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LIPSCHITZ APPROXIMATION OF HOMEOMORPHISMS

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Results

1. The aim of this paper is to prove a modification of Theorem 2.4 of Väisälä [5], and to prove, using this modification, the LIP hauptvermutung for $n \leqq 3$.

This paper is directly based on Väisälä [5]. We assume the knowledge of [5] (only Sections 1 and 2 are needed) and use freely its definitions and notations.

Let $f\colon X \to Y$ be a map between metric spaces and let $x \in X$. We say that f is *locally bilipschitz at* x if there is a neighbourhood U of x such that $f|U$ is a homeomorphism $U \to f(U)$ with $\operatorname{bilip}(f|U) < \infty$. If $A \subset X$, we say that f is *locally bilipschitz in* A if f is locally bilipschitz at x for every $x \in A$.

2. We begin by proving (cf. Theorem 2.4 of [5] and Theorem 3 below).

Theorem 1. *Let $n \leqq 3$ and $\varepsilon > 0$. Let $f\colon I^n \to R^n$ be an embedding. Then there is an embedding $f^*\colon I^n \to R^n$ such that*

(1) $f^*|\partial I^n = f|\partial I^n$,

(2) $f^*|\operatorname{int} I^n$ *is* PL,

(3) $\|f^* - f\| < \varepsilon$.

In addition, assume that f is locally bilipschitz in a closed set $C \subset \partial I^n$. Then we may assume that f^* is locally bilipschitz in C.

Thus f^*, being PL in int I^n, is locally bilipschitz in $C \cup \operatorname{int} I^n$. This observation is crucial for the following LIP hauptvermutung for $n \leqq 3$:

Theorem 2. *Let $n \leqq 3$ and let $f\colon M \to N$ be a homeomorphism between two* LIP *n-manifolds. Let $\varepsilon\colon M \to (0, \infty)$ be continuous. Then there is a lipeomorphism $g\colon M \to N$ such that $d(f(x), g(x)) < \varepsilon(x)$ for $x \in M$. Here d is the metric of N.*

In addition, if $f|U$ is a lipeomorphism for some open set $U \subset M$ and if $C \subset U$ is closed, we may assume $f|C = g|C$.

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Since every paracompact n-manifold, $n \leqq 3$, has a PL-structure which induces on it a LIP-structure, Theorem 2 has the following

Corollary. *Every* LIP *n-manifold, $n \leqq 3$, is lipeomorphic to a* PL *manifold.*

This was proved in [5] for $n \leqq 2$ by a different method (Theorem 3.8) and the LIP hauptvermutung was obtained as a corollary for $n=2$.

Finally we show, using Theorem 2, that a slightly more general version of Theorem 1 is true:

Theorem 3. *Let n, ε, and f be as in Theorem* 1. *Then we may assume that f^* of Theorem* 1 *is locally bilipschitz at all points of ∂I^n where f is locally bilipschitz.*

Since Theorem 1 seems quite satisfactory for most purposes, our proof of Theorem 3 is rather sketchy.

3. The idea underlying Theorem 2 was used in Sorvali [3, Theorem 2.1] to show that if two compact Riemann surfaces are homeomorphic, then there is a quasiconformal homeomorphism between them. The proof was based on Riemann's mapping theorem and the Beurling—Ahlfors extension of a homeomorphism $f\colon R \to R$ to a homeomorphism F of $U=\{z \in C\colon \operatorname{Im} z>0\}$. If f is quasisymmetric, F is quasiconformal, but in any case F is locally quasiconformal. Originally, the idea was due to O. Lehto.

4. Quasiconformal versions of the above theorems can most probably be proved by the same method.

5. I wish to thank J. Luukkainen and J. Väisälä for pointing out errors in the earlier versions of this paper.

6. Sullivan [4] has proved the LIP hauptvermutung for $n \neq 4$. His methods are much more advanced than ours, which are quite elementary, although our proofs are unduly complicated. In addition to the results of [5] already mentioned, the LIP hauptvermutung was proved in Luukkainen—Väisälä [1, Section 8] for a space homeomorphic either to R^n, S^n, or I^n with some restrictions on n.

