Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 4, 1978/1979, 75—83

DIFFERENTIABLE TCHEBYCHEFF SUBSPACES AND HERMITE INTERPOLATION

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1. Introduction

Let $m, n \in \boldsymbol{N}$, $1 \leqq m \leqq n$, and let $E_n = (\varepsilon_{\mu\nu})_{1 \leqq \mu \leqq m,\, 0 \leqq \nu \leqq n-1}$ be an *incidence matrix* (cf. Schoenberg [12], Mäkelä—Nevanlinna—Sipilä [8]), i.e. $\varepsilon_{\mu\nu} = 0$ or 1, and $\sum_{\mu,\nu} \varepsilon_{\mu\nu} = n$. Suppose, for simplicity, that no row is composed only of zeros. Given an interval $I \subset \boldsymbol{R}$ with nonvoid interior $\mathring{I}$, interpolation nodes $t_1 < t_2 < \ldots < t_m$ in I, and an n-dimensional subspace $U \subset C^r(I)$, $r \in \boldsymbol{N}_0 := \boldsymbol{N} \cup \{0\}$, then the incidence matrix E_n gives rise to the following *interpolation problem:* Does there exist a $u \in U$ satisfying

$$u^{(\nu)}(t_\mu) = a_{\mu\nu} \tag{1}$$

for all (μ, ν) such that $\varepsilon_{\mu\nu} = 1$ with arbitrarily given data $a_{\mu\nu} \in \boldsymbol{R}$? The problem of unique solvability of this Birkhoff type interpolation problem has been treated by several authors for the case $U = \Pi_{n-1}$ (polynomials of degree not exceeding $n-1$), see e.g. the surveys of Sharma [13] or Lorentz [7].

In a recent paper, Mäkelä—Nevanlinna—Sipilä [8] put the following question: Given a fixed type of incidence matrix E_n, what are the properties of $U \subset C^r(I)$ necessary and sufficient that the interpolation problem (1) be uniquely solvable (with respect to certain special or arbitrary nodes)? They considered various types of spaces U generalizing results of Matthews [9], Ikebe [3] (the latter treated general Birkhoff interpolation problems) as well as of [1]. In most of their results Mäkelä—Nevanlinna—Sipilä [8] make use of *"polynomial like"* spaces U for which the dimension of U is reduced by differentiation.

It is the topic of the present paper to investigate those n-dimensional subspaces $U \subset C^r(I)$, $r \in \boldsymbol{N}$, whose *dimension is not necessarily reduced by differentiation* in order to get results for Hermite interpolation incidence matrices. In [1] we showed that the *osculatory Hermite interpolation problem* is uniquely solvable with respect to differentiable Tchebycheff subspaces. When proving this, one of the main problems was that — because of the lack of sufficient differentiability — the usual multiplicity notion for zeros of a function does not work. Thus we have to use a certain *multi-*

doi:10.5186/aasfm.1978-79.0408

plicity notion for r-times differentiable functions in order to treat the general Hermite interpolation in $C^r(I)$ here (cf. Mäkelä—Nevanlinna—Sipilä [8]). Then we give a characterization of Hermite subspaces which enables us to prove the main theorem on Hermite interpolation in Section 3. Some of these results were announced in [2]. In addition we can characterize *weakly differentiable Tchebycheff subspaces* $U\subset C^r(I)$ of dimension n by means of a certain class of incidence matrices.

Other investigations on Hermite interpolation by nonpolynomials are due to Polya [10] and Karlin—Studden [6]. Some results on Birkhoff interpolation with respect to nonpolynomials go back to Karlin—Karon [4, 5].

2. A characterization of Hermite subspaces

In order to prove the results in Section 3 we need the notion of a *Hermite subspace* as well as a characterization of Hermite spaces which is established in this section.

