Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 4, 1978/1979, 41—44

A NOTE ON HURWITZ'S ZETA-FUNCTION

R. BALASUBRAMANIAN

The aim of this paper is to prove an asymptotic formula for the mean square of Hurwitz's zeta-function, defined by

$$\zeta(s;\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}\quad\text{in}\quad \operatorname{Re} s>1$$

and its analytic continuation in $\operatorname{Re} s\leqq 1$. The result is as follows:

Theorem. *If* $\zeta_1(s;\alpha)=\zeta(s;\alpha)-1/\alpha^s$, *then for* $t\geqq 30$,

$$\int_0^1|\zeta_1(1/2+it,\alpha)|^2\,d\alpha=\log t+O(\log\log t).$$

This improves the result of Koksma and Lekkerkerker [2] which states that

$$\int_0^1|\zeta_1(1/2+it,\alpha)|^2\,d\alpha=O(\log t).$$

It is very likely that the error term $O(\log\log t)$ of the theorem can be improved.

A modification of our proof gives an asymptotic formula for $\sum_\chi|L(s,\chi)|^2$, which improves a theorem of Gallagher [1] in some range of t. It will form the subject matter of another paper.

We prove the theorem by establishing six lemmas.

Lemma 1. *We have*

$$\zeta_1(1/2+it,\alpha)=\sum_{1\leqq n\leqq T}\frac{1}{(n+\alpha)^{1/2+it}}+O(t^{-1/2}),$$

where T is the nearest integer to t.

Proof. This is well known.

doi:10.5186/aasfm.1978-79.0401

Lemma 2. *We have*

$$\int_0^1 |\zeta_1(1/2+it,\alpha)|^2\,d\alpha = \log t + O(|J_1+J_2+J_3-J_4-J_5-J_6|)+O(1),$$

where

$$J_1 = \sum_{k\leqq T^{1/2}} \frac{(T-k+1)^{1/2}(T+1)^{1/2}e^{-it(\log(T-k+1)-\log(T+1))}}{tk},$$

$$J_2 = \sum_{T^{1/2}<k\leqq T^{1/2}\log^3 T} \frac{(T-k+1)^{1/2}(T+1)^{1/2}e^{-it(\log(T-k+1)-\log(T+1))}}{tk},$$

$$J_3 = \sum_{T^{1/2}\log^3 T<k\leqq T} \frac{(T-k+1)^{1/2}(T+1)^{1/2}e^{-it(\log(T-k+1)-\log(T+1))}}{tk},$$

$$J_4 = \sum_{k=1}^{T} \frac{(1+k)^{1/2}e^{it\log(1+k)}}{tk},$$

$$J_5 = \frac{1}{2}\sum_{k=1}^{T}\int_1^{T-k+1} \frac{v^{-1/2}(v+k)^{1/2}e^{-it(\log v-\log(v+k))}}{tk}\,dv,$$

and

$$J_6 = \frac{1}{2}\sum_{k=1}^{T}\int_1^{T-k+1} \frac{(v+k)^{-1/2}v^{1/2}\cdot e^{-it(\log v-\log(v+k))}}{tk}\,dv.$$

Proof. Using Lemma 1, we have

$$\int_0^1 |\zeta_1(1/2+it,\alpha)|^2\,d\alpha = \int_0^1 \sum_{n\leqq T}\sum_{m\leqq T} \frac{1}{(n+\alpha)^{1/2+it}}\frac{1}{(m+\alpha)^{1/2-it}}\,d\alpha + O(1)$$

and the terms corresponding to $m=n$ give the main term $\log t$, with an error $O(1)$. The terms $m\neq n$ give

$$\int_0^1 \sum_{\substack{n\leqq T\\ m\neq n}}\sum_{m\leqq T} \frac{1}{(n+\alpha)^{1/2+it}}\frac{1}{(m+\alpha)^{1/2-it}}\,d\alpha$$

$$= \int_0^1 \sum_{m\leqq T}\sum_{n<m} \frac{1}{(n+\alpha)^{1/2+it}}\frac{1}{(m+\alpha)^{1/2-it}}\,d\alpha$$

$$+ \int_0^1 \sum_{m\leqq T}\sum_{n>m} \frac{1}{(n+\alpha)^{1/2+it}}\frac{1}{(m+\alpha)^{1/2-it}}\,d\alpha.$$

