Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 4, 1978/1979, 279—291

LOWER BOUNDS FOR THE MODULI OF PATH FAMILIES WITH APPLICATIONS TO NON-TANGENTIAL LIMITS OF QUASICONFORMAL MAPPINGS

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1. Introduction

Given a set $E\subset R^n$ and a point $x\in R^n$, $n\geqq 2$, we denote by $\operatorname{cap}\underline{\operatorname{dens}}(E, x)$ and $\operatorname{cap}\overline{\operatorname{dens}}(E, x)$ the lower and upper n-capacity densities of E at x. These concepts will be defined in Section 2 by means of n-moduli of path families, and therefore one could as well regard these as "lower and upper n-modulus densities". For an alternative definition involving n-capacities of condensers, we refer the reader to Martio and Sarvas [7] and to Remark 2.6.

Let now E_1 and E_2 be two sets in R^n with $\operatorname{cap}\underline{\operatorname{dens}}(E_j, 0)=\delta_j>0$, $j=1, 2$, and for $r>0$ let Γ_r denote the path family whose elements join E_1 and E_2 in $R^n\setminus\bar{B}^n(r)$ in the sense of Section 2, and let $M(\Gamma_r)$ denote the n-modulus of Γ_r. Our main result is the following lower bound for $M(\Gamma_r)$: there exists a constant $c>0$ depending only on δ_1, δ_2, and n such that for small $r>0$

$$M(\Gamma_r)\geqq c\log\frac{1}{r}. \tag{1.1}$$

This lower bound is well known only in some particular cases, e.g. when E_1 and E_2 are connected sets joining 0 and the boundary of the unit ball B^n. The estimate (1.1), together with other lower bounds of Section 3, is proved by means of the so-called *comparison principle* for the modulus. The comparison principle was introduced by Näkki in [8] and it is closely related to a lemma of Martio, Rickman, and Väisälä [6, 3.11].

In Section 4 we shall use the method of Section 3 to study the following problem. Let f be a quasiconformal mapping of B^n, let $b\in\partial B^n$, let $E_j\subset B^n$ be a set with $b\in\bar{E}_j$, $j=1, 2$, and assume that $f(x)$ tends to a limit α_j as x approaches b through E_j, $j=1, 2$. How thick must the sets E_j be at b in order that $\alpha_1=\alpha_2$? It is easy to see that this is the case if E_1 and E_2 are non-degenerate connected sets. We shall show that even the considerably weaker conditions $\operatorname{cap}\underline{\operatorname{dens}}(E_1, b)>0$ and $\operatorname{cap}\overline{\operatorname{dens}}(E_2, b)>0$ imply $\alpha_1=\alpha_2$. As regards the sharpness of these conditions,

doi:10.5186/aasfm.1978-79.0430

we shall show that the former condition cannot be replaced by the weaker condition $\operatorname{cap}\overline{\operatorname{dens}}(E_1, b)>0$. Problems of this kind are related to the results of [13], and the main result of Section 4, Theorem 4.12, gives us a new proof for a quasiconformal version of J. L. Doob's theorem [1, Theorem 4] (cf. also [13, Section 5]).

The results of this paper were announced in [14], where also an application of (1.1) to quasiregular mappings was given.

2. Preliminary results

2.1. *Notation.* Throughout the paper we assume that n is a fixed integer and $n \geqq 2$. We denote the n-dimensional euclidean space by R^n and its one-point compactification by $\bar{R}^n = R^n \cup \{\infty\}$. All topological operations are performed with respect to $\bar{R}^n$ unless otherwise mentioned. Balls and spheres centered at $x \in R^n$ and with radius $r>0$ are denoted, respectively, by

$$B^n(x, r) = \{z \in R^n \colon |z-x| < r\},$$

$$S^{n-1}(x, r) = \{z \in R^n \colon |z-x| = r\}.$$

We employ the abbreviations $B^n(r)=B^n(0, r)$, $B^n=B^n(1)$, $S^{n-1}(r)=S^{n-1}(0, r)$, and $S^{n-1}=S^{n-1}(1)$. For $r>s>0$ we denote the spherical ring $B^n(r)\setminus \bar{B}^n(s)$ by $R(r, s)$.

2.2. *Path families and their modulus.* A *path* is a continuous nonconstant mapping $\gamma\colon [0, 1] \to A$, where A is a subset of $\bar{R}^n$. The point set $\gamma[0, 1]$ will be denoted by $|\gamma|$. Given sets E, F, and G in $\bar{R}^n$, we let $\Delta(E, F; G)$ denote the family of all paths γ joining E and F in G in the following sense: $\gamma(0) \in E$, $\gamma(1) \in F$ and $|\gamma| \subset G$. For the definition and basic properties of the *(n-)modulus* $M(\Gamma)$ of a path family Γ we refer the reader to Väisälä's book [10, Chapter 1]. Given a set $E \subset R^n$, $r>0$, and $x \in R^n$, we introduce the abbreviation

$$M(E, r, x) = M(\Delta(S^{n-1}(x, 2r), \bar{B}^n(x, r) \cap E; R^n)). \tag{2.3}$$

Let $u \in R^n$ and $0<a<b$ and let Γ be a path family such that $\overline{|\gamma|} \cap S^{n-1}(u, a) \neq \emptyset \neq \overline{|\gamma|} \cap S^{n-1}(u, b)$ for every $\gamma \in \Gamma$. Then the upper bound

$$M(\Gamma) \leqq \omega_{n-1}\left(\log\frac{b}{a}\right)^{1-n} \tag{2.4}$$

holds [10, 6.4, 7.5] and here ω_{n-1} is the surface area of S^{n-1}.