Let $n\in\mathbf{N}$, $r\in\mathbf{N}_0$, and $I\subset\mathbf{R}$ an interval ($\mathring{I}\neq\emptyset$). An n-dimensional subspace $U\subset C^r(I)$ is called a *Hermite subspace of* $C^r(I)$ provided that for any $m\in\mathbf{N}$, $m\leqq n$, any $(\alpha_1, \ldots, \alpha_m)\in\mathbf{N}_0^m$ satisfying $\max_{1\leqq\mu\leqq m}\alpha_\mu\leqq r$ and $\sum_{1\leqq\mu\leqq m}(\alpha_\mu+1)=n$, and arbitrary interpolation nodes $t_1<t_2<\ldots<t_m$ in I the following Hermite interpolation problem is uniquely solvable:

Given any $a_{\mu\tau}\in\mathbf{R}$ $(0\leqq\tau\leqq\alpha_\mu,\ 1\leqq\mu\leqq m)$, does there exist a $u\in U$ satisfying

$$u^{(\tau)}(t_\mu) = a_{\mu\tau} \quad (0 \leqq \tau \leqq \alpha_\mu,\ 1 \leqq \mu \leqq m)? \tag{2}$$

The set of all n-dimensional Hermite subspaces of $C^r(I)$ will be denoted by $\mathscr{H}_n^r(I)$.

In order to characterize Hermite subspaces we need the following *notion of multiplicity for zeros* of functions in $C^r(I)$ which will be defined inductively (see Mäkelä—Nevanlinna—Sipilä [8] and [2]):

(i) Let $x\in C^r(I)$, $t\in I$, then define

$$z_0(x, t) := \begin{cases} 1 & \text{if} \quad x(t) = 0 \\ 0 & \text{if} \quad x(t) \neq 0 \end{cases}$$

and, for $1\leqq\varrho\leqq r$:

$$z_\varrho(x, t) := \begin{cases} z_{\varrho-1}(x', t)+1 & \text{if} \quad x(t) = 0 \\ 0 & \text{if} \quad x(t) \neq 0. \end{cases}$$

Then $\mathscr{Z}_r(x):=\sum_{t\in I} z_r(x, t)$ is the number of all zeros of $x\in C^r(I)$ where multiple zeros are counted according to this (weak) multiplicity notion.

(ii) In order to count *multiplicities of zeros in a strict sense,* let again $x \in C^r(I)$, $t \in I$. Then define (cf. Rice [11])

$$\tilde{z}_0(x,t) := \begin{cases} 2 & \text{if } x(t)=0,\ t\in \mathring{I}, \text{ and } t \text{ is an isolated zero where } x \text{ does not change sign} \\ 1 & \text{if } x(t)=0, \text{ and } \tilde{z}_0(x,t)\neq 2 \\ 0 & \text{if } x(t)\neq 0 \end{cases}$$

and, for $1 \leqq \varrho \leqq r$:

$$\tilde{z}_\varrho(x,t) := \begin{cases} \tilde{z}_{\varrho-1}(x',t)+1 & \text{if } x(t)=0 \\ 0 & \text{if } x(t)\neq 0. \end{cases}$$

The number of all zeros counted with this strict multiplicity notion will be denoted by $\tilde{\mathscr{Z}}_r(x) := \sum_{t\in I} \tilde{z}_r(x,t)$.

Note that we have $0 \leqq z_r(x,t) \leqq r+1$, $0 \leqq \tilde{z}_r(x,t) \leqq r+2$ for $x \in C^r(I)$ and $t \in I$. In addition, for $\tilde{z}_r(x,t) \leqq r+1$, we have $\tilde{z}_r(x,t) = z_r(x,t)$.

With these preparations we are able to prove the following theorem without the use of determinants by *interpolation theoretical means* only:

Theorem 1. *Let $U \subset C^r(I)$ be an n-dimensional subspace ($r \in \mathbf{N}_0$, $n \in \mathbf{N}$, $I \subset \mathbf{R}$ an interval with nonvoid interior). Then these assertions are equivalent:*

(i) *U is an n-dimensional Hermite subspace of $C^r(I)$,*

(ii) *For any $u \in U$, $u \neq 0$, we have: $\mathscr{Z}_r(u) \leqq n-1$,*

(iii) *For any $u \in U$, $u \neq 0$, we have: $\tilde{\mathscr{Z}}_r(u) \leqq n-1$.*

Proof. (i)⇒(ii): Let U be an n-dimensional subspace of $C^r(I)$, and suppose there exists a $u_0 \in U$, $u_0 \neq 0$, such that $\mathscr{Z}_r(u_0) \geqq n$. We can assume that u_0 possesses only a finite number of zeros $t_1, \dots, t_k \in I$ ($k \leqq n-1$); otherwise u_0 would be a nontrivial solution of the Lagrange interpolation problem $u(s_\nu)=0$ for n or more points $s_\nu \in I$.

