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ON DECOMPOSITION OF SOLUTIONS OF SOME HIGHER ORDER ELLIPTIC EQUATIONS

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We study in $\boldsymbol{R}^n$ an equation of the form

$$P(A)u = f, \tag{1}$$

where $P(\xi)=\sum_{\nu=0}^{r} a_\nu \xi^\nu$ denotes a complex polynomial of the degree $r>0$ normalized with $a_r=1$ and A is a partial differential operator of the order $2m$. About the operator

$$Au = \sum_{0\leqq|\alpha|,\,|\beta|\leqq m} (-1)^{|\alpha|}\,\partial^\alpha(a_{\alpha\beta}\,\partial^\beta u)$$

we assume that $a_{\alpha\beta}=a^0_{\alpha\beta}+r_{\alpha\beta}$, where $a^0_{\alpha\beta}$ are constants and the functions $r_{\alpha\beta}$ are infinitely many times differentiable so that they vanish at infinity faster than any negative power of the radius $r=|x|$. It is also required that $a^0_{\alpha\beta}=0$ for all multi-indices $|\alpha|+|\beta|<2m$. The polynomial $P(\xi)$ can be written as the product

$$P(\xi) = \xi^{r_0}\prod_{\varrho=1}^{q}(\xi-k_\varrho^2)^{r_\varrho},$$

where $0\neq k_\varrho=\varkappa_\varrho+i\lambda_\varrho$, $0\leqq\arg k_\varrho<\pi$, $k_\varrho\neq k_\tau$, $\tau\neq\varrho$. Our aim is to reduce the problem (1) to simpler ones by decomposing any solution of (1) into a certain combination of solutions for equations like

$$A^{r_0}w = g,\quad (A-k^2)w = g,\quad k\neq 0.$$

In the cases where these equations have a unique solution in some classes of functions we obtain a unique solution for (1) by fixing the corresponding conditions for the components of u. There are some earlier articles which deal with equations of the above polynomial type in unbounded domains. Vekua [8] studied the equation $P(\Delta)u=0$ and also solved an exterior boundary value problem of the Riquier type. Paneyah [6] considered the corresponding inhomogeneous equation in the whole space (for $r_0=0$). He also pointed out that the Laplace operator could be replaced by a more general second order elliptic operator having constant coefficients. In paper [9] (for $r_0=0$) Witsch allowed A to be a uniformly strongly elliptic second order operator whose coefficients approach those of the Laplace operator at infinity. He was also able to give a Fredholm type theorem for the exterior boundary value problem with homogeneous Dirichlet boundary data.

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This article can be considered as a note to the paper of Witsch. With it we would remark that at least the whole space result remains valid if A is a certain higher order operator. Further, the decomposition given in [9] does not cover the case $r_0>0$. We make use of the general decomposition for a special case where the dimension of the space $\mathbf{R}^n$ is large enough; $n \geqq 2mr_0+1$.

The key to the factorization of the solution is a formula which shows how the operator $Q(A)$ with an arbitrary polynomial Q operates on the powers of the differential expression $\Lambda u = x_i \partial_i u$ ($\Lambda^0 u = u$). For the multi-indices $\alpha=(\alpha_1, \ldots, \alpha_n)$ and $e^i=(e^i_1, \ldots, e^i_n)$, where $e^i_j=\delta_{ij}$, we define

$$\delta[\alpha, e^i] = \begin{cases} 1 & \alpha_i \geqq 1 \\ 0 & \alpha_i = 0. \end{cases}$$

Then

$$\partial^\alpha (x_i u) = x_i \partial^\alpha u + \alpha_i \delta[\alpha, e^i] \partial^{\alpha - e_i} u$$

holds for all multi-indices α. By applying this equality we easily find

$$\partial^\alpha (\Lambda u) = \Lambda \partial^\alpha u + |\alpha| \partial^\alpha u. \tag{2}$$