If $E \subset R^n$ and $x \in R^n$, we define the *lower* and *upper (n-)capacity densities* of E at x by

$$\begin{aligned} \operatorname{cap}\underline{\operatorname{dens}}(E, x) &= \liminf_{r\to 0} M(E, r, x), \\ \operatorname{cap}\overline{\operatorname{dens}}(E, x) &= \limsup_{r\to 0} M(E, r, x). \end{aligned} \tag{2.5}$$

2.6. Remark. Martio and Sarvas considered in [7] the condition $\text{cap}\,\overline{\text{dens}}\,(E, x)=0$ for compact E. The definition in [7] was based on the use of condensers and their n-capacities. It follows from Ziemer [15] that the definition of Martio and Sarvas is, for compact E, equivalent to (2.5).

The most important lower bounds for the moduli of path families are given by the following lemma. This result is often called the (spherical) *cap-inequality* and was proved by Gehring (cf. [10, Chapter 10]).

2.7. Lemma. *Let E and F be disjoint non-empty subsets of the sphere $S=S^{n-1}(x, r)$ and let M^S be the n-modulus on S. Then*

$$M^S(\Delta(E, F; S)) \geqq c_n/r,$$

where c_n is a positive constant, as in [10, (10.11)], *depending only on n.*

Throughout the entire paper we let c_n denote this constant. The cap-inequality yields the following standard lower bounds for the quantities $M(E, r, 0)$, which will be frequently used in the sequel.

The euclidean diameter of $A\subset R^n$ is denoted by $d(A)$.

2.8. Lemma. *Let E be a set in R^n and let $r>0$. Suppose that there is a connected set $E_r\subset \bar{B}^n(r)\cap E$. Then*

$$M(E, r, 0) \geqq c_n \log\frac{4r+d(E_r)}{4r-d(E_r)}. \tag{1}$$

If $\bar{E}_r\cap S^{n-1}(r)\neq\emptyset$ and $\bar{E}_r\cap S^{n-1}(s)\neq\emptyset$ for some $s\in(0, r)$, then

$$M(E, r, 0) \geqq c_n \log\frac{2r-s}{r}. \tag{2}$$

Proof. The lemma was proved in [13]. For completeness we will prove (2). To prove the second inequality fix $u\in\bar{E}_r\cap S^{n-1}(s)$ and $v\in\bar{E}_r\cap S^{n-1}(r)$ and choose a line L through u and v. Let $w\in L\cap S^{n-1}(2r)$ be such that $|v-w|\leqq|u-w|$. Let p and q denote the lengths of the projections of $u-v$ and $v-w$ on the line through 0 and v. We get by the cap-inequality (cf. [10, 10.12])

$$M(E, r, 0) \geqq c_n \log\frac{|u-v|+|v-w|}{|v-w|} \geqq c_n \log\left(\frac{p}{q}+1\right) \geqq c_n \log\frac{2r-s}{r},$$

where we have applied the obvious estimate $p/q\geqq(r-s)/r$.

Lemma 2.8 gives us an example of a situation where one obtains a lower bound for the modulus of a path family joining two sets by means of the cap-inequality. In many cases this is not possible; see e.g. the situation described at the beginning of Section 3. In such cases we shall apply the next lemma, which, following Näkki [8, 3.3], we shall call *the comparison principle for the modulus.* Martio, Rickman, and Väisälä have used the idea behind Lemma 2.9 in the proof of Lemma 3.11 in [6].

2.9. Lemma. *Let F_1, F_2, and F_3 be three sets in $\bar{R}^n$ and write $\Gamma_{ij}=\Delta(F_i, F_j; R^n)$, $1\leqq i, j\leqq 3$. If there exist $x\in R^n$ and $0<a<b$ such that $F_1, F_2\subset \bar{B}^n(x, a)$ and $F_3\subset R^n\setminus B^n(x, b)$, the following estimate holds:*

$$M(\Gamma_{12}) \geqq 3^{-n} \min\left\{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log\frac{b}{a}\right\}.$$

3. Lower bounds for the moduli of path families

Let E_1 and E_2 be two sets in R^n with $M(E_j, s, 0)\geqq\delta_j>0$, $j=1, 2$, for some $s>0$. For the estimates of this section it is important to find a lower bound in terms of δ_1, δ_2, and n for the quantity

$$M(\Delta(E_1, E_2; A)),$$

where A is the spherical ring $R(\lambda s, s/\lambda)$ and $\lambda>1$ is an appropriately chosen number depending only on δ_1, δ_2, and n. Applying the comparison principle of Lemma 2.9 with $F_1=E_1\cap\bar{B}^n(s)$, $F_2=E_2\cap\bar{B}^n(s)$, and $F_3=S^{n-1}(2s)$, we get the lower bound

$$M(\Delta(E_1, E_2; R^n)) \geqq 3^{-n} \min\{\delta_1, \delta_2, c_n \log 2\}.$$

Utilizing this lower bound and the upper bound of (2.4) we shall now give a number $\lambda>1$ with the desired property.

