let $A = g[\bigcup_{i<\kappa}(A^i \times \{i\})]$. We shall conclude the proof of $X \in \mathrm{ND}(I)$ by showing that

$$\{\alpha \in X : A \cap \alpha = A_\alpha\} \cap \triangle_{i<\kappa} Y_i \cap C = \varnothing. \tag{6}$$

For $\alpha \in C$ we have $A \cap \alpha = g[\bigcup_{i<\alpha}((A^i \cap \alpha) \times \{i\})]$. So if $A \cap \alpha = A_\alpha$ we must have $A^i_\alpha = A^i \cap \alpha$ for every $i < \alpha$ by (4). But if $\alpha \in X$ and we fix $i = f(\alpha)$ it then follows from (5) that $\alpha \notin Y_i$ and thus (6) holds.

For $\mathrm{SD}(I)$ the proof is somewhat easier. Suppose again that $X \subseteq \kappa$ and $f : X \to \kappa$ is a regressive function such that $f^{-1}\{i\} \in \mathrm{SD}(I)$ for every $i < \kappa$. So for each $i < \kappa$ there exists a $\diamondsuit^*(I|f^{-1}\{i\})$-sequence $(W^i_\alpha : \alpha < \kappa)$.

Let A be an arbitrary set. Now $X_i = \{\alpha \in f^{-1}\{i\} : A \cap \alpha \notin W^i_\alpha\} \in I$ for every $i < \kappa$ and thus

$$\{\alpha \in X : A \cap \alpha \notin W^{f(\alpha)}_\alpha\} = \triangledown_{i<\kappa} X_i \in I$$

whereby $(W^{f(\alpha)}_\alpha : \alpha < \kappa)$ is a $\diamondsuit^*(I|X)$-sequence.

Finally it is a triviality that $I \subseteq \mathrm{ND}(I)$ and $I \subseteq \mathrm{SD}(I)$ and by Lemma 3.1.1 we have that $E \in \mathrm{SD}(I) \setminus I$ implies $E \notin \mathrm{ND}(I)$. $\square$

3.2. Subtlety and ineffability

To make some phrasings more fluent we shall talk about a *subset sequence* when we mean a sequence $(A_i : i \in X)$ such that $A_i \subseteq i$ for every $i \in X$.

Let X be a set of ordinals. A set $H \subseteq X$ is *homogeneous* for a sequence $(A_\alpha : \alpha \in X)$ if $A_\alpha = A_\beta \cap \alpha$ for every pair of ordinals $\alpha < \beta$ in H. A simple observation sometimes used in proofs is that if H is a set of ordinals, then H is homogeneous for the sequence $(A_i : i \in X)$ if and only if there exists a set A such that $H \subseteq \{i \in X : A \cap i = A_i\}$.

Let I be an ideal over κ. By $\operatorname{Sb} I$ we denote the collection of sets $X \subseteq \kappa$ with the property that there exists a set $Y \in I^*$ and a subset sequence $(A_\alpha : \alpha \in X \cap Y)$ for which there is no homogeneous set of cardinality 2. By $\operatorname{In} I$ we mean the collection of sets $X \subseteq \kappa$ such that there exists a subset sequence $(A_\alpha : \alpha \in X)$ for which every homogeneous set belongs to I.

To a large extent the following is due to Baumgartner [2].

3.2.1. Lemma. *If I is an ideal over κ and $\mathrm{NS}_\kappa \subseteq \operatorname{In} I$ then $\operatorname{In} I$ is a normal ideal extending I.*

Proof. To begin with it is not difficult to see that $\operatorname{In} I$ is an ideal whenever I is an ideal and that $I \subseteq \operatorname{In} I$. We shall now use Lemma 2.1.2 to check the normality.

Let $X \subseteq \kappa$, let $f : X \to \kappa$ be regressive and for each $\beta < \kappa$, suppose that the subset sequence $(A^\beta_\alpha : \alpha \in f^{-1}\{\beta\})$ witnesses that $f^{-1}\{\beta\} \in \operatorname{In} I$. Let $g : \kappa \times \kappa \to 2$ be a bijection. Because we already know that $\operatorname{In} I$ is an ideal such that $\mathrm{NS}_\kappa \subseteq \operatorname{In} I$ we may assume that $g[\alpha \times 2] = \alpha$ for every $\alpha \in X$. Put

$$A_\alpha = g[(A^{f(\alpha)}_\alpha \times \{0\}) \cup (f(\alpha) \times \{1\})]$$

for each $\alpha \in X$. Let A be arbitrary and put $H = \{\alpha \in X : A \cap \alpha = A_\alpha\}$. For $\xi < \alpha$ both in H we have $g^{-1}[A_\xi] = g^{-1}[A_\alpha] \cap (\xi \times \xi)$ and thus f must be constant on H. Let β be the constant value of f in H. If H is unbounded in κ then

$$g^{-1}[A] = \bigcup_{\alpha \in H} g^{-1}[A_\alpha] = \bigcup_{\alpha \in H} (A^\beta_\alpha \times \{0\}) \cup (\beta \times \{1\})$$

whereby H is homogeneous for $(A^\beta_\alpha : \alpha \in f^{-1}\{\beta\})$. It follows that $H \in I$. □

Note that we did not require in the previous lemma that I must be normal, but only that $\mathrm{NS}_\kappa \subseteq \operatorname{In} I$. This is utilised through the following fact.

3.2.2. Lemma. $\mathrm{NS}_\kappa \subseteq \operatorname{In}([\kappa]^{<\kappa})$.

Proof. Let C be a closed unbounded subset of κ. Let $H \subseteq \kappa \setminus C$ be homogeneous for the subset sequence $(\max(C \cap \alpha) : \alpha \in \kappa \setminus C)$. Now suppose that $\alpha \in H$ and $\beta = \min(C \setminus \alpha)$. Then we must have $H \subseteq \beta$ and it follows that $\kappa \setminus C \in \operatorname{In}([\kappa]^{<\kappa})$. □

Since $[\kappa]^{<\kappa}$ is the smallest possible ideal over κ by our strict definition, it follows that $\operatorname{In} I$ is always a normal ideal. The operation Sb is not as well behaved as In.

3.2.3. Lemma. *Let I be an ideal over κ and let μ be another regular cardinal. Then $\operatorname{Sb} I$ is an ideal extending I and if I is normal then $\operatorname{Sb} I$ is normal. If $\mathrm{NS}_\kappa \subseteq \operatorname{Sb} I$ and I is μ-complete then $\operatorname{Sb} I$ is μ-complete.*

Proof. As in the proof of Lemma 3.2.1 it is easy to see that $\operatorname{Sb} I$ is an ideal such that $I \subseteq \operatorname{Sb} I$. We shall deal with normality through Lemma 2.1.1.

Let $f : X \to \kappa$ be regressive and for each $\beta < \kappa$, suppose that $Y_\beta \in I^*$ and $(A^\beta_\alpha : \alpha \in f^{-1}\{\beta\} \cap Y_\beta)$ witness that $f^{-1}\{\beta\} \in \operatorname{Sb} I$. Let $g : \kappa \times \kappa \to \kappa$ be a bijection. As in the proof of the normality of $\operatorname{In} I$ we can assume that $g[\alpha \times \alpha] = \alpha$ for every $\alpha \in X$. Still as in the proof of Lemma 3.2.1, put

$$A_\alpha = g[(A^{f(\alpha)}_\alpha \times \{0\}) \cup (f(\alpha) \times \{1\})]$$

for $\alpha \in X$. Let $Y = \Delta_{\beta<\kappa} Y_\beta$. If $\{\xi, \alpha\} \subseteq X \cap Y$ and $A_\xi = A_\alpha \cap \xi$ then $f(\xi) = f(\alpha)$ and $A^{f(\xi)}_\xi = A^{f(\alpha)}_\alpha \cap \xi$. Since $\{\xi, \alpha\} \subseteq Y$ it then follows that $\{\xi, \alpha\} \subseteq Y_{f(\xi)}$ and therefore we must have $\xi = \alpha$. We conclude that $X \in \operatorname{Sb} I$.

For the μ-completeness we can use the same argument as above, if we assume that $\operatorname{ran}(f) \subseteq \mu$ and put $Y = \bigcap_{\beta<\mu} Y_\beta$. But in this case I does not necessary extend NS_κ so we need the assumption $\mathrm{NS}_\kappa \subseteq \operatorname{Sb} I$ to be able to assume that $g[\alpha \times \alpha] = \alpha$ for all $\alpha \in X$ without loosing generality. □

3.2.4. Lemma. *Let I and J be ideals over κ. If $J \subseteq \operatorname{In} I$ then $\operatorname{Sb} J \subseteq \operatorname{In} I$.*

Proof. Suppose that $X \subseteq \kappa$, $Y \in J^*$, and the subset sequence $(A_\alpha : \alpha \in X \cap Y)$ has no homogeneous sets of cardinality 2. We wish to prove that $X \in \operatorname{In} I$ but since $Y \in (\operatorname{In} I)^*$ it suffices to prove that $X \cap Y \in \operatorname{In} I$, but this of course is immediate. □

By lemmas 3.2.2 and 3.2.4 and the obvious fact that the operation In is monotone, we have

$$\mathrm{NS}_\kappa \subseteq \operatorname{Sb} \mathrm{NS}_\kappa \subseteq \operatorname{In}([\kappa]^{<\kappa}) \subseteq \operatorname{In} \mathrm{NS}_\kappa$$

where each of the four collections involved is a normal ideal over κ by lemmas 3.2.1 and 3.2.3. The cardinal κ is said to be *subtle* if $\operatorname{Sb} \mathrm{NS}_\kappa$ is proper, *almost ineffable* if $\operatorname{In}([\kappa]^{<\kappa})$ is proper, and *ineffable* if $\operatorname{In} \mathrm{NS}_\kappa$ is proper. These notions were introduced by Jensen and Kunen [11]. The letter combinations Sb and In used for the operations involved refer to these concepts. The sets that have positive measure with respect to the ideals above are also called *subtle*, *almost ineffable*, and *ineffable* respectively.

One can consider applying the operations defined in sections 3.1 and 3.2 repeatedly. Furthermore the union of a collection of ideals over a regular cardinal κ is itself an ideal over κ. Therefore let us put $\operatorname{In}^0 I = I$, $\operatorname{In}^{\alpha+1} = \operatorname{In}(\operatorname{In}^\alpha I)$, and $\operatorname{In}^\alpha I = \bigcup_{\beta<\alpha} \operatorname{In}^\beta I$ for limit ordinals α. We shall use analogous notation for the other operations defined.

If an operation is repeated as above, then sooner or later a fixpoint $\operatorname{In}^\alpha I$ must be reached for which $\operatorname{In}^{\alpha+1} I = \operatorname{In}^\alpha I$. Let $\operatorname{In}^\infty I$ denote this fixpoint. If $\operatorname{In}^\infty \mathrm{NS}_\kappa$ is proper we say that the cardinal κ is *totally ineffable* and if $\operatorname{Sb}^\infty \mathrm{NS}_\kappa$ is proper then κ is *totally subtle*. By a simple induction argument Lemma 3.2.4 also holds with Sb replaced by Sb^∞. Thus almost ineffable cardinals are totally subtle.

By virtue of Lemma 3.2.1 the ideal $\operatorname{In}^\alpha I$ is normal for any successor ordinal α regardless of whether I is normal or not. In [2] Baumgartner studied the ideals $\operatorname{In}^\alpha \mathrm{NS}_\kappa$ and proved that if κ is totally subtle and $\operatorname{In}^\alpha \mathrm{NS}_\kappa = \operatorname{In}^\infty \mathrm{NS}_\kappa$ then $\alpha \geq \kappa^+$.

Let f be a function on $[\kappa]^2$. We call f a partition as we think of the set $[\kappa]^2$ being partitioned into parts labeled by the elements in $\operatorname{ran} f$. A subset H of κ is *homogeneous* for the partition f if f is constant on the set $[H]^2$. For $X \subseteq \kappa$ and an ideal I over κ we write $X \to (I^+)^2$ if we want to say that X has the following partition property: For every function $f : [\kappa]^2 \to 2$ there exists a homogeneous set $H \subseteq X$ such that $H \in I^+$. If S is not a subset of κ but $S \subseteq \mathscr{P}(\kappa)$ then $S \to (I^+)^2$ is taken to mean that every $X \in S$ has the partition property $X \to (I^+)^2$.

3.2.5. Lemma. *Let I be an ideal over a regular cardinal κ and let $X \subseteq \kappa$. If $\mathrm{NS}_\kappa \subseteq I$ then $X \notin \operatorname{In} I$ if and only if $X \to (I^+)^2$.*

Proof. Suppose that $X \notin \operatorname{In} I$ and $f : [X]^2 \to 2$ is arbitrary. Let $A_\beta = \{\alpha \in X \cap \beta : f(\alpha, \beta) = 1\}$ for every $\beta \in X$. Fix A so that $H = \{\beta \in X : A \cap \beta = A_\beta\} \notin I$. Now either $H \cap A$ or $H \setminus A$ is the homogeneous set we are looking for.

Now suppose that $X \to (I^+)^2$ and the subset sequence $(A_\alpha : \alpha \in X)$ is arbitrary. Define $f : [X]^2 \to 2$ by letting $f(\alpha, \beta) = 1$ iff A_α lexicographically precedes A_β. (We consider the lexicographic ordering of the characteristic functions with domain κ.)