We make use of some notations in [9]. Let $\mathscr{T}$ (resp. $\hat{\mathscr{T}}$) denote the class of all functions infinitely many times differentiable which vanish at infinity faster than any negative power of r (resp. which grow more slowly than some positive power of r). A differential operator whose coefficients are of the class $\mathscr{T}$ (resp. $\hat{\mathscr{T}}$, C_0^∞) is called $\mathscr{T}$-operator (resp. $\hat{\mathscr{T}}$-, C_0^∞-operator). We write the operator Au as a sum $Au = A^0 u + Ru$ with

$$A^0 u = \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} a^0_{\alpha\beta} \partial^{\alpha+\beta} u,$$

$$Ru = \sum_{0 \leqq |\alpha|, |\beta| \leqq m} (-1)^{|\alpha|} \partial^\alpha (r_{\alpha\beta} \partial^\beta u),$$

where further

$$Ru = \sum_{|\alpha|=|\beta|=m} (-1)^m r_{\alpha\beta} \partial^{\alpha+\beta} u + \tilde{M} u$$

with a $\mathscr{T}$-operator $\tilde{M}u$ of the order $\leqq 2m-1$. By using (2) we then see that

$$A(\Lambda u) = \Lambda(Au) + 2mAu + Mu \tag{3}$$

with a $\mathscr{T}$-operator

$$M = -2m\tilde{M}u - \sum_{|\alpha|=|\beta|=m} (-1)^m (\Lambda r_{\alpha\beta}) \partial^{\alpha+\beta} u + \tilde{M}\Lambda u - \Lambda \tilde{M} u,$$

which is at most of the order $\leqq 2m$. We also see that if the functions $r_{\alpha\beta}$ belong to the class C_0^∞, then M is a C_0^∞-operator.

We denote with d the differential operator $dQ(\xi)=\xi Q'(\xi)$; $d^\mu = d(d^{\mu-1})$, $d^0 Q = Q$ in the ring of all polynomials we can generalize formula (3) as follows:

Lemma 1. *Let Q be a complex polynomial of the degree l. For an arbitrary non-negative integer μ and for a function $u\in C^\infty$ we have*

$$Q(\Lambda)\Lambda^\mu u = \sum_{\nu=0}^{\mu}\binom{\mu}{\nu}\Lambda^\nu[(2md)^{\mu-\nu}Q](\Lambda u)+M_{Q(\xi),\mu}u, \tag{4}$$

where $M_{Q(\xi),\mu}$ is a $\mathcal{T}$-operator of the order $\leqq 2ml+\mu-1$. If $r_{\alpha\beta}\in C_0^\infty$, then $M_{Q(\xi),\mu}$ is a C_0^∞-operator.

Proof. Formula (4) represents only a slight extension of a corresponding result in [9]. For completeness we give the arguments. It is enough to show the validity of (4) for all monomials $Q(\xi)=\xi^l$. But formula (3) shows the validity in the case $Q(\xi)=\xi$, $\mu=1$. The induction on l gives

$$\Lambda^l(\Lambda u) = (2ml+\Lambda)\Lambda^l u+M_{\xi^l,1}u \tag{5}$$

because in the induction step we then have

$$\Lambda^{l+1}(\Lambda u) = (2m(l+1)+\Lambda)(\Lambda^{l+1}u)+\Lambda M_{\xi^l,1}u+M_{\xi,1}\Lambda^l u,$$

where $\Lambda M_{\xi^l,1}+M_{\xi,1}\Lambda^l$ is at most of the order $2m(l+1)$. Through induction on μ we obtain

$$\Lambda^l(\Lambda^\mu u) = (2ml+\Lambda)^\mu\Lambda^l u+M_{\xi^l,\mu}u, \tag{6}$$

where in the induction step we have now by (5) and (6)

$$M_{\xi^l,\mu+1} = (2ml+\Lambda)M_{\xi^l,1}+M_{\xi^l,\mu}\Lambda,$$

which is a $\mathcal{T}$-operator of the order at most $2ml+\mu$. □

By using formula (4) we can prove an extension of a result in [9]:

Theorem 2. *If the integer r_0 is positive, then there exist the $\hat{\mathcal{T}}$-operators $N_{\mu,\nu}$, N, $M^0_{\mu,\nu}$, M^0 and the $\mathcal{T}$-operators $M^{\varrho,\sigma}_{\mu,\nu}$, $M^{\varrho,\sigma}$, where $\mu,\varrho=1,\dots,q$; $\nu=0,\dots,r_\mu-1$; $\sigma=0,\dots,r_\varrho-1$ and the numbers $B^{\mu,\nu}_{j,0}$, $j=0,\dots,\nu$ so that*