3.1. Lemma. *Let $\delta_1, \delta_2>0$ and let $\lambda>1$ be such that*

$$\omega_{n-1}(\log\sqrt{\lambda})^{1-n} \leqq t/6,$$

where $t=3^{-n}\min\{\delta_1, \delta_2, c_n\log 2\}$. If $s>0$ and E_1 and E_2 are two sets in R^n with $M(E_j, s, 0)\geqq\delta_j$ for $j=1, 2$, the following lower bound holds:

$$M(\Delta(E_1, E_2; R(\lambda s, s/\lambda))) \geqq t/2.$$

Proof. Denote by F_1, F_2, and F_3 the sets $\bar{R}(s, s/\sqrt{\lambda})\cap E_1$, $\bar{R}(s, s/\sqrt{\lambda})\cap E_2$, and $S^{n-1}(2s)$, respectively. From the choice of λ and (2.4) it follows that

$$M(\bar{B}^n(s/\sqrt{\lambda}), s, 0) \leqq t/6.$$

This implies for $j=1, 2$ that

$$M(F_j, s, 0) \geqq \delta_j-\frac{t}{6} \geqq 3^n\frac{5t}{6}.$$

Hence by the comparison principle of Lemma 2.9,

$$M(\Delta(F_1, F_2; R^n)) \geqq 3^{-n}\min\left\{3^n\frac{5t}{6}, c_n\log 2\right\} \geqq \frac{5t}{6}.$$

Since $F_j \subset \bar{R}(s, s/\sqrt{\lambda})$, $j=1, 2$, we get by (2.4) and the choice of λ,

$$M(\Delta(F_1, F_2; R(\sqrt{\lambda}s, s/\lambda))) \geqq M(\Delta(F_1, F_2; R^n)) - 2\omega_{n-1}(\log\sqrt{\lambda})^{1-n}$$

$$\geqq \frac{5t}{6} - \frac{2t}{6} = \frac{t}{2},$$

which together with the estimate

$$M(\Delta(E_1, E_2; R(\lambda s, s/\lambda))) \geqq M(\Delta(F_1, F_2; R(\sqrt{\lambda}s, s/\lambda)))$$

yields the desired lower bound.

We now prove the estimate (1.1) in the introduction.

3.2. Theorem. *Let $\delta_1, \delta_2 > 0$ and let $\lambda > 1$ be the number in Lemma 3.1. Then there exists a constant $c > 0$ depending only on δ_1, δ_2, and n with the following property: if $r \in (0, \lambda^{-1}]$ and $E_1, E_2 \subset R^n$ with $M(E_j, s, 0) \geqq \delta_j$ for $s \in [\lambda r, 1]$ and $j=1, 2$, then*

$$M(\Gamma_r) \geqq c \log \frac{1}{r},$$

where $\Gamma_r = \Delta(E_1, E_2; R^n \setminus \bar{B}^n(r))$.

Proof. Fix $r \in (0, \lambda^{-1}]$. Define $m = \max\{k \in N: \lambda^{1-2k} \geqq r\}$. Then $\lambda^{-3m} \leqq r \leqq \lambda^{1-2m}$ and m is a positive integer with $m \geqq (\log(1/r)/(3\log\lambda))$. The path families

$$\Gamma_k = \Delta(E_1, E_2; R(\lambda^{3-2k}, \lambda^{1-2k})), \quad k = 1, \ldots, m,$$

are separate in the sense of [10, 6.7] and $\bigcup_{k=1}^{m} \Gamma_k \subset \Gamma_r$. Hence

$$M(\Gamma_r) \geqq M\left(\bigcup_{k=1}^{m} \Gamma_k\right) \geqq \sum_{k=1}^{m} M(\Gamma_k).$$

From Lemma 3.1 it follows that there exists $t > 0$ depending only on δ_1, δ_2, and n such that $M(\Gamma_k) \geqq t/2$ for each k. Thus we get

$$M(\Gamma_r) \geqq \frac{t}{6\log\lambda} \log \frac{1}{r}.$$

Since λ depends only on δ_1, δ_2, and n, this estimate is of the desired type.

3.3. Corollary. *Let $E_j \subset R^n$ with $M(E_j, s, 0) \geqq \delta_j > 0$ for $s \in (0, 1]$ and $j=1, 2$, and let $\lambda > 1$ be as in Lemma 3.1. Then for $r \in (0, \lambda^{-1}]$*

$$M(\Gamma_r) \geqq c \log \frac{1}{r},$$

where $\Gamma_r = \Delta(E_1, E_2; R^n \setminus \bar{B}^n(r))$ and c is as in Theorem 3.2.

3.4. Theorem. *Let $\delta>0$ and let $\lambda>1$ be the number in Lemma 3.1 corresponding to the case $\delta_1=\delta_2=\delta$. Then there is a number $d>0$ depending only on δ and n with the following property: if $r\in(0, \lambda^{-2}]$ and $E\subset R^n$ with $M(E, s, 0)\geqq\delta$ for $s\in[\lambda r, 1]$, and F_r is a continuum joining $S^{n-1}(r)$ and S^{n-1}, then*

$$M(\Gamma_r) \geqq d\log\frac{1}{r},$$

where $\Gamma_r=\Delta(E, F_r; R^n\setminus\bar{B}^n(r))$.