Proofs

We assume $n=2$ or $n=3$ since proofs are trivial if $n=1$.

7. We begin by establishing a lemma needed in the proofs of Theorems 1 and 3.

Let $R_0=I^n$ and $R_k=I^n \setminus \operatorname{int} I^n(1-2^{2-k}/3)$, $k \geqq 1$. Then, if $k \geqq 1$, R_k is a ring-like subset of I^n and we obtain R_{k+1} from R_k by dividing R_k into two rings and discarding the inner one. There is a natural way of dividing R_k into cubes of side length $2^{2-k}/3$ if $k \geqq 1$ and of length 2 if $k=0$ in such a way that, for each cube Q, $Q \cap \operatorname{int} I^n$ is a union of cubes of K (cf. the figure in [5]). We denote the family of these cubes into which we have divided each R_k by $\overline{K}_k$, $k \geqq 0$.

Let M be an n-submanifold of I^n. We say that M is of *finite K-type* if $M \cap \operatorname{int} I^n$ is a union of cubes of K and if there is $k \geqq 0$ such that $M \cap R_k$ is a union of cubes of $\overline{K}_k$. Therefore M is a finite union of cubes of $K \cup \overline{K}_k$ for sufficiently big k. Let $\mathscr{F}$ be a family of subsets of I^n and $X \subset I^n$. We often need to consider a family of sets of the form $\{S \in \mathscr{F}\colon S \subset X\}$ which we therefore denote by $\mathscr{F}|X$.

Lemma. *Let $C \subset \partial I^n$ be compact and contained in some face F of I^n or let $C = \partial I^n$. Then there is a sequence $M_0 = I^n \supset M_1 \supset M_2 \supset \dots$ of submanifolds of I^n that are of finite K-type such that:*

(i) $\bigcap_{i \geqq 0} M_i = C$.

(ii) $M_{i+1} \subset \operatorname{int} M_i$ *(int is relative to I^n), $i \geqq 0$.*

(iii) $\operatorname{cl}(\partial M_i \cap \operatorname{int} I^n) \cap C = \emptyset$ *for $i \geqq 0$.*

(iv) *Each M_i is a union of cubes $Q \in K_j(F) = \{Q \in \overline{K}_j\colon Q \cap F \neq \emptyset\}$ for some $j \geqq 0$, or, if $C = \partial I^n$, $M_i = R_i$.*

(v) *Let X be either M_i or $\operatorname{cl}(M_i \setminus M_{i+1})$ for some $i \geqq 0$. Then there is a partition $K_1(X) \cup \dots \cup K_m(X)$ of $K|X$ (m depends on X) such that if $K_r^*(X) = K_1(X) \cup \dots \cup K_r(X)$, $F_r(X) = \cup K_r^*(X)$ and $P_Q(X) = 2I^n \cap \alpha_Q^{-1} F_{r-1}(X)$ for $Q \in K_r(X)$ (cf.* [5, 2.7]*), then the conditions* (1)—(3) *of* [5, 2.7] *remain true after the substitutions $K_r \mapsto K_r(X)$ and $P_Q \mapsto P_Q(X)$.*

(vi) *Let X be $\operatorname{cl}(M_i \setminus M_{i+1})$ for some $i \geqq 0$. Let $K_0'(X) = K\big|(\operatorname{cl}(M_{i-1} \setminus M_i) \cup \operatorname{cl}(M_{i+1} \setminus M_{i+2}))$ (if $i=0$, $M_{-1} = \emptyset$). Then there is a partition $K_1'(X) \cup \dots \cup K_{m'}'(X)$ of $K|X$ (m' depends on X) such that if $K_r'^*(X) = K_0'(X) \cup \dots \cup K_r'(X)$, $F_r'(X) = \cup K_r'^*(X)$ and $P_Q'(X) = 2I^n \cap \alpha_Q^{-1} F_{r-1}'(X)$ for $Q \in K_r'(X)$, then the conditions* (1)—(3) *for $r \geqq 1$ of* [5, 2.7] *remain true after the substitution $K_r \mapsto K_r'(X)$ and $P_Q \mapsto P_Q'(X)$.*