(i) *if $u\in C^\infty$ solves equation (1), then the functions*

$$u_{\mu,\nu} = N_{\mu,\nu}u, \tag{7a}$$

$$\zeta = Nu \tag{7b}$$

satisfy the equations

$$(\Lambda-k_\mu^2)u_{\mu,\nu} = g_{\mu,\nu}, \tag{8a}$$

$$\Lambda^{r_0}\zeta = g \tag{8b}$$

with

$$g_{\mu,\nu} = M^0_{\mu,\nu}f+\sum_{\varrho=1}^{q}\sum_{\sigma=0}^{r_\varrho-1}M^{\varrho,\sigma}_{\mu,\nu}u_{\varrho,\sigma}, \tag{9a}$$

$$g = M^0 f+\sum_{\varrho=1}^{q}\sum_{\sigma=0}^{r_\varrho-1}M^{\varrho,\sigma}u_{\varrho,\sigma}, \tag{9b}$$

and the function u has the representation

$$u=\zeta+\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{j=0}^{\nu}B_{j,0}^{\mu,\nu}\Lambda^j u_{\mu,\nu}. \tag{10}$$

(ii) *If, conversely, the functions* $u_{\mu,\nu},\zeta\in C^\infty$ *solve the system of* (8) *and* (9), *then the function u defined by* (10) *satisfies* (1), *and the functions* $u_{\mu,\nu},\zeta$ *can be calculated from* (7).

(iii) *The operators* $M_{\mu,\nu}^{\varrho,\sigma}$ *have the property* $M_{\mu,\nu}^{\varrho,\sigma}=0$ *if lexically* $(\varrho,\sigma)\geqq(\mu,\nu)$. *If* $r_{\alpha\beta}\in C_0^\infty$ *then the operators* $M_{\mu,\nu}^{\varrho,\sigma}$, $M^{\varrho,\sigma}$ *are* C_0^∞*-operators. In the case* $r_0=0$ *the operator N is absent and we must use only* (7a), (8a), (9a) *and* (10) *without the function* ζ.

Proof. Assume first that $r_0=0$. In this case the argument follows [9] without any essential modification. It is also easy to verify from the proof that the operators $M_{\mu,\nu}^{\varrho,\sigma}$ are C_0^∞-operators if the functions $r_{\alpha\beta}$ have finite supports. Suppose then that $r_0>0$ and let $u\in C^\infty$ be a function which satisfies (1). Denote $P(\xi)=\xi^{r_0}Q(\xi)$ and $v=\Lambda^{r_0}u$, which gives $Q(\Lambda)v=f$. According to the first case there exist $\hat{\mathscr{T}}$-operators $\hat N_{\mu,\nu}$, $\hat M_{\mu,\nu}^0$ and $\mathscr{T}$-operators $\hat M_{\mu,\nu}^{\varrho,\sigma}$ such that

$$v=\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\Lambda^\nu v_{\mu,\nu} \tag{11}$$

with

$$v_{\mu,\nu}=\hat N_{\mu,\nu}v,\quad (\Lambda-k_\mu^2)v_{\mu,\nu}=g_{\mu,\nu} \tag{12}$$

$$g_{\mu,\nu}=\hat M_{\mu,\nu}^0 f+\sum_{\varrho=1}^{q}\sum_{\sigma=0}^{r_\varrho-1}\hat M_{\mu,\nu}^{\varrho,\sigma}v_{\varrho,\sigma}, \tag{13}$$

where $\hat M_{\mu,\nu}^{\varrho,\sigma}=0$, $(\varrho,\sigma)\geqq(\mu,\nu)$.

We show now that one can choose the numbers $B_{j,0}^{\mu,\nu}$, $j=0,\ldots,\nu$ such that the function

$$w=\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{j=0}^{\nu}B_{j,0}^{\mu,\nu}\Lambda^j v_{\mu,\nu} \tag{14}$$

satisfies the equation

$$\Lambda^{r_0}w=v-\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\hat M^{\mu,\nu}v_{\mu,\nu}-M^0 f, \tag{15}$$

where $\hat M^{\mu,\nu}$ $(\mu=1,\ldots,q;\ \nu=0,1,\ldots,r_\mu-1)$ are $\mathscr{T}$-operators and M^0 is a $\hat{\mathscr{T}}$-operator. Using the formulae (4) and (12) we get

$$\begin{aligned}\Lambda(\Lambda^j v_{\mu,\nu})&=\sum_{\alpha=0}^{j}\binom{j}{\alpha}(2m)^{j-\alpha}\Lambda^\alpha(\Lambda v_{\mu,\nu})+M_{\xi,j}v_{\mu,\nu}\\&=k_\mu^2\sum_{\alpha=0}^{j}\binom{j}{\alpha}(2m)^{j-\alpha}\Lambda^\alpha v_{\mu,\nu}+L_j^{\mu,\nu}g_{\mu,\nu}+M_{\xi,j}v_{\mu,\nu}\end{aligned} \tag{16}$$