Pick a set $H \notin I$ that is homogeneous for f. We define $A \subseteq \kappa$ by defining $A \cap \xi$ by induction on $\xi < \kappa$ using the following requirement: For each ξ there exists an ordinal $\eta(\xi) < \kappa$ such that $A \cap \xi = A_\alpha \cap \xi$ for every $\alpha \in H \setminus \eta(\xi)$. The limit steps are trivial and the successor steps are easily handled by the properties of the lexicographic ordering. Now $H' = \{\alpha \in H \cap \operatorname{acc} \kappa : \eta[\alpha] \subseteq \alpha\} \notin I$ ($\operatorname{acc} \kappa$ is the set of limit ordinals below κ) and $A_\alpha = A \cap \alpha$ whenever $\alpha \in H'$. □

If D is a normal measure on κ then $D \to (D)^2$ whereby $\operatorname{In}^\infty I \subseteq D^*$ for any normal ideal $I \subseteq D^*$. Since we know that $\mathrm{NS}_\kappa \subseteq D^*$, Lemma 3.2.5 implies that measurable cardinals are totally ineffable.

3.3. Subtlety and diamonds

If I and J are ideals over a regular cardinal κ then the collection

$$\{X \cup Y : X \in I,\ Y \in J\}$$

is an ideal over κ. Clearly this is the ideal generated by I and J, i.e. the smallest ideal containing both I and J. We shall denote it by $\langle I \cup J \rangle$. The ideals I and J are said to be *coherent* if $\langle I \cup J \rangle$ is proper. In [1] Baumgartner proves the following result:

3.3.1. Lemma. *Let κ be a regular cardinal.*

(a) $\langle \Pi^1_1 \cup \operatorname{Sb} \mathrm{NS}_\kappa \rangle = \operatorname{In}([\kappa]^{<\kappa})$.

(*b*) $\langle \Pi^1_{n+2} \cup \mathrm{Sb}\,\mathrm{NS}_\kappa \rangle = \mathrm{In}\,\Pi^1_n$ *for all* $n < \omega$.

Especially it follows that almost ineffable sets are weakly compact and ineffable sets are Π^1_2-indescribable. By Lemma 3.2.4 and monotonicity of the operation Sb it follows that $\mathrm{Sb}\,\Pi^1_1 = \mathrm{In}([\kappa]^{<\kappa})$ and $\mathrm{Sb}\,\Pi^1_{n+2} = \mathrm{In}\,\Pi^1_n$.

3.3.2. Lemma. *Let* I *be an ideal over a regular cardinal* κ.

(*a*) $\kappa \in \mathrm{In}(I|E)$ *if and only if* $E \in \mathrm{In}\,I$.

(*b*) $\kappa \in \mathrm{Sb}(I|E)$ *implies that* $E \in \mathrm{Sb}\,I$.

Proof. Any subset sequence $(A_\alpha : \alpha < \kappa)$ witnesses that $\kappa \in \mathrm{In}(I|E)$ if and only if its restriction $(A_\alpha : \alpha \in E)$ witnesses that $E \in \mathrm{In}\,I$. Given $Y \subseteq \kappa$ put $Z = Y \cup (\kappa \setminus E)$. Then $Y \in (I|E)^*$ iff $Z \in I^*$. If the sequence $(A_\alpha : \alpha \in Y)$ witnesses that $\kappa \in \mathrm{Sb}(I|E)$ then the sequence $(A_\alpha : \alpha \in E \cap Z)$ witnesses that $E \in \mathrm{Sb}\,I$. □

As a consequence of the following lemma, $\diamondsuit^*_\kappa$ fails whenever κ is ineffable which is a result due to Jensen and Kunen [11]. They also proved that the converse holds if $V = L$. Thus in L we have that $\diamondsuit^*_\kappa$ holds if and only if κ is not ineffable.

3.3.3. Lemma. $\mathrm{SD}\,I \subseteq \mathrm{In}\,I$ *for every normal ideal* I *over a regular cardinal* κ.

Proof. Suppose that $\kappa \in \mathrm{SD}\,I$ and $(W_\alpha : \alpha < \kappa)$ is a $\diamondsuit^*(I)$-sequence. Pick a subset sequence $(A_\alpha : \alpha < \kappa)$ such that $A_\alpha \notin W_\alpha$ for every $\alpha < \kappa$. There can not exist a set A such that $\{\alpha < \kappa : A \cap \alpha = A_\alpha\} \notin I$. Therefore $\kappa \in \mathrm{In}\,I$. Since I was arbitrary it follows that $\diamondsuit^*(I|E)$ implies that $\kappa \in \mathrm{In}(I|E)$ for any $E \subseteq \kappa$. By Lemma 3.3.2 we are done. □

In [11] it was also pointed out that $\diamondsuit_\kappa$ holds at any subtle cardinal κ (see [12]). In fact $\diamondsuit_\kappa(E)$ holds for any subtle set E. Thus $\mathrm{ND}_\kappa \subseteq \mathrm{Sb}\,\mathrm{NS}_\kappa$ in terms of the notations for the ideals. The proof of this fact generalises to other levels in the Levy hierarchy in a rather straightforward way. Sun [18] proved that $\diamondsuit(\mathrm{WC}_\kappa)$ holds for almost ineffable cardinals.

3.3.4. Theorem. *Let* κ *be a regular cardinal.*

(*a*) $\mathrm{ND}(\Pi^1_1) \subseteq \mathrm{In}([\kappa]^{<\kappa})$.

(*b*) $\mathrm{ND}(\Pi^1_{n+2}) \subseteq \mathrm{In}\,\Pi^1_n$ *for all* $n < \omega$.

Proof. We shall prove (*a*). The proof of (*b*) is nearly the same. Let E be an almost ineffable subset of κ. By induction on $\alpha \leq \kappa$ we define sets $A_\alpha \subseteq \alpha$, $U_\alpha \subseteq V_\alpha$, and Π^1_1-formulae ϕ_α such that the conditions

(i) $\langle V_\alpha, \in, U_\alpha \rangle \models \phi_\alpha$

(ii) If $\xi \in E \cap \alpha$ and $\langle V_\xi, \in, U_\alpha \cap V_\xi \rangle \models \phi_\alpha$ then $A_\alpha \cap \xi \neq A_\xi$

hold whenever it is possible to make them hold at a step $\alpha \leq \kappa$ in the construction. To be precise we do not choose actual Π^1_1-formulae (whatever that means) but rather natural numbers that code the appropriate formulae.

For ordinals $\alpha \leq \kappa$ such that conditions (i) and (ii) are met, put

$$X_\alpha = \{\xi \in E \cap \alpha : \langle V_\xi, \in, U_\alpha \cap V_\xi \rangle \models \phi_\alpha\} \tag{7}$$

and let $X_\alpha = \alpha$ otherwise. By the construction $(A_\alpha : \alpha < \kappa)$ will be a $\diamondsuit_\kappa(\mathrm{WC}_\kappa|E)$-sequence if and only if $X_\kappa = \kappa$. (Note that we intend condition (ii) to imply $0 \notin X_\alpha$, although the expression in the condition may not be meaningful for $\xi = 0$). We shall complete the argument by deriving a contradiction from the assumption $X_\kappa \neq \kappa$.

We first notice that $X_\alpha \neq \alpha$ for every $\alpha \in X_\kappa$. This is because $A_\kappa \cap \alpha$, $U_\kappa \cap V_\alpha$, and ϕ_κ would satisfy the conditions (i) and (ii) if they were to be chosen as A_α, U_α, and ϕ_α respectively.

Let f be a bijection $\kappa \times V_\kappa \times \omega \to \kappa$. There is a closed unbounded set C such that $f[\alpha \times V_\alpha \times \omega] = \alpha$ for every $\alpha \in C$. Let $B_\alpha = f[A_\alpha \times U_\alpha \times \{\phi_\alpha\}]$ for $\alpha \in X_\kappa \cap C$.

As pointed out earlier, the monotonicity of the operation Sb together with Lemma 3.2.4 and Lemma 3.3.1 implies that $\mathrm{Sb}\,\mathrm{WC}_\kappa$ is the almost ineffable ideal. Since $X_\kappa \cap C$ is the intersection of $E \notin \mathrm{Sb}\,\mathrm{WC}_\kappa$ and a set in the weakly compact filter, there exist ordinals $\xi < \alpha$ both in $X_\kappa \cap C$ such that $B_\alpha \cap \xi = B_\xi$. It follows that $A_\alpha \cap \xi = A_\xi$, $U_\alpha \cap V_\xi = U_\xi$ and $\phi_\alpha = \phi_\xi$. But this means that $X_\alpha = \alpha$ which is a contradiction since $\alpha \in X_\kappa$. □

4. Weak compactness

From now on we shall concentrate on the weakly compact ideal WC_κ and the principle $\diamondsuit(\mathrm{WC}_\kappa)$ which we may call *weakly compact diamond*. We shall also consider the principles $\diamondsuit(\mathrm{WC}_\kappa|E)$ where E is a weakly compact subset of κ.

4.1. Some notes on ultraproducts

Let κ be a regular cardinal. A *κ-complete algebra of sets* is a non-empty collection of sets which contains the union of all its members and is closed under set difference and unions and intersections of cardinality less than κ. Thus a σ-algebra could also be referred to as a $\aleph_1$-complete algebra of sets. Sometimes one talks about a *field of sets* instead of an algebra of sets. A subcollection F of a κ-complete algebra of sets S is a *filter* if it is closed under finite intersections and $X \in F$ and $X \subseteq Y \in S$ implies that $Y \in F$. The notions of *ultrafilter* and a μ-complete filter where $\mu \leq \kappa$ is a regular cardinal are defined as expected.

Let M be a transitive set. Typically M would be V_κ or a model of a large enough finite fragment of ZFC such that $\kappa \in M$. By $\operatorname{def} M$ we shall denote the collection of subsets of M that are definable over M. We shall require that $\kappa \in \operatorname{def} M$ which clearly holds for the examples of M mentioned above. Let $S \subseteq \mathscr{P}(\kappa)$ be a κ-complete algebra of sets such that $\operatorname{def}(M) \cap \mathscr{P}(\kappa) \subseteq S$.

For a first order formula $\phi(x_1, \dots, x_n)$ and for functions $f_1, \dots, f_n \in {}^\kappa M$ we put

$$X_{\phi[f_1,\dots,f_n]} = \{\alpha < \kappa : \phi^M[f_1(\alpha), \dots, f_n(\alpha)]\}.$$

If the functions $f_1, \dots, f_n$ are in $\operatorname{def} M$ then $X_{\phi[f_1,\dots,f_n]} \in S$ because of the requirement $\kappa \in \operatorname{def} M$. Let U be an ultrafilter on S. The condition $X_{f=g} \in U$ defines an equivalence relation $=_U$ on the set $\operatorname{def}(M) \cap {}^\kappa M$ and the condition $X_{f\in g} \in U$ defines a binary relation $[f] \in_U [g]$ on the set of equivalence classes. Consider the structure $\mathfrak{N} = \langle(\operatorname{def}(M) \cap {}^\kappa M)/=_U, \in_U\rangle$.

4.1.1. Lemma. *Let $\phi(x_1, \dots, x_n)$ be a first order formula and let $f_1, \dots, f_n \in \operatorname{def}(M) \cap {}^\kappa M$. If $\operatorname{def} M$ contains a well-ordering of M then*

$$\mathfrak{N} \models \phi[[f_1], \dots, [f_n]] \text{ if and only if } X_{\phi[f_1,\dots,f_n]} \in U.$$

Proof. By induction on the complexity of ϕ as in the standard Łoś's Theorem. The well-ordering of M is used in the existential quantifier step where it is required to find a function $f \in \operatorname{def}(M) \cap {}^\kappa M$ such that

$$\phi^M[f(\alpha), f_1(\alpha), \dots, f_n(\alpha)]$$

holds for all $\alpha < \kappa$ with the property that there exists at least one $x \in M$ for which $\phi^M[x, f_1(\alpha), \dots, f_n(\alpha)]$ holds. □

By the above lemma $\mathfrak{N}$ is extensional. Let us now assume that U is σ-complete. Then $\mathfrak{N}$ is also well-founded. Let N be the Mostowski collapse of $\mathfrak{N}$. N is called the *definable ultrapower* of M modulo U. The corresponding *canonical embedding* $j : M \to N$ is defined by $j(x) = \pi([f_x])$ where π is the Mostowski collapse and f_x is the constant function $\kappa \to M$ with value x. By Lemma 4.1.1 the canonical embedding j is an elementary embedding.

Hereafter we shall let $[f]$ denote the element of N that is the image of the equivalence class of $f : \kappa \to M$ under the Mostowski collapse, rather than the equivalence class itself.

4.1.2. Lemma. *If U is a κ-complete ultrafilter over κ and every regressive function in $\operatorname{def}(M) \cap {}^\kappa\kappa$ is constant on a set in U, then $\kappa = [\mathrm{id}_\kappa]$ and for any $X \in M \cap \mathscr{P}(\kappa)$ it holds that $X \in U$ if and only if $\kappa \in j(X)$.*

Proof. For the second claim, just compare id_κ and the constant function f_X. □

Through lemmas 4.1.4 and 4.1.6 below we shall make extensive use of the following characterisation of weak compactness due to Baumgartner [2].

4.1.3. Theorem. *Let $S \subseteq \mathscr{P}(\kappa)$ be a κ-complete algebra of sets and F a collection of regressive functions on κ such that $|F| = |S| = \kappa$, $\kappa \in S$, and $f^{-1}\{\alpha\} \in S$ for all $f \in F$ and $\alpha < \kappa$. If $X \in S$ is weakly compact then there exists a κ-complete ultrafilter U on S such that $X \in U$ and each $f \in F$ is constant on a set in U. Conversely if such an ultrafilter U exists for every S and F as above such that $X \in S$ then X is weakly compact.*

4.1.4. Lemma. *If E is a weakly compact subset of κ and M is a transitive set such that $E \in M$ and $\kappa \in \operatorname{def} M$ then there exists a transitive set N and an elementary embedding $j : M \to N$ such that $\kappa \in j(E)$.*

What is the role of $\operatorname{def} M$ in the arguments above? We shall now consider replacing $\operatorname{def} M$ by some other collection D such that $M \subseteq D \subseteq \mathscr{P}(M)$. To be able to use Theorem 4.1.3 we must have $|D \cap \mathscr{P}(\kappa)| = \kappa$. Apart from that there are only two points to be watched. One is that for any functions $f_1, \ldots, f_n \in D \cap {}^{\kappa}M$ and any first order formula $\phi(x_1, \ldots, x_n)$ we must have $X_{\phi[f_1,\ldots,f_n]} \in D \cap \mathscr{P}(\kappa)$. The other is that the function $f : \kappa \to M$ constructed in the proof of Lemma 4.1.1 must be in D.