Proof. It is easy to see that we can find a sequence $M_0 \supset M_1 \supset \dots$ satisfying conditions (i)—(iv). In (v) and (vi) we need not care about the condition (3) of [5, 2.7] since if we can find $K_i(X)$ and $K_i'(X)$ that satisfy (1) and (2) we can always have (3) by further partitioning of the families $K_i(X)$ and $K_i'(X)$, $i \geqq 1$. Also, we assume $n=3$ since the proof is much simpler if $n=2$.

First we note the following fact. Let Y be either a cube $Q \in \overline{K}_l$, $l \geqq 0$, or of the form $\operatorname{cl}(Q \setminus (\cup K_k(F)))$ or $\operatorname{cl}(Q \setminus R_k)$ where $Q \in \overline{K}_l$ and $k > l$. Then $K_1|Y, \dots, K_M|Y$ is a partition of $K|Y$ satisfying conditions (1) and (2) of [5, 2.7], modified as in (v). This fact is fundamental to our proof of (v) and (vi).

The manifold X is a finite union $(\cup K_X') \cup (\cup K_X'')$ where $K_X' = K_k(F)|X$ (if $C = \partial I^n$, $K_X' = \overline{K}_k|X$) and $K_X'' = \{\operatorname{cl}(Q \setminus (\cup K_k(F)))\colon Q \in K_l(F)\}\big|X$ (if $C = \partial I^n$, $K_X'' = \{\operatorname{cl}(Q \setminus R_k)\colon Q \in \overline{K}_l\}$) for some $k > l \geqq 0$ (or $k=0$ and $K_X'' = \emptyset$). We show that we can arrange the elements of $K_X' \cup K_X''$ in a sequence, $K_X' \cup K_X'' = \{Q_0, \dots, Q_{s-1}\}$, in such a way that if

$$(*) \quad K_{iM+j}(X) = K_j | Q_i \quad (K = K_1 \cup \dots \cup K_M),$$

then $K_1(X), \dots, K_{sM}(X)$ is a partition of $K|X$ satisfying (v). In fact, it suffices for this that the sequence $\{Q_0, \dots, Q_{s-1}\}$ satisfies the following condition, analogous to (2) of [5, 2.7].

(∗ ∗) *For* $r<s$, $\bigcup_{i=0}^{r} Q_i$ *is an n-manifold and* $Q_r \cap (\bigcup_{i=0}^{r-1} Q_i)$ *is an* $(n-1)$-*manifold, and* $K_X''=\{Q_0, \dots, Q_t\}$ *for some* $t \geqq -1$.

If $C=\partial I^n$, $X=R_l$ or $X=\mathrm{cl}(R_l \setminus R_{l+1})$ for some $l \geqq 0$, and it is easy to see that there is a sequence $\{Q_0, \dots, Q_{s-1}\}$ satisfying (∗ ∗). Therefore we now assume that C is contained in some face F of I^n.

The family $K_k(F)$, which contains K_X', forms a two-dimensional lattice, and we denote by Q_{ij}^1 the element in i-th row and j-th column. In like manner, K_X'' is contained in $\{\mathrm{cl}(Q \setminus (\cup K_k(F))) \colon Q \in K_l(F)\}$ which forms a two-dimensional lattice. Denote by Q_{ij}^0 the element in i-th row and j-th column. We can define a linear order in $K_X' \cup K_X''$ by

(a) $Q_{ij}^k < Q_{i'j'}^{k'}$ *if* $k<k'$ *or if* $k=k'$ *and* $i<i'$,

(b) $Q_{ij}^k < Q_{ij'}^k$ *if* $j<j'$ *unless* $Q_{i-1,j}^k \not\subset X$ *and* $Q_{i-1,j'}^k \subset X$ *in which case* $Q_{ij'}^k < Q_{ij}^k$.