It is sufficient to consider one of the terms. We obtain

$$\int_0^1 \sum_{m\le T}\sum_{n<m}\frac{1}{(n+\alpha)^{1/2+it}}\,\frac{1}{(m+\alpha)^{1/2-it}}\,d\alpha$$

$$=\sum_{k=1}^{T}\sum_{n=1}^{T-k}\int_0^1\frac{1}{(n+\alpha)^{1/2+it}}\,\frac{1}{(n+k+\alpha)^{1/2-it}}\,d\alpha$$

$$=\sum_{k=1}^{T}\sum_{n=1}^{T-k}\int_n^{n+1}\frac{1}{v^{1/2+it}(v+k)^{1/2-it}}\,dv$$

$$=\sum_{k=1}^{T}\int_1^{T-k+1}\frac{1}{v^{1/2}(v+k)^{1/2}}\,e^{-it(\log v-\log(v+k))}\,dv$$

$$=-i\sum_{k=1}^{T}\frac{1}{kt}\int_1^{T-k+1}v^{1/2}(v+k)^{1/2}\,d\left(e^{-it(\log v-\log(v+k))}\right)$$

$$=-i\sum_{k=1}^{T}\frac{1}{kt}\left[v^{1/2}(v+k)^{1/2}e^{-it(\log v-\log(v+k))}\right]_1^{T-k+1}$$

$$+i\sum_{k=1}^{T}\frac{1}{kt}\int_1^{T-k+1}\left((1/2)v^{-1/2}(v+k)^{1/2}+(1/2)(v+k)^{-1/2}v^{1/2}\right)e^{-it(\log v-\log(v+k))}\,dv$$

$$=-i(J_1+J_2+J_3-J_4-J_5-J_6).$$

This proves the lemma.

Lemma 3. *If J_4 and J_2 are as defined in Lemma 2, then,*

$$J_4=O(1),\quad J_2=O(\log\log T).$$

Proof. Trivial.

Lemma 4. *If J_5 and J_6 are as defined in Lemma 2, then*

$$J_5=O(1),\quad J_6=O(1).$$

Proof. The result follows, using Lemma 4.3 (p. 61) of Titchmarsh [3].

Lemma 5. *If J_1 is as defined in Lemma 2, then* $J_1=O(1)$.

Proof. In J_1, $\log(T-k+1)-\log(T+1)$ can be replaced by $-k/(T+1)$ with a small error. Hence

$$J_1=\sum_{k\le T^{1/2}}\frac{(T-k+1)^{1/2}(T+1)^{1/2}\cdot e^{itk/(T+1)}}{tk}+O(1).$$

Since the partial sums of $\sum_k e^{itk/(T+1)}$ are bounded, the result follows from Abel's partial summation formula.

LEMMA 6. *If J_3 is as defined in Lemma 2, then*

$$J_3 = O(1).$$

Proof. We apply Theorem 5.9 (p. 90) of Titchmarsh [3] to get a good bound for the sums

$$\sum_{X \leqq k \leqq Y} e^{-it \log (T-k+1)}$$

where $Y \leqq X$; and $T^{1/2} \log^3 T \leqq X \leqq T/100$, and use Abel's partial summation formula to prove that

$$\sum_{T^{1/2}\log^3 T < k \leqq T/100} \frac{(T-k+1)^{1/2}(T+1)^{1/2} e^{-it(\log(T-k+1) - \log(T+1))}}{tk}$$

is $O(1)$. We observe that

$$\sum_{T/100 < k \leqq T} \frac{(T-k+1)^{1/2}(T+1)^{1/2} e^{-it(\log(T-k+1) - \log(T+1))}}{tk}$$

is $O(1)$. This proves the lemma.

The theorem follows from Lemmas 2 to 6.

Acknowledgement. The author wishes to express his thanks to Prof. K. Ramachandra for his encouragement and for checking the manuscript.

References

[1] GALLAGHER, P. X.: Local mean value and density estimates for Dirichlet *L*-functions. - Indag. Math. 37, 1975, 259—264.

[2] KOKSMA, J. F., and C. G. LEKKERKERKER: A mean-value theorem for $\zeta(s, w)$. - Indag. Math. 14, 1952, 446—452.

[3] TITCHMARSH, E. C.: The theory of the Riemann zeta-function. - Clarendon Press, Oxford, 1951.

Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005
India

Received 21 February 1978