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A PROBLEM ON JULIA SETS

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1. Introduction. Throughout the paper f, g will denote rational functions in the complex plane. To avoid exceptional trivial cases it will be assumed that f is neither constant nor a Moebius transformation. For $n\in\mathbf{N}$ the n-th iterate of f is written f^n. The set $N(f)=\{z; f^n$ is normal in some neighbourhood of $z\}$ and the Julia set $J(f)=\hat{\mathbf{C}}\setminus N(f)$ are fundamental in the iteration theory of f. We recall that $J(f)$ is a non-empty perfect set, that $J(f)=J(f^n)$, $n\in\mathbf{N}$, and that $J(f)$ and $N(f)$ have the property of 'complete invariance', expressed for J by $J(f)=f(J(f))=f^{-1}(J(f))$.

The functions f, g are called permutable if $f\circ g=g\circ f$. Julia [3] showed that for permutable f, g one has $J(f)=J(g)$. Indeed if $\alpha\in N(f)$ and $\delta>0$ we may choose $\varrho>0$ so small that the disc $D=D(\alpha,\varrho)\subset N(f)$ and that the spherical diameter of $f^n(D)$ is at most δ for all $n\in\mathbf{N}$. Since g is uniformly continuous in the spherical metric we can assume δ so small that the diameter of $f^n(g(D))=g(f^n(D))$ is uniformly small for all n. Thus $g(\alpha)\in g(D)\in N(f)$ and we have shown that $g(N(f))\subset N(f)$, whence $N(f)\subset N(g)$ and by symmetry $N(f)=N(g), J(f)=J(g)$.

One may ask if a converse of Julia's result holds, but it is easy to see that the cases when $J=\hat{\mathbf{C}}$ or when J is a circumference must be excepted. On the other hand in the cases when $J(f)$ is neither $\hat{\mathbf{C}}$ nor a part of a circumference or straight line it is known since Fatou [2] that J has a very complicated (non-differentiable) structure, which suggests that the class of functions g such that $J(g)=J(f)$ should be rather restricted, perhaps even that f, g are then permutable.

For polynomials the results are not difficult. The set J is said to have a rotational symmetry L if there is a linear function $L(z)=\delta(z+b)-b$, where $b\in\mathbf{C}$, $|\delta|=1$, $\delta\neq 1$, such that J is invariant under $z\to L(z)$.

Theorem 1. *If f, g are polynomials such that $J(f)=J(g)=J$, then either J has a rotational symmetry or f and g are permutable.*

A supplement to Theorem 1 will list the exceptional cases where rotational symmetry occurs and show that if $J(f)$ has a j-fold symmetry, $1<j<\infty$, then f and g are related to polynomials $\hat{F}$, $\hat{G}$ which are permutable. A consequence of Theorem 1 and its supplement is the following result.

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Theorem 2. *If f is a polynomial such that $J(f)$ is not a circumference, then the set of polynomials g such that $J(f)=J(g)$ is countably infinite.*

The method of proof of Theorem 1 does not extend to rational functions. However, a different argument can be used to deal with at least the large class of rational functions where J has a cusp.

If $\gamma(t)$, $\gamma'(t)$, $0\leqq t\leqq 1$ are two differentiable arcs which intersect only at $\alpha=\gamma(0)=\gamma'(0)$, where they are tangent, then we shall say that γ, γ' form a cusp at α. In a small disc Δ centred at α the cusp region will be the smaller of the two regions into which Δ is divided by γ, γ'. The set $J(f)$ is said to have a cusp at α if there are curves γ, γ' with the above properties such that for small Δ, $J(f)\cap\Delta$ belongs to the cusp region.

Now $J(f)$ certainly has a cusp (at the point ξ) if

(i) some iterate of f has a fixed point ξ of order $p\geqq 1$ such that $f^p(\xi)=\xi$, $(f^p)'(\xi)=1$, $(f^p)''(\xi)\neq 0$.

$J(f)$ will also have cusps at preimages of points such as ξ. It is interesting to try to characterise all the cusp points of J and we can do this at least under an additional assumption.

