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HENRI VIRTANEN

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Turku, March 2008

Henri Virtanen

Contents

1. Introduction

Analytic number theory, in the modern sense, is said to begin with the work of Dirichlet. Modifying Euler's well-known analytic proof that there are infinitely many primes, Dirichlet's idea was to generalize Euler's formula $\prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s}$, valid for $\Re(s) > 1$. He introduced the characters, now bearing his name, to prove that any arithmetic progression $a + nq$, $n = 0, 1, 2, \dots$, where a and q are relatively prime, contains infinitely many prime numbers. Ever since, these characters and the associated Dirichlet series called L-functions have been studied extensively.

Our aim is to study the mean square of quadratic Dirichlet L-functions, proving in particular the asymptotic formula

$$\sum_{\chi}{}^{*} L^2(1,\chi) = A^* X + P^*(X) X^{1/2} + O\left(X^{1/2}\omega(X)\right), \tag{1.1}$$

where the summation is taken over all real primitive non-principal characters χ (possible restricted to even or odd characters) with conductor at most X. Here A^* is a constant, $P^*(X)$ is an explicitly given function of order $\log X$ and $\omega(X)$ is a function, similar to that occurring in the error term of the prime number theorem, which tends to zero as X tends to infinity. This result, to be exactly formulated in Theorem 4.13, improves for $k = 2$ the mean value result

$$\sum_{\chi}{}^{*} L^k(1,\chi) = a(k) X + O\left(X^{1/2} (\log X)^{b(k)}\right), \tag{1.2}$$

proved by Jutila [21] in 1973.

The Dirichlet class number formula gives a connection between the value of $L(1,\chi)$ and the number of ideal classes of the corresponding quadratic number field, or the number of classes of quadratic forms for a given discriminant. This connection is briefly recalled in Chapter 5. Especially, in the case of imaginary quadratic fields or positive definite quadratic forms, (1.2) gives almost directly the corresponding moments for the class numbers.

Already Gauss made conjectures about the average order of the number of primitive classes of quadratic forms for a given discriminant. Gauss' conjecture for imaginary quadratic fields was first proved by Lipschitz in 1865. (For real quadratic fields, Gauss' conjecture concerned the average order of the product of the class number and the logarithm of the fundamental unit of the field, which was proved by Siegel in 1966.) Vinogradov was the first to obtain a mean value result with *two* main terms in 1917. The exponent of the error term in Vinogradov's result was $5/6+\varepsilon$, which he improved later on. At present the best result of this type is

$$\sum_{n\le N} h(-n) = \frac{\pi}{18\zeta(3)}N^{3/2} - \frac{2}{2\pi^2}N + O\left(N^{29/44+\varepsilon}\right) \tag{1.3}$$

due to Chamizo and Iwaniec [6]. Actually, the above formula is given in a slightly different form from that in Vinogradov's paper [33], since Vinogradov and Gauss considered the narrow class number.

Although the Dirichlet class number formula has been well known for a long time, sums of $L^k(1,\chi)$ over real primitive characters, or equivalently over fundamental discriminants, did not appear in the literature until the latter half of the twentieth century. Jutila's moment formula,

$$\sideset{}{^*}\sum_{1\le d\le X} h^k(-d) = c(k)X^{(k+2)/2}\left(1+O\left(X^{-1/2}(\log X)^{d(k)}\right)\right),$$

obtained from (1.2), with $-d$ running over fundamental discriminants, is the best known for general k. For the case $k=2$ our result (1.1) gives that

$$\sideset{}{^*}\sum_{1\le d\le X} h^2(-d) = aX^2 + b(X)X^{3/2} + O\left(X^{3/2}\omega(X)\right), \tag{1.4}$$

where a is a constant and $b(X)=O(\log X)$. A similar result for the so-called relative class numbers is also considered in Chapter 5.

We begin by giving a survey of the basic properties of characters and L-functions in Sections 2.1 and 2.2. Some known estimates for character sums are presented in Section 2.3.

The square of the L-series at $s=1$ for a non-trivial character χ can be written as

$$L^2(1,\chi) = \sum_{n=1}^{\infty}\frac{d(n)\chi(n)}{n} = \sum_{\nu=0}^{\infty}\frac{\chi(2)^\nu d(2^\nu)}{2^\nu}\sum_{\substack{n=1\\ 2\nmid n}}^{\infty}\frac{d(n)\chi(n)}{n},$$

where $d(n)$ is the number of divisors of n. Here it is understand that $\chi(2)^0=1$ even if $\chi(2)=0$. In particular, we apply this for the real character $\chi(n)=\left(\frac{q\alpha^2}{n}\right)$, where $|q|\le X$ and α is odd, and split up the series over n on the right into two parts, where the sum is taken over $n\le Y$ or $n>Y$, respectively. The latter part will give an error term when Y is chosen suitably.

When we take the average over q, the contribution of the terms with n a square is easy to deal with, and it gives the first main term in (1.1). The harder part is to treat the non-square values of n. We change the order of the summations and consider the sum over q, for a given n, by attaching a smooth weight. The resulting double sum is similar to that studied by Conrey et al in [9], up to the presence of the divisor function $d(n)$. However, the treatment of the smoothed sum over q by Poisson summation will be similar, and an analysis of the sum over n yields a term similar to the one obtained in [9], which will essentially give the second main term.

The motivation for the paper [9] originated from Soundararajan's paper [30] where he studied sums over fundamental discriminants. Our approach generalizes the method used in [9], since we allow the q-sum to run over all discriminants (of certain type), instead of restricting to the fundamental ones. In Chapter 3 we obtain the mean square result over (all) discriminants with error term $O(X^{2/5+\varepsilon})$. The sum over primitive characters, and the proof for (1.1), are considered later in Chapter 4.

The mean square of primitive L-functions is obtained by sieving out the primitive characters from the sum over all characters by using the Möbius function μ. The efficiency of this method depends on the known zero-free region for the Riemann zeta-function. In particular, assuming the Riemann hypothesis we would get an error term $O(X^{8/17+\varepsilon})$ in (1.1).

Readers interested in historical facts are referred, for example, to the article [29] or to the excellent introductions of the articles [15] and [6], and to the book [16], which all provide some historical information concerning the subject of this work.

Notations. In sums concerning characters or discriminants two notations are used: the sum $\sum'$ is taken over discriminants or non-trivial real characters, and the sum $\sum^*$ is taken over fundamental discriminants or primitive characters. The integral $\int_{(c)}$ means an integral over the complex line with real part equal to c, that is, $\int_{(c)} = \lim_{T\to\infty} \int_{c-iT}^{c+iT}$. The letter p denotes a prime number, and $\prod_p$ and $\sum_p$ are taken over all primes. Basic facts about such sums and products are assumed to be known. Familiar estimates for the Riemann zeta-function like $\zeta(s)^{-1} \ll (\sigma - 1)$ near the line $\sigma = 1$ $(\sigma > 1)$ and $\zeta(s) \ll \log t$ for $\sigma \geq 1$ can be found for example in [19] or [32]. Here and elsewhere in the thesis $s = \sigma + \mathrm{i}t$.

Quite often the properties of a function depend on whether its argument is a square or not, so for example by $\sum_{n=\square}$ we mean that the summation is taken over square values, hence the symbol $\square$ stands for an arbitrary square integer. A small positive number, which may be different from line to line, is denoted by ε. We also use the standard notation $e(x) = \mathrm{e}^{2\pi \mathrm{i}x}$. The notation $f(x) \ll g(x)$, due to Vinogradov, is used to mean that the absolute value of the function f is less than a positive constant times the function g, that is, $f(x) = O(g(x))$.

2. Preliminaries

2.1. **Dirichlet characters and Gaussian sums.** By definition a *Dirichlet character* modulo n, where n is a natural number, is a function $\chi : \mathbb{Z} \to \mathbb{C}$ which is periodic modulo n, completely multiplicative, not identically zero, and $\chi(m) = 0$ if $(m, n) > 1$. For every n there is always at least the *trivial character* $\chi_0^{(n)}(m)$ which is one if $(m,n)=1$, and zero otherwise. For every character χ modulo n there is a unique *primitive character* χ^* modulo n^* which induces it, that is, n^* is the smallest divisor of n such that χ can be written as $\chi=\chi_0^{(n)}\chi^*$. The number n^* is the *conductor* of the character χ. Characters can be divided into *even* and *odd* characters depending on the value $\chi(-1)$. If $\chi(-1)=1$, the character is even, and if $\chi(-1)=-1$, it is odd.

A Dirichlet character is *real* (or quadratic) if its values are real. It is well known that the *Kronecker symbol* $\left(\frac{q}{m}\right)$ defines a real non-principal character modulo $|q|$ if q is a *discriminant*, that is, $q\equiv 0, 1 \pmod 4$ and q is not a square. Actually, all real characters modulo $|q|$ can be formed by using this symbol and moreover all primitive real characters modulo $|q|$ are of the form $\chi_q(m)=\left(\frac{q}{m}\right)$, where q is a *fundamental discriminant,* that is, q is a squarefree discriminant or $q=4D$, where D is squarefree and $D\equiv 2$ or $3 \pmod 4$. From now on, the symbol χ_q is always used to mean the Kronecker symbol $\left(\frac{q}{\cdot}\right)$ and hence a real character modulo $|q|$. For odd positive n, the *Jacobi symbol* $\left(\frac{a}{n}\right)$ defines the real character $\left(\frac{a}{n}\right)=\chi_{\tilde{n}}(a)$, where $\tilde{n}=n$ if $n\equiv 1 \pmod 4$, and $\tilde{n}=-n$ if $n\equiv 3 \pmod 4$. This character modulo n is primitive if n is squarefree.

Remark 2.1. The theory of real characters is closely related to the theory of quadratic fields, or to the equivalent theory of the binary quadratic forms. The fundamental discriminants are just the discriminants of quadratic fields, and a real primitive character χ_q is associated with a real quadratic field or with an imaginary quadratic field, according to whether χ_q is even or odd.

In the theory of quadratic forms the discriminant of $ax^2+bxy+cy^2$ is defined to be b^2-4ac, which is clearly congruent to zero or one modulo four. A fundamental discriminant is one which has the property that all forms of that discriminant have $(a,b,c)=1$.

Basic but not elementary properties of the real characters are the *law of quadratic reciprocity* and *its supplements*: If m and n are odd relatively prime natural numbers, then for the Jacobi symbols we have

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right)=(-1)^{((m-1)/2)((n-1)/2)},$$

$$\left(\frac{-1}{n}\right)=(-1)^{(n-1)/2}=\chi_{-4}(n),$$

and

$$\left(\frac{2}{n}\right)=(-1)^{(n^2-1)/8}=\chi_8(n).$$

When m and n are primes, the first of these relations is the reciprocity law for *Legendre symbols*. While trying to generalize the quadratic reciprocity law for Legendre symbols to higher power residues, Gauss gave all in all six different proofs for this law. The fourth and sixth proof led him to study sums which are now known as Gaussian sums.

A *Gaussian sum* related to the character χ modulo n is the exponential sum

$$\tau_k(\chi) = \sum_{a \pmod n} \chi(a) e\left(\frac{ak}{n}\right),$$

where $k \in \mathbb{Z}$. If k and n are relatively prime, then $\tau_k(\chi) = \bar{\chi}(k)\tau(\chi)$, where $\tau(\chi) = \tau_1(\chi)$. Actually, Gauss introduced in 1801 the *quadratic Gauss sum* $\sum_{a=0}^{n-1} \mathrm{e}^{2\pi \mathrm{i} k a^2/n}$, which coincides with the sum $\tau_k(\chi)$ if χ is a real character, n is an odd squarefree integer, and $(n,k) = 1$ (see [2]). In this case, the character is defined by the Jacobi symbol $\left(\frac{a}{n}\right)$ and it is easy to show that

$$\tau(\chi) = \begin{cases} \pm\sqrt{n} & \text{if } n \equiv 1 \pmod 4, \\ \pm\mathrm{i}\sqrt{n} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

However, the sign here is not so easy to decide. Even Gauss needed several years before in 1805 he was able to prove that the sign is always plus.

For further use we introduce, from [30], the function

$$\begin{aligned} G_k(n) &= \left(\frac{1-\mathrm{i}}{2} + \left(\frac{-1}{n}\right)\frac{1+\mathrm{i}}{2}\right) \sum_{a \pmod n} \left(\frac{a}{n}\right) e\left(\frac{ak}{n}\right) \\ &= \left(\frac{1-\mathrm{i}}{2} + \left(\frac{-1}{n}\right)\frac{1+\mathrm{i}}{2}\right) \tau_k(n), \end{aligned}$$

where n is odd and positive, $k \in \mathbb{Z}$, and $\tau_k(n)$ is the Gaussian sum related to the real character $\left(\frac{a}{n}\right)$ modulo n. Hence this sum can be written as

$$\tau_k(n) = \left(\frac{1+\mathrm{i}}{2} + \left(\frac{-1}{n}\right)\frac{1-\mathrm{i}}{2}\right) G_k(n).$$

The main properties of the function $G_k(n)$ are listed in the next lemma (see [30, Lemma 2.3]).

Lemma 2.2. *The function $G_k(n)$ is multiplicative in n, that is, if m and n are odd natural numbers and $(m,n) = 1$, then $G_k(mn) = G_k(m)G_k(n)$. If p^a is the greatest power of the prime $p \neq 2$ which divides the number k, and $a = \infty$ if $k = 0$, then for $b \geq 1$*

$$G_k(p^b) = \begin{cases} 0 & \text{if } b \text{ is odd and } b \leq a, \\ \varphi(p^b) & \text{if } b \text{ is even and } b \leq a, \\ -p^a & \text{if } b = a+1 \text{ is even}, \\ \left(\frac{k/p^a}{p}\right) p^a \sqrt{p} & \text{if } b = a+1 \text{ is odd}, \\ 0 & \text{if } b \geq a+2. \end{cases}$$

Moreover, $G_{mk}(n) = \left(\frac{m}{n}\right) G_k(n)$ if $(m,n) = 1$, $\chi_{-4}(n) G_k(n) = G_{-k}(n)$, and $\chi_8(n) G_k(n) = G_{2k}(n)$.

2.2. **Dirichlet L-functions.** Dirichlet [10, 11] proved his famous theorem on primes in arithmetical progressions in 1837 by studying the series

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}, \tag{2.3}$$

where χ is a character modulo n. This series known as the *Dirichlet L-function* is absolutely convergent in the half plane $\sigma > 1$, where it has the *Euler product*

$$L(s,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$

It can be shown [28, Satz 3.3] that (2.3) actually converges even when $\sigma > 0$ if $\chi \neq \chi_0^{(n)}$, and by analytic continuation it is a regular function in the whole complex plane. For primitive non-principal characters, let

$$\Delta(s,\chi)=\frac{\tau(\chi)}{\mathrm{i}^a\sqrt{n}}\left(\frac{\pi}{n}\right)^{s-1/2}\frac{\Gamma(\frac{1}{2}(1-s+a))}{\Gamma(\frac{1}{2}(s+a))}, \tag{2.4}$$

where $a = 0$ if χ is even, and $a = 1$ if χ is odd. Now we have the *functional equation for the L-function*

$$L(s,\chi)=\Delta(s,\chi)L(1-s,\bar{\chi}),$$

and hence the L-function is entire for primitive non-principal characters. For the L-functions with non-primitive characters, the Euler product formula and the functional equation for the L-function with the corresponding primitive character give the analytic continuation. Only for the function $L(s,\chi_0^{(n)})$ there is a simple pole at 1 with residue $\varphi(n)n^{-1}$.

The Euler product formula shows that the L-function has no zeros if $\sigma > 1$. Therefore, there are no zeros in the plane $\sigma < 0$ either, except the trivial ones which cancel the poles of the Γ-function in (2.4). It is known that there are no zeros on the line $\sigma = 1$ and the *generalized Riemann hypothesis* (GRH) asserts that there are no zeros with real part $\sigma > 1/2$. Therefore, all non-trivial zeros should lie on the line $\sigma = 1/2$. However, like for the Riemann zeta-function, only a certain zero-free region is known.

Estimates for a single L-function at $s = 1$ have been obtained for example by Elliott [12] by showing that there exist constants c_1 and c_2 such that

$$\frac{c_1}{\log\log q}\le |L(1,\chi)|\le c_2\log\log q,$$

for almost all primes $q \le X$, where $\chi(a) = \left(\frac{a}{q}\right)$ is the Legendre symbol. Assuming GRH, Littlewood [25] has shown that there are infinitely many q such that this lower (or upper) bound is really the correct size of $L(1,\chi)$. Later Chowla [7] has shown without the assumption of GRH that

$$L(1,\chi_q)\sim\frac{\pi^2\,\mathrm{e}^{-\gamma}}{6\log\log|q|}$$

for infinitely many q with χ_q real and primitive. Albeit there are discriminants for which the value of $L(1,\chi_q)$ is near to the lower or upper bound mentioned above, the values of $L(1,\chi_q)$ are of constant size in mean, as Jutila's result (1.2) shows.

