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ON TOPOLOGICALLY AND QUASICONFORMALLY HOMOGENEOUS CONTINUA

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A subset M of the Riemann sphere is called *quasiconformally homogeneous* if for each pair of points P and Q of M there is a quasiconformal map φ defined in a neighborhood of M such that $\varphi(M)=M$ and $\varphi(P)=Q$. For information about quasiconformal mappings, see [1].

Recently the second author showed [3] that a simple closed curve is quasiconformally homogeneous if and only if it is a quasicircle (i.e., the image of a circle under a quasiconformal map). In this note we prove the following more general result.

Theorem 1. *Every non-degenerate quasiconformally homogeneous continuum is a quasicircle.*

Note that a continuum is called non-degenerate if it is an infinite proper subset of the sphere.

It can be shown by function theoretic methods that a non-degenerate quasiconformally homogeneous continuum must contain an arc. Hence by a theorem of Bing [2] such a continuum is a simple closed curve. However, we prefer an alternative method which combines the result of [3] with a purely topological theorem.

Let M be a proper subcontinuum of S^2. We say that M is *homogeneous with respect to neighborhood extensions* if for each pair of points $x, y \in M$, there exist both a neighborhood U of M in S^2 and a homeomorphism $h\colon U \to S^2$ such that (1) $h(x)=y$ and (2) $h(M)=M$.

By definition, every quasiconformally homogeneous continuum is homogeneous with respect to neighborhood extensions. Thus Theorem 1 follows by Theorem 2 and [3].

Theorem 2. *Let M be a non-degenerate proper subcontinuum of S^2 such that M is homogeneous with respect to neighborhood extensions. Then M is a simple closed curve.*

[1]) The first author is indebted to Morton Brown and F. W. Gehring, both for bringing this problem to her attention and for a number of interesting conversations.

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Proof.

(1) Clearly each point of M must be accessible.

(2) Any indecomposable plane continuum contains inaccessible points by [5].

(3) Thus M contains no indecomposable continuum and is hereditarily decomposable.

(4) Now by Theorem 2 of [4], every homogeneous hereditarily decomposable plane continuum is a simple closed curve.

The theorem follows.

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