Thus let $z_r(u_0, t_\varkappa) = \beta_\varkappa + 1$ ($1 \leqq \varkappa \leqq k$), and define

$$m := \min \Big\{ l \in \mathbf{N} \colon \sum_{1 \leqq \lambda \leqq l} z_r(u_0, t_\lambda) \geqq n \Big\}.$$

Now put $\alpha_\mu = \beta_\mu$ ($1 \leqq \mu \leqq m-1$), and $\alpha_m = n - \sum_{1 \leqq \mu \leqq m-1} z_r(u_0, t_\mu) - 1$. Then the Hermite interpolation problem

$$u^{(\tau)}(t_\mu) = 0 \quad (0 \leqq \tau \leqq \alpha_\mu,\ 1 \leqq \mu \leqq m)$$

has the nontrivial solution u_0 which contradicts the fact that a Hermite subspace yields a unique solution of any Hermite interpolation problem involving derivative conditions up to order r.

(ii)⇒(i): Now suppose $\mathscr{Z}_r(u) \leqq n-1$ for any $u \in U$, $u \neq 0$, and let an interpolation problem of type (2) with nodes $t_1 < t_2 < \dots < t_m$ in I be given. It suffices to show that the homogeneous problem corresponding to (2) has the trivial solution only.

Any solution $u \in U$ of this problem possesses zeros of weak multiplicities $\geqq \alpha_\mu + 1$ at t_μ, thus we have

$$\sum_{t \in I} z_r(u, t) \geqq \sum_{1 \leqq \mu \leqq m} z_r(u, t_\mu) \geqq \sum_{1 \leqq \mu \leqq m} (\alpha_\mu + 1) = n,$$

hence $u=0$ by (ii). This yields (i).

Since (iii)$\Rightarrow$(ii) is obvious, we only have to prove the converse direction. It turns out to be convenient to show that the equivalent statements (i) and (ii) together imply (iii).

(i)+(ii)$\Rightarrow$(iii): Assume there is a $u_0 \in U$, $u_0 \neq 0$, such that $\tilde{\mathscr{Z}}_r(u_0) \geqq n$. Since (ii) holds, u_0 has at most $k \leqq n-1$ zeros $t_1 < t_2 < \ldots < t_k$ in I. The assertion (ii) implies, too, that there is at least one of the t_μ's, say t_{μ_0}, such that $\tilde{z}_r(u_0, t_{\mu_0}) = r+2$, which means that

$$M' := \{\varkappa\colon 1 \leqq \varkappa \leqq k,\ \tilde{z}_r(u_0, t_\varkappa) = r+2\} \neq \emptyset.$$

Define

$$\sigma_\varkappa := \operatorname*{sign}_{t \in U(t_\varkappa)} u_0^{(r)}(t) \quad \text{for} \quad \varkappa \in M'$$

where $U(t_\varkappa) := \{t \in I\colon t_\varkappa - \varepsilon < t < t_\varkappa + \varepsilon\} \setminus \{t_\varkappa\}$ with a sufficiently small $\varepsilon > 0$ so that $\sigma_\varkappa$ is well defined (i.e. the restriction of $u_0^{(r)}$ to $U(t_\varkappa)$ has constant sign). Let $M := \{1, \ldots, k\} \setminus M'$, and define $v_0 \in U$ as the unique solution of the following Hermite interpolation problem (according to (i)):

$$\begin{aligned} v_0^{(\alpha)}(t_\varkappa) &= 0 && \text{for} \quad \varkappa \in M, && 0 \leqq \alpha \leqq z_r(u_0, t_\varkappa) - 1, \\ v_0^{(\alpha)}(t_\varkappa) &= 0 && \text{for} \quad \varkappa \in M', && 0 \leqq \alpha \leqq r-1, \\ v_0^{(r)}(t_\varkappa) &= -\sigma_\varkappa && \text{for} \quad \varkappa \in M', \end{aligned}$$

and, since $q := \sum_{\varkappa \in M} z_r(u_0, t_\varkappa) + \sum_{\varkappa \in M'} (r+1) < n$, choose $p := n - q$ points s_π different from each other and from the $t_\varkappa$'s, and complete the interpolation conditions above by

$$v_0(s_\pi) = 0 \quad \text{for} \quad 1 \leqq \pi \leqq p.$$

Since $M' \neq \emptyset$, we have $v_0 \neq 0$. Now we consider $w_0 := u_0 + \eta v_0$ with $\eta > 0$ sufficiently small, thus $w_0 \neq 0$. By construction, we have

