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SQUARE INTEGRABLE HARMONIC FUNCTIONS ON PLANE REGIONS

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Let R be a thin horizontal elastic plate clamped along its border. We denote by $\beta_R(z,\zeta)$ the deflection of R at $z\in R$ under a point load at $\zeta\in R$, that is, the *biharmonic Green's function* on R with the pole at ζ. The function is characterized by $\Delta_z^2\beta_R(z,\zeta)=2\pi\delta_\zeta$ on R, with δ_ζ the Dirac measure at ζ, and by the conditions $\beta_R(z,\zeta)=\partial\beta_R(z,\zeta)/\partial n_z=0$ at the boundary ∂R of R (e.g., Bergman—Schiffer [1]). Accordingly, it is customary to assume that the boundary ∂R of R relative to the complex plane C is smooth. If R is *an arbitrary plane region*, a natural procedure is to define $\beta_R(z,\zeta)$ for $z\in R$ as the directed limit

$$\lim_{\Omega\to R}\beta_\Omega(z,\zeta) \tag{1}$$

where $\{\Omega\}$ is the directed set of regular subregions, i.e., relatively compact subregions Ω of R with smooth boundaries $\partial\Omega$. We denote by O_β the family of *plane* regions R for which (1) is divergent for some $\zeta\in R$. The purpose of the present paper is to give a complete characterization of O_β as follows:

1°. A plane region $R\in O_\beta$ if and only if the complement $C-R$ of R does not contain any noncollinear triple of points (and hence e.g. $C-\{0,1,i\}\notin O_\beta$!).

2°. If a plane region $R\notin O_\beta$, then $\beta_R(z,\zeta)=\lim_{\Omega\to R}\beta_\Omega(z,\zeta)$ is symmetric and continuous on $R\times R$, the convergence is uniform on every compact subset of $R\times R$, and $z\to\beta_R(z,\zeta)$ is biharmonic on $R-\zeta$.

3°. There exist plane regions R which are "unstable" in the sense that (1) is divergent for some $(z,\zeta)\in R\times R$ but convergent for some other $(z,\zeta)\in R\times R$. Such unstable regions R are characterized by the existence of a line $l(R)$ such that $C-R$ is a proper subset of $l(R)$ consisting of at least two points, and (1) is divergent at e.g., (ζ,ζ) for any $\zeta\notin l(R)$ and convergent at every $(z,\zeta)\in(l(R)\times l(R))\cap(R\times R)$.

We denote by $H_2(R)$ the closed subspace of $L_2(R)$ consisting of square integrable harmonic functions on R. To prove 1°—3°, we shall make essential use of the follow-

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ing results obtained in [3]. Let $H(\Omega)$ be the class of harmonic functions on Ω and denote by $(\ ,\ )_\Omega$ the inner product in $L_2(\Omega)$. The function $H_\Omega(z,\zeta)=\Delta_z\beta_\Omega(z,\zeta)$ is referred to as the *β-density* on Ω, characterized by $H_\Omega(z,\zeta)+\log|z-\zeta|\in H(\Omega)$ as a function of z, and by $(H_\Omega(\cdot,\zeta),u)_\Omega=0$ for every $u\in H_2(\Omega)$. Then $\beta_\Omega(z,\zeta)=(H_\Omega(\cdot,z),\ H_\Omega(\cdot,\zeta))_\Omega$ and

(2) $$|\beta_{\Omega'}(z,\zeta)-\beta_\Omega(z,\zeta)| \leqq \|H_{\Omega'}(\cdot,z)-H_\Omega(\cdot,z)\|_R\cdot\|H_{\Omega'}(\cdot,\zeta)-H_\Omega(\cdot,\zeta)\|_R$$

on $\Omega\times\Omega$ for every regular subregion Ω' with $\Omega\subset\Omega'\subset R$; here $\|\cdot\|_R$ is the norm in $L_2(R)$, and we have set $H_\Omega(z,\zeta)=0$ for $(z,\zeta)\notin\Omega\times\Omega$. In particular, we have

(3) $$\beta_{\Omega'}(\zeta,\zeta)-\beta_\Omega(\zeta,\zeta) = \|H_{\Omega'}(\cdot,\zeta)-H_\Omega(\cdot,\zeta)\|_R^2 = \|H_{\Omega'}(\cdot,\zeta)\|_R^2-\|H_\Omega(\cdot,\zeta)\|_R^2.$$

Thus the limit (1) exists if and only if

(4) $$\lim_{\Omega\to R}\|H_\Omega(\cdot,\zeta)\|^2<+\infty.$$

This in turn is equivalent to the existence of an $H_R(\cdot,\zeta)$ on R such that $H_R(z,\zeta)+\log|z-\zeta|\in H(R)$ as a function of z and $(H_R(\cdot,\zeta),u)=0$ for every $u\in H_2(R)$; in this case, $\lim_{\Omega\to R}\|H_R(\cdot,\zeta)-H_{\Omega'}(\cdot,\zeta)\|_R=0$ and $\beta_R(z,\zeta)=(H_R(\cdot,z),H_R(\cdot,\zeta))_R$.

