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**ON A HOMOMORPHIC CHARACTERIZATION OF
RECURSIVELY ENUMERABLE LANGUAGES**

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1. Introduction

According to the well-known theorems of Medvedev and Schützenberger, every regular language is a homomorphic image of a 2-testable language and every context-free language is a homomorphic image of the intersection of a Dyck language and a 2-testable language. Many such homomorphic representations are known also for the family of recursively enumerable languages. Every recursively enumerable language is a homomorphic image of the intersection of two deterministic context-free languages, [3], and a homomorphic image of a language generated by a context-sensitive grammar in linear time, [1], as well as a homomorphic image of a language generated by a λ -free context-free programmed grammar, [6].

The purpose of this paper is to establish by a direct combinatorial argument the following result. Consider a fixed alphabet V_T . Then there exist another alphabet V' , deterministic context-free languages L_1 and L_2 over V' and a homomorphism h of $W(V')$ onto $W(V_T)$ with the following property. For every recursively enumerable language L over V_T , there is a regular noncounting language K over V' such that

$$L = h(L_1 \cap L_2 \cap K).$$

Thus, only the noncounting language K depends on L , everything else is determined by the alphabet of L alone. Essentially the same result has been proven by Fisher and Raney, [2]. Their proof, however, is based on a complicated theory of automata on networks.

Homomorphic representation can be used for proving results concerning decidability, nonclosure and generative capacity, [1], [6]–[9]. Some simple applications to decidability will be considered also in this paper. Very few homomorphic representations are known for Lindenmayer systems (cf. [5], [7], [10], [11]). This may be due to the resistance displayed by these systems against closure operations. We would like to mention, finally, that in spite of the many homomorphic representations given for recursively enumerable languages, there still is no satisfactory general theory concerning such representations.

2. Definitions and results

The reader is assumed to be familiar with the basic notions concerning automata and formal languages, [12]. As regards locally testable and non-counting languages, cf. [4].

The set of all words, including the empty word λ , over an alphabet V is denoted by $W(V)$. We use the customary notations mi , $*$, $+$ for mirror image, catenation closure and λ -free catenation closure. In the statement of the following theorem,

$$V_T = \{a_1, \dots, a_{u-4}\}, \quad u \geq 5,$$

$$V'_T = \{a'_i \mid a_i \in V_T\},$$

$$V' = V'_T \cup \{0, 1\}.$$

Theorem 1. *There exist two deterministic context-free languages L_1 and L_2 over V' and a homomorphism h of $W(V')$ onto $W(V_T)$ with the following property. For every recursively enumerable language L over V_T , there is a noncounting regular language K over V' such that*

$$(1) \quad L = h(L_1 \cap L_2 \cap K).$$

All constructions involved in the proof of Theorem 1 are effective. Since the emptiness problem is undecidable for recursively enumerable languages and since a homomorphic image of a language is empty if and only if the language itself is empty, the following theorem is an immediate corollary of Theorem 1.

Theorem 2. *There are two deterministic context-free languages L_1 and L_2 such that there is no algorithm for deciding of an arbitrary noncounting regular language K whether or not the intersection $L_1 \cap L_2 \cap K$ is empty.*

In some sense, Theorem 2 can be considered as an unsolvability result for regular languages since K is the only variable. However, one can also say that it is not a problem dealing «properly» with regular languages.

Many undecidability results similar to Theorem 2 can be obtained. We mention only the following, due to the fact that one can easily (by analyzing the proof in the next section) modify the construction in such a way that $L_1 \cap L_2 \cap K$ is nonempty if and only if it is infinite.

Theorem 3. *There are two deterministic context-free languages L_1 and L_2 such that there is no algorithm for deciding of an arbitrary noncounting regular language K whether or not the intersection $L_1 \cap L_2 \cap K$ is finite.*

3. Proof of Theorem 1

Every recursively enumerable language over V_T is generated by a type 0 grammar. Without loss of generality, we assume that the non-terminals form an initial segment of the sequence

$$(2) \quad a_{u+1}, a_{u+3}, a_{u+5}, \dots,$$

a_{u+1} being the initial letter.

In what follows, the letters

$$a_{u-3}, a_{u-2}, a_{u-1}, a_u$$

will play the role of boundary markers, and the letters in the sequence

$$a_{u+2}, a_{u+4}, a_{u+6}, \dots$$

the role of production indicators.