$$z_r(w_0, t_\varkappa) \geqq z_r(u_0, t_\varkappa) \quad \text{for} \quad \varkappa \in M$$

and

$$z_r(w_0, t_\varkappa) \geqq r \qquad \text{for} \quad \varkappa \in M',$$

and with a sufficiently small $\eta > 0$ for any $\varkappa \in M'$ there are points $\tau_\varkappa^+$ and $\tau_\varkappa^-$ (satisfying $\tau_\varkappa^- < t_\varkappa < \tau_\varkappa^+$) in a neighbourhood of $t_\varkappa$ such that $\{t\colon \tau_\varkappa^- < t < \tau_\varkappa^+\} \not\ni t_\mu, \tau_\mu^+, \tau_\mu^-, s_\pi$ for $1 \leqq \mu \leqq k$, $\mu \neq \varkappa$, and $1 \leqq \pi \leqq p$, satisfying

$$z_r(w_0, \tau_\varkappa^+) \geqq 1, \quad z_r(w_0, \tau_\varkappa^-) \geqq 1.$$

Hence we get

$$\mathscr{Z}_r(w_0) \geqq \sum_{\varkappa \in M} z_r(w_0, t_\varkappa) + \sum_{\varkappa \in M'} (r+2) = \tilde{\mathscr{Z}}_r(u_0) \geqq n$$

while $w_0 \neq 0$. Since $w_0 \in U$, this contradicts (ii). Thus $\tilde{\mathscr{Z}}_r(u) \leqq n-1$ for all $u \in U$, $u \neq 0$. □

We remark that Theorem 1 also can be obtained with the aid of results due to Mäkelä—Nevanlinna—Sipilä [8] using a characterization of Hermite subspaces *in terms of determinant conditions.*

If we assume U being an n-dimensional space of sufficiently differentiable functions (and not only $U \subset C^r(I)$) then Theorem 1 reduces to a result of Karlin—Studden [6].

3. Differentiable Tchebycheff subspaces

We are going to prove sufficient criteria to guarantee unique Hermite interpolation within the framework of *differentiable Tchebycheff subspaces.* The latter ones will be defined by properties of the number of zeros of a function without using multiplicities of zeros. But when proving the existence and uniqueness Theorem 3 the multiplicities introduced in Section 2 will play an important role.

Given a subspace $U \subset C^r(I)$, $I \subset \boldsymbol{R}$ an interval satisfying $\mathring{I} \neq \emptyset$, we define $U^{(\varrho)} := \{w : w$ is derivative of order ϱ of some $u \in U\}$ for $1 \leqq \varrho \leqq r \in \boldsymbol{N}$. In the case $\varrho = 0$ we have $U^{(0)} := U$. Finally, let $n_\varrho := \dim U^{(\varrho)}$ for $1 \leqq \varrho \leqq r$, and $n_0 := n = \dim U$.

Definition 2. Let $n, r \in \boldsymbol{N}$, $I \subset \boldsymbol{R}$ an interval with nonvoid interior, and $U \subset C^r(I)$ an n-dimensional subspace.

(i) U is called an *r-times differentiable Tchebycheff subspace of dimension n* if for $0 \leqq \varrho \leqq r$ the following condition holds: Any $u \in U^{(\varrho)}$, $u \neq 0$, possesses at most $n_\varrho - 1$ zeros in I (without counting multiplicities).

(ii) An r-times differentiable Tchebycheff subspace U of dimension n is called

α) an *r-times strictly differentiable Tchebycheff subspace* if

$$r \leqq n \quad \text{and} \quad \dim U^{(\varrho)} = n - \varrho \quad (0 \leqq \varrho \leqq r),$$

β) an *r-times weakly differentiable Tchebycheff subspace* if

$$\dim U^{(\varrho)} = n \quad (0 \leqq \varrho \leqq r).$$

In order to describe certain interpolation problems we are going to introduce incidence matrices of type $\mathscr{T}_n(s, r)$. Given an incidence matrix $E_n = (\varepsilon_{\mu\nu})_{1 \leqq \mu \leqq m,\, 0 \leqq \nu \leqq n-1}$ (see Section 1), and $s \in \boldsymbol{N}_0$, $r \in \boldsymbol{N}$ such that $0 \leqq s \leqq r \leqq n-1$, then we put

$$m_\nu := \{\mu : \varepsilon_{\mu\nu} = 1 \ (1 \leqq \mu \leqq m)\} \quad \text{for} \quad 0 \leqq \nu \leqq n-1.$$