We will see that the orthogonal complement $H_2(R)_\zeta^\perp$ of $H_2(R)$ in $H_2(R-\zeta)$ is either $\{0\}$ or $\boldsymbol{R}H_R(\cdot,\zeta)$, where $\boldsymbol{R}$ is the field of real numbers. Accordingly, the essential point is to determine the pairs (R,ζ) of plane regions R and their points $\zeta\in R$ such that $\dim H_2(R-\zeta)=\dim H_2(R)+1$. Thus we are led to study the Hilbert space $H_2(R)$. It is locally bounded and therefore has a reproducing kernel $h_R(z,\zeta)$ characterized by $u(\zeta)=(u,h_R(\cdot,\zeta))_R$ for every $u\in H_2(R)$. It is seen that $h_R(\cdot,\zeta)=\Delta_\zeta H_R(\cdot,\zeta)$ if $H_R(\cdot,\zeta)$ exists (cf. e.g. Garabedian [2]); we will, however, not make use of this fact in the present work.

In nos. **1—5** we study the dimension of $H_2(R)$ and give a complete characterization of those plane regions R for which $\dim H_2(R)=0$. We then proceed to $H_2(R-\zeta)$ and, in nos. **6—7**, characterize those plane regions R for which $\dim H_2(R-\zeta)=\dim H_2(R)+1$ for *every* $\zeta\in R$, for *some* $\zeta\in R$, or for *no* $\zeta\in R$. For the first case we study, in nos. **8—11**, the continuity of $H_R(\cdot,\zeta)$ and the uniformity of the convergence $H_\Omega(\cdot,\zeta)\to H_R(\cdot,\zeta)$ with respect to ζ. That assertions 1°—3° follow from these considerations will be briefly discussed in the final no. **12**.

We close this introduction by stressing once more that the class O_β is not conformally invariant and not even invariant under Möbius transformations. In fact, the regions $\boldsymbol{C}-\{0,1,i\}\notin O_\beta$ and $\boldsymbol{C}-\{0,1,2\}\in O_\beta$ are equivalent by the Möbius transformation $(z,0,1,i)=(w,0,1,2)$.

1. Suppose $u(z)$ is harmonic in a punctured disk $\Delta_0(\zeta,\varrho)$: $0<|z-\zeta|<\varrho$ about a point $\zeta\in\boldsymbol{C}$ (the finite complex plane). Then $u(z)$ has the Laurent expansion

$$u(z)=\mathrm{Re}\left(-c\log(z-\zeta)+\sum_{n=-\infty}^{\infty}a_n(z-\zeta)^n\right)$$

in $\Delta_0(\zeta, \varrho)$, with $c \in \boldsymbol{R}$ (the field of real numbers) and $a_n \in \boldsymbol{C}$. It is readily seen that u is square integrable in $\Delta_0(\zeta, \bar{\varrho})$ ($\bar{\varrho} \in (0, \varrho)$) if and only if $a_n = 0$ for every negative n:

$$u(z) = \mathrm{Re}\left(-c \log(z-\zeta) + \sum_{n=0}^{\infty} a_n (z-\zeta)^n\right). \tag{5}$$

Next suppose $u(z)$ is harmonic in a punctured disk $\Delta_0(\infty, \varrho)$: $\varrho < |z| < +\infty$ about the point ∞ at infinity. Then the Laurent expansion of $u(z)$ is given by

$$u(z) = \mathrm{Re}\left(c \log z + \sum_{n=-\infty}^{\infty} a_n z^{-n}\right)$$

in $\Delta_0(\infty, \bar{\varrho})$ ($\varrho < \bar{\varrho}$) where $c \in \boldsymbol{R}$ and $a_n \in \boldsymbol{C}$. Again it is clear that u is square integrable if and only if $c = a_n = 0$ for every integer $n \leqq 1$:

$$u(z) = \mathrm{Re}\left(\sum_{n=2}^{\infty} a_n z^{-n}\right), \tag{6}$$

and in this case u is also harmonic at ∞ with $u(\infty) = 0$.