All letters a_i , $i \geq 1$, will now be encoded in the alphabet $\{0, 1\}$ by defining the homomorphism

$$h_1(a_i) = 10^i 1, \quad i \geq 1.$$

To make the following definitions more readable, we also use the following abbreviations

$$\begin{aligned} h_1(a_u) &= c, & h_1(a_{u-i}) &= c_i \text{ for } 1 \leq i \leq 3, \\ h_1(a_{u+1}) &= S, & h_1(a_{u+2i}) &= d_i \text{ for } i \geq 1. \end{aligned}$$

Furthermore, we denote

$$U_T = (h_1(a_1) \cup \dots \cup h_1(a_{u-4}))^*.$$

By U we denote the language consisting of λ and of all words of the form $h_1(a_{i_1}) \dots h_1(a_{i_v})$ where $v \geq 1$ and, for each $j = 1, \dots, v$, either $1 \leq i_j \leq u-4$ or else $i_j = u + 2r + 1$ for some $r \geq 0$. (Intuitively, U_T consists of encoded words over the terminal alphabet and U of encoded words in terminal and nonterminal letters.)

We are now in the position to define the homomorphism h and the two languages L_1 and L_2 over V' . By definition

$$h(b) = \begin{cases} a_i & \text{for } b = a'_i \ (a'_i \in V'_T), \\ \lambda & \text{for } b \in \{0, 1\}, \end{cases}$$

$$\begin{aligned} L_1 &= \{P_1 c_1 \alpha d_i P_2 c \ mi(P_2) \ c_2 \ mi(\beta) \ c_3 \ mi(P_1) c\} \\ &\quad P_1, P_2, \alpha, \beta \in U, \ i \geq 1\}^+ \ (V'_T)^* \end{aligned}$$

and

$$\begin{aligned}
L_2 = & \{c_1 S d_i c \mid i \geq 1\} \{P_1 c_2 P_2 c_3 P_3 c P_4 c_1 P_5 d_j P_6 c \mid \\
& P_1, P_2, P_3, P_4, P_5, P_6 \in U, j \geq 1, P_1 P_2 P_3 = mi(P_4 P_5 P_6)\}^* \\
& \{Q_1 c_2 Q_2 c_3 Q_3 c Q_4 \mid Q_4 \in W(V'_T); \\
& Q_1, Q_2, Q_3 \in U_T, Q_1 Q_2 Q_3 = h_1 h(mi(Q_4))\}.
\end{aligned}$$

So far our definitions are based on the alphabet V_T alone. We now consider an arbitrary but fixed recursively enumerable language L over V_T , generated by a type 0 grammar $G = (V_N, V_T, a_{u+1}, F)$. Denote $V = V_N \cup V_T$. We assume that V_N consists of a finite initial segment of (2) and the production set is

$$F = \{\alpha_i \rightarrow \beta_i \mid 1 \leq i \leq k\}.$$

We now define $K = K_1 \cap K_2$, where

$$\begin{aligned}
K_1 = & \{P_1 c_1 h_1(x_i) d_i P_2 c P_3 c_2 h_1(mi(\beta_i)) c_3 P_4 c \mid \\
& P_1, P_2, P_3, P_4 \in (h_1(V))^*, 1 \leq i \leq k\}^+ (V'_T)^*
\end{aligned}$$

and

$$\begin{aligned}
K_2 = & \{c_1 S d_i c \mid 1 \leq i \leq k\} \{P_1 c_2 P_2 c_3 P_3 c P_4 c_1 P_5 d_j P_6 c \mid \\
& P_1, P_2, P_3, P_4, P_5, P_6 \in (h_1(V))^*, 1 \leq j \leq k\}^* \\
& \{Q_1 c_2 Q_2 c_3 Q_3 c Q_4 \mid Q_4 \in W(V'_T); Q_1, Q_2, Q_3 \in U_T\}.
\end{aligned}$$