Now we can enumerate $K_X' \cup K_X''$ as $\{Q_0, \dots, Q_{s-1}\}$ in such a way that $Q_i < Q_j$ if and only if $i<j$. It is a straightforward verification that (∗ ∗) is true. Consequently (∗) defines a partition of $K|X$ satisfying (v).

The proof of (vi) is similar. We use the above notation and again enumerate $K_X' \cup K_X''$ as $\{Q_0, \dots, Q_{s-1}\}$ in such a way that the right hand side of (∗) defines a partition $K_1'(X), \dots, K_{sM}'(X)$ of $K|X$ satisfying (vi). For this it is sufficient that (∗ ∗ ∗) is true where $Q_{-1}=F_0'(X)=\cup K_0'(X)$:

(∗ ∗ ∗) *For* $r<s$, $\bigcup_{i=-1}^{r} Q_i$ *is an n-manifold and* $Q_r \cap (\bigcup_{i=-1}^{r-1} Q_i)$ *is an* $(n-1)$-*manifold, and* $K_X'=\{Q_0, \dots, Q_t\}$ *for some* $t \geqq -1$.

As above, we define a linear order in $K_X' \cup K_X''$ by

(a′) $Q_{ij}^k < Q_{i'j'}^{k'}$ *if* $k>k'$ *or if* $k=k'$ *and* $i<i'$,

(b′) $Q_{ij}^k < Q_{ij'}^k$ *if* $j<j'$ *unless* $Q_{i+1,j}^k \subset X$ *and* $Q_{i+1,j'}^k \not\subset X$ *in which case* $Q_{ij'}^k < Q_{ij}^k$.

Conclusion is as above.

8. *Proof of Theorem* 1. We can assume that either $C=\partial I^n$ or that C is contained in some face F of I^n. Otherwise, if $C \neq \partial I^n$, there is a PL homeomorphism $h\colon I^n \to I^n$ such that $h(C) \subset F$ and we replace f by fh^{-1}.

In these circumstances we can use the above lemma. Let $M_0 \supset M_1 \supset \dots$ be the sequence of the lemma. We use only the portion $M_0 \supset M_1 \supset M_2 \supset M_3$ of the sequence and assume that $f|M_1$ is a lipeomorphism. Then we find a partition $K_1(M_3), \dots, K_m(M_3)$ of $K|M_3$ as in (v) of the lemma. Define numbers $\delta_1 \leqq \dots \leqq \delta_m$

as in [5, 2.8] with $L=\text{bilip}\,(f|M_1)$ and $M=m$. Now Lemma 2.9 of [5] remains true after the substitutions, when $\varepsilon'>0$,

$$L\mapsto \text{bilip}\,(f|M_1),\quad M\mapsto m,\quad F_r\mapsto F_r(M_3),\quad K_r^*\mapsto K_r^*(M_3),\quad \varepsilon\mapsto\varepsilon',$$

even if the assumption that f is a lipeomorphism $I^n\to f(I^n)$ of Lemma 2.9 is weakened into the assumption that $\text{bilip}\,(f|M_1)<\infty$.

Once this modification of Lemma 2.9 of [5] is proved, one can show as in 2.10 of [5], if $\varepsilon'>0$ is given:

There is an embedding f_3: $M_3\to R^n$ *such that*

(i) $f_3|M_3\cap\partial I^n=f|M_3\cap\partial I^n$,

(ii) $\|f_3(x)-f(x)\|\leqq\varepsilon'\lambda_Q$ *if* $Q\in K$ *and* $x\in Q\subset M_3$,

(iii) $f_3|M_3\cap\text{int}\,I^n$ *is* PL,

(iv) f_3 *is locally bilipschitz at points of* C.