For convenience we denote by $l_\zeta(z)$ the normalized logarithmic pole $-\log|z-\zeta|$ at $\zeta \in \boldsymbol{C}$. The Laurent expansion of $l_\zeta(z)$ is

$$l_\zeta(z) = \mathrm{Re}\left(-\log z + \sum_{n=1}^{\infty} \frac{\zeta^n}{n} z^{-n}\right) \tag{7}$$

in $\Delta_0(\infty, |\delta|)$. Note that the coefficient of z^{-1} is ζ.

2. Let $\boldsymbol{\zeta}$ be the set of m distinct points ζ_j in $\boldsymbol{C}$ ($j = 1, \dots, m$) and consider the region $R_{\boldsymbol{\zeta}} = \boldsymbol{C} - \boldsymbol{\zeta}$. The matrix

$$A = A(\boldsymbol{\zeta}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mathrm{Re}\,\zeta_1 & \mathrm{Re}\,\zeta_2 & \dots & \mathrm{Re}\,\zeta_m \\ \mathrm{Im}\,\zeta_1 & \mathrm{Im}\,\zeta_2 & \dots & \mathrm{Im}\,\zeta_m \end{pmatrix} \tag{8}$$

associated with the region $R_{\boldsymbol{\zeta}}$ will be instrumental in our reasoning. We shall also use the column vector $\boldsymbol{t}$ whose components are $t_1, t_2, \dots, t_m$ in $\boldsymbol{R}$. Let $S = S(\boldsymbol{\zeta})$ be the vector space of solution vectors $\boldsymbol{t}$ of the equation $A\boldsymbol{t} = \boldsymbol{0}$ where $\boldsymbol{0}$ is the transpose of $(0, 0, 0)$. Then

$$\dim S(\boldsymbol{\zeta}) = m - \mathrm{rank}\, A(\boldsymbol{\zeta}).$$

With each column vector $\boldsymbol{t}$ we associate $h_{\boldsymbol{t}} = \sum_{j=1}^{m} t_j l_{\zeta_j}$. We will show that $\boldsymbol{t} \to h_{\boldsymbol{t}}$ is a linear bijection: $S \to H_2(R_{\boldsymbol{\zeta}})$, so that

$$\dim H_2(R_{\boldsymbol{\zeta}}) = m - \mathrm{rank}\, A(\boldsymbol{\zeta}). \tag{9}$$

First we prove that $h_{\boldsymbol{t}} \in H_2(R_{\boldsymbol{\zeta}})$ if $\boldsymbol{t} \in S$. It is clear that $h_{\boldsymbol{t}}$ belongs to $H(R_{\boldsymbol{\zeta}})$ and is square integrable over some $\Delta_0(\zeta_j, \varrho_j)$ for every $j = 1, \dots, m$. By (7) we see that

$$h_t(z) = \mathrm{Re}\left(-\left(\sum_{j=1}^{m} t_j\right) \log z + \left(\sum_{j=1}^{m} \zeta_j t_j\right) z^{-1} + \sum_{n=2}^{\infty} \alpha_m z^{-n}\right)$$

where $\alpha_n = n^{-1}\sum_{j=1}^m \zeta_j^n t_j$. Since $At = \mathbf{0}$, the coefficients of $\log z$ and z^{-1} of $h_t(z)$ must vanish and we obtain $h_t(z) = \mathrm{Re}\left(\sum_{n=2}^\infty \alpha_n z^{-n}\right)$, which by (6) shows that h_t is also square integrable over some $\Delta_0(\infty, \varrho)$. Since $\boldsymbol{C} - \Delta_0(\infty, \varrho) \cup \sum_{j=1}^m \Delta_0(\zeta_j, \varrho_j)$ is compact we finally conclude that $h_t \in H_2(R_\zeta)$, that is, $\boldsymbol{t} \to h_t$ is a well defined mapping: $S \to H_2(R_\zeta)$. Since $\{l_{\zeta_j}\}_{j=1,\dots,m}$ is a linearly independent family, we see that $\boldsymbol{t} \to h_t$ is a linear injection of S into $H_2(R_\zeta)$.