with an $\hat{\mathscr{T}}$-operator $L_j^{\mu,\nu}$. Noting that the product of a $\mathscr{T}$-operator and a $\hat{\mathscr{T}}$-operator is a $\mathscr{T}$-operator we get through the formulae (14), (16) and (13)

$$\Lambda w=\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{\alpha=0}^{\nu}B_{\alpha,1}^{\mu,\nu}\Lambda^\alpha v_{\mu,\nu}-M_1 f-\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\hat M_1^{\mu,\nu}v_{\mu,\nu} \tag{17}$$

with a $\hat{\mathscr{T}}$-operator M_1 and $\mathscr{T}$-operators $\hat{M}_1^{\mu,\nu}$, where $\hat{M}_1^{\mu,\nu}$ are C_0^∞-operators if $r_{\alpha\beta}\in C_0^\infty$. Here the numbers $B_{\alpha,1}^{\mu,\nu}$ can be calculated by means of the formula

(18) $$B_{\alpha,1}^{\mu,\nu} = k_\mu^2 \sum_{j=\alpha}^{\nu} B_{j,0}^{\mu,\nu}(2m)^{j-\alpha}\binom{j}{\alpha}, \quad 0 \leqq \alpha \leqq \nu.$$

This system has the form

$$\begin{aligned} B_{\nu,1}^{\mu,\nu} &= k_\mu^2 B_{\nu,0}^{\mu,\nu}, \\ B_{\nu-1,1}^{\mu,\nu} &= k_\mu^2\left\{B_{\nu-1,0}^{\mu,\nu}+2m\binom{\nu}{\nu-1} B_{\nu,0}^{\mu,\nu}\right\}, \\ &\vdots \\ B_{\alpha,1}^{\mu,\nu} &= k_\mu^2\left\{B_{\alpha,0}^{\mu,\nu}+2m\binom{\alpha+1}{\alpha} B_{\alpha+1,0}^{\mu,\nu}+\ldots+(2m)^{\nu-\alpha}\binom{\nu}{\alpha} B_{\nu,0}^{\mu,\nu}\right\} \end{aligned}$$

and is therefore uniquely solvable; if the numbers $B_{\alpha,1}^{\mu,\nu}$ are known, then the numbers $B_{j,0}^{\mu,\nu}$ are uniquely defined. When we apply again the operator A to the equation (17) $(l-1)$-times, we can denote generally

(19) $$A^l w = \sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{\alpha=0}^{\nu} B_{\alpha,l}^{\mu,\nu} A^\alpha v_{\mu,\nu} - M_l f - \sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1} \hat{M}_l^{\mu,\nu} v_{\mu,\nu}$$

with a $\hat{\mathscr{T}}$-operator M_l and with the $\mathscr{T}$-operators $\hat{M}_l^{\mu,\nu}$ together with the coefficients

(20) $$B_{\alpha,l}^{\mu,\nu} = k_\mu^2 \sum_{j=\alpha}^{\nu} B_{j,l-1}^{\mu,\nu}(2m)^{j-\alpha}\binom{j}{\alpha}.$$

This system is uniquely solvable as above. Choosing at the stage $l=r_0$

(21) $$B_{\nu,r_0}^{\mu,\nu} = 1, \quad B_{\alpha,r_0}^{\mu,\nu} = 0, \quad 0 \leqq \alpha < \nu$$

and, accordingly, the constants $B_{\alpha,l}^{\mu,\nu}$, $l=0, \ldots, r_0-1$ such that the equation (20) is valid for every $l=1, \ldots, r_0$, we obtain from (14) the function w which satisfies the equation (15) with $M^0=M_{r_0}$, $\hat{M}^{\mu,\nu}=\hat{M}_{r_0}^{\mu,\nu}$. For the function $\zeta=u-w$ we then have

(22) $$A^{r_0}\zeta = A^{r_0}u - A^{r_0}w = \sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1} \hat{M}^{\mu,\nu} v_{\mu,\nu} + M^0 f.$$