Proof. Fix $r\in(0, \lambda^{-2}]$. Since by Lemma 2.8 $M(F_r, s, 0)\geqq c_n\log(2-\lambda^{-1})$ for $s\in[\lambda r, 1]$ it follows from Theorem 3.2 that

$$M(\Delta(E, F_r; R^n\setminus\bar{B}^n(s))) \geqq c\log\frac{1}{s}$$

for $s\in[\lambda r, 1]$, where c is the positive constant given by Theorem 3.2 for $\delta_1=\delta$ and $\delta_2=c_n\log(2-\lambda^{-1})$. Then

$$M(\Gamma_r) \geqq M(\Delta(E, F_r; R^n\setminus\bar{B}^n(s)))$$

for $s\in[\lambda r, 1]$ and hence

$$M(\Gamma_r) \geqq c\log\frac{1}{\lambda r} \geqq \frac{c}{2}\log\frac{1}{r},$$

where in the last step we have used the fact $r\in(0, \lambda^{-2}]$. We have proved the asserted estimate with $d=c/2$.

In the next result we show that one may remove the restriction $r\in(0, \lambda^{-2}]$ of Theorem 3.4 if one slightly modifies Γ_r and d.

3.5. Theorem. *Let $\delta>0$ and let E be a set in R^n with $M(E, s, 0)\geqq\delta$ for $s\in(0, 1]$. Then there is a number $d^*>0$ depending only on δ and n such that if $r\in(0, 1)$ and F_r is a continuum joining $S^{n-1}(r)$ and S^{n-1}, then*

$$M(\Gamma_r^*) \geqq d^*\log\frac{1}{r},$$

where $\Gamma_r^*=\Delta(E, F_r; R^n)$.

Proof. Let $\lambda>1$ be the number in Lemma 3.1 corresponding to the case $\delta_1=\delta_2=\delta$. If $r\in(0, \lambda^{-2}]$, the desired estimate follows from Theorem 3.4 with $d^*=d$. Fix $r\in(\lambda^{-2}, 1)$. Applying the comparison principle of Lemma 2.9 to the sets $\bar{B}^n\cap E$, $\bar{B}^n\cap F_r$, and $S^{n-1}(2)$ we get

$$M(\Gamma_r^*) \geqq 3^{-n}\min\{\delta, M(F_r, 1, 0), c_n\log 2\}.$$

In combination with the lower bound of Lemma 2.8 (2) this estimate yields

$$M(\Gamma_r^*) \geqq 3^{-n}\min\{\delta, c_n\log(2-r)\}.$$

Let $a=(3^n \log \lambda^2)^{-1} \min \{\delta, c_n \log (2-\lambda^{-2})\}$. Since $r\in(\lambda^{-2}, 1)$ we obtain

$$M(\Gamma_r^*) \geqq a \log \frac{1}{r}.$$

Hence

$$M(\Gamma_r^*) \geqq d^* \log \frac{1}{r}$$

for all $r\in(0, 1)$, where $d^*=\min \{d, a\}>0$.

3.6. Remark. In Lemma 3.1 we assumed that $M(E_j, s, 0)\geqq\delta_j>0$ for $j=1, 2$ and obtained a lower bound for $M(\Gamma(\lambda s, s/\lambda))$, where $\Gamma(\lambda s, s/\lambda)=\Delta(E_1, E_2; R(\lambda s, s/\lambda))$. If we assume that $M(E_j\cap B^n(s), s, 0)\geqq\delta_j>0$ for $j=1, 2$, we can prove a related lower bound for $M(\Gamma(s, s/\sqrt{\lambda}))$ by making use of (2.4) and the reflection principle for the modulus (cf. Lemma 4.5).

3.7. Remark. Observe that the lower bound of Theorem 3.2 follows from the cap-inequality, Lemma 2.7, in certain special cases, e.g. when both E_1 and E_2 meet $S^{n-1}(r)$ for each $r\in(0, 1]$ (cf. [10, 10.14]). However, the condition of Theorem 3.2 may be satisfied even if $(E_1\cup E_2)\cap S^{n-1}(r)=\emptyset$ for almost every $r\in(0, 1]$. In fact, by a result of Wallin there are sets E_1, E_2 with $M(E_j, r, 0)\geqq\delta_j>0$ for every $r\in(0, 1]$ $j=1, 2$, such that the Hausdorff dimension of E_j is zero, $j=1, 2$ (see [13, 2.5 (3)]). Various sufficient conditions for $\operatorname{cap}\underline{\operatorname{dens}}(E, 0)>0$ were given in [13, Section 2] and in Martio [5].

Let E_1 and E_2 be two sets with $\operatorname{cap}\underline{\operatorname{dens}}(E_j, 0)>0$, $j=1, 2$, and for $r>0$ let $\Gamma_r=\Delta(E_1, E_2; R^n\setminus\bar{B}^n(r))$. Then Theorem 3.2 shows that $M(\Gamma_r)$ tends to infinity with a certain rapidity when $r\to 0$. In the next two theorems we study the behavior of $M(\Gamma_r)$ under the more general assumptions that $\operatorname{cap}\underline{\operatorname{dens}}(E_1, 0)>0$, $\operatorname{cap}\overline{\operatorname{dens}}(E_2, 0)>0$. We show that

$$\lim_{r\to 0} M(\Gamma_r) = \infty$$

also in this case, but the convergence may take place as slowly as we wish.