Next we prove that it is surjective. Choose an arbitrary $u \in H_2(R_\zeta)$. By (5) we have

$$u(z) = \mathrm{Re}\left(-t_j \log(z-\zeta_j) + \sum_{n=0}^\infty a_{jn}(z-\zeta_j)^n\right)$$

in a certain $\Delta_0(\zeta_j, \varrho_j)$ $(j=1, \dots, m)$. This determines the column vector $\boldsymbol{t}$ whose components are $t_1, \dots, t_m$. Observe that $l_{\zeta_j} - l_{\zeta_m} \in H(\hat{\boldsymbol{C}} - \{\zeta_j, \zeta_m\})$ $(j=1, \dots, m)$ and vanishes at ∞; here $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, the extended complex plane. Consider the function

$$h(z) = u(z) - \sum_{j=1}^{m-1} t_j(l_{\zeta_j}(z) - l_{\zeta_m}(z)).$$

By (6) and the above remark, we see that $h \in H(\hat{\boldsymbol{C}} - \zeta_m)$, and

$$h(z) = \mathrm{Re}\left(\left(\sum_{j=1}^m t_j\right) l_{\zeta_m}(z) + \sum_{n=0}^\infty a_{mn}(z-\zeta_m)^n\right)$$

in a certain $\Delta_0(\zeta_m, \varrho_m)$. We denote by C the boundary of the disk $\Delta(\zeta_m, \varrho_m/2)$: $|z-\zeta_m| < \varrho_m/2$. Then h is harmonic on $\hat{\boldsymbol{C}} - \bar{\Delta}(\zeta_m, \varrho_m/2)$ and the Gauss theorem assures the vanishing of the flux of h across C. On the other hand this flux is the sum of

$$\left(\sum_{j=1}^m t_j\right) \int_C * dl_{\zeta_m}(z) = 2\pi \sum_{j=1}^m t_j$$

and $\int_C * d\left(\mathrm{Re}\sum_{n=0}^\infty a_{mn}(z-\zeta_m)^n\right) = 0$. Therefore $\sum_{j=1}^m t_j = 0$ and $h \in H(\hat{\boldsymbol{C}})$. Since $h(\infty) = 0$, we conclude that $h \equiv 0$ on $\boldsymbol{C}$, i.e.

$$u \equiv \sum_{j=1}^{m-1} t_j(l_{\zeta_j} - l_{\zeta_m}) = \sum_{j=1}^{m-1} t_j l_{\zeta_j} + \left(-\sum_{j=1}^{m-1} t_j\right) l_{\zeta_m}.$$

In view of $\sum_{j=1}^n t_j = 0$, we obtain $u = h_t$. Therefore

$$u(z) = h_t(z) = \mathrm{Re}\left(\left(\sum_{j=1}^m \zeta_j t_j\right) z^{-1} + \sum_{n=2}^\infty \alpha_n z^{-n}\right)$$

with $\alpha_n = n^{-1}\sum_{j=1}^m \zeta_j^n t_j$. Since $u \in H_2(R)$, (6) implies that $\sum_{j=1}^m \zeta_j t_j = 0$. By taking the real and imaginary parts we deduce $At = \mathbf{0}$, that is $\boldsymbol{t} \in S$. Thus $\boldsymbol{t} \to h_t$: $S \to H_2(R)$ is a linear bijection and we have established (9).

3. Suppose ζ consists of at least four points. Then, since $\operatorname{rank} A(\zeta)\leqq 3$ in any case, we conclude by (9) that $\dim H_2(R)\geqq 4-3=1$. Next suppose ζ consists of three points. Then $\dim H_2(R)=3-\operatorname{rank} A(\zeta)$. Although $\operatorname{rank} A(\zeta)\geqq 1$, the equality here cannot occur since otherwise we would have $\zeta_1=\zeta_2=\zeta_3$ in ζ. Therefore $\operatorname{rank} A=2$ or 3. In the latter case, $\dim H_2(R)=0$. In the former case, $\dim H_2(R)=1$. The relation $\operatorname{rank} A=2$ is, in the present situation, equivalent to $\det A=0$, which in turn is equivalent to ζ_1, ζ_2, and ζ_3 in ζ being collinear. If ζ consists of two points ζ_1, ζ_2, then $\operatorname{rank} A(\zeta)=2$ since $\zeta_1\neq\zeta_2$. Therefore $\dim H_2(R)=2-\operatorname{rank} A(\zeta)=0$. If ζ consists of one point, then $\operatorname{rank} A=1$ and again $\dim H_2(R)=0$. The case $\zeta=\emptyset$ (empty) may be treated directly. In this case, in view of (6), any $u\in H_2(R_\emptyset)$ must belong to $u\in H(\hat{C})$ and $u(\infty)=0$. Therefore $u\equiv 0$, that is $\dim H_2(R_\emptyset)=\dim H_2(C)=0$.