Choosing the operators

(23) $$\begin{cases} M^{\varrho,\sigma} = \hat{M}^{\varrho,\sigma} \\ N_{\mu,\nu} = \hat{N}_{\mu,\nu} A^{r_0} \\ N = 1 - \sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{j=0}^{\nu} B_{j,0}^{\mu,\nu} A^j N_{\mu,\nu} \\ M_{\mu,\nu}^0 = \hat{M}_{\mu,\nu}^0, \quad M_{\mu,\nu}^{\varrho,\sigma} = \hat{M}_{\mu,\nu}^{\varrho,\sigma} \end{cases}$$

we see that the functions $u_{\mu,\nu}=N_{\mu,\nu}u$, $\zeta=Nu$ satisfy the equations (8) and (9) and that u has the representation (10). If, conversely, $u_{\mu,\nu}$, $\zeta\in C^\infty$ are functions satisfying the equations (8) and (9), and if $u=\zeta+w$ is defined by (10), we have

$$A^{r_0}w = v-\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1} M^{\mu,\nu}u_{\mu,\nu}-M^0 f \tag{24}$$

with $v=\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\Lambda^\nu u_{\mu,\nu}$, and hence further by (9) and (24)

$$A^{r_0}u = v. \tag{25}$$

On the other hand, we have from the case $r_0=0$ the relations $Q(A)v=f$, $u_{\mu,\nu}=\hat{N}_{\mu,\nu}v$ and therefore also

$$P(A)u = Q(A)v = f$$

with $u_{\mu,\nu}=\hat{N}_{\mu,\nu}v=\hat{N}_{\mu,\nu}A^{r_0}u=N_{\mu,\nu}u$. Finally we obtain

$$\zeta = u-\sum_{\mu=1}^{q}\sum_{\nu=0}^{r_\mu-1}\sum_{j=0}^{\nu} B_{j,0}^{\mu,\nu}A^j u_{\mu,\nu} = Nu. \qquad \square$$

Let us assume that the coefficients satisfy $a_{\alpha\beta}(x)=\bar{a}_{\beta\alpha}(x)$ and that the operator A is uniformly strongly elliptic so that

$$\sum_{|\alpha|,\,|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha\xi^\beta \geqq c|\xi|^{2m}, \quad c>0$$

for every $x\in\boldsymbol{R}^n$, $\xi\in\boldsymbol{R}^n$. For a given A we define for every complex number z an operator A_0+z in $L^2=L^2(\boldsymbol{R}^n)$ with the domain

$$D(A_0+z) = \{u\in H^m|\exists f\in L^2\colon\ \forall\varphi\in C_0^\infty\ B_z(u,\varphi) = (f|\varphi)_0\}$$

(which in fact is independent of z) and define further $A_0u+zu=f$ for $u\in D(A_0+z)$. Here we have used the defining formula

$$B_z(u,\varphi) = \sum_{0\leqq|\alpha|,\,|\beta|\leqq m} (a_{\alpha\beta}\partial^\beta u|\partial^\alpha\varphi)_0+z(u|\varphi)_0$$

for the sesquilinear form $B_z\colon H^m\times H^m\to\boldsymbol{C}$ with $H^m=H^m(\boldsymbol{R}^n)=H_0^m(\boldsymbol{R}^n)$. The operator A_0 is symmetric and by the inequality of Gårding ([1])

$$\operatorname{Re}B_{-k^2}(u,u) \geqq c_1\|u\|_m^2-c_2\|u\|_0^2$$

with $c_1>0$, $c_2\geqq 0$ as well as by the formula

$$\operatorname{Im}B_{-k^2}(u,u) = -2\lambda\varkappa\|u\|_0^2$$

we get in the case $\lambda\neq 0$, $\varkappa\neq 0$, $k=\varkappa+i\lambda$ for a sufficiently small number $0<\eta\leqq 1$ for all $u\in H^m$

$$\begin{aligned}|B_{-k^2}(u,u)| &\geqq \frac{1}{\sqrt{2}}(\eta|\operatorname{Re}B_{-k^2}(u,u)|+|\operatorname{Im}B_{-k^2}(u,u)|)\\ &\geqq c_3\|u\|_m^2\end{aligned}$$

with a positive number $c_3 = c_3(\eta)$. According to well-known arguments it then holds for the ranges $R(A_0 - k^2) = L^2$ if $\varkappa \neq 0$, $\lambda \neq 0$. The operator A_0 is therefore selfadjoint ([2]). We assume that the operator A_0 is also positive, in other words, $(A_0 u|u)_0 \geqq 0$ for every $u \in D(A_0)$. If we then take $f \in L^2$ and $k = i\lambda$, $\lambda > 0$, there exists exactly one function $u \in D(A_0)$ with $A_0 u - k^2 u = f$. If in addition $f \in C^\infty$, then the function u is also regular and satisfies the equation $Au - k^2 u = f$ in the classical sense.