The next step is to consider the set $M_0'=\text{cl}\,(M_0\setminus M_2)=\text{cl}\,(I^n\setminus M_2)$. Let ε'': $M_0'\to[0,\infty)$ be a continuous function such that $\varepsilon''(x)>0$ if $x\in M_0'\cap\text{int}\,I^n$. Then we have:

There is an embedding f_0': $M_0'\to R^n$ *such that*

(i′) $f_0'|M_0'\cap\partial I^n=f|M_0'\cap\partial I^n$,

(ii′) $\|f_0'(x)-f(x)\|\leqq\varepsilon''(x)$ *if* $x\in M_0'$,

(iii′) $f_0'|M_0'\cap\text{int}\,I^n$ *is* PL.

This follows e.g. from Theorem 1 of Moise [2, Section 36]. (If this Theorem is used it is better first to seek an approximation for $f|(I^n\setminus M_3)$ and then restrict it to M_0').

We finish the proof by glueing f_3 and f_0' together by finding a function f_2': $M_2'=\text{cl}\,(M_2\setminus M_3)\to R^n$ such that

(i″) f_3, f_0' *and* f_2' *define together an embedding* $I^n\to R^n$,

(ii″) $f_2'|M_2'\cap\partial I^n=f|M_2'\cap\partial I^n$,

(iii″) $\|f_2'(x)-f(x)\|\leqq\varepsilon$ *if* $x\in M_2'$,

(iv″) $f_2'|M_2'\cap\text{int}\,I^n$ *is* PL.

If we have found such a function f_2', then $f^*=f_0'\cup f_2'\cup f_3$ satisfies the conditions of Theorem 1, provided that ε' and ε'' are sufficiently small.

Since $f|M_1$ is a lipeomorphism we can again use the construction of [5, 2.11] to prove the existence of such f_2'. Let $K_0'(M_2')$ be as in (vi) of the above lemma and let $K_1'(M_2'),\dots,K_{m'}'(M_2')$ be the partition of $K|M_2'$ in (vi) of the lemma. Since $\text{bilip}\,(f|M_1)<\infty$ we can apply Lemma 2.6 of [5] and find numbers $\delta_0\leqq\dots\leqq\delta_{m'}$ as in 2.8, with $L=\text{bilip}\,(f|M_1)$. Now, assuming that ε' and ε'' are sufficiently small, we can extend $f_0'\cup f_3$ (which is defined in a set containing $F_0'(M_2')$) first to $F_1'(M_2')$, then to $F_2'(M_2')$, ... , and finally to $F_{m'}'(M_2')$ as in [5, 2.11]. Notice that the situation is now simpler since we do not require (i) of Lemma 2.9, (ii) being sufficient. As in 2.10, it is shown that we obtain in this manner an embedding f_2': $M_2' \to R^n$ (which need not be a LIP embedding) satisfying (i″)—(iv″).

9. *Proof of Theorem* 2. We can assume ε to be chosen in such a way that, for each $x \in M$, the set $\{y \in N: d(y, f(x)) \leq \varepsilon(x)\}$ is contained in a set lipeomorphic to R^n (or to R^n_+ if $x \in \partial M$). We can find a family $\mathcal{V}$ of closed LIP n-balls in M such that $\mathcal{V}$ is locally finite in M, $\cup \mathcal{V} \subset M \setminus C$, and $\cup \{\text{int } V: V \in \mathcal{V}\} \supset M \setminus U$. There are also continuous functions η_i: $M \to (0, \infty)$, $0 \leq i \leq k$, where k depends only on n, such that:

(1) $\eta_0 + \ldots + \eta_{k-1} \leq \eta_k = \varepsilon$.

(2) *If* $x \in \partial V$, $V \in \mathcal{V}$, *x has a neighbourhood* U_x *such that* $(U_x, U_x \cap \partial V)$ *is lipeomorphic to* (R^n, R^{n-1}) *or to* (R^n_+, R^{n-1}_+), *according to whether* $x \notin \partial M$ *or* $x \in \partial M$.

(3) *There is a partition* $\mathcal{V}_1, \ldots, \mathcal{V}_k$ *of* $\mathcal{V}$ *such that each* $\mathcal{V}_i$ *consists of disjoint balls.*

(4) *Let* $V \in \mathcal{V}$ *and* $x \in V$. *Then* $\operatorname{diam} f(V) \leq \eta_0(x)$.