Note that $R\subset R'$ implies $H_2(R')\subset H_2(R)$. We have proved:

Theorem. *The space $H_2(R)$ is degenerate, i.e. $H_2(R)=\{0\}$, if and only if $C-R$ consists of at most two points or three noncollinear points.*

4. In our earlier paper [3] we considered the degenerate class O_{H_2} of Riemannian manifolds M for which $H_2(M)=\{0\}$. We also considered the class O_{SH_2} of those M which contain a subregion $N\in O_{H_2}$ with an exterior point. For *plane regions* R we have thus determined the classes O_{H_2} and O_{SH_2} as follows: $R\in O_{H_2}$ if and only if $C-R$ contains at most two points or three noncollinear points; $O_{SH_2}=\emptyset$.

5. Consider the set $\zeta=\{\zeta_1,\zeta_2,\zeta_3,\dots,\zeta_m\}$, $m\geqq 3$, with the noncollinear points $\zeta_1=1+i$, $\zeta_2=1$, $\zeta_3=0$. We have $\operatorname{rank} A=3$ and

$$\dim H_2(R_\zeta) = m-3.$$

The set $R^{(k)}=R_\zeta$ with $m=3+k$ for $k=0, 1, 2, \dots$ satisfies

$$\dim H_2(R^{(k)}) = k \quad (k = 0, 1, 2, \dots, \aleph). \tag{10}$$

Here $\aleph$ is the countably infinite cardinal number and $R^{(\aleph)}=R_\zeta=C-\zeta$ with $\zeta=\{1+i, 1, 0, \zeta_4, \zeta_5, \dots\}$ closed and all points in ζ distinct. We still have to prove (10) for $K=\aleph$. Let $\zeta(m)=\{1+i, 1, 0, \zeta_4, \zeta_5, \dots, \zeta_m\}$ $(m=4, 5, \dots)$. Observe that area integrals over R_ζ are identical with those over any $R_{\zeta(m)}$ and a fortiori $u\to u|R_\zeta$ is an isometric injection: $H_2(R_{\zeta(m)})\to H_2(R_\zeta)$. Therefore $\dim H_2(R_\zeta)\geqq\dim H_2(R_{\zeta(m)})=m-3$ for every $m=4, 5, \dots$. On the other hand, since $L_2(R)$ is separable for any subregion (and actually for any measurable subset) R of C, $\dim H_2(R)\leqq\aleph$ as a closed subspace of $L_2(R)$ with $\dim L_2(R)=\aleph$. We thus deduce (10) also for $k=\aleph$. In summary:

Theorem. *The dimension of $H_2(R)$ for any plane region R is at most countably infinite and there actually exists a plane region R such that the dimension of $H_2(R)$ is an arbitrarily preassigned countable cardinal number.*

6. Let R be a plane region and $\zeta\in R$. Then $H_2(R)$ is a closed subspace of $H_2(R-\zeta)$. We denote by $H_2(R)_\zeta^\perp$ the orthogonal complement of $H_2(R)$ in $H_2(R-\zeta)$:

$$H_2(R-\zeta)=H_2(R)\oplus H_2(R)_\zeta^\perp. \tag{11}$$

It may happen that $H_2(R)_\zeta^\perp=\{0\}$. In this case we say that (R,ζ) is a *noneffective pair*. Otherwise we assert that $\dim H_2(R)_\zeta^\perp=1$. In fact, suppose $u_1, u_2\in H_2(R)_\zeta^\perp$. In view of (5)

$$u_j(z)=\mathrm{Re}\left(-c_j\log(z-\zeta)+\sum_{n=0}^{\infty}a_{jn}(z-\zeta)^n\right)$$

in a certain $\Delta_0(\zeta,\varrho)$, where $c_j\in \boldsymbol{R}$ with $c_j\neq 0$ $(j=1,2)$. Therefore $c_2u_1-c_1u_2\in H_2(R)$ and we have $(u_j, c_2u_1-c_1u_2)=0$ $(j=1,2)$. From these it follows that $\|c_2u_1-c_1u_2\|_R^2=0$, i.e. u_1 and u_2 are linearly dependent. We say that (R,ζ) is an *effective pair* if it is not noneffective. For an effective pair (R,ζ) we have seen that $H_2(R)_\zeta^\perp$ has a single generator:

$$H_2(R)_\zeta^\perp=\boldsymbol{R}H_R(\cdot,\zeta) \tag{12}$$

where the generator $H_R(\cdot,\zeta)$ is so normalized that

$$H_R(z,\zeta)=\mathrm{Re}\left(-\log(z-\zeta)+\sum_{n=0}^{\infty}a_n(z-\zeta)^n\right). \tag{13}$$

7. We next discuss the effectiveness of a point ζ in a given plane region R. There are three cases:

Case 1. (R,ζ) is a noneffective pair for every $\zeta\in R$.