The distribution of the values of $L^k(1,\chi_q)$ have also been studied by probabilistic methods, for example in [17, 26]. However, compared with Jutila's result (1.2), these probabilistic methods give bigger error terms.

On the line $\sigma = 1/2$, the *convexity bound* [20, Th. 5.23] for the Dirichlet L-functions shows that

$$L(s,\chi) \ll (q\,|s|)^{1/4},$$

where χ is a primitive character modulo q. This implies

$$L(s,\chi_{q\alpha^2}) \ll |qs|^{1/4}\, d(\alpha) \tag{2.5}$$

for non-principal (possible non-primitive) real characters. The mean value estimate

$$\sideset{}{'}\sum_{|q|\le X} \int_{-T}^{T} \left| L\left(\frac{1}{2}+\mathrm{i}t,\chi_q\right)\right|^2 \mathrm{d}t \ll (XT)^{1+\varepsilon}. \tag{2.6}$$

proved by Jutila [23] in 1975, is essentially of the expected order. For the fourth power, we have Heath-Brown's [18] result

$$\sideset{}{'}\sum_{|q|\le X} \left| L\left(\frac{1}{2}+\mathrm{i}t,\chi_q\right)\right|^4 \ll (X(|t|+1))^{1+\varepsilon}, \tag{2.7}$$

which is stronger in the q-aspect, but the averaging in the t-aspect is missing.

2.3. **Estimates for character sums.** A central question in analytical number theory is to understand the behaviour of the character sum

$$\sum_{n\le Y} \chi_q(n),$$

where χ_q is a non-principal real character modulo $|q|$. (Indeed, this is the case for every non-trivial character, but here we consider only real characters.) Since the characters are periodic functions, it is easy to show that this sum is always $\le |q|$ in absolute value. Around 1918 Pólya and Vinogradov improved this upper bound to $O(\sqrt{|q|}\log|q|)$. On the other hand, Paley [27] has shown, in 1932, that there exist character sums of size $\sqrt{|q|}\log\log|q|$. For short character sums, Burgess [4] has shown some better estimates, but albeit plenty of work has been done, the classical upper bound of Pólya and Vinogradov is still the best known in general.

In 1973 Jutila [21] has shown that

$$\sideset{}{'}\sum_{\chi_q} \left|\sum_{n\le Y} \chi_q(n)\right|^2 \ll XY\log^8 X,$$

where the character sum is over non-trivial real characters whose modulus is at most X. In 2002 the author [34] proved the fourth power estimate

$$\sum_{\chi_q}{}' \left| \sum_{n \le Y} \chi_q(n) \right|^4 \ll XY^2 X^{\varepsilon}.$$

Jutila [22] has conjectured that a similar estimate would hold for all even powers of the character sum, but only the second and fourth power are settled. The best known estimate for the mean square is due to Armon [1], where the exponent of $\log X$ is one.

More general estimates for character sums, involving some complex coefficients a_n, are also known. For example, Heath-Brown gives in his paper [18] the following nicely symmetric character sum estimate:

$$\sum_m \left| \sum_n a_n \left(\frac{n}{m} \right) \right|^2 \ll (MN)^{\varepsilon}(M+N) \sum_n |a_n|^2,$$

where m and n are restricted to odd squarefree numbers in the intervals $[1, M]$ and $[1, N]$, respectively. In this thesis, Heath-Brown's result is used in the form [18, Corollary 2]

$$\sum_{\chi_q}{}^* \left| \sum_{n \le N} a_n \chi_q(n) \right|^2 \ll_{\varepsilon} (XN)^{\varepsilon}(X+N) \sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|,$$

where χ_q runs over real non-trivial primitive characters with conductor at most X and the coefficients a_n are arbitrary complex numbers. For $|a_n| \le 1$, the sum $\sum_{n_1 n_2 = \square} |a_{n_1} a_{n_2}|$ can be estimated using the following lemma, which is essentially from [34], but the proof presented here is more elementary.

Lemma 2.8. *If* $2 \le N_0 \le N$, *then*

$$\sum_{\substack{N - N_0 \le m, n \le N + N_0 \\ mn = \square}} 1 \ll N_0 \log N_0.$$

Proof. Since m and n run trough the same numbers, there are at most $2N_0 + 1$ trivial squares $m = n$.

For the non-trivial squares, let us write $n = n_1 a^2$ and $m = n_1 b^2$, where n_1 is squarefree and $a \ne b$. Now two different numbers a and b lie in an interval whose length is of the order $\frac{N_0}{\sqrt{n_1 N}}$ which can be used to estimate the number of the numbers a and b. Therefore, the number of the non-trivial squares is

$$\ll \sum_{n_1 \le \frac{N_0^2}{N}} \frac{N_0}{\sqrt{n_1 N}} \cdot \frac{N_0}{\sqrt{n_1 N}} \ll \frac{N_0^2}{N} \log N_0 \ll N_0 \log N_0.$$

$\square$

Corollary 2.9. *Let* $|a_n| \leq 1$ *and* $N_0 \leq N$. *Then*

$$\sum_{|q|\leq X}^{*} \left| \sum_{N-N_0<n\leq N+N_0} a_n \chi_q(n) \right|^2 \ll_\varepsilon (XN)^\varepsilon (X+N) N_0.$$

The method used later in this thesis is based on the paper [9] by Conrey, Farmer and Soundararajan, where they obtained an asymptotic formula for the sum

$$S(X,Y) = \sum_{\substack{m\leq X\\ m \text{ odd}}} \sum_{\substack{n\leq Y\\ n \text{ odd}}} \left(\frac{m}{n}\right). \tag{2.10}$$

The motivation for studying this sum originated from Soundararajan's proof [30] that $L(\frac{1}{2}, \chi_q) \neq 0$ for a positive proportion of fundamental discriminants q. The interesting case of (2.10) is when X and Y are of somewhat comparable size. In [9], a smoothed sum was considered instead of the sum (2.10), where the smoothing was done with respect to both parameters of summation. This leads to the asymptotic formula

$$S(X,Y) = \frac{2}{\pi^2} C\left(Y/X\right) X^{3/2} + O\left(\left(XY^{7/16} + YX^{7/16}\right) \log XY\right),$$

where

$$C(x) = \sqrt{x} + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^x \sqrt{y} \left(1 - \cos\left(\frac{2\pi k^2}{y}\right) + \sin\left(\frac{2\pi k^2}{y}\right)\right) \mathrm{d}y, \tag{2.11}$$

for $x \geq 0$, with the alternative expression

$$C(x) = x + x^{3/2} \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2} \int_0^{1/x} \sqrt{y} \sin\left(\frac{\pi k^2}{2y}\right) \mathrm{d}y.$$

The function $C(x)$ is quite complicated. For example, $C'(x)$ is not everywhere differentiable. Indeed, when the sum over k in the second expression of $C(x)$ is considered as a function of y, we have a function which is commonly called *"Riemann's nondifferentiable function"* since Riemann suggested it as an example of a continuous function which is not differentiable. For a discussion of this topic see [9] and its references. In [9] it was left to the reader to verify that the above two expressions for $C(x)$ agree. We give here a proof for that.

Since both expressions of $C(x)$ tend to zero as x tends to zero from above, it is sufficient to show that the derivatives of these expressions agree when $x > 0$. Differentiating the second expression for $C(x)$ and using partial integration in the remaining integral term with respect to the factor $\sqrt{y}$, we get

$$
\begin{aligned}
&1+x^{1/2}\sum_{k=1}^{\infty}\int_0^{1/x} y^{-1/2}\cos\left(\frac{\pi k^2}{2y}\right)\mathrm{d}y\\
&=1+x^{1/2}\sum_{k=1}^{\infty}\int_x^{\infty} y^{-3/2}\cos\left(\frac{\pi k^2 y}{2}\right)\mathrm{d}y\\
&=1+x^{1/2}\sum_{k=1}^{\infty}\frac{1}{2}\left(\int_x^{\infty} y^{-3/2}\,\mathrm{e}^{\frac{\pi \mathrm{i} k^2 y}{2}}\,\mathrm{d}y+\int_x^{\infty} y^{-3/2}\,\mathrm{e}^{-\frac{\pi \mathrm{i} k^2 y}{2}}\,\mathrm{d}y\right).
\end{aligned}
$$

Here the integrals are from x to infinity, but by Cauchy's integral theorem the line of integration can be turned upwards or downwards without changing the value of the integral. So we may turn the line of integration upwards in the first integral and downwards in the second integral, and change the order of integration and summation to obtain

$$
1+\frac{x^{1/2}}{2}\left(\int_x^{x+\mathrm{i}\infty} y^{-3/2}\sum_{k=1}^{\infty}\mathrm{e}^{\frac{\pi \mathrm{i} k^2 y}{2}}\,\mathrm{d}y+\int_x^{x-\mathrm{i}\infty} y^{-3/2}\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{\pi \mathrm{i} k^2 y}{2}}\,\mathrm{d}y\right). \tag{2.12}
$$

Let $\theta(z)=1+2\sum_{k=1}^{\infty}\mathrm{e}^{-\pi k^2 z}$, where $z>0$. It is well-known that

$$
\theta(1/z)=\sqrt{z}\theta(z).
$$

By analytic continuation, this holds for $\Re(z)>0$, in particular if $z=-\mathrm{i}y$ and $\Im(y)>0$, or $z=\mathrm{i}y$ and $\Im(y)<0$. By this transformation formula, the expression in the brackets in (2.12) is

$$
\begin{aligned}
&\int_x^{x+\mathrm{i}\infty} y^{-3/2}\left(-\frac{1}{2}+\frac{1}{\sqrt{-\mathrm{i}y/2}}\sum_{k=1}^{\infty}\mathrm{e}^{-2\pi \mathrm{i} k^2/y}+\frac{1}{\sqrt{-2\mathrm{i}y}}\right)\mathrm{d}y\\
&+\int_x^{x-\mathrm{i}\infty} y^{-3/2}\left(-\frac{1}{2}+\frac{1}{\sqrt{\mathrm{i}y/2}}\sum_{k=1}^{\infty}\mathrm{e}^{2\pi \mathrm{i} k^2/y}+\frac{1}{\sqrt{2\mathrm{i}y}}\right)\mathrm{d}y\\
&=-2x^{-1/2}+\frac{\sqrt{2}}{2}x^{-1}\left(\mathrm{e}^{-\pi\mathrm{i}/4}+\mathrm{e}^{\pi\mathrm{i}/4}\right)\\
&\quad+\sqrt{2}\,\mathrm{e}^{\pi\mathrm{i}/4}\sum_{k=1}^{\infty}\frac{1}{-2\pi\mathrm{i}k^2}\left(\mathrm{e}^{-2\pi\mathrm{i}k^2/x}-1\right)\\
&\quad+\sqrt{2}\,\mathrm{e}^{-\pi\mathrm{i}/4}\sum_{k=1}^{\infty}\frac{1}{2\pi\mathrm{i}k^2}\left(\mathrm{e}^{2\pi\mathrm{i}k^2/x}-1\right)\\
&=-2x^{-1/2}+x^{-1}+\sum_{k=1}^{\infty}\frac{1}{\pi k^2}\left(1-\cos\left(\frac{2\pi k^2}{x}\right)+\sin\left(\frac{2\pi k^2}{x}\right)\right).
\end{aligned}
$$

Hence (2.12) is

$$\frac{1}{2}x^{-1/2}+\frac{\sqrt{x}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(1-\cos\left(\frac{2\pi k^2}{x}\right)+\sin\left(\frac{2\pi k^2}{x}\right)\right)$$

which indeed is the derivative of (2.11), and hence the two expressions of $C(x)$ agree.

2.4. **Some methods of estimation.** In this section, we introduce some basic methods for estimating the sum

$$\sum_{n\le x} f(n), \tag{2.13}$$

where f is an arithmetical function.

If the series $\sum_{n=1}^{\infty} f(n)$ converges, we may simply separate the tail part as an error term, and write

$$\sum_{n\le x} f(n)=\sum_{n=1}^{\infty} f(n)-\sum_{n>x} f(n)=\text{main term }+\text{ error.}$$

As an example of this, see the proof of Lemma 3.6.

For a positive real number x which is not an integer, we have

$$\sum_{n\le x} f(n)=\frac{1}{2\pi\mathrm{i}}\int_{(c)} F(s)x^s\frac{\mathrm{d}s}{s}, \tag{2.14}$$

where $F(s)=\sum_{n=1}^{\infty} f(n)n^{-s}$ is the *generating function* of f, which is assumed to converge absolutely on the line $\sigma=c>0$. If x is an integer, then the term $f(x)$ must be replaced by $\frac{1}{2}f(x)$ on the left. Formula (2.14), known as *Perron's formula*, gives an opportunity to study the sum (2.13) using analytic devices. It is usually applied in a truncated form,

$$\sum_{n\le x} f(n)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T} F(s)x^s\frac{\mathrm{d}s}{s}+O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c}+A_x\left(1+\frac{x\log x}{T}\right)\right), \tag{2.15}$$

where $A_x=\max_{\frac{3}{4}x\le n\le\frac{5}{4}x}|f(n)|$. In the above formulation of Perron's formula it is essential that the generating function is absolutely convergent on the line of integration and $c>0$. Hence, if the function $f(n)$ decays rapidly, (2.15) is not the most effective form of Perron's formula. The conditions of absolute convergence and positivity of c can be retained by using two different parameters. Namely, (2.15) can be stated in the form

$$\sum_{n\le x} f(n)n^{-w}=\frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} F(s+w)x^s\frac{\mathrm{d}s}{s}+O\left(\frac{x^b}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^{b+u}}+\frac{A_x}{x^u}\left(1+\frac{x\log x}{T}\right)\right), \tag{2.16}$$

where A_x is as before and $w=u+\mathrm{i}v$, $b>0$, $u+b\ge c_a+\varepsilon$, where c_a is the abscissa of absolute convergence of the series $F(s)$. For more details, see for example [3, 28, 31].

By the theorem of residues, when the integration is moved to the left, the possible poles of the integrand give the main terms and in the best case the remaining integrals can be estimated satisfactorily.

The *partial summation* provides a method to estimate the sum

$$\sum_{y\le n\le x} f(n)g(n),$$

when an estimate for the sums

$$\sum_{y\le n\le \xi} f(n) \quad \text{with } y\le\xi\le x$$

is known and the function g is continuously differentiable in the range of summation. Namely,

$$\sum_{y\le n\le x} f(n)g(n) = g(x)\sum_{y\le n\le x} f(n) - \int_y^x \left(\sum_{y\le n\le \xi} f(n)\right) g'(\xi)\mathrm{d}\xi.$$

Another common method is to use a *smooth weight function*. When the summation in the original problem is over some interval $[M,N]$, the smooth weight function makes it possible to start and end the summation more "gently", and hence to use analytic methods to estimate the sum. When a smooth weight function is used, it usually causes an error which should be estimated satisfactorily. A smooth weight function for a character sum is introduced in Section 3.2, and the consequent error is analysed in Section 3.4.

Of course, there are other commonly used methods, too, but those mentioned above are the most used in this thesis.

3. The mean square of quadratic Dirichlet L-functions at 1

In this chapter, we apply the method taken from [30], as further developed in [9], to obtain an asymptotic mean value formula for the square of quadratic Dirichlet L-functions at $s=1$. In order to avoid some technical complications, we first study a smoothed mean value over even characters with odd modulus; the smooth weight function will be introduced in Section 3.2, and the consequent smoothing error is estimated in Section 3.4. Mean values over other types of real characters are obtained similarly. In Section 3.5, we formulate the main theorem of the thesis. The case where the characters are restricted to primitive ones will be treated separately later in Chapter 4.

3.1. **Sum over even characters with odd modulus; a preliminary decomposition.**

Let us start from the sum

$$\sum_{\substack{1<q\le X\\ q\equiv 1\,(4)\\ q\ne\square}} L^2(1,\chi_{q\alpha^2}). \tag{3.1}$$

Here and henceforth α is odd and $X > 3$. The square of an L-function is a series of multiplicative functions, so we can separate the evenness from the series by writing

$$L^2(1, \chi_{q\alpha^2}) = \sum_{\substack{n=1 \\ (n,\alpha)=1}}^{\infty} \frac{d(n)\chi_q(n)}{n} = \sum_{\nu=0}^{\infty} \frac{\chi_8(q)^\nu d(2^\nu)}{2^\nu} \sum_{\substack{n=1 \\ (n,2\alpha)=1}}^{\infty} \frac{d(n)\chi_q(n)}{n}.$$

The remaining series over n can be split up into two parts, where the sum is taken over $n \leq Y$ or $n > Y$, respectively. Here Y is a parameter at our disposal, to be chosen later suitably. Lemma 3.2 below shows that

$$\sum_{\substack{1<q\leq X \\ q\equiv 1(4) \\ q\neq\square}} \sum_{\nu=0}^{\infty} \frac{\chi_8(q)^\nu d(2^\nu)}{2^\nu} \sum_{\substack{n>Y \\ (n,2\alpha)=1}} \frac{d(n)\chi_q(n)}{n} \ll \sum_{\substack{1<q\leq X \\ q\equiv 1(4) \\ q\neq\square}} \left| \sum_{n>Y} \frac{d(n)\chi_{4q\alpha^2}(n)}{n} \right|$$
$$\ll O(XY^{-1/2} d^2(\alpha)(\log Y)^{17}).$$

Note that since we are looking for an estimate with an error smaller than $X^{1/2}$, Y should exceed X.