(5) *If* $V \in \mathcal{V}$ *and* $i < k$, $3(\max_{x \in V} \eta_i(x)) \leq \min_{x \in V} \eta_{i+1}(x)$.

It is not difficult to see that there is such a family $\mathcal{V}$ and such functions η_i. We give only some hints for (3). Choose first an open cover $\mathcal{U}$ of $M \setminus U$ such that the elements of $\mathcal{U}$ have a sufficiently small diameter. In addition we assume that $\mathcal{U}$ is locally finite and that there is a partition $\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}$ of $\mathcal{U}$ such that each $\mathcal{U}_i$ consists of disjoint sets and that $\cup \mathcal{U}_i$ is lipeomorphic to an open subset of R^n_+. We can also assume that there is a cover $\mathcal{U}' = \{C_V: V \in \mathcal{U}\}$ of $M \setminus U$ whose elements are compact sets with $C_V \subset V$. Let $C_i = \cup \{C_V: V \in \mathcal{U}_i\}$. Since now a neighbourhood of C_i is lipeomorphic to an open subset of R^n_+, we can find a locally finite (in M) cover $\mathcal{V}^i$ of C_i such that the elements of $\mathcal{V}^i$ are closed LIP n-balls (with $\cup \{\text{int } V: V \in \mathcal{V}^i\} \supset C_i$) and such that there is a partition $\mathcal{V}_{(i-1)(n+1)+1}, \ldots, \mathcal{V}_{i(n+1)}$ of $\mathcal{V}^i$ where each $\mathcal{V}_j$ consists of disjoint n-balls. Then $\mathcal{V} = \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{(n+1)^2}$ is the desired cover of $M \setminus U$. This construction gives $k = (n+1)^2$ but most certainly we could have $k = n+1$.

We can also find for each $V \in \mathcal{V}$ a closed LIP n-ball $W_V \subset \text{int } V$ such that $\{\text{int } W_V: V \in \mathcal{V}\}$ still covers $M \setminus U$. Let $V^* \subset U$ be some closed set such that $\{\text{int } V^*\} \cup \{\text{int } W_V: V \in \mathcal{V}\}$ is a cover of M. (Then $\text{int } V^* \supset C$.) Let $\mathcal{V}_0 = \{V^*\}$ and set $W_{V^*} = V^*$.

We prove Theorem 2 by showing:

There is a sequence $f = f_0, f_1, \ldots, f_k$ *of homeomorphisms* $M \to N$ *such that:*

(i) *Each* f_i *is locally bilipschitz in* $\cup \{W_V: V \in \mathcal{V}_j$ *with* $j \leq i\}$.

(ii) $\operatorname{diam} f_i(V) \leq \eta_i(x)$ *if* $x \in V \in \mathcal{V}$, $0 \leq i \leq k$.

(iii) $d(f_i(x), f_{i-1}(x)) \leq \eta_{i-1}(x)$ *for* $x \in M$ *and* $0 < i \leq k$.

(iv) *If* $x \in M \setminus (\cup \mathcal{V}_i)$, $f_i(x) = f_{i-1}(x)$ *for* $0 < i \leq k$.

We have already set $f_0=f$. Assume that we have defined f_j for $0\leqq j\leqq m-1$ satisfying (i)—(iv). Choose some ball $V\in\mathcal{V}_m$. By (ii) (since $\eta_{m-1}\leqq\varepsilon$) there is some set $W\subset N$ containing $f_{m-1}(V)$ and a lipeomorphism $\psi\colon W\to R^n$. There is also a lipeomorphism $\varphi\colon V\to I^n$. Then $\psi\circ f_{m-1}\circ\varphi^{-1}\colon I^n\to R^n$ is an embedding which is locally bilipschitz at $x\in I^n$ if f_{m-1} is locally bilipschitz at $\varphi^{-1}(x)\in V$. Let $C'=\varphi(\partial V\cap(\cup\{W_V\colon V\in\mathcal{V}_j \text{ with } j<m\}))$. Then $C'\subset\partial I^n$ and, by (i), $\psi\circ f_{m-1}\circ\varphi^{-1}$ is locally bilipschitz in C'. Therefore we may apply Theorem 1 and find an embedding $f^*\colon I^n\to R^n$ with the same boundary values as $\psi\circ f_{m-1}\circ\varphi^{-1}$ that is PL in $\operatorname{int} I^n$ and locally bilipschitz in C'. Therefore $f_V^*=\psi^{-1}\circ f^*\circ\varphi\colon V\to N$ is locally bilipschitz in $\cup\{W_{V'}\cap V\colon V'\in\mathcal{V}_j \text{ with } j<m\}\cup\operatorname{int} V$.