Case 2. (R,ζ) is an effective pair for some $\zeta\in R$, noneffective for some other $\zeta\in R$.

Case 3. (R,ζ) is an effective pair for every $\zeta\in R$.

We call R a *weak, unstable*, or *strong* region according as case 1, 2, 3 occurs. We remark that for $R\subset R'$, if R' is strong so is R.

Once more we consider $R_{\boldsymbol{\zeta}}=\boldsymbol{C}-\boldsymbol{\zeta}$ with $\boldsymbol{\zeta}$ the set $\{\zeta_1,\dots,\zeta_m\}$ of distinct points in $\boldsymbol{C}$. Let $\zeta\in R_{\boldsymbol{\zeta}}$ be an arbitrary point and let $\boldsymbol{\zeta}'=\boldsymbol{\zeta}\cup\{\zeta\}$. Similarly let $A(\boldsymbol{\zeta})$ and $A(\boldsymbol{\zeta}')$ be the matrices (8) associated with $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$. Then by (9) we deduce

$$\dim H_2(R_{\boldsymbol{\zeta}})_\zeta^\perp=1-(\mathrm{rank}\,A(\boldsymbol{\zeta}')-\mathrm{rank}\,A(\boldsymbol{\zeta})), \tag{14}$$

and conclude that $(R_{\boldsymbol{\zeta}},\zeta)$ is effective if and only if

$$\mathrm{rank}\,A(\boldsymbol{\zeta})=\mathrm{rank}\,A(\boldsymbol{\zeta}'). \tag{15}$$

Suppose $m\geqq 3$. Recall that $\mathrm{rank}\,A(\boldsymbol{\zeta})\geqq 2$. From this we can easily see that (15) is valid for every choice of $\zeta\in R_{\boldsymbol{\zeta}}$ if and only if $\mathrm{rank}\,A(\boldsymbol{\zeta})=3$, i.e. there are three noncollinear points in $\boldsymbol{\zeta}$. If $m\leqq 1$, then $\boldsymbol{\zeta}'$ contains at most two points and therefore $H_2(R_{\boldsymbol{\zeta}}-\zeta)=H_2(R_{\boldsymbol{\zeta}'})=H_2(R_{\boldsymbol{\zeta}})=\{0\}$ and a fortiori $\dim H_2(R)_\zeta^\perp=0$. In the remaining case ζ contains at least two points and ζ is collinear. Let l be the line on which every point of ζ lies. In this case $\operatorname{rank} A(\zeta)=2$. If $\zeta\in l-\zeta$, then $\operatorname{rank} A(\zeta')=2$ and (R_{ζ},ζ) is effective. If $\zeta\notin l$, then $\operatorname{rank} A(\zeta')=3$ and (R_{ζ},ζ) is noneffective.

We summarize our observations thus far:

Theorem. *A plane region R is strong if and only if $C-R$ contains three noncollinear points, R is weak if and only if $C-R$ contains at most one point, and R is unstable if and only if $C-R$ contains at least two points and is a proper subset of a line.*

8. In nos. **8**—**11** we will concentrate on *strong* regions R. We recall that they are characterized by the existence of three noncollinear points $\zeta_1, \zeta_2, \zeta_3$, in $C-R$. The region $R_0=C-\{\zeta_1,\zeta_2,\zeta_3\}$ is of course strong and by virtue of $R_0\supset R$ we have $H_2(R_0)\subset H_2(R)$. For each $\zeta\in R$ we consider the function

$$h(z,\zeta)=\left(\sum_{j=1}^{3} t_j(\zeta)\,l_{\zeta_j}(z)\right)+l_{\zeta}(z) \tag{16}$$

where the $t_j=t_j(\zeta)\in \boldsymbol{R}$ $(j=1,2,3)$ will be later so chosen that $h(\cdot,\zeta)\in H_2(R_0-\zeta)\subset H_2(R-\zeta)$. Let the Laurent expansion of $h(\cdot,\zeta)$ about ∞ be

$$h(z,\zeta)=\operatorname{Re}\left(-\left(\left(\sum_{j=1}^{3} t_j\right)+1\right)\log z+\left(\left(\sum_{j=1}^{3}\zeta_j t_j\right)+\zeta\right)z^{-1}+\sum_{n=2}^{\infty}\alpha_n z^{-n}\right).$$