Given M and an ultrafilter U, let us call N the *$\kappa\omega$-definable ultrapower* of M modulo U, if the transitive set N is defined exactly as the definable ultrapower, except that $\operatorname{def} M$ is replaced by the collection of all subsets of M that are $\mathscr{L}_{\kappa\omega}$-definable over M.

4.1.5. Lemma. *If ${}^{<\kappa}M \subseteq M$ and N is the $\kappa\omega$-definable ultrapower of M modulo U, then ${}^{<\kappa}N \subseteq N$.*

Proof. Suppose that $\alpha < \kappa$ and $(f_i : i < \alpha)$ represents an α-sequence in N. So each f_i is a function $\kappa \to M$ which is $\mathscr{L}_{\kappa\omega}$-definable over M. For $\xi < \kappa$ let s_ξ be the sequence $(f_i(\xi) : i < \alpha)$. Since ${}^{<\kappa}M \subseteq M$ we have $s_\xi \in M$ for every $\xi < \kappa$. Let $f : \kappa \to M$ be defined by $f(\xi) = s_\xi$. Clearly f is $\mathscr{L}_{\kappa\omega}$-definable over M since we may use a disjunction involving the formulae defining the functions f_i. Thus $[f]$ is an element in N and it is straightforward to see that $[f]$ is the sequence $([f_i] : i < \alpha)$. $\square$

4.1.6. Lemma. *If E is a weakly compact subset of κ and M is a transitive set such that $M^{<\kappa} \subseteq M$, $E \in M$, and $\kappa \in M$ then there exists a transitive set N and an elementary embedding $j : M \to N$ such that $N^{<\kappa} \subseteq N$ and $\kappa \in j(E)$.*

4.2. Weakly compact diamond

The following result was proved by Sun [18] in the case $E = \kappa$. Note that by the discussion preceding Lemma 3.3.3 this result is of interest only for ineffable cardinals.

4.2.1. Theorem. *If $V = L$ then $\diamondsuit(\mathrm{WC}_\kappa|E)$ holds for every weakly compact subset E of a weakly compact cardinal κ.*

Proof. We pick sets $A_\alpha \subseteq \alpha$ and $U_\alpha \subseteq V_\alpha$ and Π^1_1-sentences ϕ_α by induction on $\alpha \leq \kappa$ in the same manner as in the proof of Theorem 3.3.4. Only now we require that (A_α, U_α) is the $<_L$-least pair for which the conditions

(i) $\langle V_\alpha, \in, U_\alpha \rangle \models \phi$

(ii) If $\xi \in E \cap \alpha$ and $\langle V_\xi, \in, U_\alpha \cap V_\xi \rangle \models \phi$ then $A_\alpha \cap \xi \neq A_\xi$

can be satisfied if they can be satisfied at all for some Π^1_1-sentence ϕ. Here we have dropped the subscript on ϕ since we only need to refer to one particular sentence in the forthcoming argument. Also

$$X_\alpha = \{\xi \in E \cap \alpha : \langle V_\xi, \in, U_\alpha \cap V_\xi \rangle \models \phi\} \tag{8}$$

is defined exactly as in the proof of Theorem 3.3.4 and the theorem is proved by deriving a contradiction from the antithesis $X_\kappa = \kappa$. So suppose that there exists a Π^1_1-sentence ϕ satisfying conditions (i) and (ii) for $\alpha = \kappa$.

For each pair (A, U) of subsets of κ such that $(A, U) <_L (A_\kappa, U_\kappa)$ there exists a countable collection $W_{A,U} \subseteq V_{\kappa+1}$ which contains witnesses for that fact that the pair (A, U) was not chosen in place of (A_κ, U_κ) in the construction. More exactly if (ii) holds for $(A, U) <_L (A_\kappa, U_\kappa)$ and some Π^1_1-sentence ϕ which evaluated in $\langle V_\kappa, \in, U \rangle$ is equivalent to $\forall X \subseteq V_\kappa(\langle V_\kappa, \in, U \rangle \models \psi[X])$ then there exists a set $X \in W_{A,U}$ such that $\langle V_\kappa, \in, U \rangle \not\models \psi[X]$. The set

$$W = \bigcup \{W_{A,U} : (A, U) \in \mathscr{P}(\kappa) \times V_{\kappa+1}, (A, U) <_L (A_\kappa, U_\kappa)\}$$

has cardinality κ since $\mathscr{P}(\kappa) \times V_{\kappa+1} \subseteq L_{\kappa^+}$ and $|L_\alpha| = |\alpha|$ for every infinite ordinal α. Let M be a transitive set such that $|M| = \kappa$, $((A_\alpha, U_\alpha) : \alpha \leq \kappa) \in M$, $E \in M$, $V_\kappa \cup W \subseteq M$, and M satisfies a large enough finite fragment of ZFC.

Because E is weakly compact it is immediate from (8) that X_κ is weakly compact. Clearly $X_\kappa \in \operatorname{def} M$. By Lemma 4.1.4 there exists a transitive set N and an elementary embedding $j : M \to N$ such that $\kappa \in j(X_\kappa)$. Because $V^M_{\kappa+1} \subseteq V^N_{\kappa+1}$ the set W is also contained in N. Thus the construction is absolute up to the point that $(j((A_\alpha : \alpha \leq \kappa)))_\kappa = A_\kappa$. But $j(A_\kappa) \cap \kappa = A_\kappa$ and $\kappa \in j(X_\kappa)$ by our choice of the embedding j. This is a contradiction by (ii) and (8) and the elementarity of j. $\square$

5. Weak Compactness in generic extensions

5.1. Forcing preliminaries

We use the Boolean algebraic convention regarding the ordering relation of forcing notions i.e. a condition p is *stronger* than a condition q if $p \leq q$. Trees are tacitly reversed as soon as they are used as forcing notions.

If we say that τ is a *name for* some object x in the generic extension then we shall take it to mean that any condition forces τ to have whatever properties we have attributed to x.

When we say that P is an iteration of the sequence $(\dot{Q}_\alpha : \alpha < \eta)$ we mean that P is an iteration of length η and $\dot{Q}_\alpha$ is the name of the forcing notion used to construct $P_{\alpha+1}$ from P_α. Since there for a condition $p \in P_\alpha$ usually is a proper class of P_α-names τ such that $p \Vdash$ "$\tau \in \dot{Q}_\alpha$", one must somehow choose a representative set of such names that will then be the set

$$\{q(\alpha) : q \in P,\ q|\alpha = p\}.$$

Often one picks all possible names up to equivalence without further specifying the choice, but we need to be more restrictive in some cases. Therefore we take the following somewhat nonstandard approach. With each $\dot{Q}_\alpha$ we associate a set of P_α-names that we denote by $\operatorname{dom} \dot{Q}_\alpha$. Then we require that for every $p \in P$ and $\alpha \in \operatorname{dom} p$, we have $p(\alpha) \in \operatorname{dom} \dot{Q}_\alpha$ and $p|\alpha \Vdash$ "$p(\alpha) \in \dot{Q}_\alpha$".

One may think of $\operatorname{dom} \dot{Q}_\alpha$ as a property of the name $\dot{Q}_\alpha$ and even as the actual domain of the set $\dot{Q}_\alpha$ if names are formally defined Shoenfield style. But one may also, if one prefers, think of $\operatorname{dom} \dot{Q}_\alpha$ as merely a notational convention for specifying how the definition of an iteration is to be interpreted in detail. Anyhow we shall require that for any P_α-generic set G the interpretation of the name $\dot{Q}_\alpha$ is $\bigcup_{p \in G}\{\tau_G : \tau \in \operatorname{dom} \dot{Q}_\alpha,\ p \Vdash \text{“}\tau \in \dot{Q}_\alpha\text{”}\}$.

We consider the conditions of an iteration of length α to be partial functions on α. The *support* of a condition p is identified with the domain of p. This is convenient in in many situations, but as a technicality (see the proof of Lemma 5.1.2, where this approach is less convenient) we must require that if $\alpha = \max \operatorname{dom} p$ and $p(\alpha)$ is a name for a maximal element of $\dot{Q}_\alpha$ then $p \leq p|\alpha \leq p$.

As exemplified in the last paragraph, iterations are not partial orders in the strict sense, since they are not antisymmetric. But antisymmetry of forcing notions is not needed for the development of the theory of forcing. We consider a *forcing notion* to be a base set endowed with a transitive and reflexive binary relation $\leq$. Such objects are also called *preorders*. The inequality $p \leq q \leq p$ defines an equivalence relation which furthermore is a congruence with respect to the relation $\leq$. Thus there is a canonical partial ordering of the equivalence classes.

We shall say that two forcing notions P and Q are *isomorphic* if there exists a surjective function $h : P \to Q$ such that $h(p) \leq h(q)$ if and only if $p \leq q$ for every p and q in P. If P is antisymmetric then such a function h is necessarily injective. If P is not antisymmetric h still induces an injective mapping when we pass to equivalence classes, making the use of the term "isomorphism" permissible.

The following fact is well known. Note that the premises of the lemma imply that P_α is a direct limit.

5.1.1. Lemma. *Let α be a limit ordinal of cofinality κ and let P be an iteration of length α such that P_β satisfies the κ-cc for every $\beta < \alpha$. If there exists a stationary set of ordinals $\beta < \alpha$ such that P_β is a direct limit then P_α satisfies the κ-cc.*

For an iteration P of length α, an ordinal $\beta < \alpha$, and a P_β-generic set G_β we put

$$P^\beta = \{p|[\beta,\alpha) : p \in P \text{ and } p|\beta \in G_\beta\}$$

and define the ordering of P^β as follows: $f \leq g$ if there exists a condition $p \in G_\beta$ such that $p \cup f$ and $p \cup g$ are conditions in P and $p \cup f \leq p \cup g$. We may write P^β_γ for $\beta < \gamma < \alpha$ without fearing ambiguity since

$$(P_\gamma)^\beta = \{f|\gamma : f \in P^\beta\} = \{f \in P^\beta : \operatorname{dom} f \subseteq \gamma\}$$

and the orderings of $(P_\gamma)^\beta$ and P^β agree. If H is P^β-generic over $V[G_\beta]$ then the set

$$G = \{p \in P : p|\beta \in G_\beta \text{ and } p|[\beta,\alpha) \in H\}$$

is P-generic over V and $G_\beta = G \cap P_\beta$. It is useful to note that when choosing a condition $p \in P$ to represent a particular condition $p|[\beta,\alpha) \in H$, we may assume that $p \in G$. In the situation described above we shall denote G by $G_\beta * H$. If we identify a condition $p \in P$ with the pair $(p|\beta, p|[\beta,\alpha))$ then this is in conformance with standard notation.

Another well known fact is that P^β is isomorphic to an iteration R of a sequence $(\dot{S}_\gamma : \gamma < \eta)$ through an isomorphism $h : P^\beta \to R$ such that the conditions

(i) $\operatorname{dom}(h(f)) = \{\gamma : \beta + \gamma \in \operatorname{dom} f\}$

(ii) The interpretations of $\dot{Q}_{\beta+\gamma}$ and $\dot{S}_\gamma$ coincide for every $\gamma < \eta$

hold. We shall say that P^β is a direct limit whenever R is a direct limit and similarly for inverse limits. By our definition of isomorphism R is unique only when we pass to equivalence classes. Therefore we shall say that a forcing notion P of length α is an *inverse limit* if there for every partial function p on α such that $p|\beta \in P_\beta$ for every $\beta < \alpha$ exists a condition $q \in P$ such that $q|\beta \leq p|\beta \leq q|\beta$ for every $\beta < \alpha$. Then there will be no ambiguity in the statement that P^β is an inverse limit.

5.1.2. Lemma. *Suppose that P is an iteration of $(\dot{Q}_\alpha : \alpha < \eta)$, $\beta < \eta$, and κ is a regular cardinal. If P satisfies the conditions*

(i) *For every P_β-name τ and condition $p \in P_\alpha$ where $\beta \leq \alpha < \eta$, if $p \Vdash$ "$\tau \in \dot{Q}_\alpha$" then there exists a name $\sigma \in \operatorname{dom} \dot{Q}_\alpha$ such that $p \Vdash$ "$\sigma = \tau$"*

(ii) *P_α is either a direct or an inverse limit for every limit ordinal α such that $\beta < \alpha \leq \eta$*

(iii) *P_β satisfies the* κ-cc

(iv) *P_α is an inverse limit for every limit ordinal $\alpha > \beta$ such that* $\operatorname{cf}\alpha \leq \kappa$

then for every limit ordinal $\alpha > \beta$, P^β_α is a direct limit iff P_α is a direct limit and P^β_α is an inverse limit iff P_α is an inverse limit.

Proof. The only implication which is not straightforward is the fact that if P_α is an inverse limit then P^β_α is an inverse limit.

So let $\dot{f}$ be a P_β-name for partial function on α such that $\Vdash$ "$\dot{f}|\gamma \in \dot{P}^\beta$" for all $\gamma < \alpha$. By condition (i) we can define a partial function p on $\alpha \setminus \beta$ such that

(v) $p(\gamma) \in \operatorname{dom} \dot{Q}_\gamma$ for every $\gamma \in \operatorname{dom} p$

(vi) If $\Vdash$ "$\dot{f}(\gamma)$ is undefined" then $p(\gamma)$ is undefined

(vii) $\Vdash$ "$\gamma \in \operatorname{dom} \dot{f} \to \dot{f}(\gamma) = p(\gamma)$"

(viii) $\Vdash$ "$p(\gamma)$ is maximal in $\dot{Q}_\gamma$ if defined when $\dot{f}(\gamma)$ is undefined".