An incidence matrix E_n (or the corresponding interpolation problem with respect to some interpolation nodes t_μ $(1 \leqq \mu \leqq m)$) will be called of *type* $\mathscr{T}_n(s, r)$ if the following conditions hold:

$$m_0 \quad = m_1 \quad = \ldots = m_{s-1} = \emptyset,$$
$$m_s \quad \supset m_{s+1} \supset \ldots \supset m_r, \quad m_s \neq \emptyset,$$
$$m_{r+1} = m_{r+2} = \ldots = m_{n-1} = \emptyset.$$

For $s=0$ we have a *Hermite interpolation incidence matrix,* whereas $s>0$ yields the incidence matrix of a *"shifted" Hermite problem.*

We are now coming to state the main result of this paper, where we shall use the following notation. Let E_n be an incidence matrix, and $t_1<t_2<\ldots<t_m$ interpolation nodes in the interval $I\subset \mathbf{R}$. Then we define

$$M_\nu := \{t_\mu\colon \varepsilon_{\mu\nu} = 1 \ (1 \leqq \mu \leqq m)\} \quad \text{for} \quad 0 \leqq \nu \leqq n-1.$$

Thus in the case of an incidence matrix of type $\mathcal{T}_n(0, r)$, $0\leqq r\leqq n-1$, we have

$$\emptyset \neq M_0 \supset M_1 \supset \ldots \supset M_r, \quad M_{r+1} = \ldots = M_{n-1} = \emptyset.$$

Theorem 3. *Let $n\in \mathbf{N}$, $r\in \mathbf{N}_0$, and $U\subset C^r(I)$ be an n-dimensional subspace ($I\subset \mathbf{R}$ an interval satisfying $\mathring{I}\neq\emptyset$). Suppose one of the following conditions holds:*

(i) *U is an r-times strictly differentiable Tchebycheff subspace of $C^r(I)$.*

(ii) *U is an r-times differentiable Tchebycheff subspace, and we have $M_r\subset \mathring{I}$.*

(iii) *U is obtained as the restriction to I of an r-times differentiable Tchebycheff subspace $\tilde{U}\subset C^r(J)$ of dimension n where $I\subset \mathring{J}$.*

Then in either case U is an n-dimensional Hermite subspace of $C^r(I)$.

Proof. (i) According to Theorem 1 we have to show (by induction) that if U is an r-times strictly differentiable Tchebycheff subspace of dimension n then for any $u\in U$, $u\neq 0$, we have $\mathcal{Z}_r(u)\leqq n-1$.

The case $r=0$ is obvious. Thus let $r>0$, and suppose that the foregoing implication holds true for any ϱ-times strictly differentiable Tchebycheff subspace $(0\leqq\varrho\leqq r-1)$ of any fixed dimension $d\in \mathbf{N}$ satisfying $\varrho\leqq d$.

Let U be an r-times strictly differentiable Tchebycheff subspace of dimension n, and suppose there exists a $u\in U$, $u\neq 0$, satisfying $\mathcal{Z}_r(u)=p\geqq n$. Then u possesses only a finite number of zeros, say $t_1, \ldots, t_l$, in I. Introducing $\beta_\lambda := z_r(u, t_\lambda)$ for $1\leqq\lambda\leqq l$, we have $p=\sum_{1\leqq\lambda\leqq l}\beta_\lambda$. By definition of the weak multiplicity of zeros, the derivative u' satisfies

$$z_{r-1}(u', t_\lambda) = \beta_\lambda - 1 \quad (1 \leqq \lambda \leqq l).$$

In addition, Rolle's theorem yields the existence of $l-1$ further zeros in the open intervals $]t_\lambda, t_{\lambda+1}[$ for $1\leqq\lambda\leqq l-1$. Thus

$$\mathcal{Z}_{r-1}(u') \geqq \sum_{1\leqq\lambda\leqq l} (\beta_\lambda - 1) + l - 1 = p - 1 \geqq n-1.$$

Now $U^{(1)}$ is an $(r-1)$-times strictly differentiable Tchebycheff subspace of dimension $n-1$ which yields $u'=0$, hence $u=0$, which is a contradiction. Thus $U\in\mathcal{H}_n^r(I)$.

The result that condition (i) implies U being an n-dimensional Hermite subspace of $C^r(I)$ is covered by a theorem of Ikebe [3].