Lemma 3.2. *Let X and Y be greater than 3 and β a natural number. Then*

$$\sum_{|q|\leq X}{}' \left| \sum_{n>Y} \frac{d(n)\chi_{q\beta^2}(n)}{n} \right| \ll XY^{-1/2} d^2(\beta) \log^{17}(XY),$$

where the sum over q is taken over discriminants.

Proof. First the sum over n is divided into dyadic parts,

$$\sum_{j=0}^{\infty} \sum_{2^j Y < n \leq 2^{j+1} Y} \frac{d(n)\chi_{q\beta^2}(n)}{n}. \tag{3.3}$$

By partial summation, the absolute value of the sum over n in (3.3) summed over q is

$$\frac{1}{2^{j+1}Y} \sum_{|q|\leq X}{}' \left| \sum_{2^j Y < n \leq 2^{j+1}Y} d(n)\chi_{q\beta^2}(n) \right| + \int_{2^j Y}^{2^{j+1}Y} y^{-2} \sum_{|q|\leq X}{}' \left| \sum_{2^j Y < n \leq y} d(n)\chi_{q\beta^2}(n) \right| \mathrm{d}y.$$

Now the lemma follows by using the mean value estimate

$$\sum_{|q|\leq X}{}' \left| \sum_{n\leq N} d(n)\chi_0^{(\beta)}(n)\chi_q(n) \right| \ll XN^{1/2} d^2(\beta) \log^{17}(XN)$$

proved by Jutila [24], and noticing that the summation over j in (3.3) converges. $\square$

To the remaining sum

$$\sum_{\substack{1<q\le X\\ q\equiv 1\,(4)\\ q\neq\square}}\sum_{\nu=0}^{\infty}\frac{\chi_8(q)^\nu d(2^\nu)}{2^\nu}\sum_{\substack{n\le Y\\ (n,2\alpha)=1}}\frac{d(n)\chi_q(n)}{n} \tag{3.4}$$

it is useful to introduce a smooth weight function. In [9] the smoothing was done in both n and q aspects, but we use it here only for the sum over q. Weighting requires that the summation should be taken over *all* values of $q \equiv 1 \pmod 4$. Therefore, we add the extra terms $q = \square$ into (3.4), and then from the final result we subtract the influence of these terms which can be evaluated by the following lemma, since

$$\sum_{\substack{q\le X^{1/2}\\ 2\nmid q}}\chi_0^{(q)}(n) = \sum_{\substack{q\le X^{1/2}\\ (q,2n)=1}} 1 = X^{1/2}\frac{\varphi(n)}{2n} + O(d(n))$$

for odd n, where $\varphi(n)$ is the *Euler totient function*.

Lemma 3.5.

$$\sum_{\substack{n\le Y\\ (n,2\alpha)=1}}\frac{d(n)\varphi(n)}{n^2} = b_1(\alpha)\log^2 Y + b_2(\alpha)\log Y + b_3(\alpha) + O(Y^{-1/2}\log^2 Y d(\alpha)),$$

where $0 < b_1(\alpha) < 1$, $b_2(\alpha) = O(\log\log 3\alpha)$, *and* $b_3(\alpha) = O((\log\log 3\alpha)^2)$.

Proof. The generating function of $d(n)\varphi(n)n^{-1}\chi_0^{(2\alpha)}$ is

$$\prod_{p\nmid 2\alpha}\left(1-\frac{1}{p^s}\right)^{-2}\left(1-\frac{2}{p^{s+1}}+\frac{1}{p^{2s+1}}\right) = \zeta^2(s)Q(s)Q_2(s)Q_\alpha(s),$$

where, for $\sigma \ge \sigma_0 > 0$,

$$\begin{aligned} Q_2(s) &= 1-\frac{2^{s+1}-1}{2^{2s+1}-2^{s+1}+1} = O(1),\\ Q_\alpha(s) &= \prod_{p\mid\alpha}\left(1-\frac{(p-1)(2p^s-1)}{p^{2s+1}-2p^s+1}\right)\\ &= \prod_{p\mid\alpha}\left(1-\frac{2}{p^s}+O\left(\frac{1}{p^{\sigma+1}}\right)\right) = O(d(\alpha)), \end{aligned}$$

and

$$Q(s) = \prod_p\left(1-\frac{2}{p^{s+1}}+\frac{1}{p^{2s+1}}\right)$$

converges. By Perron's formula (2.16) we have

$$\begin{aligned}\sum_{\substack{n\le Y\\(n,2\alpha)=1}}\frac{d(n)\varphi(n)}{n^2}=&\frac{1}{2\pi\mathrm{i}}\int_{\varepsilon-\mathrm{i}T}^{\varepsilon+\mathrm{i}T}\zeta^2(s+1)Q(s+1)Q_2(s+1)Q_\alpha(s+1)\frac{Y^s}{s}\mathrm{d}s\\&+O\left(\frac{Y^\varepsilon}{T}+Y^{\varepsilon-1}\right).\end{aligned}$$

Let us choose $T=Y$ and move the integration to the line $\sigma=-1/2$. The pole $s=0$ of order three produces the main terms $b_1(\alpha)\log^2 Y+b_2(\alpha)\log Y+b_3(\alpha)$. By the theorem of residues we obtain

$$b_1(\alpha)=\frac{1}{5}\prod_p\left(1-\frac{2}{p^2}+\frac{1}{p^3}\right)\prod_{p\mid\alpha}\left(1-\frac{2p-1}{p^2+p-1}\right)<1,$$

$b_2(\alpha)=b_{21}Q'_\alpha(1)+b_{20}Q_\alpha(1)$, and $b_3(\alpha)=b_{32}Q''_\alpha(1)+b_{31}Q'_\alpha(1)+b_{30}Q_\alpha(1)$, where the coefficients b_{ij} are constants, $Q_\alpha(1)<1$ and

$$\begin{aligned}Q'_\alpha(1)=&O\left(\sum_{p\mid\alpha}\frac{\log p}{p}\right)=O(\log\omega(\alpha))=O(\log\log 3\alpha),\\Q''_\alpha(1)=&O\left(\sum_{p\mid\alpha}\frac{\log^2 p}{p}\right)+O\left(\left(\sum_{p\mid\alpha}\frac{\log p}{p}\right)^2\right)=O(\log^2\omega(\alpha))=O((\log\log 3\alpha)^2),\end{aligned}$$

where $\omega(\alpha)$ is the number of distinct prime factors of α. (Note that the symbol ω has here different meaning than elsewhere in the thesis.)

Since $\zeta^2(\sigma+\mathrm{i}T)\ll T^{1-\sigma}\log^2 T+1$, when $\sigma\ge 0$, and $\int_{-T}^{T}\left|\zeta(\frac12+\mathrm{i}t)\right|^2\mathrm{d}t\ll T\log T$, see [19, p. 29], the remaining integrals are

$$\begin{aligned}\ll& d(\alpha)Y^{-1/2}\int_{1/2-\mathrm{i}T}^{1/2+\mathrm{i}T}\frac{|\zeta^2(s)|}{|s|}|\mathrm{d}s|+d(\alpha)T^{-1}\log^2 T\\\ll& Y^{-1/2}d(\alpha)\log^2 Y.\end{aligned}$$

□

The contribution of the terms $n=\square$ in (3.4), with the condition $q\ne\square$ omitted, is

$$\frac12\sum_{\nu=0}^{\infty}\frac{d(2^\nu)}{2^\nu}\sum_{\substack{n\le Y^{1/2}\\(n,2\alpha)=1}}\frac{d(n^2)}{n^2}\left(\sum_{\substack{q\le X\\(q,n)=1}}\chi_8^\nu(q)+\sum_{\substack{q\le X\\(q,n)=1}}\chi_8^\nu(q)\chi_{-4}(q)\right),$$

with the convention $\chi_8^0=\chi_0^{(8)}$. Here the condition $q\equiv 1 \pmod 4$ is rewritten by using characters modulo 4. This leads to "full" character sums, which are easier to handle. Those sums over q, which are non-trivial character sums, give a contribution

$O(1)$. However, for even ν the first sum over q gives a main term by the following lemma.

Lemma 3.6.

$$\sum_{\substack{n\le Y^{1/2}\\(n,2\alpha)=1}}\frac{d(n^2)}{n^2}\sum_{\substack{1<q\le X\\(q,2n)=1}}1=a(\alpha)X+O\left(XY^{-1/2}\log^2 Y\right)+O(1),$$

where

$$a(\alpha)=\frac{1}{2}\sum_{\substack{n=1\\(n,2\alpha)=1}}^{\infty}\frac{d(n^2)\varphi(n)}{n^3}\le a(1).$$

Proof. Since

$$\sum_{\substack{1<q\le X\\(q,2n)=1}}1=X\frac{\varphi(n)}{2n}+O(d(n)),$$

for odd n, we need to estimate the sum

$$\sum_{\substack{n\le Y^{1/2}\\(n,2\alpha)=1}}^{\infty}\frac{d(n^2)\varphi(n)}{n^3}=\sum_{\substack{n=1\\(n,2\alpha)=1}}^{\infty}\frac{d(n^2)\varphi(n)}{n^3}-\sum_{\substack{n>Y^{1/2}\\(n,2\alpha)=1}}^{\infty}\frac{d(n^2)\varphi(n)}{n^3}.$$

The main term is a convergent series and it has the Euler product

$$\frac{18}{29}\zeta^2(2)\prod_p\left(1+\frac{1}{p^2}-\frac{3}{p^3}+\frac{1}{p^5}\right)\prod_{p|\alpha}\left(1-\frac{3p^2-1}{p^4+p^3+2p^2-p-1}\right).$$

Clearly the product over the prime divisors of α is positive and less than one, and the error term is

$$\sum_{\substack{n>Y^{1/2}\\(n,2\alpha)=1}}^{\infty}\frac{d(n^2)\varphi(n)}{n^3}\ll\frac{\log^2 Y}{Y^{1/2}},$$

owing to the estimate $\sum_{n\le x}d(n^2)\ll x\log^2 x$. $\square$

Since the summations over ν in (3.4) converge, and $\frac{1}{2}\sum_{\nu=0}^{\infty}\frac{d(4^\nu)}{4^\nu}=\frac{10}{9}$ and $\sum_{\nu=0}^{\infty}\frac{d(2^\nu)}{2^\nu}=4$, we have shown that (3.1) is

$$\begin{aligned}&\frac{10}{9}a(\alpha)X+\sum_{\nu=0}^{\infty}\frac{d(2^\nu)}{2^{\nu+1}}\sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\ne\square}}\frac{d(n)}{n}\sum_{1<q\le X}\chi_8^\nu(q)(1+\chi_{-4}(q))\left(\frac{q}{n}\right)\\&-2b_1(\alpha)X^{1/2}\log^2 Y-2b_2(\alpha)X^{1/2}\log Y-2b_3(\alpha)X^{1/2}\\&+O(XY^{-1/2}d^2(\alpha)(\log Y)^{17})+O(\log^2 X d(\alpha)),\end{aligned}\tag{3.7}$$

To the remaining sum, we apply a smooth weight function, which is introduced next. The error caused by the weighting will be estimated in Section 3.4.

3.2. **Smoothing and transforming the q-sums.** In [9] the authors consider the sum of Jacobi symbols $\left(\frac{m}{n}\right)$ for given n (and then the sum over n), which leads them to the sum $S_{0,n}(X)$ below. We generalize their method to the sums of $\chi_t(q)\left(\frac{q}{n}\right)$ over q, where χ_t is a character modulo 8 and n is a given odd number.

Let H be a smooth function compactly supported in $(0,1)$. Let us assume also that $H(y)=1$ for $y\in(1/U,1-1/U)$, and $H^{(j)}(y)\ll_j U^j$ for all integers $j\geq 0$. The parameter $U>2$ will be chosen later. Let

$$\hat{H}(\xi)=\int_{-\infty}^{\infty}H(y)e(-\xi y)\mathrm{d}y$$

be the Fourier transform of the function H.

We define

$$\tilde{H}_{\pm}(\xi)=\frac{1+\mathrm{i}}{2}\hat{H}(\xi)\pm\frac{1-\mathrm{i}}{2}\hat{H}(-\xi).$$

It is an easy calculation to see that

$$\mathrm{i}\tilde{H}_{+}(-\xi)=\tilde{H}_{-}(\xi)\text{ and }\mathrm{i}\tilde{H}_{-}(-\xi)=-\tilde{H}_{+}(\xi).$$

By partial integration and the estimate $H^{(j)}(t)\ll U^j$ we get that

$$\left|\hat{H}(\xi)\right|,\quad\left|\tilde{H}_{\pm}(\xi)\right|,\quad\left|(\tilde{H}_{\pm}(\xi))'\right|\ll U^{j-1}\left|\xi\right|^{-j}\tag{3.8}$$

for all integers $j\geq 1$ and for all real numbers ξ. Furthermore,

$$\tilde{H}_{+}(\xi)=\frac{1-\cos(2\pi\xi)+\sin(2\pi\xi)}{2\pi\xi}+O\left(\frac{1}{U}\right).\tag{3.9}$$

When ξ is small, $\tilde{H}_{+}(\xi)=1+\pi\xi+O(\xi^2)+O(U^{-1})$ is always positive, but for large ξ the non-oscillating term $1/(2\pi\xi)$ changes sign with ξ.

Let

$$S_{t,n}(X)=\sum_{q=1}^{\infty}\chi_t(q)\left(\frac{q}{n}\right)H\left(\frac{q}{X}\right),\tag{3.10}$$

where n is an odd non-square integer, and $t=0,-4,8$ or -8. So χ_t is a real character modulo 8 and (3.10) is a smoothed version of the character sum

$$\sum_{q\leq X}\chi_t(q)\left(\frac{q}{n}\right)$$

for odd n. When t is zero, Lemma 2.6 of [30] gives

$$S_{0,n}(X)=\frac{X}{2n}\left(\frac{2}{n}\right)\sum_{k=-\infty}^{\infty}(-1)^kG_k(n)\tilde{H}_{+}\left(\frac{kX}{2n}\right).$$

For $t \neq 0$, we get by the Poisson summation that

$$\begin{aligned} S_{t,n}(X) &= \sum_{q=1}^{\infty} \chi_t(q) \left(\frac{q}{n}\right) H\left(\frac{q}{X}\right) = \sum_{b \pmod{|t|n}} \chi_t(b) \left(\frac{b}{n}\right) \sum_{d=0}^{\infty} H\left(\frac{(|t|\, nd + b)}{X}\right) \\ &= \frac{X}{|t|\, n} \sum_{b \pmod{|t|n}} \chi_t(b) \left(\frac{b}{n}\right) \sum_{k=-\infty}^{\infty} \hat{H}\left(\frac{kX}{|t|\, n}\right) e\left(\frac{bk}{|t|\, n}\right). \end{aligned}$$

Since n is odd, the numbers b can be written in the form $un + v\,|t|$, where u goes through the residue classes modulo $|t|$ and v goes through the residue classes modulo n. Hence

$$\begin{aligned} \sum_{b \pmod{|t|n}} \chi_t(b) \left(\frac{b}{n}\right) e\left(\frac{bk}{|t|\, n}\right) &= \chi_t(n) \left(\frac{|t|}{n}\right) \sum_{u \pmod{|t|}} \chi_t(u) e\left(\frac{uk}{|t|}\right) \sum_{v \pmod{n}} \left(\frac{v}{n}\right) e\left(\frac{vk}{n}\right) \\ &= \chi_t(n) \left(\frac{|t|}{n}\right) \chi_t(k) \tau(\chi_t) \tau_k(n), \end{aligned}$$

which means that

$$S_{t,n}(X) = \frac{X}{|t|\, n} \chi_t(n) \tau(\chi_t) \left(\frac{|t|}{n}\right) \sum_{k=-\infty}^{\infty} \hat{H}\left(\frac{kX}{|t|\, n}\right) \chi_t(k) \tau_k(n)$$

for $t \neq 0$. Now

$$\begin{aligned} S_{-4,n}(X) &= \frac{2\mathrm{i}X}{4n} \chi_{-4}(n) \sum_{k=-\infty}^{\infty} \hat{H}\left(\frac{kX}{4n}\right) \chi_{-4}(k) \left(\frac{1+\mathrm{i}}{2} G_k(n) + \frac{1-\mathrm{i}}{2} G_{-k}(n)\right) \\ &= \frac{2\mathrm{i}X}{4n} \chi_{-4}(n) \sum_{k=-\infty}^{\infty} \chi_{-4}(k) G_k(n) \tilde{H}_-\left(\frac{kX}{4n}\right) \\ &= \frac{2\mathrm{i}X}{4n} \sum_{k=-\infty}^{\infty} \chi_{-4}(k) G_{-k}(n) \tilde{H}_-\left(\frac{kX}{4n}\right) \\ &= \frac{-2\mathrm{i}X}{4n} \sum_{k=-\infty}^{\infty} \chi_{-4}(k) G_k(n) \tilde{H}_-\left(\frac{-kX}{4n}\right) \\ &= \frac{2X}{4n} \sum_{k=-\infty}^{\infty} \chi_{-4}(k) G_k(n) \tilde{H}_+\left(\frac{kX}{4n}\right). \end{aligned}$$