We choose such a mapping $f_V^*\colon V\to N$ for each $V\in\mathcal{V}_m$. Now we can define

$$f_m(x)=\begin{cases} f_V^*(x) & \text{if } x\in V\in\mathcal{V}_m,\\ f_{m-1}(x) & \text{otherwise.}\end{cases}$$

Since for each $V\in\mathcal{V}_m$, $f_V^*|\partial V=f_{m-1}|\partial V$, f_m is a homeomorphism $M\to N$. Let $V\in\mathcal{V}_m$ and let $x\in\partial V\cap(\cup\{W_V\colon V\in\mathcal{V}_j \text{ with } j<m\})$. Then f_{m-1} is locally bilipschitz at x, as is $f_m|V$. Therefore, since $\mathcal{V}$ is locally finite, f_m is locally Lipschitz at x. Since $f_{m-1}(\partial V)$ has a local LIP collar at $f_{m-1}(x)$, also f_m^{-1} is locally Lipschitz at $f_{m-1}(x)=f_m(x)$. Thus f_m is locally bilipschitz in $\cup\{W_V\colon V\in\mathcal{V}_j,\ j\leqq m\}$. We have shown that f_m satisfies (i). Clearly, it satisfies (iii), (iv), and, by (5), also (ii).

Consider the function $g=f_k$. Since $(\cup\mathcal{V})\cap C=\emptyset$ and $f_0=f$, we have by (iv) $g|C=f|C$. By (i), g is a lipeomorphism $M\to N$. By (iii) and (1), $d(g(x), f(x))\leqq\varepsilon(x)$ for $x\in M$. Thus g is the lipeomorphism $M\to N$ sought for.

10. *Proof of Theorem* 3. We first show that we may assume $f|I^n\setminus X$ to be a LIP embedding where $X=\{x\in\partial I^n\colon f \text{ is not locally bilipschitz at } x\}$. Then X is a closed subset of ∂I^n.

Let $A=\partial I^n\setminus X$ and let U be an open subset of I^n containing A such that $f|U$ is a lipeomorphism $U\to f(U)$ and let $C\subset U$ be a closed neighbourhood of A in $I^n\setminus X$. We regard $\operatorname{int} I^n$ and $f(\operatorname{int} I^n)$ as LIP-manifolds, the LIP-structures being those of open submanifolds of R^n. Let $\varepsilon'\colon \operatorname{int} I^n\to(0,\varepsilon/2)$ be a continuous function such that if $h\colon \operatorname{int} I^n\to R^n$ is a continuous function and $\|h(x)-f(x)\|<\varepsilon'(x)$ for $x\in\operatorname{int} I^n$, then we can extend h to a continuous function $I^n\to R^n$ by setting $h|\partial I^n=f|\partial I^n$. Now we can apply Theorem 2 with the substitutions

$$M\mapsto \operatorname{int} I^n,\quad N\mapsto f(\operatorname{int} I^n),\quad U\mapsto U\cap\operatorname{int} I^n,\quad C\mapsto C\cap\operatorname{int} I^n,\quad \varepsilon\mapsto\varepsilon'.$$

Let $g'\colon \operatorname{int} I^n\to f(\operatorname{int} I^n)$ be the lipeomorphism obtained. Then we can extend g' to a continuous map $g\colon I^n\mapsto R^n$ by setting $g|\partial I^n=f|\partial I^n$. But then g must be an embedding $I^n\to R^n$ such that $g|I^n\setminus X$ is a LIP embedding. Clearly, if f^* is the map given by Theorem 3 with substitutions $\varepsilon\mapsto\varepsilon/2$ and $f\mapsto g$, then f^* satisfies the conditions of Theorem 3 also with respect to the original f and ε.