(ii) Again, we prove the assertion by induction with respect to r. The case $r=0$ is immediate. Thus assume $r>0$, and suppose the statement holds true for $0\leqq\varrho\leqq r-1$ and arbitrary dimension of U.

Now suppose U is an r-times differentiable Tchebycheff subspace of dimension n. Let m interpolation nodes $t_1<t_2<\ldots<t_m$ in I be given as well as nonnegative integers α_μ $(1\leqq\mu\leqq m)$ satisfying $\max_{1\leqq\mu\leqq m}\alpha_\mu\leqq r$ and $\sum_{1\leqq\mu\leqq m}(\alpha_\mu+1)=n$. It is sufficient to show that the homogeneous interpolation problem

$$u^{(\tau)}(t_\mu)=0 \quad (0\leqq\tau\leqq\alpha_\mu,\ 1\leqq\mu\leqq m) \tag{3}$$

has the trivial solution $u_0=0$ only.

For any solution u of (3) we have $z_r(u,t_\mu)\geqq\alpha_\mu+1$. Since u vanishes at $t_1,\ldots,t_m$ in I, by Rolle's theorem, u' possesses $m-1$ further zeros $\tau_\mu\in]t_\mu,t_{\mu+1}[$ $(1\leqq\mu\leqq m-1)$. Thus we have

$$\tilde{\mathscr{Z}}_{r-1}(u')\geqq\sum_{1\leqq\mu\leqq m}\alpha_\mu+(m-1)=n-1.$$

Now we have to consider two cases:

(α) If $\dim U^{(1)}=n-1$, then as in the proof of (i) we have $u=0$.

(β) Thus suppose $\dim U^{(1)}=n$. If there is a $t_{\mu_0}\in M_r$ $(\subset\mathring{I}$ by hypothesis) such that $\tilde{z}_{r-1}(u',t_{\mu_0})=r+1$, then we have

$$\tilde{\mathscr{Z}}_{r-1}(u')\geqq\sum_{\substack{\mu=1\\ \mu\neq\mu_0}}^{m}\alpha_\mu+(m-1)+(r+1)\geqq n$$

since $\alpha_{\mu_0}=r$. By Theorem 1 and the induction hypothesis we have $u'=0$, hence $u=0$. Therefore we can assume that for any $t_\mu\in M_r$ we have

$$\tilde{z}_{r-1}(u',t_\mu)=r.$$

Hence

$$\tilde{z}_r(u,t_\mu)=r+1,$$

which (since $M_r\subset\mathring{I}$) implies that

$$\tilde{z}_{r-1}(u,t_\mu)=r+1 \quad (t_\mu\in M_r).$$

In addition, for all $1\leqq\mu\leqq m$

$$\tilde{z}_{r-1}(u,t_\mu)\geqq\alpha_\mu+1$$

holds true. From this we get

$$\tilde{\mathscr{Z}}_{r-1}(u)=\sum_{1\leqq\mu\leqq m}\tilde{z}_{r-1}(u,t_\mu)\geqq\sum_{t_\mu\notin M_r}(\alpha_\mu+1)+\sum_{t_\mu\in M_r}(r+1)=\sum_{1\leqq\mu\leqq m}(\alpha_\mu+1)=n.$$

Hence $u=0$ by induction hypothesis and Theorem 1.

(iii) In this case it follows that $M_r\subset I\subset\mathring{J}$. Thus (ii) is applicable for $\tilde{U}$ instead of U. Hence there exists a unique $\tilde{u}\in\tilde{U}$ which solves the given interpolation problem of type $\mathcal{T}_n(0, r)$. Its restriction to I, $u := \tilde{u}|I$, is the unique solution of (3) with respect to U. □

Theorem 3 yields the following generalization of a theorem of Mäkelä—Nevanlinna—Sipilä [8]:

Corollary. *Let U be an n-dimensional subspace of $C^r(I)$, I an open real interval. Suppose $U, U^{(1)}, \dots, U^{(r)}$ provide a unique solution for any interpolation problem of type $\mathcal{T}_n(0, 0)$. Then any interpolation problem of type $\mathcal{T}_n(0, r)$ possesses a unique solution with respect to U.*

4. A characterization of r-times weakly differentiable Tchebycheff subspaces

The following *interpolation theoretical characterization of Tchebycheff subspaces* of $C(I)$ is well known (it is an immediate consequence of Theorem 1 for $r=0$):

An n-dimensional subspace $U \subset C(I)$, I a nontrivial interval, is a Tchebycheff subspace if and only if any interpolation problem of type $\mathcal{T}_n(0, 0)$ (i.e. Lagrange interpolation problem) with nodes in I possesses a unique solution.