Similarly,

$$S_{8,n}(X) = \frac{2\sqrt{2}X}{8n} \sum_{k=-\infty}^{\infty} \chi_8(k) G_k(n) \tilde{H}_+\left(\frac{kX}{8n}\right)$$

and

$$S_{-8,n}(X) = \frac{2\sqrt{2}X}{8n} \sum_{k=-\infty}^{\infty} \chi_{-8}(k) G_k(n) \tilde{H}_+ \left(\frac{kX}{8n} \right).$$

Let us put henceforth $\tilde{H} = \tilde{H}_+$, and define

$$S_n(x) = \frac{x}{2n} \sum_{k=1}^{\infty} G_{k^2}(n) \tilde{H} \left(\frac{k^2 x}{n} \right) \quad \text{and} \quad S_n^{(odd)}(x) = \frac{x}{2n} \sum_{\substack{k=1 \\ 2 \nmid k}}^{\infty} G_{k^2}(n) \tilde{H} \left(\frac{k^2 x}{n} \right).$$

Now we can use $S_n(X)$ to get a main term from $S_{0,n}(X)$ by separating the terms $k = 2\square$ from the sum over k. Since $\left(\frac{2}{n}\right) G_k(n) = G_{2k}(n)$, we have

$$\begin{aligned} S_{0,n}(X) =& \frac{X}{2n} \sum_{k=1}^{\infty} G_{k^2}(n) \tilde{H} \left(\frac{k^2 X}{n} \right) + \frac{X}{2n} \sum_{\substack{k=-\infty \\ k \neq 2\square}}^{\infty} (-1)^k G_{2k}(n) \tilde{H} \left(\frac{kX}{2n} \right) \\ =& S_n(X) + R_0(X, n). \end{aligned}$$

Similarly, extracting the terms $k = \square$ from the sums $S_{t,n}(X)$ for $t \neq 0$, we may separate a main term in terms of $S_n^{(odd)}(x)$ from these sums;

$$\begin{aligned} S_{8,n}(X) =& 4\sqrt{2} S_n^{(odd)} \left(\frac{X}{8} \right) + \frac{\sqrt{2}X}{4n} \sum_{\substack{k=-\infty \\ k \neq \square}}^{\infty} \chi_8(k) G_k(n) \tilde{H} \left(\frac{kX}{8n} \right) \\ =& 4\sqrt{2} S_n^{(odd)} \left(\frac{X}{8} \right) + R_8(X, n), \\ S_{-4,n}(X) =& 4 S_n^{(odd)} \left(\frac{X}{4} \right) + \frac{X}{2n} \sum_{\substack{k=-\infty \\ k \neq \square}}^{\infty} \chi_{-4}(k) G_k(n) \tilde{H} \left(\frac{kX}{4n} \right) \\ =& 4 S_n^{(odd)} \left(\frac{X}{4} \right) + R_{-4}(X, n), \\ S_{-8,n}(X) =& 4\sqrt{2} S_n^{(odd)} \left(\frac{X}{8} \right) + \frac{\sqrt{2}X}{4n} \sum_{\substack{k=-\infty \\ k \neq \square}}^{\infty} \chi_{-8}(k) G_k(n) \tilde{H} \left(\frac{kX}{8n} \right) \\ =& 4\sqrt{2} S_n^{(odd)} \left(\frac{X}{8} \right) + R_{-8}(X, n). \end{aligned}$$

So instead of (3.10), it is enough to deal with the sums $S_n(x)$ and $S_n^{(odd)}(x)$. Note that the remainders $R_t(X, n)$ are also similar to each other. We study these remainder terms more carefully in Section 3.4.

3.3. **Summing the smoothed sums.** We are now ready to analyse the double sum

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\neq\square}} \frac{d(n)}{n} \sum_{1\le q\le X} \chi_8^{\nu}(q)\left(1+\chi_{-4}(q)\right)\left(\frac{q}{n}\right), \tag{3.11}$$

which remained to be handled in (3.7). The character sum in (3.11) is a sum of two sums of the form $\sum \chi_t(q)\left(\frac{q}{n}\right)$, where χ_t is a character modulo 8. Using the weight function H introduced in Section 3.2, transforming the sum over q into a sum over k as in Section 3.2, and taking into account the contribution involving $S_n^{(odd)}$, we end up with the sum

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\neq\square}} \frac{d(n)}{n} S_n^{(odd)}(x) = x\sum_{\substack{k=1\\2\nmid k}}^{\infty} \sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\neq\square}} \frac{d(n)}{n^2} G_{k^2}(n)\tilde{H}\left(\frac{k^2x}{n}\right) =: x\sum_{\substack{k=1\\2\nmid k}}^{\infty} M_\alpha(k), \tag{3.12}$$

for $x = X/8$ or $X/4$. Note that $M_\alpha(k)$ depends on the parameters X and Y, although this is not indicated. The contribution of the sums S_n is a similar sum without the oddness condition for k. Our aim is to show that from (3.12) we get a main term, plus error terms which are smaller than $X^{1/2}$, and likewise for the sums involving S_n. Hence, up to the error arising from the use of the smooth weight and from the terms $R_t(X,n)$, the sum (3.11) reduces to (3.12). The following lemmas give us useful tools to deal with the sums $M_\alpha(k)$.

Lemma 3.13. *Let β be a natural number. Then*

$$\sum_{\substack{n\le x\\(n,\beta)=1}} \mu^2(n)d(n) = A_1(\beta)x\log x + A_2(\beta)x + O\left(d(\beta)x^{1/2}\log^5 x\right),$$

where

$$A_1(\beta) = \frac{1}{\zeta^2(2)}\prod_p\left(1-\frac{1}{(p+1)^2}\right)\prod_{p\mid\beta}\frac{p}{p+2} < 1,$$

and $A_2(\beta) = A_1(\beta)B(\beta)$, where

$$B(\beta) = 2\gamma - 1 - 4\frac{\zeta'}{\zeta}(2) + 2\sum_p \frac{\log p}{(p+1)(p+2)} + 2\sum_{p\mid\beta}\frac{\log p}{p+2} = O(\log\log 3\beta).$$

Proof. The Euler product of the generating function of $\mu^2(n)d(n)$ is

$$\prod_p\left(1+\frac{2}{p^s}\right) = \frac{\zeta^2(s)}{\zeta^2(2s)}\prod_p \frac{1+\frac{2}{p^s}}{\left(1+\frac{1}{p^s}\right)^2}.$$

Thus

$$\sum_{\substack{n=1\\(n,\beta)=1}}^{\infty}\frac{\mu^2(n)d(n)}{n^s}=\frac{\zeta^2(s)}{\zeta^2(2s)}\prod_{p|\beta}\left(\frac{1}{1+\frac{2}{p^s}}\right)\prod_{p}\left(1-\frac{1}{(p^s+1)^2}\right)$$
$$=\frac{\zeta^2(s)}{\zeta^2(2s)}Q_\beta(s)Q(s),$$

where

$$Q_\beta(s)=\prod_{p|\beta}\left(\frac{1}{1+\frac{2}{p^s}}\right)=\prod_{p|\beta}\left(1+O\left(\frac{1}{p^\sigma}\right)\right)=O(d(\beta))\qquad\text{for }\sigma\geq\varepsilon,$$

and

$$Q(s)=\prod_{p}\left(1-\frac{1}{(p^s+1)^2}\right)=\prod_{p}\left(1-\frac{1}{p^{2s}}+O\left(\frac{1}{p^{3\sigma}}\right)\right)\ll\frac{1}{|\zeta(2s)|}\ll\frac{1}{2\sigma-1},$$

when $1/2<\sigma\ll 1$.

By Perron's formula

$$\sum_{\substack{n\leq x\\(n,\beta)=1}}\mu^2(n)d(n)=(2\pi i)^{-1}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}Q_\beta(s)Q(s)\frac{\zeta^2(s)}{\zeta^2(2s)}\frac{x^s}{s}\mathrm{d}s+O\left(\frac{x^{1+\varepsilon}}{T}+x^\varepsilon\right).$$

Moving the line of integration to the line $\sigma=1/2+(\log x)^{-1}$ and setting $T=x$, the horizontal parts of integration give $O(d(\beta)x^\varepsilon)$ and on the line $\sigma=1/2+(\log x)^{-1}$ the integral is $O(d(\beta)x^{1/2}\log^5 x)$, since $\int_1^T|\zeta(\sigma+it)|^2\,\mathrm{d}t=O(T\log T)$, when $1/2\leq\sigma\leq 1$.

The main terms, which come from the double pole at $s=1$, can be calculated by using the following Laurent series:

$$\frac{x^s}{s}=x+(x\log x-x)(s-1)+O\left((s-1)^2\right),$$
$$\frac{\zeta^2(s)}{\zeta^2(2s)}=\frac{1}{\zeta^2(2)}(s-1)^{-2}+\frac{2\gamma-4\frac{\zeta'}{\zeta}(2)}{\zeta^2(2)}(s-1)^{-1}+O\left(1\right),$$
$$Q(s)=Q(1)\left(1+2\sum_p\frac{\log p}{(p+1)(p+2)}(s-1)\right)+O\left((s-1)^2\right),$$
$$Q_\beta(s)=Q_\beta(1)\left(1+2\sum_{p|\beta}\frac{\log p}{(p+2)}(s-1)\right)+O\left((s-1)^2\right).$$

Hence, by the theorem of residues, the main terms are

$$\frac{Q(1)Q_\beta(1)}{\zeta^2(2)} x \log x$$
$$+\frac{Q(1)Q_\beta(1)}{\zeta^2(2)} x \left(2\gamma - 1 - 4\frac{\zeta'}{\zeta}(2) + \sum_p \frac{2\log p}{(p+1)(p+2)} + \sum_{p\mid\beta} \frac{2\log p}{p+2} \right).$$

□

Lemma 3.14. *Let k be a natural number. Then*

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1}} \frac{G_{k^2}(n)d(n)}{\sqrt{n}} = A_3(\alpha,k)Y\log Y + A_4(\alpha,k)Y$$
$$+ O\left(Y^{1/2}(\log Y)^5 d(\alpha k)d^3(k)\right),$$

where $A_3(\alpha,k) = O(\log\log 3k)$ and $A_4(\alpha,k) = O\left((\log\log 3\alpha k)^2\right)$.

Proof. Let $n = n_1 n_2$, where $(n_1,k)=1$ and n_2 is divisible only by primes dividing k. Hence $(n_1,n_2)=1$ and

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1}} \frac{G_{k^2}(n)d(n)}{\sqrt{n}} = \sum_{\substack{n_2\le Y\\(n_2,2\alpha)=1}} \frac{G_{k^2}(n_2)d(n_2)}{\sqrt{n_2}} \sum_{\substack{n_1\le \frac{Y}{n_2}\\(n_1,2\alpha k)=1}} \frac{G_{k^2}(n_1)d(n_1)}{\sqrt{n_1}}.$$

Since $G_{k^2}(n_1) = \mu^2(n_1)\sqrt{n_1}$ by Lemma 2.2, the second sum is by Lemma 3.13

$$\sum_{\substack{n_1\le \frac{Y}{n_2}\\(n_1,2\alpha k)=1}} \mu^2(n_1)d(n_1) = A_1(2\alpha k)\frac{Y}{n_2}\log\frac{Y}{n_2} + A_2(2\alpha k)\frac{Y}{n_2}$$
$$+ O\left(\left(\frac{Y}{n_2}\right)^{1/2}\left(\log\frac{Y}{n_2}\right)^5 d(\alpha k)\right),$$

where $A_1(2\alpha k)$ and $A_2(2\alpha k)$ are as in Lemma 3.13. Therefore, it is enough to estimate the sums

$$S_1 = \sum_{\substack{n_2\le Y\\(n_2,2\alpha)=1}} \frac{G_{k^2}(n_2)d(n_2)}{n_2^{3/2}}, \qquad S_2 = \sum_{\substack{n_2\le Y\\(n_2,2\alpha)=1}} \frac{G_{k^2}(n_2)d(n_2)\log n_2}{n_2^{3/2}}$$

$$\text{and} \qquad S_3 = \sum_{\substack{n_2\le Y\\(n_2,2\alpha)=1}} \frac{G_{k^2}(n_2)d(n_2)}{n_2}\log^5\frac{Y}{n_2}.$$

By Lemma 2.2 the generating function of $G_{k^2}(n_2)d(n_2)\chi_0^{(2\alpha)}(n_2)$ is

$$F(s)=\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\frac{\varphi(p^2)d(p^2)}{p^{2s}}+\cdots+\frac{\varphi(p^{2a})d(p^{2a})}{p^{2as}}+\frac{p^{2a+1/2}d(p^{2a+1})}{p^{(2a+1)s}}\right)$$
$$=\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\left(1-\frac{1}{p}\right)\left(\frac{3}{p^{2(s-1)}}+\cdots+\frac{2a+1}{p^{2a(s-1)}}\right)+\frac{2a+2}{p^{(2a+1)(s-1)}p^{1/2}}\right)$$
$$=\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\frac{3(1-p^{-1})}{p^{2(s-1)}-1}+\frac{p-1}{(p^{2(s-1)}-1)^2}\left(\frac{2}{p}+\frac{p^{2(s-1)}+1}{p^{2(s-1)a+1}}\right)\right.$$
$$\left.+\frac{2a+2}{p^{(2a+1)(s-1)}}\left(\frac{1}{p^{1/2}}-\frac{(p-1)p^{s-2}}{p^{2(s-1)}-1}\right)\right),$$

for $s\neq 1$, and

$$F(1)=\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\left(1-\frac{1}{p}\right)(a+2)a+\frac{2a+2}{p^{1/2}}\right).$$

Since $G_{k^2}(n_2)$ is always non-negative, we may estimate the sum S_3 by using the generating function:

$$S_3\le F(1)\log^5 Y\le \log^5 Y\prod_{p^a\|k}((a+1)(a+1)+2(a+1))$$
$$\le \log^5 Y\prod_{p^a\|k}(a+1)^3=O\left(d^3(k)\log^5 Y\right),$$

which gives the error term $O\left(Y^{1/2}(\log Y)^5 d(\alpha k)\, d^3(k)\right)$.