It is immediately verified that L_1 and L_2 are deterministic context-free languages over the alphabet V' . It is also obvious that K_1 and K_2 are denoted by star-free regular expressions (involving intersections and complements) and, consequently, K is a noncounting regular language. We shall now prove that (1) holds. For this purpose, we introduce two auxiliary languages L'_1 and L'_2 over the alphabet V' as follows:

$$\begin{aligned}
L'_1 = & \{P_1 h_1(x_i) d_i P_2 c \ mi(P_2) \ h_1(mi(\beta_i)) \ mi(P_1) c \mid \\
& 1 \leq i \leq k; P_1, P_2 \in (h_1(V))^* \}^+ (V'_T)^*
\end{aligned}$$

and

$$\begin{aligned}
L'_2 = & \{S d_i c \mid 1 \leq i \leq k\} \{mi(P_1 P_2) c P_1 d_j P_2 c \mid \\
& P_1, P_2 \in (h_1(V))^*, 1 \leq j \leq k\}^* \{Q_1 c Q_2 \mid \\
& Q_2 \in W(V'_T), Q_1 = h_1 h(mi(Q_2))\}.
\end{aligned}$$

We claim that

$$(3) \quad L = h(L'_1 \cap L'_2).$$

To prove the equation (3), we show first that the left side is included in the right side. Assume that $P \in L$. Since G generates L , there is an integer m , words $R_{j_1}, R_{j_2} \in W(V)$ and indices $g(j)$ with $1 \leq g(j) \leq k$, defined for all $j = 0, \dots, m$, such that the following conditions are satisfied. For every $j = 0, \dots, m-1$,

$$(4) \quad R_{j_1} \beta_{g(j)} R_{j_2} = R_{(j+1)_1} \alpha_{g(j+1)} R_{(j+1)_2}.$$

Furthermore,

$$(5) \quad R_{0_1} = R_{0_2} = \lambda, \quad \alpha_{g(0)} = a_{u+1}, \quad R_{m_1} \beta_{g(m)} R_{m_2} = P.$$

In other words, we consider the following derivation according to G :

$$\begin{aligned} a_{u+1} &= R_{0_1} \alpha_{g(0)} R_{0_2} \Rightarrow R_{0_1} \beta_{g(0)} R_{0_2} = R_{1_1} \alpha_{g(1)} R_{1_2} \\ &\Rightarrow R_{1_1} \beta_{g(1)} R_{1_2} = R_{2_1} \alpha_{g(2)} R_{2_2} \Rightarrow \dots \Rightarrow \\ R_{(m-1)_1} \beta_{g(m-1)} R_{(m-1)_2} &= R_{m_1} \alpha_{g(m)} R_{m_2} \Rightarrow R_{m_1} \beta_{g(m)} R_{m_2} = P. \end{aligned}$$

We now define, for any $P_1, P_2 \in W(V)$ and $1 \leq i \leq k$,

$$t(P_1, i, P_2) = h_1(P_1) h_1(\alpha_i d_i h_1(P_2)) c \, mi(h_1(P_2)) \, mi(h_1(\beta_i)) \, mi(h_1(P_1)) c$$

and consider the word

$$(6) \quad R = t(R_{0_1}, g(0), R_{0_2}) t(R_{1_1}, g(1), R_{1_2}) \dots t(R_{m_1}, g(m), R_{m_2}) P',$$

where P' is obtained from P by replacing every letter with the corresponding primed one. (Thus, $h(P') = P$.) By the definition of L'_1 , we have $R \in L'_1$. (Note that the operators mi and h_1 commute). Using the notation

$$s(P_1, i, P_2) = mi(h_1(P_1 P_2)) c h_1(P_1) d_i h_1(P_2) c,$$

we may also write (by (4) and (5))

$$\begin{aligned} R &= S d_{g(0)} c \, mi(h_1(R_{0_1} \beta_{g(0)} R_{0_2})) \, c h_1(R_{1_1}) h_1(\alpha_{g(1)}) d_{g(1)} \\ &\quad h_1(R_{1_2}) c \dots mi(h_1(R_{(m-1)_1} \beta_{g(m-1)} R_{(m-1)_2})) c \\ &\quad h_1(R_{m_1}) h_1(\alpha_{g(m)}) d_{g(m)} h_1(R_{m_2}) c \, mi(h_1(R_{m_1} \beta_{g(m)} R_{m_2})) c P' \\ &= S d_{g(0)} c \, mi(h_1(R_{1_1} \alpha_{g(1)} R_{1_2})) c h_1(R_{1_1}) h_1(\alpha_{g(1)}) d_{g(1)} \\ &\quad h_1(R_{1_2}) c \dots mi(h_1(R_{m_1} \alpha_{g(m)} R_{m_2})) c \\ &\quad h_1(R_{m_1}) h_1(\alpha_{g(m)}) d_{g(m)} h_1(R_{m_2}) c \, mi(h_1(R_{m_1} \beta_{g(m)} R_{m_2})) c P' \\ &= S d_{g(0)} c s(R_{1_1} \alpha_{g(1)}, g(1), R_{1_2}) \\ &\quad \dots s(R_{m_1} \alpha_{g(m)}, g(m), R_{m_2}) \, mi(h_1(P)) c P'. \end{aligned}$$

From the last expression we see that $R \in L'_2$ and, hence, $R \in L'_1 \cap L'_2$. On the other hand, $h(R) = P$. This implies that $P \in h(L'_1 \cap L'_2)$.