Now we are going to consider a corresponding *characterization of r-times weakly differentiable Tchebycheff subspaces* defined on an open interval. The interpolation problems which are needed for this characterization turn out to be of *Birkhoff type* (see e.g. Schoenberg [12]).

Theorem 4. *Let $U \subset C^r(I)$ be an n-dimensional subspace, $I \subset \mathbf{R}$ an open interval. U is an r-times weakly differentiable Tchebycheff subspace if and only if any interpolation problem of type $\mathcal{T}_n(s, r)$ possesses a unique solution with respect to U for any $0 \leqq s \leqq r$.*

Proof. Let U be an r-times weakly differentiable Tchebycheff subspace of dimension n. Then for $0 \leqq s \leqq r$ the spaces $U^{(s)}$ are $(r-s)$-times weakly differentiable Tchebycheff subspaces of dimension n. By Theorem 3, any interpolation problem of type $\mathcal{T}_n(0, r-s)$ possesses a unique solution with respect to $U^{(s)}$ given any nodes $t_1 < t_2 < \dots < t_m$ in I and arbitrary interpolation data. This means that any interpolation problem of type $\mathcal{T}_n(s, r)$ has a unique solution with respect to U for $0 \leqq s \leqq r$.

Conversely, let any interpolation problem of type $\mathcal{T}_n(s, r)$ be uniquely solvable with respect to U $(0 \leqq s \leqq r)$. Then, in particular, any *shifted Lagrange interpolation* problem

$$u^{(s)}(t_\nu) = a_\nu \quad (1 \leqq \nu \leqq n)$$

possesses a unique solution with respect to U, hence $U^{(s)}$ is an n-dimensional Tchebycheff subspace for $0 \leqq s \leqq r$ which follows from Theorem 1 and Definition 2. □

Examples of r-times weakly differentiable Tchebycheff subspaces are given by certain *families of exponential functions* (see [2]).

References

[1] Haussmann, W.: On interpolation with derivatives. - SIAM J. Numer. Anal. 8, 1971, 483—485.

[2] Haussmann, W.: Hermite-Interpolation mit Čebyšev-Unterräumen. - Numerische Methoden der Approximationstheorie 1, Internat. Schriftenreihe zur Numer. Math. 16, 49—55, Birkhäuser, Basel—Stuttgart, 1972.

[3] Ikebe, Y.: Hermite—Birkhoff interpolation problems in Haar subspaces. - J. Approximation Theory 8, 1973, 142—149.

[4] Karlin, S., and J. M. Karon: Poised and non-poised Hermite—Birkhoff interpolation. - Indiana Univ. Math. J. 21, 1972, 1131—1170.

[5] Karlin, S., and J. M. Karon: On Hermite—Birkhoff interpolation. - J. Approximation Theory 6, 1972, 90—115.

[6] Karlin, S., and W. J. Studden: Tchebycheff systems: with applications in analysis and statistics. - Interscience Publishers, New York—London—Sydney, 1966.

[7] Lorentz, G. G.: Birkhoff interpolation problem. - University of Texas, Center for Numerical Analysis, Report CNA-103, Austin, 1975.

[8] Mäkelä, M., O. Nevanlinna, and A. H. Sipilä: On some generalized Hermite—Birkhoff interpolation problems. - Ann. Acad. Sci. Fenn. Ser. A I 563, 1974, 1—13.

[9] Matthews, J. W.: Interpolation with derivatives. - SIAM Rev. 12, 1970, 127—128.

[10] Polya, G.: On the mean value theorem corresponding to a given linear homogeneous differential equation. - Trans. Amer. Math. Soc. 24, 1922, 312—324.

[11] Rice, J. R.: The approximation of functions. Vol. 1. Linear theory. - Addison-Wesley, Reading, Mass.—Palo Alto—London, 1964.

[12] Schoenberg, I. J.: On Hermite—Birkhoff interpolation. - J. Math. Anal. Appl. 16, 1966, 538—543.

[13] Sharma, A.: Some poised and nonpoised problems of interpolation. - SIAM Rev. 14, 1972, 129—151.

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Received 12 December 1977