3.8. Theorem. *Let E_1 and E_2 be two sets with* $\operatorname{cap}\underline{\operatorname{dens}}(E_1, 0)=\delta_1>0$ *and* $\operatorname{cap}\overline{\operatorname{dens}}(E_2, 0)=\delta_2>0$, *and let* $\Gamma_r=\Delta(E_1, E_2; R^n\setminus\bar{B}^n(r))$ *for* $r>0$. *Then* $M(\Gamma_r)\to\infty$ *as* $r\to 0$.

Proof. Choose a sequence (r_k) tending to zero such that $M(E_j, r_k, 0)\geqq\delta_j/2$, $j=1, 2$, for every $k=1, 2, \ldots$. Let $\lambda>1$ and $t>0$ be the constants corresponding to $\delta_1/2$, $\delta_2/2$, and n given by Lemma 3.1. Passing to a subsequence and relabeling if necessary, we may assume that the rings $R(\lambda r_k, r_k/\lambda)$, $k=1, 2, \ldots$ are separate. Let $\Gamma_k=\Delta(E_1, E_2; R(\lambda r_k, r_k/\lambda))$, $k=1, 2, \ldots$. Since the families Γ_k are separate and $M(\Gamma_k)\geqq t/2>0$ for all k, the assertion follows from [10, 6.7].

3.9. Theorem. *Let* $h\colon (0, 1]\to(0, \infty)$ *be a non-increasing function with* $\lim_{t\to 0+} h(t)=\infty$. *Then there exist sets E and F with* $\operatorname{cap}\underline{\operatorname{dens}}(E, 0)>0$ *and* $\operatorname{cap}\overline{\operatorname{dens}}(F, 0)>0$ *such that* $M(\Gamma_r)\leqq h(r)$ *for all* $r\in(0, 1]$, *where* $\Gamma_r=\Delta(E, F; R^n\setminus\bar{B}^n(r))$.

Proof. Let $E_k=S^{n-1}(2^{-2k})$, $F_k=S^{n-1}(2^{-2k+1})$, $k=1, 2, \dots$, and $E=\cup E_k$. Then $\operatorname{cap}\underline{\operatorname{dens}}(E, 0)\geqq c_n \log(5/3)$ by Lemma 2.8 (1). We shall now choose an infinite set $P\subset N$ such that the set $F=\cup\{F_k\colon k\in P\}$ has the desired property. Observe that for any infinite set $P\subset N$ $\operatorname{cap}\overline{\operatorname{dens}}(F, 0)\geqq c_n \log 3$ by Lemma 2.8 (1). If $k\geqq 2$, then by [10, 7.5, 6.2, 6.4]

$$M(\Delta(F_k, E; R^n)) = 2\omega_{n-1}(\log 2)^{1-n} = b.$$

For $k\geqq 1$ and $0<r<2^{-2k+1}$

$$M(\Delta(F_k, E; R^n\setminus \bar{B}^n(r))) \leqq b.$$

Let

$$p_1 = \min\{k\in N\colon h(2^{-2k+2}) \geqq b\}$$

$$p_{m+1} = \min\{k\in N,\ k > p_m\colon h(2^{-2k+2}) \geqq (m+1)b\}$$

$$m = 1, 2, \dots.$$

We show that the set $P=\{p_k\colon k\in N\}$ has the asserted property. Fix $r\in(0, 1]$. If $r\geqq 2^{-2p_1+1}$, there is nothing to prove, since then $M(\Gamma_r)=M(\emptyset)=0<h(r)$. Hence we may assume $r\in(0, 2^{-2p_1+1})$. Let

$$s = \max\{k\in N\colon 2^{-2p_k+1} \geqq r\} \geqq 1.$$

Then by [10, 6.2]

$$M(\Gamma_r) \leqq sb \leqq h(2^{-2p_s+2}) \leqq h(r)$$

as desired.

3.10. Remark. In Theorem 3.8 one may not replace the assumptions by $\operatorname{cap}\overline{\operatorname{dens}}(E_j, 0)>0$, $j=1, 2$. To show this we construct for a given $\varepsilon>0$ sets E_1 and E_2 with $M(\Delta(E_1, E_2; R^n))<\varepsilon$ and $\operatorname{cap}\overline{\operatorname{dens}}(E_j, 0)>0$, $j=1, 2$.

Let $\varepsilon>0$ and $r_1=1$. Choose $r_{k+1}\in(0, r_k/2)$, $k=1, 2, \dots$ such that

$$\omega_{n-1}\left(\log\frac{r_k}{r_{k+1}}\right)^{1-n} < \varepsilon 2^{-k}.$$

Then it follows from (2.4) that the sets $E_1=\bigcup_{k=0}^{\infty} S^{n-1}(r_{2k+1})$ and $E_2=\bigcup_{k=1}^{\infty} S^{n-1}(r_{2k})$ satisfy $M(\Delta(E_1, E_2; R^n))<\varepsilon$. From Lemma 2.8 (1) it follows that $\operatorname{cap}\overline{\operatorname{dens}}(E_j, 0)>0$, $j=1, 2$.

4. Non-tangential absolute values of quasiconformal maps

In the present section we shall use the method of Section 3 to study boundary behavior of quasiconformal mappings. A homeomorphism $f\colon G\to G'$, where G and G' are domains in R^n, is *quasiconformal* if there exists a constant $K\in[1, \infty)$ such that for every path family Γ in G

$$M(\Gamma)/K \leqq M(f\Gamma) \leqq KM(\Gamma), \tag{4.1}$$

where $f\Gamma=\{f\circ\gamma\colon \gamma\in\Gamma\}$. The smallest possible K is denoted by $K(f)$.