We claim that p is a condition in P_α. It is clear that $\Vdash$ "$p(\gamma) \in \dot{Q}_\gamma$" for every $\gamma \in \operatorname{dom} p$. It remains to check that $\operatorname{dom} p$ conforms with the definition of P_α. Let γ be a limit ordinal such that $\beta < \gamma \leq \alpha$. If P_γ is not a direct limit then P_γ must be an inverse limit and everything is fine. Suppose now that P_γ is a direct limit. We must check that $\operatorname{dom}(p) \cap \gamma$ is a bounded subset of γ. But

$$\Vdash \text{“}\operatorname{dom}(\dot{f}) \cap \gamma \text{ is a bounded subset of } \gamma\text{”}$$

and thus using conditions (iii) and (iv) we can find an ordinal $\delta < \gamma$ such that $\Vdash$ "$\operatorname{dom}(\dot{f}) \cap \gamma \subseteq \delta$". By condition (vi) we have now seen that $p \in P_\alpha$.

Fix a P_β-generic set G_β and let f be the interpretation of $\dot{f}$. Now $p \in P^\beta_\alpha$ since $p|\beta$ is the empty function. Fix an arbitrary ordinal γ such that $\beta < \gamma < \alpha$. Since $f|\gamma \in P^\beta$ there exists a condition $q \in P$ such that $q|\beta \in G_\beta$ and $f|\gamma = q|[\beta, \alpha)$. Also $(q|\beta) \cup (p|\gamma)$ is a condition in P and $q \leq (q|\beta) \cup (p|\gamma) \leq q$. Therefore $f|\gamma \leq p|\gamma \leq f|\gamma$. $\square$

Suppose now that M and N are transitive classes or sets and $j : M \to N$ is an elementary embedding. Also assume that $P \in M$ where P is an iteration of length α and $j(P)_\beta = P$ where $\beta < j(\alpha)$ and the models M and N satisfy a large enough finite fragment of ZFC to make the use of the forcing theorem possible. Let G be P-generic over both M and N and let H be $j(P)^\beta$-generic over $N[G]$. As noted earlier $N[G][H]$ is a generic extension by the forcing notion $j(P)$. If it in addition holds that $j(p)|[\beta, \alpha) \in H$ for every $p \in G$ then the equation

$$j(\tau_G) = j(\tau)_{G*H}$$

holds for every P-name τ such that $\tau_G \in M$. Furthermore, if the equation is applied to the class of all P-names τ, taking the right hand side as a definition of the left hand side, it defines an elementary embedding $j : M[G] \to N[G][H]$. It is natural and convenient to denote this extended embedding by j too, since its restriction to M is the original j. Obviously $j(G) = G * H$.

Consider the following game between players I and II on a forcing notion P. The length of the game is an ordinal α. The players take turns picking conditions in P that will eventually form a decreasing sequence $(p_i : i < \alpha)$ if player II is to win the game. The conditions p_i are picked in increasing order of the index so that player I picks p_i for i even and player II for i odd.

If the game successfully goes on for α rounds then player II wins. The game ends in the victory of player I if the moves played at some point form a decreasing sequence that has length less than α but no lower bound in P, making the next move impossible. If κ is a regular cardinal and player II has a winning strategy in this game for every $\alpha < \kappa$ then the forcing notion P is said to be *κ-game closed*. Let $\mathscr{G}_\alpha(P)$ denote the game described above.

5.1.3. Lemma. *Let P be an iteration of the sequence $(\dot{Q}_\alpha : \alpha < \eta)$ and let κ be a regular cardinal. If P satisfies the conditions*

(i) *For every $\alpha < \eta$ and $\beta < \kappa$ there exists a name τ_β for a winning strategy for player II in $\mathscr{G}_\beta(\dot{Q}_\alpha)$ such that given any sequence $(\rho_i : i < \xi)$ in the ground model of P_α-names for elements of $\dot{Q}_\alpha$, there exists a name $\rho \in \operatorname{dom} \dot{Q}_\alpha$ such that $\Vdash$ "$\rho = \tau_\beta(\rho_i : i < \xi)$"*

(ii) *If $\gamma \leq \eta$ is a limit ordinal such that P_γ is a direct limit then $\operatorname{cf} \gamma \geq \kappa$*

(iii) *If $\gamma \leq \eta$ is a limit ordinal such that P_γ is an inverse limit then there exists a strictly increasing continuous sequence $(\gamma_i : i < \zeta)$ such that $\sup_{i<\zeta} \gamma_i = \gamma$ and P_{γ_i} is an inverse limit for every limit ordinal $i < \zeta$*

then P is κ-game closed.

Proof. Fix $\beta < \kappa$. By induction on $\alpha \leq \eta$ we shall construct winning strategies $\sigma_\alpha : [P_\alpha]^{<\kappa} \to P_\alpha$ for player II in $\mathscr{G}_\beta(P_\alpha)$ such that

(iv) $(\sigma_\alpha(p_i : i < \xi))|\gamma = \sigma_\gamma(p_i|\gamma : i < \xi)$ for all $\gamma < \alpha$

(v) $\operatorname{dom}(\sigma_\alpha(p_i : i < \xi)) = \bigcup_{i<\xi} \operatorname{dom} p_i$.

Suppose P_α is a direct limit. Then $\operatorname{cf} \alpha \geq \kappa$ whereby $[P_\alpha]^{<\kappa} = \bigcup_{\gamma<\alpha}([P_\gamma]^{<\kappa})$. By conditions (iv) and (v) the winning strategies constructed at earlier stages extend each other. Thus we can put $\sigma_\alpha = \bigcup_{\gamma<\alpha} \sigma_\gamma$. This is also the only possible way to construct σ_α if conditions (iv) and (v) are to remain true.

Let us then deal with the case where P_α is an inverse limit. Now we must have $\bigcup_{\gamma<\alpha} \sigma_\gamma \subseteq \sigma_\alpha$ but the inclusion is proper. Suppose that $(p_i : i < \xi)$ is a decreasing sequence such that $\bigcup_{i<\xi} \operatorname{dom} p_i$ is an unbounded subset of α. It is straightforward to check that putting

$$\sigma_\alpha(p_i : i < \xi) = \bigcup_{\gamma<\alpha} \sigma_\gamma(p_i|\gamma : i < \xi)$$

does the job.

Finally suppose that $\alpha = \gamma + 1$. We shall define $\sigma_\alpha(p_i : i < \xi)$ by induction on ξ. Since player II moves on odd indexes ξ is a successor ordinal in all relevant cases. Condition (i) is tailored for this step. We consider $(\rho_i : i < \xi)$ to be $(p_i(\gamma) : i < \xi)$ and put

$$\sigma_\alpha(p_i : i < \xi) = \sigma_\gamma(p_i|\gamma : i < \xi)^\frown(\rho)$$

where ρ is as in condition (i). □

It is evident that if P is κ-game closed then P is α-Baire for every $\alpha < \kappa$. That P is α-Baire is equivalent to the fact that forcing with P does not add any new functions with domain α. Obviously κ-closed forcing notions are κ-game closed.

Note that from the last proof we can extract a somewhat stronger result than the one appearing in the formulation of Lemma 5.1.3. Namely conditions (iv) and (v) can be taken as properties of the final winning strategy σ_η that are often useful in applications of the lemma. Let us call a winning strategy with the above mentioned properties *uniform*. In Section 5.4 we shall need to refer to a uniform winning strategy.

For a regular cardinal κ and a cardinal $\lambda \geq \kappa$, let $C_\kappa(\lambda)$ denote the standard forcing notion for adding λ many Cohen subsets of κ. Formally we shall view $C_\kappa(\lambda)$ as the set of all partial functions $f : \lambda \to 2$ such that $|f| < \kappa$, ordered by reverse inclusion. Sometimes it is convenient, for technical reasons, to replace λ by a set S of cardinality λ. Of course $C_\kappa(S)$ is then isomorphic to $C_\kappa(\lambda)$.

An iterated forcing notion P is said to have *Easton support* if the only restriction placed on the supports is that $|\operatorname{dom}(p) \cap \mu| < \mu$ for every $p \in P$ and every regular cardinal μ. In terms of limits, Easton support means taking direct limits at regular cardinals and inverse limits otherwise.

5.2. Weakly compact diamond by forcing

Let κ be a weakly compact cardinal and let $P = P_{\kappa+1}$ be the Easton support iteration of $(\dot{Q}_\alpha : \alpha \leq \kappa)$ where $\dot{Q}_\alpha$ is a name for $C_\alpha(\alpha)$ whenever α is an Mahlo cardinal and a name for the trivial forcing notion $\{1\}$ otherwise.

The forcing notion P defined above is due to Jack Silver (unpublished) who used ideas originating from unpublished work of Ronald Jensen (see [14]). This technique is known as reverse Easton forcing and was originally used to violate the GCH at various large cardinals. It is well known that κ remains weakly compact in the generic extension V^P. The proof of Lemma 5.2.3 is essentially a proof of the preservation of weak compactness augmented with the ideas from the standard proof of the fact that $\diamondsuit_\kappa$ holds in an extension by $C_\kappa(\kappa)$.

Let us assume that the sets $\operatorname{dom}\dot{Q}_\alpha$ contain all relevant names up to equivalence and have the lowest possible rank.

5.2.1. Lemma. *If $\alpha \leq \kappa$ is inaccessible then $P_\alpha \subseteq V_\alpha$ and if α is a Mahlo cardinal then P_α is α-cc, $\operatorname{dom}\dot{Q}_\alpha \subseteq V_\alpha$, and $|P_{\alpha+1}| = \alpha$.*

Proof. Let $\alpha \leq \kappa$ be inaccessible. Using the assumption made about the sets $\dot{Q}_\beta$ it is straightforward to check by induction that $P_\beta \in V_\alpha$ for all $\beta < \alpha$. Since P_α is a direct limit $P_\alpha \subseteq V_\alpha$.

Suppose now that α is a Mahlo cardinal. Then the set of ordinals $\beta < \alpha$ such that P_β is a direct limit is stationary. By Lemma 5.1.1 P_α satisfies the α-cc. Using this fact and the assumption about $\operatorname{dom}\dot{Q}_\alpha$ being of lowest possible rank, it is again straightforward to see that $\operatorname{dom}\dot{Q}_\alpha \subseteq V_\alpha$. It follows that $|P_{\alpha+1}| = \alpha$. □

It follows from the previous lemma that any Mahlo cardinal $\alpha \leq \kappa$ remains a regular cardinal in an extension by P_α. In fact we were a bit sloppy with the definition of P since the use of the notation $C_\alpha(\alpha)$ assumed α to be regular.

5.2.2. Lemma. *If $\alpha < \kappa$ is a Mahlo cardinal then $V^{P_{\alpha+1}}$ and V^P have the same subsets of α.*

Proof. By Lemma 5.2.1 $P_{\alpha+1}$ satisfies the α^+-cc and thus by Lemma 5.1.2 for limit ordinals γ such that $\alpha+1 < \gamma \leq \kappa$ it holds that $P^{\alpha+1}_\gamma$ is a direct limit when γ is regular and an inverse limit otherwise. Still by the fact that $P_{\alpha+1}$ satisfies the α^+-cc all regular cardinals above α remain regular in $V^{P_{\alpha+1}}$. Since

$$\Vdash \text{“}\dot{Q}_\gamma \text{ is } \alpha^+\text{-closed”}$$

for all γ such that $\alpha+1 \leq \gamma < \kappa$ we see that $P^{\alpha+1}$ is isomorphic to an iteration that is α^+-game closed by Lemma 5.1.3. Note that condition (i) of Lemma 5.1.3 is trivially fulfilled since the names $\operatorname{dom}\dot{Q}_\gamma$ contains all relevant names up to equivalence. □

5.2.3. Lemma. *If E is a weakly compact subset of κ in the ground model then $\diamondsuit(\mathrm{WC}_\kappa|E)$ holds in V^P.*

Proof. Given a P-generic set G we obtain a function $g : \kappa \to 2$ by putting $g = \bigcup_{p\in G} p(\kappa)_{G_\kappa}$. Define a function F on κ by

$$F(\alpha) = \{\beta < \alpha : g(\sum_{i<\alpha} i + \beta) = 1\}$$

and let $\dot{F}$ be a P-name for F. (The sums appearing in the definition of F are ordinal sums.) We shall show that F is the required sequence.

Let ϕ be a Π^1_1-sentence, let $\dot{U}$ be a P-name for a subset of V_κ of the generic extension, and suppose that

$$p \Vdash \text{“}\langle V_\kappa, \in, \dot{U}\rangle \models \phi\text{”} \tag{9}$$

for some $p \in P$. Let $\dot{A}$ be a P-name for an arbitrary subset of κ. The lemma will be proved if we can find a condition $q \leq p$ such that

$$q \Vdash \text{“}\dot{A} \cap \alpha = \dot{F}(\alpha),\ \alpha \in \check{E},\ \text{and } \langle V_\alpha, \in, \dot{U} \cap V_\alpha\rangle \models \phi\text{”} \tag{10}$$

for some ordinal $\alpha < \kappa$. Let us assume that the names $\dot{F}$, $\dot{U}$, and $\dot{A}$ discussed above have the lowest possible rank.