The sum S_1 can be written in the form

$$\sum_{\substack{n_2\le Y\\ (2\alpha,n_2)=1}}\frac{G_{k^2}(n_2)d(n_2)}{n_2^{3/2}}=\sum_{\substack{n_2=1\\ (2\alpha,n_2)=1}}^{\infty}\frac{G_{k^2}(n_2)d(n_2)}{n_2^{3/2}}+O\left(\sum_{\substack{n_2>Y\\ (2\alpha,n_2)=1}}\frac{G_{k^2}(n_2)d(n_2)}{n_2^{3/2}}\right).$$

Since

$$\sum_{\substack{n_2>Y\\ (2\alpha,n_2)=1}}\frac{G_{k^2}(n_2)d(n_2)}{n_2^{3/2}}\le\frac{1}{Y^{1/2}}\sum_{\substack{n_2>Y\\ (2\alpha,n_2)=1}}\frac{G_{k^2}(n_2)d(n_2)}{n_2}\le\frac{F(1)}{Y^{1/2}},$$

we have

$$S_1=F(3/2)+O\left(\frac{d^3(k)}{Y^{1/2}}\right),$$

where

$$F(3/2)=\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\frac{3}{p}+\frac{2}{p(p-1)}-\frac{p+1}{p^{a+1}(p-1)}\right),$$

and the error multiplied by $Y\log Y$ is still dominated by $O(Y^{1/2}(\log Y)^5 d(\alpha k)d^3(k))$. Hence the coefficient of $Y\log Y$ is

$$\begin{aligned}A_3(\alpha,k):&=A_1(2\alpha k)F(3/2)\\ &\ll\prod_{\substack{p\mid k\\ p\nmid 2\alpha}}\frac{p}{p+2}\prod_{\substack{p^a\|k\\ p\nmid 2\alpha}}\left(1+\frac{3}{p}+\frac{2}{p(p-1)}-\frac{p+1}{p^{a+1}(p-1)}\right)\\ &\ll\prod_{\substack{p\mid k\\ p\nmid 2\alpha}}\left(\frac{p}{p+2}\cdot\frac{p^2+2p-1}{p(p-1)}\right)\ll\prod_{\substack{p\mid k\\ p\nmid 2\alpha}}\left(1+\frac{1}{p}\right)\ll\log\log 3k.\end{aligned}$$

The sum S_2 can be estimated similarly. When we differentiate the function $F(3/2-s)$ and set $s=0$, we get a sum with the factor $\log n_2$. Let us write $F(s)=\prod F_p(s)$, then

$$\begin{aligned}S_2&=\frac{\mathrm{d}}{\mathrm{d}s}F\left(\frac{3}{2}-s\right)\Big|_{s=0}+O\left(\frac{d^3(k)\log Y}{Y^{1/2}}\right)\\ &=-F\left(\frac{3}{2}\right)\sum_{\substack{p^a\|k\\ p\nmid 2\alpha}}\frac{F_p'(3/2)}{F_p(3/2)}+O\left(\frac{d^3(k)\log Y}{Y^{1/2}}\right),\end{aligned}$$

where

$$F_p'\left(\frac{3}{2}\right)=-\frac{2\log p}{(p-1)^2}\left(3p+1-\frac{1}{p^{a+1}}\left(2(a+1)p^2+3p-(2a+1)\right)\right)\ll\frac{\log p}{p},$$

and the contribution of the error is dominated by $O\left(Y^{1/2}(\log Y)^5 d(\alpha k)d^3(k)\right)$. Hence the coefficient of Y is

$$\begin{aligned}A_4(\alpha,k):&=A_1(2\alpha k)\frac{\mathrm{d}}{\mathrm{d}s}F\left(\frac{3}{2}-s\right)\Big|_{s=0}+A_2(2\alpha k)F(3/2)\\ &=A_3(\alpha,k)\left(-\sum_{\substack{p^a\|k\\ p\nmid 2\alpha}}\frac{F_p'(3/2)}{F_p(3/2)}+B(2\alpha k)\right)\ll(\log\log 3k)(\log\log 3\alpha k),\end{aligned}$$

where $B(2\alpha k)$ is as in Lemma 3.13. □

Remark 3.15. If the sum over n is taken just over the square terms, then the sum over n_1 is one, and there are less than $d(k^2)$ terms in the sum over n_2, and each term is

$$\frac{G_{k^2}(n_2)d(n_2)}{\sqrt{n_2}}\le\frac{\varphi(n_2)d(n_2)}{\sqrt{n_2}}\le d(n_2)\sqrt{n_2}\ll Y^{1/2+\varepsilon}.$$

The contribution of the square terms is therefore $O(Y^{1/2+\varepsilon}k^{\varepsilon})$, so Lemma 3.14 holds, with a slightly weaker error term, even if the square terms are left out.

Remark 3.16. Although, the coefficients $A_3(\alpha,k)$ and $A_4(\alpha,k)$ are of the size of some $\log\log$ factors, they are nevertheless of constant size in mean over k (or over odd k's). Indeed,

$$\sum_{k\le K}|A_3(\alpha,k)| \ll \sum_{k\le K}\prod_{p|k}\left(1+\frac{1}{p}\right) = \sum_{k\le K}\sum_{n|k}\frac{\mu^2(n)}{n} \ll \sum_{n\le K}\frac{\mu^2(n)}{n}\cdot\frac{K}{n} \ll K.$$

Similarly,

$$\begin{aligned}\sum_{k\le K}|A_4(\alpha,k)| &\ll \sum_{k\le K}\left(\prod_{p|k}\left(1+\frac{1}{p}\right)\left(\sum_{p|k}\frac{\log p}{p}+\sum_{p|2\alpha k}\frac{\log p}{p}\right)\right)\\ &\ll \sum_{k\le K}\sum_{n|k}\frac{\mu^2(n)}{n}\left(\sum_{p|k}\frac{\log p}{p}+\log\log 3\alpha\right)\\ &\ll \sum_{n,p\le K}\frac{\mu^2(n)}{n}\frac{\log p}{p}\frac{K}{[n,p]}+\log\log 3\alpha\sum_{n\le K}\frac{\mu^2(n)}{n}\frac{K}{n},\end{aligned}$$

where $[n,p]$ is the least common multiple of n and p, which is np or n. Since in both cases the sums over n and p converge, we have

$$\sum_{k\le K}|A_4(\alpha,k)| \ll K\log\log 3\alpha.$$

Let us now return to the sum $M_\alpha(k)$ defined in (3.12). By Lemma 3.14 and Remark 3.15 we obtain by using partial summation that

$$\begin{aligned}M_\alpha(k) &= \sum_{\substack{n\le Y\\(2\alpha,n)=1\\ n\ne\square}}\frac{G_{k^2}(n)d(n)}{\sqrt{n}}\tilde{H}\left(\frac{k^2x}{n}\right)\frac{1}{n^{3/2}}\\ &= \big(A_3(\alpha,k)Y\log Y+A_4(\alpha,k)Y+O\big(Y^{1/2+\varepsilon}d(\alpha k)d^3(k)\big)\big)\,\tilde{H}\left(\frac{k^2x}{Y}\right)\frac{1}{Y^{3/2}}\\ &\quad-\int_1^Y\big(A_3(\alpha,k)y\log y+A_4(\alpha,k)y+O\big(y^{1/2+\varepsilon}d(\alpha k)d^3(k)\big)\big)\left(\tilde{H}\left(\frac{k^2x}{y}\right)\frac{1}{y^{3/2}}\right)'\,\mathrm{d}y.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}M_\alpha(k) &= \int_1^Y\frac{A_3(\alpha,k)\log y+A_3(\alpha,k)+A_4(\alpha,k)}{y^{3/2}}\tilde{H}\left(\frac{k^2x}{y}\right)\mathrm{d}y\\ &\quad+O\left(\frac{UY^{\varepsilon}d(\alpha k)d^3(k)}{k^2x}\right),\end{aligned}\tag{3.17}$$

where the error term is estimated using (3.8) for $j=1$ or 2.

Let $A(k,y,\alpha) = A_3(\alpha,k)\log y + A_3(\alpha,k) + A_4(\alpha,k)$. Using (3.8) when $k > (Uy/x)^{1/2}$ $(j=1)$, and (3.9) for smaller k, we get

$$\begin{aligned}&\sum_{\substack{k=1\\2\nmid k}}^{\infty} A(k,y,\alpha)\tilde{H}\left(\frac{k^2x}{y}\right)\\ =&\frac{y}{2\pi x}\sum_{\substack{k\le\sqrt{Uy/x}\\2\nmid k}}\frac{A(k,y,\alpha)}{k^2}\left(1-\cos\left(\frac{2\pi k^2x}{y}\right)+\sin\left(\frac{2\pi k^2x}{y}\right)\right)\\ &+O\left(\left(\frac{y}{Ux}\right)^{1/2}\left(\frac{Uy\alpha}{x}\right)^{\varepsilon}\right).\end{aligned}$$

Actually, the summation on the right can be taken over all values of k with the same error. Since the summation over k in the error term of (3.17) converges, we have

$$\begin{aligned}\sum_{\substack{k=1\\2\nmid k}}^{\infty} M_\alpha(k) =&\frac{1}{2\pi x}\int_1^Y\frac{1}{\sqrt{y}}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\frac{A(k,y,\alpha)}{k^2}\left(1-\cos\left(\frac{2\pi k^2x}{y}\right)+\sin\left(\frac{2\pi k^2x}{y}\right)\right)\mathrm{d}y\\ &+O\left(\frac{UY^{\varepsilon}d(\alpha)}{x}\right)+O\left(\left(\frac{1}{Ux}\right)^{1/2}\left(\frac{UY\alpha}{x}\right)^{\varepsilon}\right).\end{aligned}$$

The integral can be extended to be from zero to Y with an error $O(1/x)$, hence

$$x\sum_{\substack{k=1\\2\nmid k}}^{\infty} M_\alpha(k) = C_\alpha(x,Y)\sqrt{x}+O\left(\left(\frac{x}{U}\right)^{1/2-\varepsilon}(\alpha Y)^{\varepsilon}\right)+O(UY^{\varepsilon}\alpha^{\varepsilon}), \tag{3.18}$$

where we define

$$C_\alpha(x,Y)=\frac{1}{2\pi}\int_0^{Y/x}\left(\frac{1}{\sqrt{y}}\sum_{\substack{k=1\\2\nmid k}}^{\infty}\frac{A(k,xy,\alpha)}{k^2}\left(1-\cos\left(\frac{2\pi k^2}{y}\right)+\sin\left(\frac{2\pi k^2}{y}\right)\right)\right)\mathrm{d}y$$

for $x>1$.

Remark 3.19. For $k^2>y$ the values of the trigonometric functions in $C_\alpha(x,Y)$ are strongly oscillating, so we approximate them trivially to be $\ll 1$, but for $k^2<y$ the integrand is more stable and the trigonometric functions can estimated with their Taylor series. Since $x>1$ and by Remark 3.16 $A(k,xy,\alpha)$ is $O(\log xy + \log\log 3\alpha)$

in mean over k, we have

$$\begin{aligned}C_\alpha(x,Y) &\ll \sum_{k\le\sqrt{Y/x}}\frac{1}{k^2}\left(\int_0^{k^2}\frac{|A(k,xy,\alpha)|}{y^{1/2}}\mathrm{d}y+\int_{k^2}^{Y/x}\frac{|A(k,xy,\alpha)|}{y^{1/2}}\left(\frac{k^4}{y^2}+\frac{k^2}{y}\right)\mathrm{d}y\right)\\ &\quad+\sum_{k>\sqrt{Y/x}}\frac{1}{k^2}\int_0^{Y/x}\frac{|A(k,xy,\alpha)|}{y^{1/2}}\mathrm{d}y\\ &\ll \sum_{k\le\sqrt{Y/x}}\frac{|A(k,Y,\alpha)|}{k}+\left(\frac{Y}{x}\right)^{1/2}\sum_{k>\sqrt{Y/x}}\frac{|A(k,Y,\alpha)|}{k^2}\\ &\ll \log\frac{Y}{x}\left(\log Y+\log\log 3\alpha\right).\end{aligned}$$

3.4. **Error terms.** When we started to estimate (3.12) instead of (3.11), we made errors of two kind. First, an error was caused by the smoothing, and then, by the approximation of the sums $S_{t,n}$ in terms of the sum S_n or $S_n^{(odd)}$.

3.4.1. *The error from the approximation.* As mentioned in Section 3.2, all the error terms arising from the transformations from $S_{t,n}$ to S_n or $S_n^{(odd)}$ are quite similar. The term containing $R_0(X,n)$ is almost the same as the one treated in [9], and the others can be estimated similarly. Here we handle in detail the term

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\ne\square}}\frac{d(n)}{n}R_{-4}(X,n)=\frac{X}{2}\sum_{\substack{k=-\infty\\k\ne\square}}^{\infty}\chi_{-4}(k)\sum_{\substack{n\le Y\\(n,2\alpha)=1\\n\ne\square}}\frac{d(n)}{n^2}G_k(n)\tilde{H}\left(\frac{kX}{4n}\right). \tag{3.20}$$

Truncating the k-series at $k=[Y]$, estimating trivially $G_k(n)\ll n$, and using (3.8), the tail part of (3.20) is seen to be

$$\ll X\sum_{|k|>Y}\sum_{n\le Y}\frac{d(n)n^{j-1}U^{j-1}}{k^jX^j}\ll\left(\frac{U}{X}\right)^{j-1}Y^{1+\varepsilon}, \tag{3.21}$$

which may be ignored by choosing suitable large j, and assuming that $U<X^{\delta_1}$ and $Y<X^{\delta_2}$ for some positive constants $\delta_1<1$ and $\delta_2>1$. To estimate the sum over smaller values of k, we start with the sum

$$\sum_{\substack{n\le x\\(2\alpha,n)=1\\n\ne\square}}\frac{d(n)}{\sqrt{n}}G_k(n), \tag{3.22}$$

where $x\le Y$, and write $n=n_1n_2$, where n_1 and n_2 are odd, $(n_1,k)=1$, and all prime divisors of n_2 divide k. Since $G_k(n)$ is multiplicative in n, we have $G_k(n)=G_k(n_2)G_k(n_1)$ and by Lemma 2.2 $G_k(n_1)=\mu^2(n_1)\sqrt{n_1}\left(\frac{k}{n_1}\right)$. Furthermore, $G_k(n_2)=$

0 unless $n_2 \mid k^2$, and in any case $|G_k(n_2)| \le n_2$. Therefore, (3.22) is

$$\ll \sum_{\substack{n_2 | k^2 \\ n_2 \le x \\ (n_2, 2\alpha)=1}} d(n_2)\sqrt{n_2} \left| \sum_{\substack{n_1 \le x/n_2 \\ (n_1, 2\alpha)=1}} d(n_1)\mu^2(n_1)\left(\frac{k}{n_1}\right) \right| + x^{1/2+\varepsilon} d(k).$$

The last term comes from the extra terms $n = \square$, since in that case the sum over n_1 is one.

The inner sum above is similar to the sum studied in Lemma 3.13. Now the related generating function is

$$\prod_{p>2} \left(1 + 2\chi(p)p^{-s}\right) = \frac{L^2(s,\chi)}{L^2(2s,\chi)} \prod_p \frac{1+2\chi(p)p^{-s}}{(1+\chi(p)p^{-s})^2} \left(1+\chi(2)2^{1-s}\right)^{-1},$$

where $\chi = \chi_{k\alpha^2}$ if $k \equiv 1 \pmod 4$, and $\chi = \chi_{-k\alpha^2}\chi_{-4}$ if $k \equiv 3 \pmod 4$. Thus

$$\sum_{\substack{n=1 \\ (n,2)=1}}^{\infty} \frac{\mu^2(n)d(n)\chi(n)}{n^s} = \frac{L^2(s,\chi)}{L^2(2s,\chi)} Q(s,k,\alpha),$$

where

$$\begin{aligned} |Q(s,k,\alpha)| &\ll \prod_{p \nmid k\alpha} \left| 1 - \frac{1}{p^{2s}} + O\left(\frac{1}{p^{3\sigma}}\right) \right| \\ &\ll \prod_p \left(1 + \frac{1}{p^{2\sigma}}\right) \ll |\zeta(2\sigma)| \ll \frac{1}{2\sigma - 1}, \end{aligned}$$

when $\sigma \ge 1/2 + \varepsilon$. Since the L-function is regular at $s = 1$, no main term appears when Perron's formula is applied and the integration is moved to the line $\sigma = 1/2+\varepsilon$.