In the case $k > 0$ we utilize a result of Vainberg ([7]). Let

$$Q_k(\xi) = \sum_{|\alpha| = |\beta| = m} a^0_{\alpha\beta} \xi^{\alpha+\beta} - k^2$$

be the characteristic polynomial of the operator $A^0 - k^2$ and suppose that A^0 is elliptic with $a^0_{\alpha\beta} = \overline{a^0_{\alpha\beta}}$. Denote by N_k the set of the zeros for Q_k in $\boldsymbol{R}^n$. It is then easy to see that N_k is compact, connected and non-empty, so that $\operatorname{grad}_\xi Q(\xi) \neq 0$ if $\xi \in N_k$. Hence N_k is also a smooth $(n-1)$-dimensional surface. In order to use [7] we must consider operators where the part Ru containing variable coefficients depends on a parameter. Let therefore R^0 be a differential operator of the order at most $2m$ such that its coefficients are in C_0^∞. Denote with D the open set of points $\varepsilon \in \boldsymbol{C}$ where the operator $A^0 + \varepsilon R^0$ is uniformly strongly elliptic and let D_0 stand for the connected component of D which contains the origin. If the total curvature (Gaussian curvature) vanishes at no point of N_k, then for any $f \in C_0^\infty$ the equation

$$(A^0 + \varepsilon R^0) u - k^2 u = f$$

has for almost all values $\varepsilon \in D_0$ (apart from a discrete set), especially including $\varepsilon = 0$, a unique solution $u \in C^\infty$ satisfying for $r > 0$

$$|u(x)| \leqq Cr^{(1-n)/2}, \quad \left| \frac{\partial}{\partial r} u - i\mu(\omega) u(x) \right| \leqq Cr^{-n/2} \tag{26}$$

with a $C > 0$. Here $\mu(\omega) = (\sigma(\omega)|\omega)$ with $\omega = x/r$ and $\sigma(\omega)$ is the point on the surface where a continuously chosen normal has the same direction as ω.

To put all the foregoing things together we require the following:

(27) 1) $A^0 u = (-1)^m \sum_{|\alpha| = |\beta| = m} a^0_{\alpha\beta} \partial^{\alpha+\beta} u, \quad a^0_{\alpha\beta} = \overline{a^0_{\beta\alpha}},$

is strongly elliptic.

2) The total curvature of the surfaces N_k, $k > 0$ does not vanish.

3) $R^0 u = \sum_{|\alpha|, |\beta| \leqq m} (-1)^{|\alpha|} \partial^\alpha (r^0_{\alpha\beta} \partial^\beta u), \quad r^0_{\alpha\beta} = \overline{r^0_{\beta\alpha}} \in C_0^\infty$

satisfies $(R^0 \varphi | \varphi)_0 \geqq 0$ for every $\varphi \in C_0^\infty$.

In particular, assumptions 1) and 3) imply that the corresponding operator $(A^0 + \varepsilon R^0)_0$ in L^2 is positive for every $\varepsilon \geqq 0$.

Theorem 3. *Let $A=A^0+\varepsilon R^0$ be a differential operator such that assumptions 1)—3) are valid. Then apart from a discrete set of points $\varepsilon\geqq 0$ including $\varepsilon=0$ the equation $P(A)u=f\in C_0^\infty$ has for any polynomial $P(\xi)$ which does not vanish at the origin, a unique solution $u\in\hat{\mathscr{T}}$ such that $N_{\mu\nu}u\in H^m$ for* $\operatorname{Im} k_\mu>0$ *and $N_{\mu\nu}u$ satisfies* (26) *for $k_\mu>0$.*

Proof. It suffices to note that the solution u for the equation $(A-k^2)u=g\in C_0^\infty$, where $u\in H^m$ if $\operatorname{Im} k>0$ and where u satisfies (26) if $k>0$, belongs also to the class $\hat{\mathscr{T}}$. In the first case the characteristic polynomial for A^0-k^2 does not have real zeros and there exists a fundamental solution E for the equation $A^0u-k^2u=h$ which approaches zero exponentially at infinity ([5]). In the second case the equation $A^0u-k^2u=h$ has a fundamental solution E which satisfies (26) for $|x|\geqq R>0$ so that every solution u which satisfies also (26) has the form $u=E*h$ ([7]). Convolving the equation $A^0u(x)+\varepsilon R^0u(x)=g(x)$ with E we obtain

$$u(x)=-\varepsilon\int_{|y|\leqq R_0}E(x-y)R^0u(y)\,dy+\int_{|y|\leqq R_0}E(x-y)g(y)\,dy$$

if the functions $r_{\alpha\beta}$, g vanish for $|x|\geqq R_0$. For $|x|\geqq 2R_0$ we get