Let f be a quasiconformal mapping of B^n, let $b\in\partial B^n$, and let $E\subset B^n$ be a set with $\operatorname{cap}\underline{\operatorname{dens}}(E,b)>0$. The first theorem of this section shows that each non-tangential lim sup of the absolute value of f is bounded by the lim sup of the absolute value of f through the set E. As a consequence we get an extension of Tord Hall's theorem [4, Theorem II], which was proved in [13, 4.4] by different methods. The second and the last theorem of this section gives an alternative proof for the quasiconformal counterpart of J. L. Doob's theorem in [13, 5.5].

4.2. *The hyperbolic metric.* The *hyperbolic metric* ϱ in B^n is defined by the element of length

$$d\varrho=\frac{|dx|}{1-|x|^2}.$$

If a and b are points of B^n, then $\varrho(a,b)$ denotes the geodesic distance between a and b corresponding to this element of length. For $b\in B^n$ and $M\in(0,\infty)$ we let $D(b,M)$ denote the *hyperbolic ball* $\{x\in B^n\colon \varrho(b,x)<M\}$. Let $r_b=\min\{|z-b|\colon z\in\partial D(b,M)\}$. By integrating we get

$$r_b=\frac{(1-|b|^2)\tanh M}{1+|b|\tanh M}. \tag{4.3}$$

The next result follows from the proof of [13, 6.5].

4.4. Lemma. *Let $f\colon B^n\to G'$ be a quasiconformal mapping and let (b_k) be a sequence in B^n with $|b_k|\to 1$ as $k\to\infty$. If $M\in(0,\infty)$ and $E=\cup D(b_k,M)$, then*

$$\limsup_{\substack{|x|\to 1\\ x\in E}}|f(x)|=\limsup_{k\to\infty}|f(b_k)|.$$

A corresponding result holds for lim inf.

We shall need the following symmetry property for the modulus, which was proved in [13, Section 4].

4.5. Lemma. *Let E and F be two subsets of B^n. Then $M(\Delta(E,F;B^n))\geqq M(E,F;R^n))/2$.*

For $b\in\partial B^n$ and $\varphi\in(0,\pi/2)$ we let $K(b,\varphi)$ denote the cone $\{z\in R^n\colon (b|b-z)>|b-z|\cos\varphi\}$.

4.6. Theorem. *Let $f\colon B^n\to G'$ be a quasiconformal mapping, let $b\in\partial B^n$, and let $E\subset B^n$ be a set with* $\operatorname{cap}\underline{\operatorname{dens}}(E,b)>0$. *Then for every $\varphi\in(0,\pi/2)$*

$$\limsup_{\substack{x\to b\\ x\in K(b,\varphi)}}|f(x)|\leqq\limsup_{\substack{x\to b\\ x\in E}}|f(x)|,$$

$$\liminf_{\substack{x\to b\\ x\in E}}|f(x)|\leqq\liminf_{\substack{x\to b\\ x\in K(b,\varphi)}}|f(x)|.$$

Proof. Fix $\varphi \in (0, \pi/2)$. It suffices to prove the first inequality, since the second one can be proved in the same way. Denote by $\tilde{s}$ and $\tilde{t}$ the left and right hand sides of the first inequality, respectively. Assume that $\tilde{s} > \tilde{t}$. Choose $t, s \in (\tilde{t}, \tilde{s})$ with $t < s$. By Lemma 4.4 there is a sequence (a_k) in $B^n \cap K(b, \varphi)$ with $a_k \to b$ as $k \to \infty$ and with $|f(x)| > s$ for all $x \in \cup D(a_k, 1) = F$. Choose $r_1 \in (0, 1)$ such that $|f(x)| < t$ for $x \in E_1 = E \cap B^n(b, r_1)$. Since $a_k \in K(b, \varphi)$ and $a_k \to b$, there exists an integer k_0 such that for $k \geqq k_0$

$$\frac{1-|a_k|}{|a_k - b|} \geqq (\cos\varphi)/2 > 0.$$

Write $r_k = \min\{|z - a_k| \colon z \in \partial D(a_k, 1)\}$. For $k \geqq k_0$ we obtain by (4.3)

$$\frac{r_k}{|a_k - b|} \geqq \frac{r_k}{1-|a_k|}(\cos\varphi)/2 \geqq (\tanh 1 \cos\varphi)/2.$$

By Lemma 2.8 (2) this implies that $\operatorname{cap} \overline{\operatorname{dens}}\,(F, b) > 0$. Let $\Gamma = \Delta(E_1, F; B^n)$. It follows from Lemma 4.5 and Theorem 3.8 that $M(\Gamma) = \infty$. This conclusion contradicts (4.1) and the upper bound

$$M(f\Gamma) \leqq \omega_{n-1}\left(\log\frac{s}{t}\right)^{1-n}$$

given by (2.4).

