A MINIMUM PRINCIPLE FOR POSITIVE HARMONIC FUNCTIONS

BY

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To ROLF NEVANLINNA on his 70th birthday

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A minimum principle for positive harmonic functions

Let \( D \) be a simply-connected region in the plane \( z = x + iy \), \( \zeta_0 \) a given boundary point of \( D \) and \( S = \{ z_n \}_1^\infty \subset D \) a sequence of points tending to \( \zeta_0 \) as \( n \to \infty \). Let \( \varphi \) denote a Martin harmonic function corresponding to \( \zeta_0 \), i.e. \( \varphi \) is positive in \( D \) and vanishes at each boundary point \( \neq \zeta_0 \). The object of this article is to characterize the sequences \( S \) which possess the property that for each positive harmonic \( u \) in \( D \), the inequalities

\[
(1) \quad u(z_n) \geq \lambda \varphi(z_n), \quad n = 1, 2, \ldots, \lambda > 0,
\]

imply

\[
(2) \quad u(z) \geq \lambda \varphi(z), \quad z \in D.
\]

If the implication \( (2) \) is true we shall call \( S \) an equivalence sequence for \( \zeta_0 \), therewith and henceforth allowing this paper some freedom from orthodox notions and terminologies.

**Theorem 1.** \( S \) is an equivalence sequence for \( \zeta_0 \) if and only if it contains a subset \( \{ z_n \}_1^\infty \) with the properties

\[
(3) \quad \sup_{\mu \neq r} g(z_n, z_n) < \infty
\]

\[
(4) \quad \sum_{r=1}^\infty g(z, z_n) \varphi(z_n) = \infty, \quad z \in D,
\]

where \( g \) is the Green function for \( D \).

It is convenient to restate and to prove the theorem for the upper half-plane \( \Omega, z = x + iy, y > 0 \), letting \( \zeta_0 \) be the infinite boundary point and \( \varphi = y \). When \( z = re^{i\Theta} \) tends to \( \infty \) in \( \Omega \) we have

\[
g(i, z) \sim \frac{2 \sin \Theta}{r},
\]

\[
g(i, z) \varphi(z) \sim 2 \sin^2 \Theta.
\]

By virtue of these relations the theorem can be reformulated as follows: \( \delta \)The points

\[
z_n = x_n + iy_n = r_n e^{i\Theta_n}, \quad n = 1, 2, \ldots
\]

\[
\delta
\]
form an equivalence sequence for the infinite boundary point of the upper half-plane if and only if they contain a subset \( \{ z_{n_{\mu}} \} \) satisfying the separation condition

\[
\text{in } \prod_{\mu \neq r} \left| \frac{z_{n_{\mu}} - z_{n_{r}}}{\overline{z_{n_{\mu}}} - \overline{z_{n_{r}}}} \right| > 0 ,
\]

and such that

\[
\sum_{\nu=1}^{\infty} \sin^2 \theta_{n_{\nu}} = \infty .
\]

The necessity of the conditions is easily established. To each \( z_0 \in \Omega \) and to each \( \varepsilon , 0 < \varepsilon < 1 \), we assign the circular disc

\[
A (z_0, \varepsilon) = \left\{ z : \frac{|z - z_0|}{z - \overline{z_0}} < \varepsilon \right\}
\]

and we recall that Harnack's inequalities for positive harmonic functions in \( \Omega \) can be written

\[
\frac{1 - \varepsilon}{1 + \varepsilon} \leq u(z) \leq \frac{1 + \varepsilon}{1 - \varepsilon} , z \in A(z_0, \varepsilon) .
\]

If (5) and (6) were not necessary conditions there would exist an equivalence sequence \( S \) such that each of its subsets satisfying the separation condition would make the series (6) convergent. However, from any given \( S \) it is always possible to select a subsequence \( \{ z_{n_{\nu}} \} \) such that, \( \varepsilon \) being given, the union \( \bigcup_{\nu=1}^{\infty} A(z_{n_{\nu}}, \varepsilon) \) covers \( S \), whereas each \( z_{n_{\nu}} \) is contained in the sole disc \( A(z_{n_{\nu}}, \varepsilon) \). The separation condition is therefore satisfied. If (6) were convergent the same would be true of the series

\[
u(z) = \sum_{\nu=1}^{\infty} \frac{y y_{n_{\nu}}^2}{(x - x_{n_{\nu}})^2 + y^2} ,
\]

and \( u \) would represent a positive harmonic function in \( \Omega \) with the properties

\[
u(z_{n_{\nu}}) > y_{n_{\nu}} , \nu = 1, 2, \ldots
\]

\[
u(i y) = o(y) , y \to + \infty .
\]

On applying (8) both to \( u \) and to \( \varphi = y \) we find that in each \( A(z_{n_{\nu}}, \varepsilon) \),

\[
u(z) > \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^2 y = \lambda y .
\]
This inequality would therefore remain valid on $S$, but violated at other points of $\Omega$, in view of (11). This proves the necessity of the stated conditions. The sufficiency will be derived from this more precise result:

Lemma I. Let $u$ be positive and harmonic in $\Omega$ and let $E(\lambda)$ denote the set

$$E(\lambda) = \{ z = x + iy : y > 0, \quad u(z) \geq \lambda y \}.$$  

Then the divergence of the integral

$$\int_{E(\lambda)} \frac{dx \, dy}{1 + |z|^2}$$

implies $E(\lambda) = \Omega$.