Having established that the left side of (3) is included in the right side, we now prove the reverse inclusion. Assume that $P \in h(L'_1 \cap L'_2)$. Consequently, there is a word $R \in L'_1 \cap L'_2$ such that $P = h(R)$. Since $R \in L'_1$, it can be expressed in the form (6), for some numbers m , $g(i)$ and words R_{ij} . (Recall that P' is obtained from P by replacing every letter with the corresponding primed one.) Since $R \in L'_2$, the word R can be expressed in the form

$$(7) \quad R = Sd_{f(0)}cs(Q_{11}, f(1), Q_{12}) \dots s(Q_{n1}, f(n), Q_{n2}) mi(h_1(P))cP',$$

for some numbers n , $f(i)$ and words Q_{ij} . Comparing the number of occurrences of the boundary marker c in (6) and (7), we see that $m = n$. It is also clear that $g(i) = f(i)$, for $i = 0, \dots, m$. A further comparison between (6) and (7) gives the equations

$$(8) \quad \begin{aligned} R_{01} = R_{02} = \lambda, \quad \alpha_{g(0)} = a_{u+1}, \quad R_{i1\alpha_{g(i)}} = Q_{i1}, \\ R_{i2} = Q_{i2}, \quad R_{(i-1)1\beta_{g(i-1)}}R_{(i-1)2} = Q_{i1}Q_{i2} = R_{i1\alpha_{g(i)}}R_{i2}, \end{aligned}$$

for $1 \leq i \leq m$, and also the equation

$$P = R_{m1\beta_{g(m)}}R_{m2}.$$

Thus, we obtain the following derivation according to G :

$$\begin{aligned} a_{u+1} &= R_{01\alpha_{g(0)}}R_{02} \Rightarrow R_{01\beta_{g(0)}}R_{02} = R_{11\alpha_{g(1)}}R_{12} \\ &\Rightarrow \dots \Rightarrow R_{(m-1)1\beta_{g(m-1)}}R_{(m-1)2} = R_{m1\alpha_{g(m)}}R_{m2} \\ &\Rightarrow R_{m1\beta_{g(m)}}R_{m2} = P. \end{aligned}$$

Therefore, $P \in L$. Thus, we have shown that the equation (3) is correct.

For a language L' , let $M(L')$ be the language obtained from L' by erasing from all words all occurrences of c_1 , c_2 and c_3 . (Note that M is not a homomorphism since c_i is a sequence of 0's and 1's. Note also that, for any language L' , $h(L') = h(M(L'))$.) Comparing the positions of the boundary markers c_i and the production indicators d_j , we obtain the equations

$$(9) \quad M(L_1 \cap K_1) = L'_1, \quad M(L_2 \cap K_2) = L'_2.$$

The inclusion

$$(10) \quad L'_1 \cap L'_2 \subseteq M(L_1 \cap L_2 \cap K_1 \cap K_2)$$

is established by (i) considering an arbitrary word R belonging to the left side, (ii) noting that R can be expressed in the forms (6) and (7) from which (8) can be inferred, (iii) inserting the markers c_1 , c_2 , c_3 in R at proper places which are immediately seen from the t -expressions, and (iv)

noting that the resulting word belongs to all four languages in the intersection on the right side. By (9) and (10),

$$M(L_1 \cap L_2 \cap K_1 \cap K_2) = L'_1 \cap L'_2 .$$

Hence,

$$\begin{aligned} (11) \quad h(L_1 \cap L_2 \cap K) &= h(M(L_1 \cap L_2 \cap K)) \\ &= h(M(L_1 \cap L_2 \cap K_1 \cap K_2)) = h(L'_1 \cap L'_2) . \end{aligned}$$

The equation (1) is now an immediate consequence of (3) and (11). This completes the proof.

We note that K is not, in general, locally testable. It does not seem likely that the construction could be modified to yield a locally testable language.

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