When $\sigma > 1/2$, (2.5) gives by convexity that $L^2(\sigma + \mathrm{i}T, \chi) \ll \alpha^{\varepsilon}(kT)^{1-\sigma+\varepsilon}$, and $\left(L^2(2s,\chi)\right)^{-1} \ll (\sigma - 1)^{-1}$ which means that the integrand and hence also the horizontal parts of integration are

$$\ll (\alpha k T)^{\varepsilon} \left(\frac{x}{n_2 T} + \left(\frac{xk}{n_2 T}\right)^{1/2} \right).$$

Since x and k are at most Y, we get by choosing $T = Y$ that

$$\sum_{\substack{n_1 \le x/n_2 \\ (n_1, 2\alpha)=1}} \mu^2(n_1)d(n_1)\left(\frac{k}{n_1}\right) \ll \left(\frac{x}{n_2}\right)^{1/2+\varepsilon} \left(\int_{-Y}^{Y} \frac{\left|L(\frac{1}{2}+\varepsilon+\mathrm{i}t,\chi)\right|^2}{|1/2+\varepsilon+\mathrm{i}t|} \mathrm{d}t + (Yk\alpha)^{\varepsilon} \right).$$

Hence (3.22) is

$$\ll x^{1/2+\varepsilon} k^{\varepsilon} \left(I(Y,\chi) + (Y\alpha)^{\varepsilon}\right),$$

where $I(Y,\chi)$ is the above integral. Now summing by parts and using (3.8), with $j=1$ or 2, for the function $\tilde{H}$, we have

(3.23)
$$X \sum_{\substack{n\le Y\\(n,2\alpha)=1\\ n\neq\square}} \frac{d(n)}{n^2} G_k(n)\tilde{H}\left(\frac{kX}{4n}\right)$$
$$\ll X(k\alpha)^{\varepsilon}\left(I(Y,\chi)+Y^{\varepsilon}\right)\left(\frac{Y^{\varepsilon}}{Y}\left|\tilde{H}\left(\frac{kX}{4Y}\right)\right| + \int_1^Y y^{1/2+\varepsilon}\left|\left(\tilde{H}\left(\frac{kX}{4y}\right)\frac{1}{y^{3/2}}\right)'\right| \mathrm{d}y\right)$$
$$\ll \frac{(I(Y,\chi)+Y^{\varepsilon})}{k} U(Yk\alpha)^{\varepsilon}.$$

Summing by parts and using (2.6), which holds also for $\sigma>1/2$ by convexity, we get

$$\sum_{\substack{|k|\le Y\\ 2\nmid k}} \frac{I(Y,\chi)+Y^{\varepsilon}}{k^{1-\varepsilon}} \ll \sideset{}{'}\sum_{\substack{|k|\le Y\\ k\equiv 1\ (\mathrm{mod} 4)}} \frac{I(Y,\chi)+Y^{\varepsilon}}{k^{1-\varepsilon}} + \sideset{}{'}\sum_{\substack{|k|\le Y\\ k\equiv 3\ (\mathrm{mod} 4)}} \frac{I(Y,\chi)+Y^{\varepsilon}}{k^{1-\varepsilon}} \ll Y^{\varepsilon}. \tag{3.24}$$

Hence equations (3.21), (3.23) and (3.24) show that (3.20) is

$$\ll U(Y\alpha)^{\varepsilon}. \tag{3.25}$$

3.4.2. *The error from the smoothing.* The error caused by the use of a smooth weight function appears in the q-intervals $[0,\frac{X}{U}]$ and $[X-\frac{X}{U},X]$. Let us consider the sum

$$\sum_{\substack{n\le Y\\(n,2\alpha)=1\\ n\neq\square}} \frac{d(n)}{n}\left|\sum_{X-\frac{X}{U}\le q\le X} \chi_t(q)\left(\frac{q}{n}\right)\left(1-H\left(\frac{q}{X}\right)\right)\right|,$$

which is by partial summation

$$\ll \max_{X-\frac{X}{U}\le \eta\le X} \sum_{\substack{n\le Y\\(n,2\alpha)=1\\ n\neq\square}} \frac{d(n)}{n}\left|\sum_{X-\frac{X}{U}\le q\le \eta} \chi_t(q)\left(\frac{q}{n}\right)\right|.$$

The other interval can be treated similarly.

By the classical Pólya–Vinogradov inequality we have

$$\sum_{\substack{n\le Z\\(n,2\alpha)=1\\ n\neq\square}} \frac{d(n)}{n}\left|\sum_{X-\frac{X}{U}\le q\le \eta} \chi_t(q)\left(\frac{q}{n}\right)\right| \ll \sum_{\substack{n\le Z\\(n,2\alpha)=1\\ n\neq\square}} \frac{d(n)}{n^{1/2}}\log n \ll Z^{1/2+\varepsilon}.$$

On the other hand, by Corollary 2.9,

$$\begin{aligned}\sum_{\substack{n\le x\\(n,2\alpha)=1\\n\neq\square}}\left|\sum_{X-\frac{X}{U}\le q\le\eta}\chi_t(q)\left(\frac{q}{n}\right)\right| &\ll \sum_{\substack{j\le x^{1/2}\\(j,2\alpha)=1}}\sum_{\substack{1<n\le x/j^2\\(n,2\alpha)=1\\\mu^2(n)=1}}\left|\sum_{\substack{X-\frac{X}{U}\le q\le\eta\\(q,j)=1}}\chi_t(q)\left(\frac{q}{n}\right)\right|\\ &\ll \sum_{\substack{j\le x^{1/2}\\(j,2\alpha)=1}} j^{-1}\sqrt{x}\sqrt{(\eta+x/j^2)(\eta x)^\varepsilon XU^{-1}}\\ &\ll \sqrt{x}\sqrt{(\eta+x)XU^{-1}}(\eta x)^\varepsilon.\end{aligned}$$

Restricting n from below by Z, partial summation gives

$$\begin{aligned}&\sum_{\substack{Z<n\le Y\\(n,2\alpha)=1\\n\neq\square}}\frac{d(n)}{n}\left|\sum_{X-\frac{X}{U}\le q\le\eta}\chi_t(q)\left(\frac{q}{n}\right)\right|\\ &\qquad\ll\left(\frac{(\eta+Y)X}{YU}\right)^{1/2}(XY)^\varepsilon+\int_Z^Y\left(\frac{y(\eta+y)X}{U}\right)^{1/2}\frac{(Xy)^\varepsilon}{y^2}\mathrm{d}y\\ &\qquad\ll\left(\frac{\eta X}{ZU}\right)^{1/2}Y^\varepsilon,\end{aligned}$$

assuming that $Z<X<Y$. Combining these results and choosing $Z=XU^{-1/2}$, the error induced by the smoothing is seen to be

$$\ll Z^{1/2+\varepsilon}+\frac{X}{\sqrt{ZU}}Y^\varepsilon\ll\frac{X^{1/2}}{U^{1/4}}Y^\varepsilon. \tag{3.26}$$

Gathering together all the error terms from (3.7), (3.18), (3.25) and (3.26), we find three different shapes of the dominating error terms, namely

$$\frac{X}{Y^{1/2}}Y^\varepsilon d^2(\alpha),\quad \frac{X^{1/2}}{U^{1/4}}Y^\varepsilon,\quad\text{and}\quad U(Y\alpha)^\varepsilon.$$

The last two error terms are balanced by choosing $U=X^{2/5}$, which means that Y must be greater than $X^{6/5}$, in order to obtain the error term

$$O(X^{2/5+\varepsilon}\alpha^\varepsilon).$$

Here the order of Y is quite flexible, as far as it is less than some fixed power of X, and hence $\log Y\ll\log X$.

3.5. **Main theorem.** In the preceding sections, we have studied the mean square over even characters with odd modulus. Similar results can be obtained for the mean squares over other types of real characters. For example, the mean square over even positive discriminants is

$$\sideset{}{'}\sum_{\substack{1<q\le X\\ q\equiv 0(4)}} L^2(1,\chi_q) = \sum_{n=1}^{\infty}\frac{d(n)}{n}\sum_{i=1}^{\infty}\left(\sum_{\substack{q\le X/4^i\\ 2\nmid q\\ q\ne\square}}\left(\frac{4q}{n}\right)+\sum_{\substack{q\le X/(2\cdot 4^i)\\ 2\nmid q}}\left(\frac{8q}{n}\right)\right).$$

Here it is enough to study the case $i=1$, since the other cases follow by substituting $X/4^i$ for X. The square terms in the n-series give the first main term, the extra terms from the possible square values of q give the B-part (that is, the terms with coefficients $B_i(\alpha)$), and the tail part of the n-series can be estimated by Lemma 3.2. Since here n is automatically odd, the smoothed sums over q reduce to the sum of terms $S_{0,n}(X/4)$ or $S_{0,n}(X/8)$, and the character $\chi_8(n)$ can be included to $G_k(n)$ by changing a bit the error term $R_0(X/8,n)$.

For the sum over negative odd discriminants, the B-part does not appear, and the smoothed sum is

$$\sum_{q=1}^{\infty}\chi_t(-q)\left(\frac{-q}{n}\right)H\left(\frac{q}{X}\right),$$

which gives us sums similar to $S_n(x)$ and $S_n^{(odd)}(x)$ except that $\tilde{H}(\xi)$ is replaced by $\tilde{H}(-\xi)$. This leads to the function

$$\frac{1}{2\pi}\int_0^{Y/x}\left(\frac{1}{\sqrt{y}}\sum_k\frac{A(k,xy,\alpha)}{k^2}\left(\sin\left(\frac{2\pi k^2}{y}\right)-\left(1-\cos\left(\frac{2\pi k^2}{y}\right)\right)\right)\right)\mathrm{d}y$$

instead of $C_\alpha(x,Y)$, with k running over all or only odd positive integers. This is, however, of the same order as $C_\alpha(x,Y)$. The negative even discriminants can be treated analogously.

Moreover, the appearing error terms are similar to those studied earlier, and since the remaining ν-sums converge, we have the following general theorem.

Theorem 3.27. *Let α be odd. Then*

$$\sideset{}{'}\sum_{q} L^2(1,\chi_q\chi_0^{(\alpha)}) = A(\alpha)X+P_\alpha(X)X^{1/2}+O(X^{2/5+\varepsilon}\alpha^{\varepsilon}),$$

where the summation is taken over all positive discriminants not exceeding X, $A(\alpha)$ is a constant depending only on α, and $P_\alpha(X)\ll\log^2 X$. The same result holds also for the sum over negative discriminants $-X\le q<-1$.

Especially, for the mean square over even characters with odd modulus we have:

Theorem 3.28. *Let α be odd and $X^{6/5} \le Y \le X^{\delta}$, for some constant δ. Then*

$$\sum_{\substack{1<q\le X\\ q\equiv 1\,(4)\\ q\neq\square}} L^2(1,\chi_q\chi_0^{(\alpha)}) = A(\alpha)X + P_\alpha(X)X^{1/2} + O(X^{2/5+\varepsilon}\alpha^{\varepsilon}),$$

where

$$A(\alpha) = \frac{5}{9}\sum_{(n,\alpha)=1}\frac{d(n^2)\varphi(n)}{n^3}$$

and

$$\begin{aligned} P_\alpha(X) =& c_1 C_\alpha(X,Y) + c_2 C_\alpha^{(odd)}\left(\frac{X}{4},Y\right) + c_3 C_\alpha^{(odd)}\left(\frac{X}{8},Y\right)\\ &- B_1(\alpha)\log^2 Y - B_2(\alpha)\log Y - B_3(\alpha). \end{aligned}$$

Here c_1, c_2 and c_3 are constants and

$$\begin{aligned} C_\alpha(x,Y) &= \frac{1}{2\pi}\int_0^{Y/x}\frac{1}{\sqrt{y}}\sum_{k=1}^{\infty}\frac{A(k,xy,\alpha)}{k^2}\left(1-\cos\left(\frac{2\pi k^2}{y}\right)+\sin\left(\frac{2\pi k^2}{y}\right)\right)\mathrm{d}y,\\ &\ll \log\frac{Y}{x}(\log Y + \log\log 3\alpha), \end{aligned}$$

where

$$A(k,xy,\alpha) = A_3(\alpha,k)(\log xy + 1) + A_4(\alpha,k),$$

and $A_3(\alpha,k)$ and $A_4(\alpha,k)$ are as in Lemma 3.14. *The function $C_\alpha^{(odd)}(x,Y)$ is similar except that the sum over k is restricted to odd numbers. The coefficients $B_i(\alpha)$ are*

$$B_i(\alpha) = \sum_{j=0}^{i-1} b_{ij}\frac{\mathrm{d}^j}{\mathrm{d}s^j}\left(\prod_{p\mid\alpha}\left(1-\frac{(p-1)(2p^s-1)}{p^{2s+1}-2p^s+1}\right)\right)\Bigg|_{s=1},$$

where the b_{ij} are constants. Especially, $B_1(\alpha) < 1$, $B_2(\alpha) \ll \log\log 3\alpha$ and $B_3(\alpha) \ll (\log\log 3\alpha)^2$.

While referring later on to the function $P_\alpha(X)$ we will speak about C- and B-parts, where C-part refers to the terms $C_\alpha(x,Y)$ and $C_\alpha^{(odd)}(x,Y)$, and B-part to the terms $B_i(\alpha)$.

Remark 3.29. As mentioned earlier, the parameter Y can be chosen quite freely. We have also shown that the sizes of the B- and C-parts are $O(\log^2 Y)$. We may now wonder whether our sum depends on Y? Of course, this cannot be the case, since Y is just an extra parameter which we used to derive our formula, and it does not appear in the original sum (3.1).

The answer is that there has to be some connection between the B- and C-parts, when q is positive. Indeed, they necessarily compensate each other as both parts depend on Y but their sum does not. When q is negative, the situation is a bit different since there is no B-part. Hence the balance must be obtained inside the C-part. This happens, because in this case the sum over small values of k in $C_\alpha(x, Y)$ and the tail part of the k-series have opposite signs, as mentioned in Section 3.2.

Remark 3.30. The above ideas can be also used to study the sum over the values $q \equiv 3 \pmod 4$, which appears when the sum over primitive characters χ_{4q} is studied in Chapter 4.

4. The mean square of primitive quadratic Dirichlet L-functions at 1

In the preceding chapter, a formula for the mean square of Dirichlet L-function over all non-trivial real characters was developed. However, in many applications of L-functions, the formula over primitive characters is needed instead. The sum over primitive characters can be sieved out from the sum over all real characters by using the *Möbius function* μ.

4.1. **Mean value estimate for the Möbius function.** Let $M(X) = \sum_{n \le X} \mu(n)$. The exact order of the function $M(X)$ is not known. A trivial estimate is $|M(X)| \le X$ and the *Riemann hypothesis* is equivalent to the estimate $M(X) \ll X^{1/2+\varepsilon}$. The best known result is

$$M(X) \ll X \exp\left(-C \log^{3/5} X (\log\log X)^{-1/5}\right), \tag{4.1}$$

where C is a suitable constant. Hence we may write $M(X) \ll X\omega(X)$, where $\omega(X)$ is of the same form as the exponential function above, which tends to zero when X goes to infinity. Note that all positive powers of $\log X$ can be embedded into the term $\omega(X)$ by changing the constant C.

The proof for (4.1) is based on Perron's formula. Since the generating function of μ is $\zeta^{-1}(s)$, Perron's formula gives that

$$\begin{aligned} M(X) = \sum_{n \le x} \mu(n) &= (2\pi \mathrm{i})^{-1} \int_{b-\mathrm{i}T}^{b+\mathrm{i}T} \frac{x^s \mathrm{d}s}{s\zeta(s)} + O(xT^{-1}\log x) \\ &= (2\pi\mathrm{i})^{-1} \left(\int_{b-\mathrm{i}T}^{a-\mathrm{i}T} + \int_{a-\mathrm{i}T}^{a+\mathrm{i}T} + \int_{a+\mathrm{i}T}^{b+\mathrm{i}T} \right) \frac{x^s \mathrm{d}s}{s\zeta(s)} + O(xT^{-1}\log x), \end{aligned}$$

where $b = 1 + (\log x)^{-1}$ and $a = 1 - c(\log T)^{-2/3}(\log\log T)^{-1/3}$ with a suitable positive constant c. For $\sigma \ge a$,

$$1/\zeta(s) = O\left((\log T)^{2/3}(\log\log T)^{1/3}\right),$$

so (4.1) follows by choosing $T = \exp\left(\log^{3/5} x (\log\log x)^{-1/5}\right)$. For more details, see [19]. This estimate depends vitally on the known zero-free region for the Riemann zeta-function since the integration can be safely moved to the left only if we are sure that there are no zeros inside the region. The largest known zero-free region of the Vinogradov–Korobov type is due to Ford [13, 14] which gives the constants $C = 0.2098$ and $c = 1/57.54$ above.

Similar ideas can be used to estimate the sum

$$\sum_{y<n\le x} \mu(n)f(n).$$

If $f(n)$ is a multiplicative function, then the generating function of $\mu(n)f(n)$ is

$$F(s) = \frac{1}{\zeta(s)} \prod_p \left(1 + \frac{1-f(p)}{p^s-1}\right).$$

Now if the product over primes converges this generating function has similar properties as the function $\zeta^{-1}(s)$ when Perron's formula is applied.

4.2. **Restriction to primitive characters.** For a non-principal primitive character χ_q, either $q \equiv 1 \pmod 4$ and $q \neq 1$ is squarefree, or $q = 4D$ and $D \equiv 2, 3 \pmod 4$ is squarefree. Hence

$$\sum_{q\le X}{}^{*} L^2(1,\chi_q) = \sum_{\substack{1<q\le X\\ q\equiv 1(4)}} \mu^2(q)L^2(1,\chi_q) + \sum_{\substack{q\le X/4\\ q\equiv 3(4)}} \mu^2(q)L^2(1,\chi_{4q}) + \sum_{\substack{q\le X/8\\ 2\nmid q}} \mu^2(q)L^2(1,\chi_{8q}),$$

and similarly for negative discriminants. Since $\mu^2(q) = \sum_{\alpha^2|q}\mu(\alpha)$, it remains to study the sum

(4.2)

$$\sum_{\substack{1\le\alpha\le X^{1/2}\\ 2\nmid\alpha}} \mu(\alpha)\left(\sum_{\substack{1<q\le\frac{X}{\alpha^2}\\ q\equiv 1(4)\\ q\neq\square}} L^2(1,\chi_{q\alpha^2}) + \sum_{\substack{1<q\le\frac{X}{4\alpha^2}\\ q\equiv 3(4)}} L^2(1,\chi_{4q\alpha^2}) + \sum_{\substack{1<q\le\frac{X}{8\alpha^2}\\ 2\nmid q}} L^2(1,\chi_{8q\alpha^2})\right)$$

All these sums can be estimated by using the ideas presented in Chapter 3. Below the sum over $q \equiv 1 \pmod 4$ is studied in detail, the others are obtained similarly.