The particular value of $\lambda$ is immaterial and we may therefore assume $\lambda = 1$ and set $E(1) = E$. As a consequence of Harnack's inequalities we have

$$\frac{\partial u}{\partial y} \leq \frac{u}{y}, \quad y > 0,$$

where the sign of equality is excluded unless $u = ay$, in which case the lemma is trivially true. We may therefore assume that the upper sign holds throughout $\Omega$. This implies that $u(x + iy)/y$ for fixed $x$ is strictly decreasing with increasing $y$. If not void the open set $\Omega - E$ has thus a boundary which meets vertical lines in at most one finite point. Each component of $\Omega - E$ is therefore an unbounded simply-connected region. Let $D$ be a component and $\Gamma$ its boundary. Without loss of generality we assume that $D$ contains a point $z = iy_0$ on the imaginary axis. The function $v(z) = y - u(z)$ is by assumption harmonic and strictly positive in $D$, vanishes at all finite boundary points and is thus a Martin function for $D$. We shall prove that this implies that (14) converges.

In the sequel we shall denote by $C_r$, $r > 1 + y_0$, the region

$$C_r = \{ z = x + iy : y > 0, \quad |z + i| < r \}$$

and by $\gamma_r$ the largest open arc of the circle $|z + i| = r$ contained in $D$ and containing the point $i (r - 1)$. Together with $\Gamma$ the arc $\gamma_r$ forms the boundary of a well defined simply-connected subregion $D_r$ of $C_r$.

In the continuation of the proof we shall use the fruitful notion of harmonic measure which plays such a prominent role in the work of Rolf Nevanlinna. The harmonic measure, $h(z_0, \gamma_r)$, of $\gamma_r$ is by definition the value at $z_0$ of the bounded harmonic function in $D_r$ which equals 1 on $\gamma_r$ and vanishes elsewhere on the boundary. By the minimum principle for harmonic functions,
\[(15) \quad v(z_0) \leq h(z_0, \gamma_r) \max_{z \in \gamma_r} v(z) < h(z_0, \gamma_r) \cdot r.\]

In order to estimate \(h\) we recall this result ([1], p. 10).

**Lemma II.** Let \(D\) be simply-connected, \(z_0\) a point in \(D\) and \(\gamma\) a boundary continuum. Let \(\psi\) be harmonic in \(D\) and have the properties: \(\psi(z_0) = 0\), \(\psi(z) \geq L > 0\) on \(\gamma\),

\[A = \int_D |\nabla \psi|^2 \, dx dy < \infty\]

Then
\[(16) \quad h(z_0, \gamma) < e^{-\frac{\pi L^2}{A}}.\]

For the region \(D\), the choice
\[\psi(z) = \log \left| \frac{z + i}{z_0 + i} \right|\]
yield
\[L = \log r - \log (1 + y_0) .\]

Define \(E_r = E \cap C_r\), let \(m(r)\) be determined by the relation
\[\pi m(r) = \int_{E_r} |\nabla \psi|^2 \, dx \, dy = \int_{E_r} \frac{dx \, dy}{|z + i|^2},\]
and observe that
\[\int_{C_r} \frac{dx \, dy}{|z + i|^2} < \pi \log r .\]

Hence,
\[A < \pi (\log r - m(r))\]
and
\[\frac{\pi L^2}{A} > \frac{(\log r - \log (1 + y_0))^2}{\log r - m(r)} \geq \log r - 2 \log (1 + y_0) + m(r) \left[ 1 + O \left( \frac{1}{\log r} \right) \right] .\]

If (14) diverges, then \(m(r)\) will increase to \(\infty\) with \(r\) and we would have \(h = o \left( \frac{1}{r} \right)\), and consequently \(v(z_0) = 0\), contradictory to the assumption \(v(z_0) > 0\). This proves Lemma I.
We can now continue the proof of the sufficiency of the conditions in Theorem I. In order to simplify the notations we let \( \{z_n\} \) denote the subsequence of \( S \) satisfying (5) and (6). By virtue of the separation condition (5) the discs \( \Delta(z_n, \varepsilon) \) are disjoint if \( \varepsilon \) is small enough, and they are contained, according to (12), in the set \( E(\lambda') \) if

\[
\lambda' = \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^2 \lambda.
\]

The divergence of

\[
\sum_1^\infty \sin^2 \Theta_n
\]

therefore implies that the integral (14) for \( E(\lambda') \) diverges, the radius of \( \Delta(z_n, \varepsilon) \) being \( > 2 \varepsilon y_n \). Lemma I asserts that everywhere in \( \Omega \), \( u(z) \geq \lambda' y \), and this concludes the proof since \( \lambda' \) can be taken arbitrary close to \( \lambda \).

We want to point out that Lemma I remains true also for positive superharmonic functions. The proof is the same except for one important difference. The region replacing \( D \), will be multiply-connected and Lemma II not valid. The proof can however be carried through by means of the following more general but still unpublished result.

Let \( D \) be limited by a finite number of Jordan curves \( \{I^*_\} \), and let \( \gamma \) be a closed boundary set carried by one and the same boundary component, say \( I_1' \). Let \( \alpha \) be an arc joining the given point \( z_0 \) with some point belonging to the set \( I_1' - \gamma \). Then

\[
\max_{z \in \alpha} h(z, \gamma) < 5 e^{-\alpha \lambda}
\]

where \( \lambda \) stands for the extremal length of the family of curves joining \( \alpha \) and \( \gamma \) within \( D \).

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References