$$\partial^\alpha u(x)=-\varepsilon\int_{|y|\leqq R_0}E(x-y)\,\partial^\alpha R^0u(y)\,dy+\int_{|y|\leqq R_0}E(x-y)\,\partial^\alpha g(y)\,dy$$

and therefore

$$|\partial^\alpha u(x)|\leqq c\,|x|^{-(n-1)/2},$$

which implies $u\in\hat{\mathscr{T}}$. In the case $\operatorname{Im} k>0$ we conclude even $u\in\mathscr{T}$. We can now obtain the unique solution $u\in\hat{\mathscr{T}}$ by solving the system of (8a) and (9a), starting from the indices $(\mu,\nu)=(1,0)$ and moving in the general step from the pair (μ,ν) to the pair $(\mu,\nu+1)$ if $\nu\leqq r_\mu-2$ and to $(\mu+1,0)$ if $\nu=r_\mu-1$. Because of (iii) the function $g_{\mu\nu}$ can always be calculated from the known functions and $g_{\mu\nu}$ belongs to the class C_0^∞ since $f\in C_0^\infty$ and $M_{\mu,\nu}^{\varrho,\sigma}$ are C_0^∞-operators. □

We are not able to solve an equation of the type $A^{r_0}u=f$ uniquely for a general $r_0>0$. In the following we assume that the dimension of the space $\boldsymbol{R}^n$ is sufficiently large; $n\geqq 2mr_0+1$. Let H_k denote the completion of C_0^∞ with respect to the norm $|||\cdot|||_k$,

$$|||u|||_k^2=\sum_{\nu=0}^{k}\sum_{|\alpha|=\nu}\left\|\frac{\partial^\alpha u}{(1+|x|)^{k-\nu}}\right\|_0^2,$$

and let $|\cdot|_{k,G}$ be the usual seminorm,

$$|u|_{k,G}^2=\sum_{|\alpha|=k}\|\partial^\alpha u\|_{0,G}^2.$$

With $|u|_k=|u|_{k,\boldsymbol{R}^n}$ we get

Lemma 4. *Let $n\geqq 2k+1$, $k>0$. Then there exists a constant $\gamma=\gamma(n,k)>0$ such that the inequality*

(28) $$|||u|||_k\leqq\gamma\,|u|_k$$

is valid for every $u\in H_k$.

Proof. For the technique of the following argumentation see [4]. Denote $B(r, \varrho)=\{x\in \mathbf{R}^n \mid r<|x|<\varrho\}$ with $0\leqq r<\varrho\leqq\infty$. Take $1\leqq l\leqq k$, $B=B(1/2, \infty)$ and $v\in C_0^\infty(B)$. Partial integration gives

$$\left\| \frac{\nabla v}{|x|^{l-1}}+s\frac{x}{|x|}\frac{v}{|x|^l}\right\|_{0,B}^2=\left\|\frac{\nabla v}{|x|^{l-1}}\right\|_{0,B}^2+s[s-(n-2l)]\left\|\frac{v}{|x|^l}\right\|_{0,B}^2$$

for every real number s. Choosing $s=n-2l\neq 0$ we get

$$s\left\|\frac{v}{|x|^l}\right\|_{0,B}\leqq\left\|\frac{\nabla v}{|x|^{l-1}}+s\frac{x}{|x|}\frac{v}{|x|^l}\right\|_{0,B}+\left\|\frac{\nabla v}{|x|^{l-1}}\right\|_{0,B}\leqq 2\left\|\frac{\nabla v}{|x|^{l-1}}\right\|_{0,B}$$

and further by induction

$$\left\|\frac{v}{|x|^k}\right\|_{0,B}\leqq c_1|v|_{k,B}. \tag{29}$$

We fix a test function φ supported in $B_1(0, 2)=\{x\in\mathbf{R}^n \mid |x|<2\}$ such that $\varphi(x)\equiv 1$, $|x|\leqq 1$. By using (29) and ([4]: Lemma 3.6) we obtain

$$\begin{aligned}\left\|\frac{(1-\varphi)u}{(1+|x|)^k}\right\|_0 &\leqq c_2\left\|\frac{(1-\varphi)u}{|x|^k}\right\|_{0,B(1/2,\infty)}\leqq c_3|(1-\varphi)u|_{k,B(1/2,\infty)}\\ &\leqq c_4(\|u\|_{k-1,B(1,2)}+|u|_k)\leqq c_5|u|_k\end{aligned} \tag{30}$$

for $n\geqq 2k+1$. On the other hand, the Poincaré inequality gives

$$\left\|\frac{\varphi u}{(1+|x|)^k}\right\|_0\leqq c_6|\varphi u|_{k,B_1(0,2)}\leqq c_7(\|u\|_{k-1,B(1,2)}+|u|_{k,B_1(0,2)})\leqq c_8|u|_k. \tag{31}$$