Using the reflection principle we can pick a set M' of cardinality κ that reflects the formulae (9) and (10). Let us also assume that

$$V_\kappa \cup \{P, E, \dot{F}, \dot{U}, \dot{A}\} \subseteq M'$$

and M' satisfies a large enough finite fragment of ZFC. Let $\pi : M' \to M$ be the Mostowski collapse of M'. By Lemma 5.2.1 and similar arguments the sets P, E, $\dot{F}$, $\dot{U}$, and $\dot{A}$ and their respective elements remain fixed under the isomorphism π.

Using Lemma 4.1.4 we pick a transitive set N and an elementary embedding $j : M \to N$ such that $\kappa \in j(E)$. The property of being a Mahlo cardinal is obviously preserved when moving to submodels. Therefore κ is a Mahlo cardinal in N.

By Lemma 5.2.1 applied on $j(P)$ in N we have $j(P)_\kappa = j(P) \cap V_\kappa = P_\kappa$. We can assume that the sets $\operatorname{dom} \dot{Q}_\alpha$ used in the definition of P are picked in a canonical way using some well ordering in $V_{\kappa+1}$. Absoluteness then gives $j(P)_{\kappa+1} = P_{\kappa+1} = P$ whereby $P \in N$. Let $p \in P$ be arbitrary. Since P_κ is a direct limit there exists an ordinal $\alpha < \kappa$ such that $p|\kappa = p|\alpha$. By the properties of j and the fact that $p|\alpha \in V_\kappa$ we then have $j(p)|j(\kappa) = j(p|\kappa) = j(p|\alpha) = p|\alpha$.

Fix some P-generic set G. From the above discussion we know that $j(p)|(\kappa+1) \in G$ for every $p \in G$ and therefore

$$D = \{j(p)|[\kappa + 1, j(\kappa) + 1) : p \in G\}$$

is a subset of $j(P)^{\kappa+1}$. In fact every nonempty function in D is defined only on the singleton $\{j(\kappa)\}$. Using the fact that G is a filter it is straightforward to see that D is a directed set.

We define a function d on $\{j(\kappa)\}$ by letting $d(j(\kappa))$ be a P-name for a function $h : \kappa + \kappa \to 2$ such that $\bigcup_{s \in G} s(\kappa)_{G_\kappa} \subseteq h$ and

$$\dot{A}_G = \{\alpha < \kappa : h(\kappa + \alpha) = 1\}.$$

This is possible since $\{s(\alpha)_{G_\kappa} : s \in G\}$ is $C_\kappa(\kappa)$-generic over $V[G_\kappa]$. Because P-names are also $j(P)_{j(\kappa)}$-names, we can require that $d(j(\kappa)) \in \operatorname{dom}(j(\dot{Q}_\kappa))$. So d is in fact a condition in $j(P)$. Because $d = d|[\kappa + 1, j(\kappa) + 1)$ we even have $d \in j(P)^{\kappa+1}$.

Let $f = j(p)|[\kappa+1, j(\kappa)+1)$ be an arbitrary element of D. Thus we naturally assume that $p \in G$. As noted before $\operatorname{dom} f = \{j(\kappa)\}$ unless f is the empty function. By the definition of d we have $p(\kappa)_{G_\kappa} \subseteq d(j(\kappa))_G$ and thus there is a condition $q \in G$ such that $q \Vdash$ "$p(\kappa) \subseteq d(j(\kappa))$". We may assume that $q \leq p$. Now both $q \cup d$ and $q \cup f$ are conditions in $j(P)$ and $q \cup d \leq q \cup f$. Thus $d \leq f$ in the ordering of $j(P)^{\kappa+1}$.

Let H be a $j(P)^{\kappa+1}$-generic set over $V[G]$ such that $d \in H$. Since d extends every condition in D we have $D \subseteq H$ and thus by the definition of D it is clear that $j(p)|[\kappa+1, j(\kappa)+1) \in H$ for every $p \in G$. So the embedding j can be extended to an embedding $j : M[G] \to N[G][H]$ via the definition discussed in Section 5.1.

Let us now proceed with the argument outlined in the beginning of the proof. Suppose that G is generic over V and $p \in G$ where p is a condition satisfying (9).

In $V[G]$ it holds that $\langle V_\kappa, \in, \dot{U}_G \rangle \models \phi$. By absoluteness of V_κ and downward absoluteness of universal statements this holds in $N[G]$ too. Lemma 5.2.2 can be applied with M as the ground model and by elementarity it can therefore be applied on $j(P)$ with N as the ground model. Since κ is a Mahlo cardinal $N[G][H]$ contains no subsets of κ besides those in $N[G]$. Furthermore V_κ is absolute with respect to the generic extensions under discussion. We can conclude that the statement $\langle V_\kappa, \in, \dot{U}_G \rangle \models \phi$ holds in $N[G][H]$.

In addition $j(\dot{A})_{G*H} \cap \kappa = j(\dot{A}_G) \cap \kappa = \dot{A}_G$. Since $d \in G*H$ we have $j(\dot{A})_{G*H} \cap \kappa = j(\dot{F})_{G*H}(\kappa)$. Similarly $j(\dot{U}_G) = j(\dot{U})_{G*H}$ and we have $j(\dot{U})_{G*H} \cap V_\kappa = \dot{U}_G$. Thus it holds in $N[G][H]$ that there exists an ordinal $\alpha < j(\kappa)$ such that

$$j(\dot{A})_{G*H} \cap \alpha = j(\dot{F})_{G*H}(\alpha),\ \alpha \in j(E) \text{ and } \langle V_\alpha, \in, j(\dot{U})_{G*H} \cap V_\alpha \rangle \models \phi.$$

By elementarity it is true in $M[G]$ that there exists an ordinal $\alpha < \kappa$ such that

$$\dot{A}_G \cap \alpha = \dot{F}_G(\alpha),\ \alpha \in E \text{ and } \langle V_\alpha, \in, \dot{U}_G \cap V_\alpha \rangle \models \phi$$

and therefore it is true in M that there exists an ordinal $\alpha < \kappa$ and a condition $q \leq p$ such that (10) holds.

Through the fact that the collapsing isomorphism $\pi : M' \to M$ does not move the parameters involved, (10) holds in M'. As M' was chosen to reflect (10) the same holds in V. □

Thus we have proved the following result that was announced without proof in [16] in the case $E = \kappa$. Hamkins [7] has independently obtained related results.

5.2.4. Theorem. *If E is a weakly compact subset of κ then there exists a generic extension in which $\diamondsuit(\mathrm{WC}_\kappa|E)$ holds.*

Note that in the above proof we needed to argue that (10) holds in V since we started out by assuming that (9) holds in V. Had we only assumed that (9) holds in M, we would have had trouble arguing that (9) holds in N, since for all we know N might contain subsets of V_κ that are not present in M.

There is a well known version of Lemma 5.1.3, where κ-game closed forcing notions are replaced by forcing notion that are α-directed closed for all $\alpha < \kappa$. For Lemma 5.2.3 this result would have sufficed. Furthermore, if we were only interested in the preservation of weak compactness, we could have picked the condition d in the proof of Lemma 5.2.3 just by referring to the fact that D is a small enough directed set.

5.3. Killing a weakly compact set

It follows from Lemma 5.2.3 that the forcing notion of the previous section does not only preserve weak compactness of κ, but preserves every weakly compact set. Can one kill a weakly compact set by forcing while preserving the weak compactness of κ? We shall now give an affirmative answer to this question. It even turns out that there are no extraneous requirements on the set to be killed. In the spirit of [3] it follows that a set is not weakly compact because of some intrinsic property of the set itself.

For $A \subseteq \kappa$, we shall denote the tree consisting of bounded 1-closed subsets of A by $T^1(A)$. (We consider $T^1(A)$ to be ordered by end extension.) Let E be a weakly compact subset of κ. Let $P = P_{\kappa+1}$ be the Easton support iteration of $(\dot{Q}_\alpha : \alpha \leq \kappa)$ where $\dot{Q}_\alpha$ is a name for $T^1(E \cap \alpha)$ whenever α is Mahlo and $E \cap \alpha$ is an unbounded subset of α. For other ordinals $\alpha \leq \kappa$ we let $\dot{Q}_\alpha$ be a name for the trivial forcing notion. Be aware that we are reusing the letter P and some other symbols, but there should be no confusion since we shall not make any explicit references to the earlier forcing notion. Rather it is convenient since we can refer to the fact that some of the lemmas in Section 5.2 hold verbatim for the forcing notion defined above.

We are interested in the case were both E and its complement are weakly compact. We claim that forcing with P kills the weak compactness of $\kappa \setminus E$ while preserving the weak compactness of κ. We would like to emphasise that no other requirements are placed on E. Note also that the situation at hand is drastically different from killing stationary sets by forcing. Stationary sets can never, at a later stage, regain the property of being stationary once it is lost. This is not the case with weak compactness.

5.3.1. Lemma. *If α is regular and A is an unbounded subset of α then the set $D_\beta = \{p \in T^1(A) : \sup(p) \geq \beta\}$ is dense in $T^1(A)$ for every $\beta < \alpha$ and $T^1(A)$ is α-game closed.*

Proof. It is immediately clear that the sets D_β are dense since it suffices to add one point to a condition in order to extend it to a condition in one of the sets D_β.

Player II can ensure a victory already in the first round of the game $\mathscr{G}_\beta(T^1(A))$. If p_0 is the first move made by player I and there exists an inaccessible cardinal γ such that $\beta \leq \gamma < \alpha$ then player II chooses $p_1 \in D_\gamma$. It is then impossible for player II to loose since any limit ordinal $j \leq \beta$ has small enough cofinality so that

$\bigcup_{i<j}\sup(p_i)$ can not be inaccessible unless the sequence $(p_i : i < j)$ is eventually constant. In either case $\bigcup_{i<j} p_i$ is a condition in $T^1(A)$ and a valid move for player I. □

As in Section 5.2 we shall assume that the sets $\operatorname{dom}\dot{Q}_\alpha$ used in the definition of P contain all relevant names up to equivalence but are of lowest possible rank. Both the formulation and the proof of Lemma 5.2.1 hold verbatim for the forcing notion P of this section. The same goes for Lemma 5.2.2 except that in the proof we must refer to Lemma 5.3.1 rather than the fact that $C_\gamma(\gamma)$ is α-closed for every regular γ above α.

5.3.2. Lemma. *Every set $S \subseteq E$ that is weakly compact in the ground model is weakly compact in V^P.*

Proof. The argument we give is very similar to the proof of Lemma 5.2.3 if we omit the references to the name $\dot{A}$. Thus, given a Π^1_1-sentence ϕ and a P-name $\dot{U}$ for a subset of V_κ, then for every condition $p \in P$ such that

$$p \Vdash \text{“}\langle V_\kappa, \in, \dot{U}\rangle \models \phi\text{”} \tag{11}$$

we shall find a condition $q \leq p$ and an ordinal $\alpha < \kappa$ such that

$$q \Vdash \text{“}\alpha \in \check{S},\ \text{and } \langle V_\alpha, \in, \dot{U} \cap V_\alpha\rangle \models \phi\text{”}. \tag{12}$$

We choose the models M and N and the elementary embedding $j : M \to N$ exactly as in the proof of Lemma 5.2.3 except that let S take the role of E so that $\kappa \in j(S)$.

Again it is clear that $j(P)_\kappa = P_\kappa$ and we use absoluteness to argue that $j(P)_{\kappa+1} = P$. This time we have the parameter E involved in the definition, but it causes no trouble since $j(E) \cap \kappa = E$.

Now we define the function d on $\{j(\kappa)\}$ by letting $d(j(\kappa))$ be a P-name for $\bigcup_{s\in G} s(\kappa)_{G_\kappa} \cup \{\kappa\}$. Fixing a P-generic set G we let C denote $d(j(\kappa))_G$. Working in $V[G]$ we shall conclude the proof by checking that $C \in T^1(j(E))$. This suffices since one can then proceed as in the proof of Lemma 5.2.3, picking a $j(P)^{\kappa+1}$-generic set H over $V[G]$ such that $d \in H$ and defining the extended elementary embedding $j : M[G] \to N[G][H]$.

Directly by the definition it is clear that $C \subseteq j(E)$. Since being 1-closed is upwards absolute $C \cap \gamma$ is 1-closed for every $\gamma < \kappa$. It follows that C is 1-closed as a subset of $j(\kappa)$ since $C \cap \kappa$ is an unbounded subset of κ by Lemma 5.3.1. □

5.3.3. Lemma. *The weakly compact filter of V^P contains E and every set that was in the weakly compact filter of the ground model.*

Proof. Let G be a P-generic set. The 1-club subsets of κ in the ground model remain 1-closed in V^P. Likewise $C = \bigcup_{p\in G} p(\kappa)_{G_\kappa}$ is 1-closed as a subset of κ by similar arguments as in the proof of the previous lemma. It remains to prove that the above mentioned sets are stationary in $V[G]$ which in fact implies that they are weakly compact.

If X is in the weakly compact filter in the ground model then $X \cap E$ is weakly compact in the ground model and remains weakly compact in V^P by Lemma 5.3.2. In the setting of the proof of Lemma 5.3.2 it holds that

$$j(C) = \bigcup_{p \in G*H} p(j(\kappa))_{(G*H)_{j(\kappa)}}$$

and thus $d(j(\kappa))_G \subseteq j(C)$. It follows that $\kappa \in j(C)$. We can then redo the argument yielding Lemma 5.3.2 with C replacing S, and letting $\dot{C}$ be a name for C we obtain

$$q \Vdash \text{“}\alpha \in \dot{C},\ \text{and}\ \langle V_\alpha, \in, \dot{U} \cap V_\alpha \rangle \models \phi\text{”}$$

rather than (12). □

Summarising what was said in Lemmas 5.3.2 and 5.3.3 we have the following result:

5.3.4. Theorem. *If both S and its complement are weakly compact subsets of κ then there exists a generic extension in which the weakly compact ideal is proper, extends the weakly compact ideal of the ground model, and contains the set S.*

5.4. Killing weakly compact diamond

Let E denote the set of regular cardinals below κ and let m and n be positive natural numbers. Hauser [8] has proved that it is consistent that $\diamondsuit_\kappa(E)$ fails although κ is Π^m_n-indescribable and GCH holds. Of course the consistency of the existence of a Π^m_n-indescribable cardinal must be assumed, and the result is an equiconsistency result, to be more accurate. When κ is weakly compact, E is in the weakly compact filter, so $\mathrm{NS}_\kappa|E \subseteq \mathrm{WC}_\kappa$ whereby $\diamondsuit_\kappa(\mathrm{WC})$ implies $\diamondsuit_\kappa(E)$. Thus $\diamondsuit_\kappa(\mathrm{WC})$ may fail at a weakly compact cardinal. In this section we shall prove the consistency of the failure of the weakly compact diamond in a more explicit way.