In order that $h(\cdot,\zeta)\in H_2(R_0-\zeta)$ it is necessary and sufficient that the $t_j=t_j(\zeta)$ $(j=1,2,3)$ satisfy the equation

$$\begin{pmatrix}1 & 1 & 1\\ \operatorname{Re}\zeta_1 & \operatorname{Re}\zeta_2 & \operatorname{Re}\zeta_3\\ \operatorname{Im}\zeta_1 & \operatorname{Im}\zeta_2 & \operatorname{Im}\zeta_3\end{pmatrix}\begin{pmatrix}t_1\\ t_2\\ t_3\end{pmatrix}=-\begin{pmatrix}1\\ \operatorname{Re}\zeta\\ \operatorname{Im}\zeta\end{pmatrix}. \tag{17}$$

The determinant of the first matrix is nonzero since ζ_1,ζ_2,ζ_3 are not collinear. Therefore by the Cramer formula the solution vector with components $t_j=t_j(\zeta)$ $(j=1,2,3)$ in $\boldsymbol{R}$ satisfying (17) is uniquely determined, and continuous (in fact harmonic) with respect to ζ. The function $h(\cdot,\zeta)$ in (16) with the $t_j=t_j(\zeta)$ $(j=1,2,3)$ thus determined by (17) belongs to $H_2(R_0-\zeta)\subset H_2(R-\zeta)$ and we also have $h(z,\cdot)\in H(R-z)$.

9. We next study the continuity of the mapping $\zeta\to h(z,\zeta)$: $R\to L_2(R)$. We can write

$$h(z,\zeta)=\operatorname{Re}\left(\sum_{n=2}^{\infty} n^{-1}\alpha_n(\zeta)z^{-n}\right)$$

in some $\Delta_0(\infty,\varrho)$ with $\alpha_n(\zeta)=\sum_{j=1}^{3} t_j(\zeta)\zeta_j^n+\zeta^n$ $(n=2,3,\ldots)$. Suppose $\zeta,\zeta_j\in\Delta(0,\sigma)$: $|z|<\sigma$ $(j=1,2,3)$. Then the continuity of $\zeta\to t_j(\zeta)$ implies the existence of a constant $K_\sigma>0$ such that $|\alpha_n(\zeta)|\leqq K_\sigma\sigma^n$ $(n=2,3,\ldots)$, and we obtain $|h(z,\zeta)|\leqq K_\sigma\sum_{n=2}^{\infty} n^{-1}\sigma^n|z|^{-n}$ on $\Delta_0(\infty,\sigma)$ with $\varrho>\sigma$. Therefore

$$\|h(\cdot,\zeta)\|_{\Delta_0(\infty,\sigma)}\leqq\sqrt{\pi/2}\,K_\sigma\sigma^2/(\varrho-\sigma). \tag{18}$$

Let $\zeta, \zeta' \in R$ be contained in $\Delta(0, \sigma)$, $\varrho > \sigma$, and $A = \bar{\Delta}(0, \varrho) - \Delta(0, \sigma)$. Then $\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_R^2 \leqq \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_C^2$, which is dominated by the sum of $\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{\Delta(0,\sigma)}^2$, $\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_A^2$, and $\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_{\Delta_0(\infty,\varrho)}^2$. The first of these three terms is dominated by

$$\left(\sum_{j=1}^{3} |t_j(\zeta) - t_j(\zeta')| \, \|l_{\zeta_j}\|_{\Delta(0,\sigma)} + \|l_\zeta - l_{\zeta'}\|_{\Delta(0,\sigma)} \right)^2,$$

the second by

$$\pi(\varrho^2 - \sigma^2) \left(\sup_{z \in A} |h(z, \zeta) - h(z, \zeta')| \right)^2,$$

and the last by $(\|h(\cdot, \zeta)\|_{\Delta_0(\infty,\varrho)} + \|h(\cdot, \zeta')\|_{\Delta_0(\infty,\varrho)})^2$, which is not greater than $2\pi K_\sigma^2 \sigma^4 / (\varrho - \sigma)^2$ by (18). We conclude that

$$\limsup_{\zeta' \to \zeta} \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_R \leqq \sqrt{2\pi} K_\sigma \sigma^2 / (\varrho - \sigma).$$

On letting $\varrho \to +\infty$ we obtain $\lim_{\zeta' \to \zeta} \|h(\cdot, \zeta) - h(\cdot, \zeta')\|_R = 0$.