If α is near $X^{1/2}$, then the sum over q is very short. Since the L-functions are $O(1)$ in mean square, we get by changing the order of the summations, for the values $X_0 < \alpha \le X^{1/2}$, that

$$\begin{aligned}
&\sum_{\substack{X_0<\alpha\le X^{1/2}\\ 2\nmid\alpha}} \mu(\alpha) \sum_{\substack{1<q\le X/\alpha^2\\ q\equiv 1(4)\\ q\neq\square}} L^2(1,\chi_{q\alpha^2})\\
&= \sum_{\substack{1<q\le X/X_0^2\\ q\equiv 1(4)\\ q\neq\square}} L^2(1,\chi_q) \sum_{\substack{X_0\le\alpha\le\sqrt{X/q}\\ 2\nmid\alpha}} \mu(\alpha)\prod_{p|\alpha}\left(1-\frac{\chi_q(p)}{p}\right)^2\\
&\ll \sqrt{\frac{X}{X_0^2}}\sqrt{X}\omega(X_0) \ll \frac{X}{X_0}\omega(X_0),
\end{aligned}$$

since (4.1) holds also for this slightly modified sum over α as we observed in the remark at the end of preceding section.

For the values $\alpha \leq X_0 < X^{1/2}$ we follow the proof of Theorem 3.28. Clearly the first main term is

$$X \sum_{\substack{1 \leq \alpha \leq X_0 \\ 2 \nmid \alpha}} \frac{\mu(\alpha) A(\alpha)}{\alpha^2} = A^* X + O\left(X \sum_{\substack{\alpha > X_0 \\ 2 \nmid \alpha}} \frac{\mu(\alpha) A(\alpha)}{\alpha^2} \right) = A^* X + O\left(\frac{X}{X_0} \omega(X_0) \right), \tag{4.3}$$

where $A(\alpha) = O(1)$ is a product over the prime factors of α, and A^* is a constant. The multiplier of $X^{1/2}$ is determined in the next section.

4.3. **The second main term in the primitive case.** When the ideas of the proof of Theorem 3.28 are applied to the first sum over q in (4.2), for $\alpha \leq X_0 < X^{1/2}$, the multiplier of $X^{1/2}$ is seen to be

$$\sum_{2 \nmid \alpha \leq X_0} P_\alpha \left(\frac{X}{\alpha^2} \right) \frac{\mu(\alpha)}{\alpha}. \tag{4.4}$$

Let us assume that $X_0 > X^{1/4}$, and rewrite (4.4) as

$$\sum_{2 \nmid \alpha \leq X^{1/4}} P_\alpha \left(\frac{X}{\alpha^2} \right) \frac{\mu(\alpha)}{\alpha} + \sum_{0 \leq i \ll \log(X_0/X^{1/4})} \sum_{\substack{2 \nmid \alpha \\ 2^i X^{1/4} < \alpha \leq 2^{i+1} X^{1/4}}} P_\alpha \left(\frac{X}{\alpha^2} \right) \frac{\mu(\alpha)}{\alpha}, \tag{4.5}$$

where the last sum may be incomplete. In order to apply the ideas of Chapter 3 we choose $Y = X^{1/4} \cdot (X/\alpha^2)$ in the first sum in (4.5), and show that it will give a main term of the order $\log X$. The second term in (4.5) consist of dyadic sums where $X_1 < \alpha \leq 2X_1$, and $X_1 = 2^i X^{1/4}$. Choosing $Y = (X/X_1^2)^{1/5} \cdot X/\alpha^2$, we are going to see that each of these dyadic sums is small, and hence the second term in (4.5) will give an error term.

The function P_α consists of two parts; the B-part, appearing when q is positive, is a sum of the terms

$$B_i(\alpha) \log^{3-i} Y, \text{ for } i = 1, 2, 3,$$

and the C-part is written in terms of

$$C_\alpha \left(\frac{X}{\alpha^2}, Y \right), \quad C_\alpha^{(odd)} \left(\frac{X}{4\alpha^2}, Y \right) \quad \text{or} \quad C_\alpha^{(odd)} \left(\frac{X}{8\alpha^2}, Y \right).$$

Terms with the multipliers $B_i(\alpha)$ appeared when the trivial characters were added to our sum. Recall from Lemma 3.5 that these coefficient functions are $B_1(\alpha) = b_{10} Q_\alpha(1)$, $B_2(\alpha) = b_{21} Q'_\alpha(1) + b_{20} Q_\alpha(1)$ and $B_3(\alpha) = b_{32} Q''_\alpha(1) + b_{31} Q'_\alpha(1) + b_{30} Q_\alpha(1)$, where the b_{ij} are constants, and $Q_\alpha(r)$ is a product over the prime factors of α. So we need estimates for the sums

$$\Sigma_1(x) = \sum_{2\nmid\alpha\le x} \frac{Q_\alpha(1)\mu(\alpha)}{\alpha}, \qquad \Sigma_2(x) = \sum_{2\nmid\alpha\le x} \frac{Q'_\alpha(1)\mu(\alpha)}{\alpha},$$

$$\Sigma_3(x) = \sum_{2\nmid\alpha\le x} \frac{Q''_\alpha(1)\mu(\alpha)}{\alpha}, \qquad \Sigma_4(x) = \sum_{2\nmid\alpha\le x} \frac{Q_\alpha(1)\mu(\alpha)\log\alpha}{\alpha},$$

$$\Sigma_5(x) = \sum_{2\nmid\alpha\le x} \frac{Q_\alpha(1)\mu(\alpha)\log^2\alpha}{\alpha}, \qquad \Sigma_6(x) = \sum_{2\nmid\alpha\le x} \frac{Q'_\alpha(1)\mu(\alpha)\log\alpha}{\alpha}.$$

Let us extend the definition of $Q_\alpha(r)$ by the convention $Q_\alpha(r) = 0$ if α is even. Then the generating function related to the sum $\Sigma_1(x)$ is

$$\sum_{\alpha=1}^{\infty} \frac{\mu(\alpha)Q_\alpha(r)}{\alpha^{s+1}} = \frac{1}{\zeta(s+1)} \prod_p \left(1 + \frac{1-Q_p(r)}{p^{s+1}-1}\right), \tag{4.6}$$

when $r = 1$. Let $Q(r,s)$ be the above product over primes. The series (4.6) converges for $\Re(r) > 0$ and $\Re(s) > 0$, and it can be differentiated with respect to r. So

$$\sum_{\alpha=1}^{\infty} \frac{\mu(\alpha)Q'_\alpha(r)}{\alpha^{s+1}} = \frac{Q(r,s)}{\zeta(s+1)} \sum_p \frac{Q'_p(r)}{Q_p(r) - p^{s+1}} \tag{4.7}$$

and

$$\begin{aligned}\sum_{\alpha=1}^{\infty} \frac{\mu(\alpha)Q''_\alpha(r)}{\alpha^{s+1}} = \frac{Q(r,s)}{\zeta(s+1)} \Bigg(\bigg(\sum_p \frac{Q'_p(r)}{Q_p(r) - p^{s+1}} \bigg)^2 \\ + \sum_p \left(\frac{Q''_p(r)}{Q_p(r) - p^{s+1}} - \frac{Q'_p(r)^2}{(Q_p(r) - p^{s+1})^2} \right) \Bigg).\end{aligned} \tag{4.8}$$

Setting $r = 1$ in (4.7) and (4.8) we obtain the generating functions related to the sums $\Sigma_2(x)$ and $\Sigma_3(x)$.

By the estimates of $Q_\alpha(1)$, $Q'_\alpha(1)$ and $Q''_\alpha(1)$, presented at the end of the proof of Lemma 3.5, we see that

$$Q(1,s) = \prod_p \left(1 + O\left(\frac{1}{p^{\sigma+2}}\right)\right)$$

and the sums over p in (4.7) or (4.8) are of the order

$$\sum_p \frac{\log p}{p^{\sigma+2}} \qquad \text{or} \qquad \sum_p \frac{\log^2 p}{p^{\sigma+2}}.$$

Hence these sums converge in the region $\sigma > -c(\log T)^{-2/3}(\log\log T)^{-1/3}$, and by Perron's formula $\Sigma_i(x) = O(\omega(x))$ for $i = 1, 2, 3$. By partial summation

$$\begin{aligned}\Sigma_4(x) &= \lim_{Z\to\infty}\left(\Sigma_4(Z) - \left(\Sigma_1(Z)\log Z - \Sigma_1(x)\log x - \int_x^Z \Sigma_1(t)t^{-1}\mathrm{d}t\right)\right)\\ &= a_4 + O(\omega(x)) + O\left(\int_x^\infty \omega(t)t^{-1}\mathrm{d}t\right)\end{aligned}$$

where $a_4 = \lim_{Z\to\infty}\Sigma_4(Z)$ is a constant. Since $\int_x^\infty \omega(t)t^{-1}\mathrm{d}t \ll \sum_{i=0}^\infty \omega(2^i x) \ll \omega(x)$, we have $\Sigma_4(x) = a_4 + O(\omega(x))$. Similarly $\Sigma_5(x) = a_5 + O(\omega(x))$ and $\Sigma_6(x) = a_6 + O(\omega(x))$.

Remark 4.9. Differentiating the series (4.6) and (4.7) with respect to s would give us the generating functions related to the sums Σ_i, when $i = 4, 5, 6$. These generating functions have no zeros nor poles at $s = 0$, so when Perron's formula is used the pole, coming from the term s^{-1}, gives the constant main terms a_4, a_5 and a_6.

In the dyadic sums on the left in (4.5), the contribution of the constants a_4, a_5 and a_6 cancels out, and since from these main terms only the one coming from Σ_4 is multiplied by $\log X$, the contribution of the B-part in (4.4) is

$$\text{constant}\cdot\log X + \text{constant} + O(\omega(X)).$$

since the extra logarithmic functions can be embedded into the term $\omega(X)$.

Let us then consider the C-part. We start by studying the sums

$$\Sigma_7(x,k) = \sum_{2\nmid\alpha<x}\frac{\mu(\alpha)}{\alpha}A_3(\alpha,k), \qquad \Sigma_8(x,k) = \sum_{2\nmid\alpha<x}\frac{\mu(\alpha)}{\alpha}A_4(\alpha,k),$$

$$\text{and} \qquad \Sigma_9(x,k) = \sum_{2\nmid\alpha<x}\frac{\mu(\alpha)}{\alpha}A_3(\alpha,k)\log\alpha,$$

where $A_3(\alpha,k)$ and $A_4(\alpha,k)$ are as in Lemma 3.14. Let $e = (\alpha,k)$, then by the proof of Lemma 3.14

$$\begin{aligned}A_3(\alpha,k) &= A\prod_{p\mid\alpha k}\frac{p}{p+2}\prod_{\substack{p^a\|k\\ p\nmid\alpha}}\left(1+\frac{3}{p}+\frac{2}{p(p-1)}-\frac{p+1}{p^{a+1}(p-1)}\right)\\ &= A\frac{\prod_{p\mid\alpha}\frac{p}{p+2}\prod_{p\mid k}\frac{p}{p+2}}{\prod_{p\mid e}\frac{p}{p+2}}\frac{\prod_{p^a\|k}\left(1+\frac{3}{p}+\frac{2}{p(p-1)}-\frac{p+1}{p^{a+1}(p-1)}\right)}{\prod_{\substack{p^a\|k\\ p\mid e}}\left(1+\frac{3}{p}+\frac{2}{p(p-1)}-\frac{p+1}{p^{a+1}(p-1)}\right)}\\ &= A\prod_{p\mid\alpha}\frac{p}{p+2}\frac{\prod_{p^a\|k}\left(1+\frac{p+1}{(p+2)(p-1)}\left(1-\frac{1}{p^a}\right)\right)}{\prod_{\substack{p\mid e\\ p^a\|k}}\left(1+\frac{p+1}{(p+2)(p-1)}\left(1-\frac{1}{p^a}\right)\right)},\end{aligned}$$

where A is a constant. Hence

$$\Sigma_7(x,k) = A\sum_{e|k} \frac{\prod_{p^a||k}\left(1+\frac{p+1}{(p+2)(p-1)}\left(1-\frac{1}{p^a}\right)\right)}{\prod_{\substack{p|e\\ p^a||k}}\left(1+\frac{p+1}{(p+2)(p-1)}\left(1-\frac{1}{p^a}\right)\right)} \sum_{\substack{2\nmid\alpha<x\\ (\alpha,k)=e}} \frac{\mu(\alpha)}{\alpha}\prod_{p|\alpha}\frac{p}{p+2}$$

$$= A\sum_{e|k}\mu(e)\frac{\prod_{p^a||k}\left(1+\frac{p+1}{(p+2)(p-1)}\left(1-\frac{1}{p^a}\right)\right)}{\prod_{\substack{p|e\\ p^a||k}}\left(p+2+\frac{p+1}{p-1}\left(1-\frac{1}{p^a}\right)\right)} \sum_{\substack{2\nmid\alpha<x/e\\ (\alpha,k)=1}} \frac{\mu(\alpha)}{\alpha}\prod_{p|\alpha}\frac{p}{p+2}.$$

The generating function related to the remaining α-sum is

$$\frac{1}{\zeta(s+1)}\prod_p\left(1+O\left(\frac{1}{p^{\sigma+2}}\right)\right)\prod_{p|k}\left(1+O\left(\frac{1}{p^{\sigma+1}}\right)\right).$$

When Perron's formula is applied this generating function acts like $1/\zeta(s+1)$. Since the products over the divisors of k are $O(d(k))$,

$$\Sigma_7(x,k) \ll \sum_{e|k}\omega(1+x/e)d(k) \ll \omega(2+x/k)d^2(k).$$

However, $\Sigma_7(x,k)$ is of constant size in mean over k, like $A_3(\alpha,k)$ (see Remark 3.16).

Similar arguments holds for $A_4(\alpha,k)$. By definition $A_4(\alpha,k)$ is

$$A_3(\alpha,k)\left(\frac{1}{2}\sum_{\substack{p^a||k\\ p\nmid 2\alpha}}\log p\left(\frac{3p+1-\frac{1}{p^a}\left(2p(a+1)+3-\frac{2a+1}{p}\right)}{p(p-1)\left(p^2+2p-1-\frac{p-1}{p^a}\right)}\right)+B+2\sum_{p|2\alpha k}\frac{\log p}{p+2}\right),$$

where B is a constant. Separating sums over α and k gives

$$A_4(\alpha,k) = A_3(\alpha,k)\left(\sum_{p^a||k} - \sum_{\substack{p|e\\ p^a||k}} + \sum_{p|\alpha} + \sum_{p|k} - \sum_{p|e} + \text{constant}\right),$$

where $e=(\alpha,k)$. For the sum over α, the only difference which appears compared with the case of $A_3(\alpha,k)$ is the sum with the extra factor $\sum_{\substack{q|\alpha\\ q \text{ prime}}}\frac{\log q}{q+2}$, but in this case

$$\sum_{2\nmid\alpha<x}\frac{\mu(\alpha)}{\alpha}\prod_{p|\alpha}\frac{p}{p+2}\sum_{q|\alpha}\frac{\log q}{q+2} = -\sum_{\substack{q<x\\ q\neq 2}}\frac{\log q}{(q+2)^2}\sum_{\substack{\beta<x/q\\ (\beta,2q)=1}}\frac{\mu(\beta)}{\beta}\prod_{p|\beta}\frac{p}{p+2}.$$

The sum over β can be estimated by using the above ideas, and since the sum over q converges rapidly, we get

$$\Sigma_8(x,k) \ll \omega(2+x/k)d(k)^2.$$

By partial summation, we get

$$\begin{aligned}\Sigma_9(x,k) &= \sum_{2\nmid\alpha} \frac{\mu(\alpha)A_3(\alpha,k)\log\alpha}{\alpha} + O(\omega(2+x/k)d(k)^2\log x) \\ &=: c(k) + O(\omega(2+x/k)d(k)^2\log x).\end{aligned} \tag{4.10}$$

Hence

$$\begin{aligned}\sum_{2\nmid\alpha\le x} \frac{\mu(\alpha)}{\alpha} A(k, yX/\alpha^2, \alpha) =&\Sigma_7(x,k)(\log Xy+1) + \Sigma_8(x,k) - 2\Sigma_9(x,k) \\ =& -2c(k) + O\left(\omega\left(2+x/k\right)d^2(k)(\log yX+1)\right).\end{aligned} \tag{4.11}$$

Let us now consider the first sum in (4.5). Since also $c(k)$ is of constant size in mean over k, the contribution of the main term $-2c(k)$ gives the term

$$\frac{-1}{\pi}\int_0^{X^{1/4}} \frac{1}{\sqrt{y}} \sum_{k=1}^{\infty} \frac{c(k)}{k^2}\left(1-\cos\left(\frac{2\pi k^2}{y}\right)+\sin\left(\frac{2\pi k^2}{y}\right)\right)\mathrm{d}y$$

of the order $O(\log X)$, or a similar term where the sum over k is taken over odd numbers.

