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A CONFORMAL SELF-MAP WHICH FIXES THREE POINTS IS THE IDENTITY

ERNST PESCHL and MATTI LEHTINEN

Recently I. Lieb raised the question of fixed points of conformal self-maps of plane domains with arbitrary connectivity. He conjectured the following

Theorem. *Let A be a plane domain and f: $A \to A$ a conformal map. If f has three fixed points, then f is the identity map.*

In fact, this theorem is an immediate consequence of a result by B. Maskit [2], which states that one may associate with A another domain A', conformally equivalent to A, such that all conformal self-maps of A' are Möbius transformations. Maskit's proof is rather elaborate, as it depends heavily on the structure of the group of conformal self-maps of A. In this note we attempt to give a straightforward and relatively short proof of the theorem.

We first remark that there exist domains of arbitrarily high connectivity allowing conformal self-maps with two fixed points, other than the identity. As an example one may consider any domain A obtained from the plane when the origin and an arbitrary closed set of points on the unit circle, symmetric with respect to the real axis and not containing the points -1 and 1, are deleted. Then f, $f(z)=1/z$, maps A onto itself and fixes -1 and 1.

If A is simply or doubly connected, consideration of all possible standard domains and their conformal self-maps [3, pp. 226—236] shows that the theorem is true in this case. In the sequel, therefore, we may assume that A is at least triply connected. The unit disc D can then be taken to be the universal covering surface of A, and the cover transformation group G of D relative to A is a non-elementary properly discontinuous group of Möbius transformations.

In what follows we let f be a fixed conformal self-map of A and assume that $f(w_0)=w_0$ for some w_0 in A. Denote the projection map of D onto A by h; there is no loss of generality in assuming $h(0)=w_0$. Also, we may fix a lifting $\tilde{f}$: $D \to D$ of f such that $\tilde{f}(0)=0$. Denote the n-th iterate of $\tilde{f}$ by $\tilde{f}^n$. We then have

Lemma. *There exists a natural number p such that $\tilde{f}^p = \mathrm{id}$.*

Proof. Evidently $\tilde{f}(z)=\exp(ia)z$ for some real a. Choose a g_0 in G, $g_0 \neq \mathrm{id}$. Then $g_n = \tilde{f}^n \circ g_0 \circ (\tilde{f}^n)^{-1}$ is in G. If the number of distinct g_n's is infinite, a sequence (g_{n_j}) converges to a conformal g. The discontinuity of G makes this impossible.

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Consequently $g_n = g_{n+p}$ for some n and $p \geqq 1$. In particular, then, $\exp(ia(n+p))g_0(0) = \exp(ian)g_0(0)$. But 0 is not a fixed point of g_0, and we may cancel by $\exp(ian)g_0(0)$ to obtain the assertion of the Lemma.

To proceed in the proof of the Theorem, let us denote by F the set of points z in D, such that $h(z)$ is a fixed point of f, but $h(z) \neq w_0$. Assume F to be non-empty. The set F has a positive Euclidean distance r from 0. Choose z_1 in F such that $|z_1| = r$, and let p be the smallest natural number such that $\tilde{f}^p = \mathrm{id}$. Set $z_k = \tilde{f}(z_{k-1})$, $k = 2, \ldots, p$, and join every z_k to 0 by the line segment $\tilde{\alpha}_k$. Then $\alpha_k = h(\tilde{\alpha}_k)$ is a path joining $w_1 = h(z_1)$ to w_0. Moreover, α_k is a Jordan arc. For assume $h(t_1) = h(t_2)$, where $t_1, t_2 \in \tilde{\alpha}_k$, and $|t_1| < |t_2|$. Then w_0 and w_1 are joined by $h(\beta)$, where β consists of the segment $(0, t_1)$ and a circular arc congruent modulo G with the segment (t_2, z_k). The hyperbolic length of β is strictly smaller than that of $\tilde{\alpha}_k$, but the Euclidean distance of 0 from the other end-point of β is at least r. This is in contradiction with the fact that rays issuing from the origin are geodesics in the hyperbolic metric.

A similar argument shows that for $k \neq j$, α_k and α_j cannot meet except at w_0 and w_1. Assume $h(t_k) = h(t_j)$ with $t_k \in \tilde{\alpha}_k$, $t_j \in \tilde{\alpha}_j$. Because $h(t_k)$ is not a fixed point of f, we may suppose $|t_k| < |t_j|$. It follows that w_0 and w_1 are joined by $h(\gamma)$, where γ is composed of the segment $(0, t_k)$ and a circular arc congruent modulo G with the segment (t_j, z_j). Again, the hyperbolic length of γ is strictly less than that of $\tilde{\alpha}_k$, while the Euclidean distance of 0 from the other end-point of γ is at least r.

To complete the proof, assume w_2 is a fixed point of f, distinct from w_0 and w_1. One may join w_2, which does not lie on any α_k, to w_0 by an arc α which does not meet any α_k. Then α lies in the Jordan domain B (of the extended plane) bounded by, say, α_k and α_j, where α_k and α_j are adjacent segments. Consider the lifting $\tilde{\alpha}$ of α, with initial point 0. Since h is a local homeomorphism at 0, we see that $f(\alpha)$ must emerge from w_0 into the complement of B. Then $f(\alpha)$ must have a point in common either with α_k or α_j. But this is impossible, since f carries the set of arcs α_k onto itself [4].

Remark. Another proof for the Theorem above has been found by K. Leschinger (Bonn) [1].

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Universität Bonn
Mathematisches Institut
D-5300 Bonn
BRD

University of Helsinki
Department of Mathematics
SF-00100 Helsinki 10
Finland

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