4.7. Corollary. *Let $f \colon B^n \to G'$ be a quasiconformal mapping, let $b \in \partial B^n$, and let $E, F \subset B^n$ be two sets with* $\operatorname{cap} \underline{\operatorname{dens}}\,(E, b) > 0$ *and* $\operatorname{cap} \overline{\operatorname{dens}}\,(F, b) > 0$. *Suppose that $f(x)$ tends to a limit α as x approaches b through the set F. Then* $|\alpha| \leqq \limsup_{x \to b,\, x \in E} |f(x)|$.

Proof. The proof follows from the proof of Theorem 4.6.

4.8. Remark. It is not possible to replace the condition $\operatorname{cap} \underline{\operatorname{dens}}\,(E, b) > 0$ of Corollary 4.7 by $\operatorname{cap} \overline{\operatorname{dens}}\,(E, b) > 0$. We shall now show this with the aid of the following argument, which resembles the reasoning in [13, 6.6].

Let $f \colon B^2 \to G'$ be a conformal mapping which does not possess a radial limit at $e_1 = (1, 0) \in \partial B^2$. We may assume that $0, \alpha \in C_{\mathrm{rad}}(f, e_1)$, where $\alpha \neq 0$ and $C_{\mathrm{rad}}(f, e_1)$ is the cluster set of f on the radius $(0, e_1)$. Choose sequences (a_k) and (b_k) in $(0, e_1)$ with $a_k \to e_1$ and $b_k \to e_1$ such that $f(a_k) \to 0$ and $f(b_k) \to \alpha$ as $k \to \infty$. Write $E = \cup D(a_k, 1)$ and $F = \cup D(b_k, 1)$. From Lemma 4.4 it follows that $f(x) \to 0$ as $x \to e_1$ through the set E and $f(x) \to \alpha$ as $x \to e_1$ through F. Lemma 2.8 (2) implies that $\operatorname{cap} \overline{\operatorname{dens}}\,(E, e_1) > 0$ and $\operatorname{cap} \overline{\operatorname{dens}}\,(F, e_1) > 0$. Hence the assumption $\operatorname{cap} \underline{\operatorname{dens}}\,(E, b) > 0$ of Corollary 4.7 cannot be replaced by $\operatorname{cap} \overline{\operatorname{dens}}\,(E, b) > 0$.

We now give a consequence of Theorem 4.6, which was proved in [13] by different methods. This consequence extends Tord Hall's theorem [4, Theorem II] on bounded analytic functions (see [13, Section 4]). See also F. W. Gehring's result in [2, p. 21].

4.9. Corollary. *Let $f\colon B^n \to G'$ be a quasiconformal mapping and let $f(x)$ tend to a limit α as x approaches $b \in \partial B^n$ through a set E in B^n with* $\operatorname{cap}\underline{\operatorname{dens}}(E, b) > 0$. *Then f has the angular limit α at b.*

4.10. *Cluster values.* Given a continuous mapping $f\colon B^n \to R^n$, $\varepsilon > 0$, and $\alpha \in \bar{R}^n$, we denote by E_ε the set $f^{-1} B^n(\alpha, \varepsilon)$ when $\alpha \neq \infty$ and $f^{-1}(R^n \setminus \bar{B}^n(1/\varepsilon))$ when $\alpha = \infty$. Then the *cluster set* $C(f, b)$ of f at b (cf. [10, p. 52]) can be alternatively defined as the set of all points $\alpha \in \bar{R}^n$ such that $b \in \bar{E}_\varepsilon$ for all $\varepsilon > 0$.

Let now $f\colon B^n \to G'$ be quasiconformal and $b \in \partial B^n$. Then Corollary 4.9 gives us a sufficient condition for the fact that a point α is the angular limit of f at b. The next theorem provides us with a more general result of this kind, and for this purpose we introduce some terminology (cf. [13, Section 5]). Let $\alpha \in C(f, b)$ and for $\varepsilon > 0$ write $\delta_\varepsilon = \operatorname{cap}\underline{\operatorname{dens}}(E_\varepsilon, b)$. Then α is a *capacity cluster value* of f at b if for some $d > 0$

$$\lim_{\varepsilon \to 0} \varepsilon^{\delta_\varepsilon^d} = 0. \tag{4.11}$$

The least upper bound of numbers d for which condition (4.11) holds, is called the *order* of α. Adopting this terminology we shall now prove the following theorem, which extends Doob's theorem [1, Theorem 4] to the case of quasiconformal mappings. Theorem 4.12 was proved in [13, 5.5] by a different method involving a normal family argument. For a comparison between Doob's original theorem and 4.12, see [13, Section 5].

4.12. Theorem. *Let $f\colon B^n \to G'$ be a quasiconformal mapping, let $b \in \partial B^n$, and let f have a capacity cluster value α of order greater than $1/(n-1)$ at b. Then f has the angular limit α at b.*