Suppose that α is a Mahlo cardinal, $\bar{A}$ is an α-sequence $(A_i : i < \alpha)$, and S is a set of ordinals. Let $K^1_S(\bar{A})$ be the collection of pairs (C, A) such that

(i) C is a 1-closed and bounded subset of α

(ii) $A \subseteq \sup C$

(iii) If $i \in C \cap S$ then $A \cap i \neq A_i$.

We order $K^1_S(\bar{A})$ by letting $(C', A') \leq (C, A)$ if the conditions

(iv) $C' \cap (\sup(C) + 1) = C$

(v) $A' \cap \sup C = A$

hold.

Assume that $V = L$ and S is a weakly compact subset of κ. We shall define an iteration P of $(\dot{Q}_\alpha : \alpha < \kappa^+)$. The limit P_α is direct if α is regular or if $\operatorname{cf}\alpha$ is Mahlo and $\alpha < (\operatorname{cf}\alpha)^+$. All other limits are inverse.

Simultaneously with the sequence $(\dot{Q}_\alpha : \alpha < \kappa^+)$ we shall define a sequence $(\tau_\alpha : \alpha < \kappa^+)$. If there exists a Mahlo cardinal $\beta \leq \kappa$ such that $\beta \leq \alpha < \beta^+$ then $(\tau_\alpha, \dot{Q}_\alpha)$ is the $<_L$-least pair such that

(vi) τ_α is a P_α-name for a subset sequence indexed by β

(vii) $(\tau_\alpha, \dot{Q}_\alpha) \neq (\tau_\xi, \dot{Q}_\xi)$ whenever $\beta \leq \xi < \alpha$

(viii) $\dot{Q}_\alpha$ is a name for $K^1_S(\tau_\alpha)$

(ix) $\operatorname{dom}\dot{Q}_\alpha$ is the set of all relevant P_β-names up to equivalence and is of lowest possible rank

if such a pair exists. Otherwise τ_α is arbitrary and $\dot{Q}_\alpha$ is the canonical name for the trivial forcing notion. If the conditions above hold then for any $s \in P_{\alpha+1}$, $C_{s(\alpha)}$ and $A_{s(\alpha)}$ denote P_α-names such that

$$s|\alpha \Vdash \text{“}\check{s}(\alpha) = (C_{s(\alpha)}, A_{s(\alpha)})\text{”}.$$

By conditions (viii) and (ix) $C_{s(\alpha)}$ and $A_{s(\alpha)}$ can actually be assumed to be P_β-names and the condition $s|\alpha$ could be replaced by $s|\beta$ above.

We shall often encounter sets written as $\bigcup_{s\in G_{\alpha+1}} C_{s(\alpha)}$ or $\bigcup_{s\in G_{\alpha+1}} A_{s(\alpha)}$ where G is a fixed generic set. In this setting $C_{s(\alpha)}$ and $A_{s(\alpha)}$ do not denote the names but rather their interpretations by G_α where G is the same generic set as in the subscript of the union symbol.

Again both the statement and the proof of Lemma 5.2.1 holds verbatim for our latest forcing notion. But now Lemma 5.2.1 does not tell us anything about P itself since P has length κ^+ and not $\kappa + 1$ as was the case with the forcing notions dealt with earlier. Nevertheless our forthcoming arguments rely on the fact that P to some extent resembles an iteration of length $\kappa + 1$. The idea is that for a Mahlo cardinal $\beta \leq \kappa$ one can think of the steps between β and β^+ as a single step consisting of a product forcing.

5.4.1. Lemma. *Let $\beta \leq \kappa$ be a Mahlo cardinal. In V^{P_β} there is an embedding $P^\beta_{\beta^+} \to C_\beta(\beta^+)$ that preserves incompatibility.*

Proof. We shall find an embedding into $C_\beta((\beta^+ \setminus \beta) \times 2 \times \beta)$ rather than $C_\beta(\beta^+)$. Fix a P_β-generic set G_β. Given a condition $p \in P_{\beta^+}$ such that $p|\beta \in G_\beta$ we shall define a partial function $f : (\beta^+ \setminus \beta) \times 2 \times \beta \to 2$. For $\alpha \in \operatorname{dom} p$ let C_α and A_α denote the interpretations of the P_β-names $C_{p(\alpha)}$ and $A_{p(\alpha)}$ respectively.

If $\alpha \in \mathrm{dom}(p) \setminus \beta$ then $f|(\{\alpha\} \times \{0\} \times \beta)$ is like the characteristic function of C_α with domain $\{\alpha\} \times \{0\} \times \sup(C_\alpha + 1)$. Furthermore $f|(\{\alpha\} \times \{1\} \times \beta)$ has domain $\{\alpha\} \times \{1\} \times \sup(C_\alpha)$ and is like the characteristic function of A_α. If $\alpha \geq \beta$ but $\alpha \notin \mathrm{dom}\, p$ then $\mathrm{dom}(f|(\{\alpha\} \times 2 \times \beta))$ is empty. The embedding i is defined by letting $p|[\beta, \beta^+)$ be mapped to f.

Suppose p and q are conditions in $P_{\beta+}$ such that $p|\beta$ and $q|\beta$ are in G_β. Let $f = p|[\beta, \beta^+)$ and $g = q|[\beta, \beta^+)$ and suppose that $i(f)$ and $i(g)$ are compatible in $C_\beta((\beta^+ \setminus \beta) \times 2 \times \beta)$. Then for $\alpha \in (\mathrm{dom}(p) \cap \mathrm{dom}(q)) \setminus \beta$ we have either

$$i(f)|(\{\alpha\} \times 2 \times \beta) \subseteq i(g)|(\{\alpha\} \times 2 \times \beta)$$

or vice versa. We define a function h by letting $h(\alpha) = f(\alpha)$ if the situation is as above and $h(\alpha) = g(\alpha)$ if the subset relation is reversed. For ordinals $\alpha \geq \beta$ such that only one of $f(\alpha)$ and $g(\alpha)$ is defined we let $h(\alpha)$ equal the defined one.

There exists is common extension $r \in G_\beta$ of $p|\beta$ and $q|\beta$ such that $r \cup h$ is a common extension of $r \cup f \leq r \cup g$ and thereby a common extension of p and q. For $P^\beta_{\beta+}$ it follows that h is a common extension of f and g. □

5.4.2. Lemma. *Let $\beta \leq \kappa$ be a Mahlo cardinal. $P_{\beta+}$ satisfies the β^+-cc.*

Proof. Since $C_\beta(\beta^+)$ satisfies the β^+-cc, Lemma 5.4.1 implies that P^β satisfies the β^+-cc in V^{P_β}. On the other hand P_β even satisfies the β-cc by the analogue of Lemma 5.2.1. □

5.4.3. Lemma. *If α is regular and $\bar{A}$ is an α-sequence then the set $D_\beta = \{(C, A) \in K^1_S(\bar{A}) : \sup(C) \geq \beta\}$ is dense in $K^1_S(\bar{A})$ for every $\beta < \alpha$ and $K^1_S(\bar{A})$ is α-game closed.*

5.4.4. Lemma. *If $\beta < \kappa$ is a Mahlo cardinal then $V^{P_{\beta+}}$ and V^P have the same subsets of β.*

Proof. Because $P_{\beta+}$ is β^+-cc by Lemma 5.4.2, we can apply Lemma 5.1.2 to P^{β^+}. Thus for limit ordinals γ such that $\beta^+ < \gamma < \kappa^+$ the limit $P^{\beta^+}_\gamma$ is direct if γ is regular or $\mathrm{cf}\,\gamma$ is Mahlo and $\gamma < (\mathrm{cf}\,\gamma)^+$. We are going to finish the proof by showing that an application of Lemma 5.1.3 yields that P^{β^+} is b^+-game closed.

Clearly conditions (ii) and (iii) of Lemma 5.1.3 are fulfilled. For condition (i) we need to note that the winning strategy is definable using parameters only from the ground model, but the definition is indeed very simple, as was seen in the proof of Lemma 5.4.3. □

As an immediate corollary of the previous lemma is the following:

5.4.5. Lemma. *The name $\dot{Q}_\alpha$ represents a non-trivial forcing notion for every ordinal $\alpha < \kappa^+$ such that $\mathrm{cf}\,\alpha$ is Mahlo and $\alpha < (\mathrm{cf}\,\alpha)^+$.*

We shall develop an argument similar to the proofs of Lemmas 5.2.3 and 5.3.2 to show that P preserves weakly compact sets. Since the details are slightly more involved we shall break it into a sequence of lemmas and fix some mathematical objects for the duration of this section as we go along.

Assume that E is weakly compact subset of κ and

$$p \Vdash_{\kappa^+} \text{“}\langle V_\kappa, \in, \dot{U}\rangle \models \phi\text{”} \tag{13}$$

where $p \in P$, $\dot{U}$ is a nice name for a subset of κ and ϕ is a Π^1_1-sentence. By Lemma 5.4.2 there exists an ordinal $\eta < \kappa^+$ such that $p \in P_\eta$ and $\dot{U}$ is a P_η-name. Fix an arbitrary $\eta < \kappa^+$ with these properties. For convenience we shall assume that $\eta \geq \kappa$.

5.4.6. Lemma. *$p \Vdash_\zeta$ “$\langle V_\kappa, \in, \dot{U}\rangle \models \phi$” whenever $\eta \leq \zeta \leq \kappa^+$.*

Proof. This is because V_κ is absolute w.r.t. V^{P_ζ} and $V^{P_{\kappa^+}}$, $p \in G_\zeta$ implies $p \in G_{\kappa^+}$, and Π^1_1-truth is downward absolute. □

Pick a set M' of cardinality κ that reflects the formulae (13) and

$$q \Vdash \text{“}\alpha \in \check{S}, \text{ and } \langle V_\alpha, \in, \dot{U} \cap V_\alpha\rangle \models \phi\text{”} \tag{14}$$

such that ${}^{<\kappa}M' \subseteq M'$ and M' contains a bijection $\kappa \to \eta$. Furthermore make sure that M' satisfies a large enough finite fragment of ZFC, $V_\kappa \subseteq M'$, $\eta + 1 \subseteq M'$, and E, S, P_η, and $\dot{U}$ are elements of M'. Let $\pi : M' \to M$ be the Mostowski collapse.

5.4.7. Lemma. *For every $\alpha \leq \eta$ we have*

(*a*) $P_\alpha \subseteq M'$

(*b*) $\pi | P_\alpha = \mathrm{id}_{P_\alpha}$

(*c*) $\pi(P_\alpha) = P_\alpha$

and furthermore E, S, and $\dot{U}$ are mapped on themselves by π.

Proof. (*c*) always follows from the others, and all is clear for $\alpha \leq \kappa$. For $\alpha > \kappa$ we proceed by induction. Inverse limits work since ${}^{<\kappa}M' \subseteq M'$. For $\alpha = \beta + 1$ we (just) need to show that $\operatorname{dom} \dot{Q}_\beta \subseteq M'$ and $\pi | \operatorname{dom} \dot{Q}_\beta = \mathrm{id}$. But this is clear since M' contains all antichains of P_κ. □

By Lemma 4.1.6 we can find a transitive set N and an elementary embedding $j : M \to N$ so that ${}^{<\kappa}N \subseteq N$ and $\kappa \in j(E)$. Let $\zeta = (\kappa^+)^N$. By our choice of M' we have $\eta < \zeta < j(\kappa)$.

5.4.8. Lemma. *$j(P_\eta)_\alpha = P_\alpha$ for all $\alpha \leq \zeta$.*

Proof. The conclusion is immediate for $\alpha \leq \kappa$. We proceed by induction. Since ${}^{<\kappa}N \subseteq N$ cofinality is absolute for ordinals $\alpha < \zeta$. For the same reason $\operatorname{cf}\zeta = \kappa$ ($|N| = \kappa$ by its definition). Thus the choices between direct and inverse limits are the same for $j(P_\eta)_\alpha$ and P_α.

Suppose that P_α is an inverse limit and $p \in P_\alpha$. Again ${}^{<\kappa}N \subseteq N$ together with the induction hypothesis implies that $p \in j(P_\eta)_\alpha$. We have now handeled the limit steps of the induction.

Now suppose that $\alpha = \beta + 1 > \kappa$. Here we just need to check that the conditions (vi)–(ix) in the definition of P are absolute for N. But these conditions can be rendered to refer to formulae in the forcing language only in quantifier free ways. □

We shall need to refer to the above lemma for $\alpha = \eta$, but note that η can be arbitrarily large.

We define a function d on $\{j(\alpha) : \kappa \leq \alpha < \eta\}$ by letting $d(j(\alpha))$ be a $P_{\alpha+1}$-name for the pair $(\bigcup_{s \in G_{\alpha+1}} C_{s(\alpha)} \cup \{\kappa\}, \bigcup_{s \in G_{\alpha+1}} A_{s(\alpha)})$. Furthermore we choose $d(j(\alpha))$ from $\operatorname{dom}(j(\dot{Q}_\alpha))$ whenever possible.