The contribution of the error term of (4.11) is

$$\begin{aligned}&\sum_{k\le X^{1/8}} \frac{\omega(X^{1/4}/k)d^2(k)}{k^2}\left(\int_0^{k^2}\frac{\log yX+1}{y^{1/2}}\mathrm{d}y + \int_{k^2}^{X^{1/4}}\frac{\log yX+1}{y^{1/2}}\left(\frac{k^4}{y^2}+\frac{k^2}{y}\right)\mathrm{d}y\right) \\ &\qquad + \sum_{k>X^{1/8}} \frac{\omega(X^{1/4}/k+2)d^2(k)}{k^2}\int_0^{X^{1/4}}\frac{\log yX+1}{y^{1/2}}\mathrm{d}y \\ &\ll \sum_{k\le X^{1/8}} \frac{\omega(X^{1/4}/k)d^2(k)\log(k^2X)}{k} + X^{1/8}\log X \sum_{k>X^{1/8}} \frac{\omega(X^{1/4}/k+2)d^2(k)}{k^2} \\ &\qquad \ll\omega\left(X^{1/8}\right)\log^2 X.\end{aligned}$$

In the dyadic sums in the second terms of (4.5), the contribution of the constant a_9 cancels out, as in the B-part. Hence the contribution of the C-part in (4.4) is a main term of order $O(\log X)$ plus an error term of order $O(\omega(X))$. (The extra logarithmic functions can be once more embedded into the term $\omega(X)$ by choosing a suitable constant C.)

Hence, (4.4) is

$$P^*(X) + O(\omega(X)), \tag{4.12}$$

where $P^*(X) \ll \log X$ does not depend on X_0.

4.4. **The mean square over primitive characters.** By partial summation, it is easy to see that the error term in (4.2) is $O(X^{2/5+\varepsilon}X_0^{1/5-2\varepsilon})$. Choosing $X_0 = X^{1/2}\omega(X)^{5/6}$ this and the error term from (4.3) and (4.12) give $O(X^{1/2}\omega(X)^{1/6})$. The exponent 1/6 can also be omitted by changing the constant C. Hence we have proved the following theorem:

Theorem 4.13.

$$\sum_{1<q\leq X}^{*} L^2(1,\chi_q) = A^*X + P^*(X)X^{1/2} + O\left(X^{1/2}\omega(X)\right)$$

where A^* is a constant, $P^*(X) \ll \log X$ and $\omega(X)$ tends to zero as X tends to infinity. A similar formula holds also when the summation is taken over fundamental discriminants $-X \leq q < 1$ or over all fundamental discriminants $|q| \leq X$.

Especially, for the mean square over odd positive fundamental discriminants we have:

Theorem 4.14.

$$\sum_{\substack{1<q\leq X \\ q\equiv 1\,(4) \\ q\neq\square}}^{*} L^2(1,\chi_q) = A^*X + P^*(X)X^{1/2} + O\left(X^{1/2}\omega(X)\right),$$

where

$$A^* = \sum_{2\nmid\alpha} \frac{\mu(\alpha)}{\alpha^2} \sum_{(n,\alpha)=1} \frac{d(n^2)\varphi(n)}{n^3}$$

is a constant,

$$P^*(X) = B_1 \log X + B_2 - c_1 C(X) - c_2 C^{(odd)}(X/4) - c_3 C^{(odd)}(X/8) \ll \log X,$$

where the B_i and c_i are constants and

$$\omega(X) = \exp(-C \log^{3/5} X (\log\log X)^{-1/5}),$$

with a suitable constant C. Here

$$C(x) = \frac{1}{\pi}\int_0^{x^{1/4}} \frac{1}{\sqrt{y}} \sum_{k=1}^{\infty} \frac{c(k)}{k^2}\left(1 - \cos\left(\frac{2\pi k^2}{y}\right) + \sin\left(\frac{2\pi k^2}{y}\right)\right) \mathrm{d}y,$$

where

$$c(k) = \sum_{2\nmid\alpha} \frac{\mu(\alpha)A_3(\alpha,k)\log\alpha}{\alpha},$$

and $A_3(\alpha,k)$ is as in Lemma 3.14. The function $C^{(odd)}(x)$ is similar except that the sum over k is taken over odd numbers.

Corollary 4.15. *If the Riemann hypothesis is true, then*

$$\sum_{1<q\leq X}^{*} L^2(1,\chi_q) = A^*X + P^*(X)X^{1/2} + O\left(X^{8/17+\varepsilon}\right)$$

where A^* and $P^*(X)$ are as in Theorem 4.14. A similar formula hold also for the corresponding sum over negative discriminants.

Proof. The Riemann hypothesis is equivalent to the estimate

$$M(X) \ll X^{1/2+\varepsilon}.$$

Assuming this, we see that the dominating error terms above are $O(X^{2/5+\varepsilon}X_0^{1/5+\varepsilon})$ coming from the error term of Theorem 3.27, and $O(XX_0^{-3/2+\varepsilon})$ corresponding the error term $O(XX_0^{-1}\omega(X_0))$. (The error term $O(X^{3/8})$ corresponding the contribution of the error term of (4.11) is clearly smaller.) Hence the assertion follows by choosing $X_0 = X^{6/17}$. □

5. An application to algebraic number theory

We recall here basics of the *ideal theory in quadratic fields* as a preparation for applications of Theorems 3.27 and 4.13 to the class numbers of imaginary quadratic fields. More details can be found for example in [8].

5.1. **Algebraic integers and ideal classes.** *A quadratic field* $\mathbb{Q}(\sqrt{D})$ is an extension over $\mathbb{Q}$ of degree two. All numbers of $\mathbb{Q}(\sqrt{D})$ can be represented in the form $a + b\sqrt{D}$, where the coefficients a and b are rational numbers. *Rational integers* in $\mathbb{Q}(\sqrt{D})$ are just the normal integers, and the integers of the field $\mathbb{Q}(\sqrt{D})$ mean as usual *algebraic integers*, that is, numbers which are roots of some monic polynomial with rational integer coefficients.

The integers of the field $\mathbb{Q}(\sqrt{D})$ form a ring $\mathfrak{O}$. *Units* of the field $\mathbb{Q}(\sqrt{D})$ are those integers which are invertible in $\mathfrak{O}$. In *imaginary quadratic fields* there are only the trivial units ± 1 when $D < -4$. In $\mathbb{Q}(\sqrt{-4})$ there are four units $\pm 1, \pm \mathrm{i}$ and in $\mathbb{Q}(\sqrt{-3})$ six units ± 1 and $\frac{\pm 1 \pm \sqrt{-3}}{2}$. In *real quadratic fields* there is a *fundamental unit* η and all the other units are of the form $\pm\eta^n$ with $n \in \mathbb{Z}$.

All integers in $\mathfrak{O}$ can be represented in the form $a + b\rho$, where a and b are rational integers and

$$\rho = \begin{cases} \sqrt{D} & \text{if } D \equiv 2, 3 \pmod 4, \\ \frac{1+\sqrt{D}}{2} & \text{if } D \equiv 1 \pmod 4). \end{cases}$$

So $\{1, \rho\}$ is the base of the ring $\mathfrak{O}$. Actually the same base generates the whole field. With this base we can define the *discriminant* of the field

$$d = d(1, \rho) = \det \begin{pmatrix} 1 & \rho \\ 1 & \rho' \end{pmatrix}^2 = \begin{cases} D & \text{if } D \equiv 1 \pmod 4, \\ 4D & \text{if } D \equiv 2, 3 \pmod 4, \end{cases}$$

where ρ' is the conjugate of ρ.

For the ideals of $\mathfrak{O}$ we can define an equivalence relation. Two ideals $\mathfrak{A}$ and $\mathfrak{B}$ are said to be equivalent if they differ by a *principal ideal,* that is, if there exists a principal ideal generated by an element a, such that

$$\mathfrak{A} = (a)\mathfrak{B}.$$

This equivalence relation divides the ideals into equivalence classes. The number of these classes is the *class number* $h(d)$ of the field. Clearly this number is always at least one and it can be shown that it is finite. The class number has a connection to L-functions by *the class number formula*

$$h(d) = \begin{cases} \dfrac{\sqrt{d}}{2\ln\eta} L(1,\chi_d) & \text{if } d > 0, \\ \dfrac{w\sqrt{|d|}}{2\pi} L(1,\chi_d) & \text{if } d < 0, \end{cases}$$

where w is the number of units. The fundamental unit η is usually hard to determine, but when $d < -4$ this formula is a practical device to calculate the number of classes.

These class number formulas and Theorem 4.13 give by partial summation the following theorem:

Theorem 5.1. *Letting $-d$ run over negative fundamental discriminant, we have*

$$\sum_{1\le d\le X}^{*} h^2(-d) = aX^2 + b(X)X^{3/2} + O\left(X^{3/2}\omega(X)\right),$$

where a is a constant, and $b(X)$ is a function which can be made explicit and it satisfies $b(X) \ll \log X$, and $\omega(X)$ tends to zero when X tends to infinity.

Since the square divisors of D do not change the field $\mathbb{Q}(\sqrt{D})$, the class number $h(d)$ of the field $\mathbb{Q}(\sqrt{D})$ always refers to the fundamental discriminant d. However, Theorem 3.27 can be applied to the mean square of the class number when confined to the certain subrings of $\mathfrak{O}$.

5.2. **Subrings $\mathfrak{O}_n$.** Trivially the ring $\mathfrak{O}$ contains $\mathbb{Z}$ as a subring. All other subrings of $\mathfrak{O}$ can be characterized as follows: *A subring of $\mathfrak{O}$ which does not consist only of rational integers is a set of integers of $\mathfrak{O}$ which are congruent to a rational integer modulo some fixed positive rational integer n.* This subring $\mathfrak{O}_n$ is called an *order*.

As mentioned above, all integers were represented in the form $a + b\rho$. Since the integers in $\mathfrak{O}_n$ must be congruent to some rational integer modulo n, they must be in the set $\mathbb{Z} + n\mathfrak{O}$. It can be shown that the numbers of $\mathfrak{O}_n$ can be represented in the form $a + bn\rho$, so $\{1, n\rho\}$ is the base of $\mathfrak{O}_n$. Hence the discriminant of $\mathfrak{O}_n$ is $d(1, n\rho) = n^2 d$. In this subring we define a similar equivalence relation and have a similar structure of ideal classes as in $\mathfrak{O}$. The number of these classes $h(n^2 d)$ is the so-called *relative class number*.

Let us choose a unit η_1 from the field $\mathbb{Q}(\sqrt{D})$ as follows

$$\eta_1 = \begin{cases} \text{fundamental unit} & \text{if } d > 0, \\ -1 & \text{if } d < -4, \\ \mathrm{i} & \text{if } d = -4, \\ \frac{1+\mathrm{i}\sqrt{3}}{2} & \text{if } d = -3. \end{cases}$$

The connection between the class number and the relative class number [8, p.217] is

$$h(n^2 d) = h(d)\lambda(n)n\prod_{p|n}\left(1-\frac{\chi_d(p)}{p}\right),$$

where $\lambda(n)$ is the inverse of the smallest natural number ν such that η_1^ν lies in $\mathfrak{O}_n$. Especially for an imaginary quadratic field, $\lambda(n) = 1$ if $d \le -4$. Hence,

$$h(d) = \frac{\sqrt{|d|}}{\pi}L(1,\chi_d),$$

where $d \le -4$ is a discriminant but not necessarily a fundamental one.

Now partial summation together with Theorem 3.27 gives the following result:

Theorem 5.2. *Let $h(-d)$ be a relative class number for a discriminant $-d$, then*

$$\sum_{1\le d\le X} h^2(-d) = aX^2 + b(X)X^{3/2} + O\left(X^{7/5+\varepsilon}\right),$$

where a is a constant, and $b(X)$ is a function which can be made explicit and $b(X) \ll \log^2 X$.

6. Concluding remarks

6.1. **The mean square over a short interval.** In [5] Chamizo and Iwaniec deduced the following mean value result over a short interval in the linear case:

$$\sum_{\substack{X<q\le X+N\\ q\equiv\nu\,(8)}} L(1,\chi_q) = \frac{3\zeta(2)}{28\zeta(3)}N + O\left(N^{7/8}X^\varepsilon + N^{2/3}X^{1/32+\varepsilon}\right),$$

where $1 < N < X^{1/2}$. A similar mean square estimate over a short interval can be deduced from Theorem 3.28, since

$$\sum_{\substack{X<q\le X+N\\ q\equiv 1\,(4)\\ q\ne\square}} L^2(1,\chi_{q\alpha^2}) = \sum_{\substack{1<q\le X+N\\ q\equiv 1\,(4)\\ q\ne\square}} L^2(1,\chi_{q\alpha^2}) - \sum_{\substack{1<q\le X\\ q\equiv 1\,(4)\\ q\ne\square}} L^2(1,\chi_{q\alpha^2}) \tag{6.1}$$

where $N \le X$. Clearly the main term in (6.1) is $A(\alpha)N$, and the error is $O(X^{2/5+\varepsilon}\alpha^\varepsilon)$. However, the behaviour of the middle terms is not so obvious.

The possible middle terms in (6.1) are comprised of a B-part and a C-part, where the B-part contains the terms with $B_i(\alpha)$, and the C-part is the sum of the terms with C_α or $C_\alpha^{(odd)}$. Let us apply Theorem 3.28, choosing $Y = X(X+N)$ in the first sum, and $Y = X^2$ in second sum. This can be done since the choice of Y could be done quite freely. Now it is easy to see that the B-part is $O(NX^{-1/2}\sum_{i=1}^3 B_i(\alpha)(\log X)^{3-i})$, since

$$\log(X+N) - \log X = \log\left(1+\frac{N}{X}\right) = O\left(\frac{N}{X}\right).$$

The order of the C-part is the same as the order of

$$
\begin{aligned}
(6.2)\qquad & C_\alpha\left(X+N, X(X+N)\right)(X+N)^{1/2} - C_\alpha(X,X^2)X^{1/2} \\
& \ll \left(C_\alpha(X+N, X(X+N)) - C_\alpha(X,X^2)\right)X^{1/2} + NX^{-1/2}\log^2 X.
\end{aligned}
$$

Now the upper limit of integration is same in both terms $C_\alpha(X,X^2)$ and $C_\alpha(X+N, X(X+N))$, and

$$A(k,t(X+N),\alpha) - A(k,tX,\alpha) = A_3(\alpha,k)\log(1+N/X) \ll NX^{-1}\log\log 3k.$$

Hence the order of (6.2) is $O(NX^{-1/2}\log^2 X)$, and we have the following theorem:

Corollary 6.3. *Let $\alpha < X^a$ for some positive constant a, and $1 < N \le X$. Then*

$$\sum_{\substack{X<q\le X+N \\ q\equiv 1\,(4) \\ q\ne\square}} L^2(1,\chi_{q\alpha^2}) = A(\alpha)N + O\left(X^{2/5+\varepsilon}\alpha^\varepsilon\right) + O\left(NX^{-1/2}\log^2 X\right).$$

6.2. **Other moments.** The method applied in this thesis to the mean square of quadratic Dirichlet L-functions at 1 can be used also to obtain a similar formula in the linear case. The only difference is that the function $d(n)$ is missing. Instead of Lemma 3.14 we can use the formula, which was mentioned in [9] with too optimistic error term,

$$\sum_{\substack{n\le Y \\ (n,2\alpha)=1}} \frac{G_{k^2}(n)}{\sqrt{n}} = \frac{2}{3\zeta(2)}\prod_{p|\alpha}\left(1-\frac{1}{p+1}\right)Y + O\left(Y^{1/2}d(k)\right),$$

which can be proved similarly. Hence we have an asymptotic formula for the sum of $L(1,\chi_{q\alpha^2})$ which is similar to Theorem 3.28, but where in C_α the multiplier $A(\alpha,tX,k)$ is replaced by

$$\frac{2}{3\zeta(2)}\prod_{p|\alpha}\left(1-\frac{1}{p+1}\right).$$

However, this leads to a bigger error term than in (1.3) in the contents of the class numbers.

It seems that the method itself could be used also to the fourth or higher moments. However, in order to get sufficiently sharp estimates, that is, to get an error term smaller than $X^{1/2}$, we would need some estimates which are not known so far. For example, an estimate for

$$\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \mathrm{d}t$$

was used a few times while proving the lemmas. Studying the higher moments would require the estimate

$$\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t \ll T^{1+\varepsilon},$$

which is not known for $k > 2$, and even though this is known in the case of the fourth moment, also a generalization of Jutila's result (2.6) to the fourth moment would be needed, since (2.7) is too weak over t.

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