We now assume that $f|I^n\setminus X$ is a lipeomorphism onto $f(I^n\setminus X)$. As in the proof of Theorem 1, we can assume that X is contained in some face F of I^n, otherwise either $X=\partial I^n$ and there is nothing to prove in addition to Theorem 1, or we can replace f by another map. Let $M_0 \supset M_1 \supset \dots$ be the sequence of the lemma of Section 7 where we replace C by X. Then $\operatorname{bilip}(f|\operatorname{cl}(M_i \setminus M_{i+1})) < \infty$ for $i \geqq 0$.

Let $M' = \bigcup_{i \geqq 0} \operatorname{cl}(M_{2i} \setminus M_{2i+1})$ and let $M'' = \bigcup_{i \geqq 0} \operatorname{cl}(M_{2i+1} \setminus M_{2i+2})$. Then $I^n \setminus X = M' \cup M''$. Let $\varepsilon_i > 0$ be given for $i=0, 2, 4, \dots$. We can now apply separately to each subset $\operatorname{cl}(M_{2i} \setminus M_{2i+1})$ of M' the method of [5] in 2.10 and 2.11 using (v) of the lemma of Section 7 to obtain a map $f'\colon M' \to R^n$ such that

(i) $f'|\operatorname{cl}(M_{2i} \setminus M_{2i+1})$ *is a* LIP *embedding*, $i \geqq 0$,

(ii) $\|f'(x)-f(x)\| \leqq \varepsilon_{2i}\lambda_Q$ *if* $x \in Q \in K$ *and* $Q \subset \operatorname{cl}(M_{2i} \setminus M_{2i+1})$, $i \geqq 0$,

(iii) $f'|M' \cap \partial I^n = f|M' \cap \partial I^n$,

(iv) $f'|M' \cap \operatorname{int} I^n$ *is* PL,

(v) *If* $i \geqq 0$, $\{\beta_Q f' \alpha_Q | 2I^n \cap \alpha_Q^{-1} M' : Q \in K | \operatorname{cl}(M_{2i+1} \setminus M_{2i+2})\}$ *is finite.*

If the ε_i's are suitably chosen, f' is an embedding. Again, given $\varepsilon_i > 0$, $i=1, 3, \dots$, and assuming that ε_i: s, $i=0, 2, \dots$, are sufficiently small, we can find as above, using (vi) of the lemma of Section 7, a map $f''\colon M'' \to R^n$ that satisfies conditions similar to (i)—(iv) and such that f' and f'' together define a LIP embedding $f_0\colon M' \cup M'' = I^n \setminus X \to R^n$ (resembling the construction of f_2' in the proof of Theorem 1). If $\varepsilon_i \to 0$ sufficiently fast as $i \to \infty$, we can extend f_0 to an embedding $f^*\colon I^n \to R^n$ with the desired properties.

References

[1] LUUKKAINEN, J., and J. VÄISÄLÄ: Elements of Lipschitz topology. - Ann. Acad. Sci. Fenn. Ser. A I 3, 1977, 85—122.

[2] MOISE, E. E.: Geometric topology in dimensions 2 and 3. - Springer-Verlag, New York—Heidelberg—Berlin, 1977.

[3] SORVALI, T.: The boundary mapping induced by an isomorphism of covering groups. - Ann. Acad. Sci. Fenn. Ser. A I 526, 1972, 1—31.

[4] SULLIVAN, D.: Hyperbolic geometry and homeomorphisms. - To appear in the Proceedings of the Georgia Topology Conference 1977.

[5] VÄISÄLÄ, J.: Piecewise linear approximation of lipeomorphisms. - Ann. Acad. Sci. Fenn. Ser. A I 3, 1977, 377—383.

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