5.4.9. Lemma. *Suppose that $\kappa \leq \nu \leq \eta$. If there exists a condition $p \in G_{j(\nu)}$ such that* $\operatorname{dom} d \subseteq \operatorname{dom} p \subseteq [\zeta, j(\eta))$ *and*

$$p|j(\alpha) \Vdash \text{“}d(j(\alpha)) \in \dot{Q}_{j(\alpha)} \text{ and } p(j(\alpha)) \leq d(j(\alpha))\text{”}$$

for every $\alpha \in [\kappa, \nu)$ then j extends to an elementary embedding $j : M[G_\nu] \to N[G_{j(\nu)}]$.

Proof. For notational simplicity we shall assume that $\nu = \eta$. For other ordinals ν the proof is the same apart from taking the appropriate restrictions. We need to show that $j(q) \in G_{j(\eta)}$ whenever $q \in G_\eta$. We shall first show that $q \cup p \leq j(q)$ and then finish the proof by checking that $q \cup p \in G_{j(\eta)}$ (we know that $q \cup p \in j(P_\eta)$ since $\eta < j(\kappa)$).

First note that $|\operatorname{dom} q| < \kappa$ whereby $\operatorname{dom}(j(q)) = j[\operatorname{dom} q]$. We also have $j(q)|j(\kappa) = j(q)|\kappa = q|\kappa$. Thus $q \leq j(q)|j(\kappa)$ and $\operatorname{dom}(j(q)) \subseteq \operatorname{dom}(q \cup d)$. Since $j(q(\alpha)) = q(\alpha)$ for all $\alpha \in \operatorname{dom} q$ we have $q \cup p|j(\alpha) \Vdash$ “$d(j(\alpha)) \leq j(q(\alpha))$” for all $\alpha \in \operatorname{dom}(q) \cap [\kappa, \eta)$. Thus $q \cup p \leq j(q)$. Finally there is a common extension of q and p in $G_{j(\eta)}$ and since $q \cup p = \inf\{q, p\}$ it is clear that $q \cup p \in G_{j(\eta)}$. □

5.4.10. Lemma. *Suppose that $\kappa \notin j(S)$ and $\kappa \leq \nu \leq \eta$. Then $d|j(\nu) \in j(P_\nu)$.*

Proof. First note that $\{j(\alpha) : \kappa \leq \alpha < \eta\} = j(f)|\kappa$ where f is any function in M that maps κ onto $[\kappa, \eta)$, and therefore $\operatorname{dom} d \in N$. It is also easy to check that $\operatorname{dom} d$ is bounded below every $j(\kappa)$-cofinal cardinal.

By Lemma 5.4.8 the $P_{\alpha+1}$-name $d(j(\alpha))$ is also a $j(P_\nu)_{j(\kappa)}$-name for all $\alpha \in [\kappa, \eta)$. We shall now proceed by induction on ν. The limit steps are clear by the facts mentioned above. Suppose that $\nu = \alpha + 1$. By the induction hypothesis $d|j(\alpha) \in j(P_\alpha)$ and by Lemma 5.4.9 it follows that

$$d|j(\alpha) \Vdash \text{“}j(\tau_\alpha)_i = (\tau_\alpha)_i\text{”} \tag{15}$$

for all $i < \kappa$. Here the subscript i is used to indicate the ith element of the sequence that is the interpretation of the name in question. By the induction hypothesis we only need to worry about $d(j(\alpha))$ to see that $d|j(\nu) \in j(P_\nu)$. By (15) and the fact that $\kappa \notin j(S)$ we are basically done. We are left with checking that

$$d|j(\alpha) \Vdash \text{“}\textstyle\bigcup_{s\in G_{\alpha+1}} C_{s(\alpha)} \cup \{\kappa\} \text{ is 1-closed”}$$

but this is similar to the last paragraph of the proof of Lemma 5.3.2 and relies on Lemma 5.4.3. □

5.4.11. Lemma. *If $E \subseteq \kappa$ is disjoint from S and weakly compact in the ground model then $\diamondsuit(\mathrm{WC}_\kappa|E)$ holds in V^P.*

Proof. Let $\dot{A}$ be an arbitrary P-name for a subset of κ. Since η was chosen to be arbitrary large we can assume that $\dot{A}$ is a P_η-name. Since we are now assuming that E is disjoint from S we have $\kappa \notin j(S)$ and therefore d is a condition in $j(P_\eta)$ by Lemma 5.4.10.

Fix a P_η-generic set G and consider the set $B \subseteq \kappa+\kappa$ such that $B \cap \kappa = (A_{d(j(\kappa))})_G$ and $\dot{A}_G = \{\alpha < \kappa : \kappa + \alpha \in B\}$. Fix an ordinal $\gamma < j(\kappa)$ such that $\kappa + \kappa \leq \gamma$ and $\gamma \notin j(S)$. Such ordinals certainly exists since $j(E)$ and $j(S)$ are disjoint. Let q be a condition that is exactly like d except that $q(j(\kappa))$ is a name for the pair $(B, (C_{d(j(\kappa))})_G \cup \{\gamma\})$. Clearly $q \leq d$.

Now the proof proceeds as the proof of Lemma 5.4.18 except that we assume that $q \in G_{j(\eta)}$. Using Lemma 5.4.10 the claim follows in similar fashion as we obtained the result of Lemma 5.2.3; the diamond sequence is essentially the set $\bigcup_{s\in G_{\kappa+1}} A_{s(\kappa)}$. □

One could hope to be able to prove the preservation of every weakly compact set using Lemma 5.4.9. The argument would be similar as the one in the proof of Lemma 5.4.10. But now we face the possibility that $\kappa \in j(S)$, when trying to inductively extend the condition p of Lemma 5.4.9. So in addition we would have to find a condition $q \in j(P_\nu)$ such that $q \leq p$, $\mathrm{dom}(q) \cap \zeta$ is empty, and

$$q \Vdash \text{“}\textstyle\bigcup_{s\in G_{\alpha+1}} A_{s(\alpha)} \neq j(\tau_\alpha)_\kappa\text{”}$$

so that we could define an extended condition p' by letting $p'|j(\nu) = q$ and $p'(j(\nu)) = d(j(\nu))$.

However, without any substantial restrictions on which names τ_α can be chosen in the definition of P, we will inevitably face a situation in which

$$p \Vdash \text{“}\textstyle\bigcup_{s\in G_{\alpha+1}} A_{s(\alpha)} = j(\tau_\alpha)_\kappa\text{”}$$

making the choice of q above impossible. On the other hand, any narrowed down set of possible names τ_α which enables us to put $q = p$ is too restrictive. A simple argument as in the proof of Lemma 5.4.11 would then show that the forcing does not perform the intended task. Notice that a diamond sequence could equally well be "read" from some other coordinate besides κ, the coordinate that was used in the proof of Lemma 5.4.11.

Restricting the set of possible names τ_α does not seem to be fruitful. Instead we shall use ideas developed by Hauser [8] to find an extension of the embedding j by replacing a part of the generic set with an isomorphic copy.

We say that a subset S of κ^+ is a *complete set of coordinates* (with respect to P) if $p|S \in P$ for every condition $p \in P$. In this setting $P|S$ denotes $\{p|S : p \in P\}$.

5.4.12. Lemma. *If S and S' are complete sets of coordinates such that $\kappa \subseteq S \subseteq S'$ then $P|S$ is a complete suborder of $P|S'$.*

Proof. Let p and q be conditions in P and let $\alpha \in \operatorname{dom}(p) \cap \operatorname{dom}(q) \setminus \kappa$. By condition (ix) in the definition of P we have $p|\alpha \Vdash$ "$p(\alpha) \le q(\alpha)$" if and only if $p|\kappa \Vdash$ "$p(\alpha) \le q(\alpha)$". Thus it is clear that $p|S' \le q|S'$ implies $p|S \le q|S$. Using the observation above it is straightforward to check the claim of the lemma. $\square$

Let π be a bijective partial function $\kappa^+ \to \kappa^+$. If

(i) $\pi|\kappa = \mathrm{id}_\kappa$

(ii) Both the domain and range of π are complete sets of coordinates

(iii) $p \mapsto p \circ \pi^{-1}$ defines an isomorphism between $P|\operatorname{dom}\pi$ and $P|\operatorname{ran}\pi$

then we say that π is *P-complete.* Note that the definition is symmetric in the sense that π is P-complete if and only if π^{-1} is P-complete.

5.4.13. Lemma. *Suppose that π is P-complete, $\alpha \subseteq \operatorname{dom}\pi$, $\alpha \notin \operatorname{dom}\pi$, $|\pi| < \kappa^+$ and $\beta < \kappa^+$. Then there exists an ordinal γ such that $\beta \le \gamma < \kappa^+$ and the extended function $\pi \cup \{(\alpha, \gamma)\}$ is a P-complete bijection.*

Proof. Consider the P_α-name τ_α. Since $\alpha \subseteq \operatorname{dom}\pi$ we can look upon τ_α as a $(P|\operatorname{dom}\pi)$-name. Here we have to note that P_α is a complete suborder of $P|\operatorname{dom}\pi$, but this is obvious since we are restricting to an initial segment α of $\operatorname{dom}\pi$.

Let τ_α^π denote the $(P|\operatorname{ran}\pi)$-name that corresponds to τ_α under the isomorphism $p \mapsto p \circ \pi^{-1}$. By Lemma 5.4.12 we can look upon τ_α^π as a P_γ-name for a subset sequence indexed by κ, where γ is any ordinal such that $\sup\operatorname{ran}\pi \le \gamma < \kappa^+$. Condition (vii) in the definition of P is taken to talk of the names literally as sets. Therefore there is unboundedly many ordinals γ below κ^+ such that τ_γ is equivalent to τ_α^π. Let us fix such an ordinal $\gamma \ge \max\{\beta, \sup\operatorname{ran}\pi\}$. We shall prove that γ is as desired. Let us denote the extended mapping by π too, since there can be no confusion. Thus we have $\pi(\alpha)=\gamma$.

Let $p \in P|\operatorname{dom}(\pi)$ be arbitrary. We shall first show that $p\circ\pi^{-1} \in P_{\kappa^+}$. Now suppose that $\alpha \in \operatorname{dom} p$. (It should be clear from the argument that follows that other ordinals in $\operatorname{dom} p$ do not present any problems.) We must prove that

$$p(\alpha) \in \operatorname{dom} \dot{Q}_{\pi(\alpha)} \tag{16}$$

where $\dot{Q}_{\pi(\alpha)}$ is a name for $K^1_{\check{S}}(\tau_{\pi(\alpha)})$ and that

$$(p\circ\pi^{-1})|\pi(\alpha) \Vdash \text{“}p(\alpha) \in K^1_{\check{S}}(\tau_{\pi(\alpha)})\text{”}. \tag{17}$$

But by our choice of $\pi(\alpha)$ we have $(p\circ\pi^{-1})|\pi(\alpha) = p|(\kappa^+\setminus\{\alpha\})\circ\pi^{-1}$. Furthermore $\tau_{\pi(\alpha)}$ corresponds essentially to τ_α under the isomorphism between $P|(\operatorname{dom}(\pi)\setminus\{\alpha\})$ and $P|(\operatorname{ran}(\pi)\setminus\{\pi(\alpha)\})$, and $p(\alpha)$, being a P_κ-name corresponds to itself. Thus

$$p|(\kappa^+\setminus\{\alpha\}) \Vdash \text{“}p(\alpha) \in K^1_{\check{S}}(\tau_\alpha)\text{”} \tag{18}$$

is in fact equivalent to (17). But τ_α is a P_α-name and therefore the forcing language statement in (18) depends only on the P_α-generic part of a given generic set. So in fact

$$p|\alpha \Vdash \text{“}p(\alpha) \in K^1_{\check{S}}(\tau_\alpha)\text{”} \tag{19}$$

is equivalent to (17), and (19) holds directly by the definition of P.

Since the equivalence between (17) and (19) holds not only for $p(\alpha)$ but also for other P_κ-names, (16) holds by absoluteness of the choice of the names $\dot{Q}_\alpha$. Using nearly the same argument we can see that the mapping $p\mapsto p\circ\pi^{-1}$ from $P|\operatorname{dom}\pi$ into $P|\operatorname{ran}\pi$ is surjective. That the ordering is preserved by the mapping is obvious by arguments similar to those in the proof of Lemma 5.4.12.

The domain of the extended function π is clearly complete, since $\alpha\subseteq\operatorname{dom}\pi$. Now we are left with proving that $\operatorname{ran}\pi$ is complete, but this also becomes clear when one considers the equivalence of (17) and (19). □

5.4.14. Lemma. *Suppose that S is a collection of P-complete partial functions $\kappa^+\to\kappa^+$ such that $\pi\subseteq\pi'$ or $\pi'\subseteq\pi$ for every pair of functions π and π' from S. Then $\bigcup S$ is a P-complete function.*

Proof. Let $A=\bigcup_{\pi\in S}\operatorname{dom}\pi$ and let $p\in P$. We shall show that $p|A\in P$. Let $\alpha\in\operatorname{dom}(p)\cap A$ be arbitrary. We are using induction on α so let us assume that $p|(A\cap\alpha)\in P$. There exists a function $\pi\in S$ such that $\alpha\in\operatorname{dom}\pi$. Now $p|(\operatorname{dom}(\pi)\cap\alpha)\Vdash$ “$p(\alpha)\in\dot{Q}_\alpha$”. Since $p|(A\cap\alpha)\le p|(\operatorname{dom}(\pi)\cap\alpha)$ we are done.