10. Let $u_\zeta(z) = h(z, \zeta) - H_R(z, \zeta)$. Clearly $u_\zeta \in H_2(R)$ and thus the projection of $h(\cdot, \zeta) \in H_2(R - \zeta)$ on $H_2(R)_\zeta^\perp = \boldsymbol{R} H_R(\cdot, \zeta)$ is $H_R(\cdot, \zeta)$. Observe that

$$\|h(\cdot, \zeta) - h(\cdot, \zeta')\|_R^2 = \|H_R(\cdot, \zeta) - H_R(\cdot, \zeta')\|_R^2 + \|u_\zeta - u_{\zeta'}\|^2.$$

We have obtained the following

Theorem. *On a strong region R*

$$\lim_{\zeta \to \zeta'} \|H_R(\cdot, \zeta) - H_R(\cdot, \zeta')\| = 0. \tag{19}$$

11. Let R' and R'' be subregions of a strong region R, hence strong regions. If $R' \subset R''$, then $H_{R''}(\cdot, \zeta) - H_{R'}(\cdot, \zeta)$ belongs to $H_2(R')$ and is orthogonal to $H_{R'}(\cdot, \zeta)$ over R'. Therefore

$$\|H_{R''}(\cdot, \zeta) - H_{R'}(\cdot, \zeta)\|_{R'}^2 = \|H_{R''}(\cdot, \zeta)\|_{R'}^2 - \|H_{R'}(\cdot, \zeta)\|_{R'}^2. \tag{20}$$

We denote by $\{\Omega\}$ the family of regular subregions Ω of R, a directed set by inclusion, and set $H_\Omega(z, \zeta) = 0$ for $z \in R - \Omega$. In view of (20), $\{\|H_\Omega(\cdot, \zeta)\|_R\}_{\Omega \in \{\Omega\}}$ is an increasing net and $\{H_\Omega(\cdot, \zeta)\}_{\Omega \in \{\Omega\}}$ is a Cauchy net in $L_2(R)$. It is easy to check that the limit is $H_R(\cdot, \zeta)$ and

$$\|H_R(\cdot, \zeta) - H_\Omega(\cdot, \zeta)\|_R^2 = \|H_R(\cdot, \zeta)\|_R^2 - \|H_\Omega(\cdot, \zeta)\|_R^2.$$

On any compact subset E of R, both $\|H_\Omega(\cdot, \zeta)\|_R^2$ $(\Omega \supset E)$ and $\|H_R(\cdot, \zeta)\|_R^2$ are continuous (see (19)). Thus the Dini theorem implies:

Theorem. *If R is strong, then for any compact subset E of R*

$$\lim_{\Omega \to R} \left(\sup_{\zeta \in E} \|H_R(\cdot, \zeta) - H_\Omega(\cdot, \zeta)\|_R \right) = 0. \tag{21}$$

12. We remark that the biharmonic Green's function of the disk $\Delta(0, \varrho)$ is (cf. e.g. Garabedian [2])

$$(22) \qquad \beta_{\Delta(0,\varrho)}(z, \zeta) = \frac{1}{8\pi}\left[|z-\zeta|^2 \log\left|\frac{\varrho(z-\zeta)}{\varrho^2-\bar{\zeta}z}\right| + \frac{1}{2\varrho^2}(|z|^2-\varrho^2)(|\zeta|^2-\varrho^2)\right]$$

on $\Delta(0, \varrho)\times\Delta(0, \varrho)$. Hence, clearly $C\in O_\beta$. Similarly, the biharmonic Green's function of the punctured disk $\Delta_0(0, \varrho)$ is (cf. [4], [5])

$$(23) \qquad \beta_{\Delta_0(0,\varrho)}(z, \zeta) = \beta_{\Delta(0,\varrho)}(z, \zeta) - |6\pi\varrho^{-2}\beta_{\Delta(0,\varrho)}(z, 0)\beta_{\Delta(0,\varrho)}(\zeta, 0)|$$

on $\Delta_0(0, \varrho)\times\Delta_0(0, \varrho)$. Therefore $C-\{0\}\in O_\beta$.

In view of the above, relation (4), and the theorem in no. **7**, the assertions 1° and 3° can easily be deduced. By nos. **9** and **10** the first three assertions in 2° are clear. Since $\beta_\Omega(\cdot, \zeta)$ is biharmonic on $\Omega-\zeta$ and uniformly convergent to $\beta_R(\cdot, \zeta)$ on every compact subset of $R-\zeta$, $\beta_R(\cdot, \zeta)$ is also biharmonic on $R-\zeta$.

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