From (30), (31) we get

$$\left\|\frac{u}{(1+|x|)^k}\right\|_0\leqq c_9|u|_k,$$

which easily implies (28) in C_0^∞ and thus in H_k. □

To solve the equation $A^{r_0}u=f$ we define a $|||\cdot|||_{mr_0}$-bounded sesquilinear form $B_{r_0}\colon H_{mr_0}\times H_{mr_0}\to\mathbf{C}$ by the formulae

$$B_1(u, v)=\sum_{0\leqq|\alpha|,|\beta|\leqq m}(a_{\alpha\beta}\partial^\beta u|\partial^\alpha v)_0,$$

$$B_{2l}(u, v)=(A^l u|A^l v)_0,\quad l=1, 2, \dots,$$

$$B_{2l+1}(u, v)=B_1(A^l u, A^l v),\quad l=1, 2, \dots.$$

When we write $A=A^0+\varepsilon R^0$ and $r_0=2l$ the form $B_{r_0}(u, v)$ becomes

$$B_{r_0}(u, v)=((A^0)^l u|(A^0)^l v)_0+\varepsilon\tilde{B}_{r_0}(u, v; \varepsilon),$$

where $\tilde{B}_{r_0}(u, v; \varepsilon)$ is a $|||\cdot|||_{mr_0}$-bounded sesquilinear expression with

$$\begin{aligned}\sup_{0\leqq\varepsilon\leqq 1}|\tilde{B}_{r_0}(u, v; \varepsilon)| &\leqq c_{10}|||u|||_{mr_0}|||v|||_{mr_0}\\ &\leqq c_{11}|u|_{mr_0}|v|_{mr_0}\end{aligned}$$

for every $u, v \in C_0^\infty$. On the other hand, we have in C_0^∞

$$((A^0)^l u | (A^0)^l u)_0 \geqq c_{12} |u|^2_{mr_0}$$

([1]: Lemma 7.7). Hence for a sufficiently small $\varepsilon_0 > 0$

$$B_{r_0}(u, u) \geqq (c_{12} - \varepsilon c_{10}) |u|^2_{mr_0} \geqq c_{13} |||u|||^2_{mr_0} \tag{32}$$

holds with a positive number c_{13} for every $0 \leqq \varepsilon \leqq \varepsilon_0$, $u \in H_{mr_0}$. We can prove the inequality (32) analogously in the case $r_0 = 2l+1$, $l \geqq 0$. If $f \in \mathscr{T}$, then the scalar product $(u|f)_0$ is continuous in H_{mr_0}:

$$|(u|f)_0| \leqq \|(1+|x|)^{-mr_0} u\|_0 \|(1+|x|)^{mr_0} f\|_0$$
$$\leqq |||u|||_{mr_0} \|(1+|x|)^{mr_0} f\|_0;$$

for this reason the equation $A^{r_0} u = f$ has a unique solution $u \in H_{mr_0} \cap C^\infty$ by the theorem of Lax—Milgram.

Theorem 5. *Let $A = A^0 + \varepsilon R^0$ be a differential operator such that assumptions 1)—3) are valid and let $n \geqq 2mr_0 + 1$. Then apart from a discrete set of points $0 \leqq \varepsilon \leqq \varepsilon_0$ including $\varepsilon = 0$ the equation $P(A)u = f \in C_0^\infty$ for a sufficiently small $\varepsilon_0 > 0$ has a unique solution $u \in \hat{\mathscr{T}}$; for $\operatorname{Im} k_\mu > 0$ $N_{\mu\nu} \in H^m$ and for $k_\mu > 0$ $N_{\mu\nu} u$ satisfies (26) and Nu belongs to H_{mr_0}.*

Proof. We only have to show that a solution $u \in H_{mr_0} \cap C^\infty$ of $A^{r_0} u = f \in C_0^\infty$ belongs to $\hat{\mathscr{T}}$. But this follows exactly as in [9] because we have

$$|||u|||_{mr_0} \leqq c \|(1+|x|)^{mr_0} f\|_0$$

with a number c independent of u. □

It may be pointed out that if [3] is used instead of [7], a stronger result can be obtained in the second order case.

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