Proof. Performing a preliminary Möbius transformation if necessary, we may assume that $\alpha \neq \infty$. Suppose that f does not have the angular limit α at b. Then there is $\varphi \in (0, \pi/2)$ and a sequence (b_k) in $K(b, \varphi) \cap B^n$ with $b_k \to b$ and $f(b_k) \to \beta \neq \alpha$ as $k \to \infty$. Fix $r_0 > 0$ such that $\beta \in \bar{R}^n \setminus B^n(\alpha, 2r_0)$. For $\varepsilon \in (0, r_0)$ let $E_\varepsilon = f^{-1} B^n(\alpha, \varepsilon)$. Since $b_k \in K(b, \varphi)$ and $b_k \to b$, there is k_1 such that $1 - |b_k| \geqq |b_k - b|(\cos\varphi)/2$ for $k \geqq k_1$. By Lemma 4.4 there is $k_0 \geqq k_1$ such that $fD(b_k, 1) \subset R^n \setminus B^n(\alpha, r_0)$ for $k \geqq k_0$. Let $E = \bigcup_{k \geqq k_0} D(b_k, 1)$. By (4.3) $B^n(b_k, (\tanh 1)(1 - |b_k|)) \subset D(b_k, 1)$ for all $k = 1, 2, \ldots$. Hence it follows from Lemma 2.8 (2) that for $k \geqq k_0$

$$M(E, |b_k - b|, b) \geqq c(n, \varphi) = c_n \log\bigl(1 + (\tanh 1 \cos\varphi)/2\bigr).$$

For $\varepsilon \in (0, r_0)$ write $\Gamma_\varepsilon = \Delta(E, E_\varepsilon; B^n)$. Let $\delta_\varepsilon = \operatorname{cap}\underline{\operatorname{dens}}(E_\varepsilon, b)$. Then for $\varepsilon \in (0, r_0)$ there is $k_\varepsilon \geqq k_0$ such that $M(E_\varepsilon, |b_{k_\varepsilon} - b|, b) \geqq \delta_\varepsilon/2$. For $\varepsilon \in (0, r_0)$ let $F_1^\varepsilon = E \cap \bar{B}^n(b, |b_{k_\varepsilon} - b|)$, $F_2^\varepsilon = E_\varepsilon \cap \bar{B}^n(b, |b_{k_\varepsilon} - b|)$, and $F_3^\varepsilon = S^{n-1}(b, 2|b_{k_\varepsilon} - b|)$. Because $\Delta(F_1^\varepsilon, F_2^\varepsilon; B^n) \subset \Gamma_\varepsilon$ we get by the comparison principle of Lemma 2.9 and by Lemma 4.5

$$M(\Gamma_\varepsilon) \geqq 2^{-1} \cdot 3^{-n} \min\{\delta_\varepsilon/2, c(n, \varphi), c_n \log 2\}$$

for $\varepsilon \in (0, r_0)$. By (2.4) we get

$$M(f\Gamma_\varepsilon) \leqq \omega_{n-1}\left(\log \frac{r_0}{\varepsilon}\right)^{1-n}$$

for $\varepsilon \in (0, r_0)$. This together with the preceding lower bound for $M(\Gamma_\varepsilon)$ and (4.1) shows that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence there exists $r_1 \in (0, r_0)$ such that $M(\Gamma_\varepsilon) \geqq 3^{-n-2}\delta_\varepsilon$ for $\varepsilon \in (0, r_1)$. This lower bound, together with the above upper bound for $M(f\Gamma_\varepsilon)$ and (4.1), yields for $\varepsilon \in (0, r_1)$

$$0 < (3^{n+2}K(f)\omega_{n-1})^{-1} \leqq (\log r_0^{\beta_\varepsilon} - \log \varepsilon^{\beta_\varepsilon})^{1-n}, \tag{4.13}$$

where $\beta_\varepsilon = \delta_\varepsilon^{1/(n-1)}$. Since α is a capacity cluster value of order greater than $1/(n-1)$, condition (4.11) is satisfied with $d = 1/(n-1)$ and thus (4.13) yields a contradiction when ε tends to zero.

If we examine the proof of Theorem 4.12 we see that the following result holds.

4.14. Corollary. *Let* $f\colon B^n \to G'$ *be a quasiconformal mapping, let* $b \in \partial B^n$, *and let* $E_\varepsilon = f^{-1}B^n(\varepsilon)$, $\delta_\varepsilon = \operatorname{cap} \underline{\operatorname{dens}}\, (E_\varepsilon, b)$. *If* $\limsup_{\varepsilon \to 0} \delta_\varepsilon (\log (1/\varepsilon))^{n-1} = \infty$, *then* f *has angular limit* 0 *at* b.

4.15. Remarks. (1) The assumption of Theorem 4.12 implies that

$$\lim_{\varepsilon \to 0} \delta_\varepsilon (\log (1/\varepsilon))^{n-1} = \infty .$$

Hence the assumption of Corollary 4.14 is slightly more general.

(2) For further results connected with Corollary 4.9 and Theorem 4.12 we refer the reader to [13]. Observe that these results can be easily generalized to cover the case of *closed* quasiregular mappings as well (cf. [12]). For the theory of general quasiregular mappings we refer the reader to the papers of Martio, Rickman, and Väisälä (cf. [6] and the references in [11]) and for the theory of closed quasiregular mappings to [11, Chapter II].

(3) It is possible to extend Corollary 4.9 to the case when the set E is a *compact* set on the boundary of B^n. Perhaps the most natural way to do this is to introduce the *asymptotic extension* $\mathring{f}$ of a quasiconformal mapping f of B^n (cf. Näkki [8]) and then to define the values of f on E in terms of $\mathring{f}$. Since E is compact, we can use a result of Gehring [3, Lemma 1] in place of Lemma 4.5. We can also extend Theorem 4.12 in the same way.

Acknowledgements. This research was done during the academic year 1977—78, when the author was visiting the Mittag-Leffler Institute. The author is grateful to the Royal Swedish Academy of Sciences for its financial support.

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Received 17 November 1978
Revision received 5 February 1979