The above argument works for unions of P-complete sets of coordinates in full generality. Thus also $B=\bigcup_{\pi\in S}\operatorname{ran}\pi$ is P-complete. Now let $\pi=\bigcup S$ and suppose that $p\in P|\operatorname{dom}\pi$. It is not difficult to show that $p\circ\pi^{-1}\in P$. Note that just by considering properties of sets of ordinals, every strictly increasing sequence $(\gamma_i : i<\kappa)$ of ordinals in $\operatorname{ran}\pi$ contains a subsequence $(\gamma_i : i\in I)$ such that $|I|=\kappa$ and $(\pi^{-1}(\gamma_i) : i\in I)$ is strictly increasing. Thus $\operatorname{dom}(p\circ\pi^{-1})$ conforms to the definition of P. □

5.4.15. Lemma. *Any P-complete partial function $\pi : \kappa^+ \to \kappa^+$ such that $|\pi| < \kappa^+$ can be extended to a P-complete total function $\kappa^+ \to \kappa^+$.*

Proof. The before-mentioned symmetry allows us to apply Lemma 5.4.13 not only to a P-complete function π but also to the inverse π^{-1}. Thus using a standard back and forth type argument it is straightforward to use Lemmas 5.4.13 and 5.4.14 to complete a given partial function to a total one. $\square$

In the argument that follows we shall cheat slightly. Actually Lemmas 5.4.16 and 5.4.17 are intertwined in an induction argument in the same way that Lemmas 5.4.9 and 5.4.10 are. But we shall actually refer to a modified version of Lemma 5.4.17 before proving it, in order not to break the natural flow of the presentation.

Fix a P_ζ-name σ for a uniform winning strategy for player II in $\mathscr{G}_\zeta(j(P_\eta)^\zeta)$. We shall inductively define a P-complete function $\pi : \eta \to \zeta$ simultaneously with a sequence $(\dot{f}_\nu : \kappa \leq \nu < \eta)$ of P_ζ-names for forcing conditions in $j(P_\eta)^\zeta$ such that the conditions introduced below are upheld for as long as possible. As one might expect we shall then prove that the conditions indeed can be realised throughout the entire construction. The construction takes place in N.

The induction is on ν where $\kappa \leq \nu < \eta$. We shall try to make sure that

(iv) $\pi|(\nu+1)$ is P-complete

and that the following holds in any generic extension by P_ζ:

(v) $(f_i : i < \nu)$ is a decreasing sequence of conditions in $j(P_\eta)^\zeta$ that form the moves of player I in some instance of $\mathscr{G}_\zeta(j(P_\eta)^\zeta)$ where player II plays according to the interpretation of σ

(vi) $j(\nu) = \max \operatorname{dom} f_\nu$ and $f_\nu(j(\nu))$ is a $P_{\pi(\nu)+1}$-name such that

$$\Vdash \text{“}f_\nu(j(\nu)) = (\textstyle\bigcup_{s \in \dot{G}_{\pi(\nu)+1}} C_{s(\pi(\nu))} \cup \{\kappa\}, \bigcup_{s \in \dot{G}_{\pi(\nu)+1}} A_{s(\pi(\nu))})\text{”}.$$

Note that condition (vi) closely resembles the definition of the function d apart from the coordinates being shifted by π.

5.4.16. Lemma. *There exist a function $\pi : \eta \to \zeta$ and a sequence $(\dot{f}_\nu : \kappa \leq \nu < \eta)$ satisfying the conditions in the definition for all ordinals ν such that $\kappa \leq \nu < \eta$.*

Proof. Suppose that $(\rho_i : \kappa \leq i < \nu)$ and $\pi|\nu$ are defined successfully maintaining conditions (iv)–(vi) for ordinals smaller than ν. We shall define $\pi(\nu)$ and $\dot{f}_\nu$ so that the conditions hold for ν.

Arguing in an arbitrary generic extension of N by P_ζ, we can use the induction hypothesis about condition (v) to conclude that there exists a lower bound f for $\{f_i : i < \nu\}$. Fix such a condition f and if ν happens to be a successor ordinal, let f be the choice of player II by the winning strategy. (Recall that player I plays first at limit ordinals.) Since the winning strategy is uniform we can require that $f \in j(P_\nu)^\zeta$.

Consider a further generic extension by $j(P_\nu)^\zeta$ such that f is in the generic set. Let $j(\tau_\nu)_\kappa$ denote the κth element in the sequence that is the interpretation of the $j(P_\nu)$-name $j(\tau_\nu)$. By standard density arguments there exist at most one ordinal $\alpha < \zeta$ for which $\bigcup_{s \in G_{\alpha+1}} A_{s(\alpha)} = j(\tau_\nu)_\kappa$.

Let us now go back to argue in the generic extension by P_ζ and let $j(\tau_\nu)_\kappa$ denote a $j(P_\eta)^\zeta$-name for the object under discussion. By the previous remarks we can find a condition $f^* \in j(P_\nu)^\zeta$ and a fixed ordinal α_ν such that $f^* \leq f$ and

$$f^* \Vdash \text{“}\textstyle\bigcup_{s \in G_{\alpha+1}} A_{s(\alpha)} \neq j(\tau_\nu)_\kappa\text{”} \tag{20}$$

for all ordinals α such that $\alpha_\nu < \alpha < \zeta$.

Continuing the argument in N we consider P_ζ-names for the objects α_ν and f^*. Since P_ζ satisfies the ζ-cc by Lemma 5.4.2 applied inside N, the possible values of α_ν are bounded below ζ. Now we fix $\pi(\nu)$ above this upper bound so that

$$\alpha_\nu < \pi(\nu) \tag{21}$$

holds in any generic extension by P_ζ where α_ν is the realisation of the name we chose and $\pi(\nu)$ is fixed in the ground model N.

At the same time we can make sure that $\pi|(\nu+1)$ is P-complete which is possible by Lemma 5.4.13. If ν is a limit ordinal we shall also need to refer to Lemma 5.4.14 to first conclude that $\pi|\nu$ is P-complete.

To deal with condition (vi) we first need to note that $\{j(\nu) : \kappa \leq \nu < \eta\} \in N$, but this was already dealt with at the beginning of the proof of Lemma 5.4.10. Now pick $\dot{f}_\nu$ such that $f^* = f_\nu|j(\nu)$ in any generic extension by P_ζ, and condition (vi) is satisfied. The only thing left to verify is that f_ν actually is a condition in $j(P_\eta)^\zeta$.

At this point we are going refer to the fact that f^* behaves for $j(P_\nu)^\zeta$ as f_η does for $j(P_\eta)^\zeta$ in Lemma 5.4.17. The suspicious reader may read Lemma 5.4.17 and the definitions before and after it, replacing η by ν. So similarly as in the proof of Lemma 5.4.10 we may extend j to see that

$$f^* \Vdash \text{“}j(\tau_\nu)_i = (\tau_\nu)_i\text{”}$$

for all $i < \kappa$. Furthermore

$$f^* \Vdash \text{“}\textstyle\bigcup_{s \in G_{\pi(\nu)+1}} C_{s(\pi(\nu))} \cup \{\kappa\} \text{ is 1-closed”}$$

but now also the potential problem that occurs if $\kappa \in j(S)$ is handled by (20) and (21). □

Let p be the condition fixed by (13). Fix a P_ζ-generic set G_ζ such that $p \circ \pi^{-1} \in G_\zeta$. Since π is P-complete $\{q \circ \pi^{-1} : q \in P_\eta\} = P|\operatorname{ran}\pi$ is a complete suborder of P_ζ. Thus by putting

$$G_\eta^\pi = \{q \in P_\eta : q \circ \pi^{-1} \in G_\zeta\}$$

we obtain a P_η-generic set G_η^π such that $p \in G_\eta^\pi$. Consider the forcing notion $j(P_\eta)^\zeta$ in the extension $N[G_\zeta]$ and let f_η be a lower bound for the sequence $(f_\nu : \nu < \eta)$. Fix a $j(P_\eta)^\zeta$-generic set H over $N[G_\zeta]$ such that $f_\eta \in H$.

5.4.17. Lemma. *$q \circ \pi^{-1} \cup f_\eta$ is a condition in $j(P_\eta)$ such that $q \circ \pi^{-1} \cup f_\eta \leq j(q)$ for every $q \in P_\eta$ and therefore $j(q) \in G_\zeta * H$ for every condition $q \in G_\eta^\pi$.*

Proof. Let $q \in P_\eta$. As before we know that $\mathrm{dom}(j(q)) = j[\mathrm{dom}\, q]$. Since $j(q)|j(\kappa) = q|\kappa = (q \circ \pi^{-1})|\kappa$, we can conclude that $q \circ \pi^{-1} \leq j(q)|j(\kappa)$ and $\mathrm{dom}(j(q)) \subseteq \mathrm{dom}((q \circ \pi^{-1}) \cup f_\eta)$. Let $\alpha \in \mathrm{dom}(q) \cap [\kappa, \eta)$ be arbitrary. We first note that $j(q)(j(\alpha)) = q(\alpha) = (q \circ \pi^{-1})(\pi(\alpha))$. By condition (vi) in the definition of the sequence $(f_\alpha : \alpha < \eta)$ we have

$$(q \circ \pi^{-1})|(\pi(\alpha) + 1) \Vdash \text{“}f_\alpha(j(\alpha)) \leq (p \circ \pi^{-1})(\pi(\alpha))\text{”}$$

and since $f_\eta \leq f_\alpha$ we can conclude that $q \circ \pi^{-1} \cup f_\eta \leq j(q)$. Since $(q \circ \pi^{-1}) \cup f_\eta \in G_\eta * H$ the second claim is obviously true. □

By Lemma 5.4.17 the equation

$$j(\tau_{G_\eta^\pi}) = j(\tau)_{G_\zeta * H}$$

is well behaved and can be used to define an extended elementary embedding $j : M[G_\eta^\pi] \to N[G_\zeta][H]$. Let us fix such an embedding. Let us also extend π to a total P-complete bijection $\zeta \to \zeta$. This is possible by applying Lemma 5.4.15 inside N. We shall denote the extended function by π too, so that our original function is $\pi|\eta$. Now we are set to prove the crucial lemma.

5.4.18. Lemma. *Every set $E \subseteq \kappa$ that is weakly compact in the ground model is weakly compact in V^P.*

Proof. Recall that we had fixed a weakly compact set E such that $\kappa \in j(E)$. Since this set E was completely arbitrary we can let it exemplify the E of the lemma. Let $\dot{U}^\pi$ denote the P_ζ-name that corresponds to the P_η-name $\dot{U}$ under the automorphism $p \mapsto p \circ \pi^{-1}$ of P_ζ. Since

$$p \Vdash_\zeta \text{“}\langle V_\kappa, \in, \dot{U}\rangle \models \phi\text{”}$$

by (13) and Lemma 5.4.6, it then clearly follows that

$$p \circ \pi^{-1} \Vdash_\zeta \text{“}\langle V_\kappa, \in, \dot{U}^\pi\rangle \models \phi\text{”}.$$

Now $\dot{U}^\pi_{G_\zeta} = \dot{U}_{G_\eta^\pi}$ and since the condition $p \circ \pi^{-1} \in G_\zeta$ we have that $\langle V_\kappa, \in, \dot{U}_{G_\eta^\pi}\rangle \models \phi$ holds in $V[G_\zeta]$ and thus in $N[G_\zeta]$.

Since it is true (Lemma 5.4.4) in $N[G_\zeta]$ that $j(P_\eta)^\zeta$ does not add subsets of κ the statement holds in $N[G_\zeta][H]$. By definition $j(\dot{U}_{G_\eta^\pi}) = j(\dot{U})_{G_\zeta * H}$. Thus by elementarity and the fact that $\dot{U}_{G^\pi_\eta} = j(\dot{U}_{G^\pi_\eta}) \cap V_\kappa$ it holds in $M[G^\pi_\eta]$ that there exists an ordinal $\alpha \in E$ such that $\langle V_\alpha, \in, \dot{U}_{G^\pi_\eta} \cap V_\alpha \rangle \models \phi$.

Since $p \in G^\pi_\eta$ it holds in M that there exists an ordinal $\alpha \in E$ and a condition $q \leq p$ in P_η such that

$$q \Vdash_\eta \text{“}\langle V_\alpha, \in, \dot{U} \cap V_\alpha \rangle \models \phi\text{”}$$

and by the choice of M' this holds in V too. Finally by absoluteness with respect to the generic extensions involved, the above holds with η replaced with κ^+. □

Having proved that weak compactness is preserved, we move on to checking that forcing with P really kills the desired version of weakly compact diamond.

5.4.19. Lemma. *For every $\alpha < \kappa^+$ the interpretation of τ_α is not a $\diamondsuit(\mathrm{WC}_\kappa|S)$-sequence in V^P.*

Proof. In similar fashion as for Lemma 5.3.3 we need to show that

$$C = \bigcup_{s \in G_{\alpha+1}} C_{s(\alpha)}$$

is stationary in V^P. Since $\eta < \kappa^+$ may be arbitrary large, we may assume that $\alpha < \eta$. By Lemma 5.4.2 it is enough to show that C is stationary in V^{P_η}. But $C_{d(j(\kappa))} \subseteq j(C)$ so $\kappa \in j(C)$ and the argument of Lemma 5.4.18 yields that C is weakly compact in V^{P_η}. □

5.4.20. Lemma. *$\diamondsuit(\mathrm{WC}_\kappa|S)$ fails in V^P.*

Proof. By the previous lemma we only have to check that every subset sequence indexed by κ in V^P is the realisation of τ_α for some $\alpha < \kappa^+$. □

The result of this section is a combination of Lemmas 5.4.11, 5.4.18, and 5.4.20.

5.4.21. Theorem. *If S is a weakly compact subset of κ then there exists a generic extension that preserves every weakly compact subset of κ and in which $\diamondsuit(\mathrm{WC}_\kappa|S)$ fails and $\diamondsuit(\mathrm{WC}_\kappa|E)$ holds for every weakly compact E in the ground model such that E is disjoint from S.*

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