Theorem 3. *If f is a rational function such that all the critical points of f belong to $N(f)$ and α is a cusp of $J(f)$ then α is preperiodic.*

Clearly the set of cusps for a function which satisfies the assumptions of Theorem 3 is at most countably infinite. A simple geometric argument, which will be given later, shows that the assumption that the critical points of f belong to $N(f)$ is not in fact necessary to ensure the countability of the cusps. This is the key fact needed to prove

Theorem 4. *If the rational function f is such that $J(f)$ has infinitely many cusps then the set of all rational g such that $J(f)=J(g)$ is countably infinite.*

If α is a cusp of $J(f)$, then the backwards orbit 0^- of α is infinite, and, provided that

(ii) the critical points of f belong to $N(f)$, then 0^- consists entirely of cusps. Without (ii) this is false, e.g. $J(z^2-2)$ has just two cusps at ± 2.

Denote by C the class of rational functions (of degree at least two) such that (i) and (ii) above hold.

Such functions satisfy the assumptions of both Theorems 3 and 4. The functions of class C are in a sense much less restricted than polynomials. Only the restriction that $(f^p)'=1$ at some fixed point lowers the dimension of the family to one less than the full dimension $(2d+1)$ in the case of rational functions of degree d. Thus, for example, if $d=2$ we may find 4 dimensional complex manifolds of functions of the type

$$f(z) = \xi + \frac{(a+\lambda)(z-\xi)^2 + b(z-\xi)}{(z-\xi)^2 + a(z-\xi) + b},$$

which belong to C with $d=2$, $p=1$, for arbitrary complex ξ, a, b, λ provided that $b \neq 0$, $a+\lambda-a/b \neq 0$, so long as $|f'(\xi+\lambda)|<1$. One has $f(\xi)=\xi$, $f'(\xi)=1, f''(\xi)\neq 0$, $f(\xi+\lambda)=\xi+\lambda$. The two critical points of f are then in $N(f)$, one each in the region of attraction of ξ and of $\xi+\lambda$.

2. The case of polynomials. Suppose now that f is a polynomial of degree $n>1$, with leading term az^n. Denote by D the unbounded component of $N(f)$, in which the iterates $f^k \to \infty$. $J(f)$ is the boundary of D. It is classical (see e.g. Fatou [2]) that there is a function B univalent in some $D' = \{z\colon |z|>K\} \subset D$,

$$B(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots \quad \text{in} \quad D', \tag{1}$$

such that $|B(z)|>1$ in D' and

$$B \circ f \circ B^{-1}(t) = at^n.$$

B is closely related to the Green's function $g(z, \infty)$ of D; in fact

$$g_D(z, \infty) = \log|B(z)| + \left(\frac{1}{n-1}\right)\log|a|. \tag{2}$$

The connection has been already noted e.g. by A. Douady [1]. We include an explanation for completeness. A preprint by J. L. Fernandez uses this Green's function to study a related question about the relation of a monic polynomial to its Julia set.

Indeed if F is any spherically compact subset of D we have, for sufficiently large k, that $f^k(F) \subset D'$ and then

$$B(f^k) = a_k B^{n^k}, \quad a_k = a^{((n^k-1)/(n-1))},$$

is analytic and non-zero in F so that

$$\log|B(z)| = n^{-k}\{\log|B(f^k(z))| - \log|a_k|\} \tag{3}$$

gives a harmonic continuation of the left hand side to F. Fix a large value $H>K$ and denote $\gamma = \{z\colon |z|=H\}$. Let $M = \sup|B(z)|$ in the region between γ, $f(\gamma)$. Let $\varkappa$ be the smallest value of k such that $|f^k(z)| \geqq H$. Since $f(\partial D) = \partial D$ we see that $\varkappa \to \infty$ as $z \to \partial D$ in D. Since $1 < |B(f^\varkappa(z))| \leqq M$ we obtain (2) from (3).

Proof of Theorem 1. Suppose that f is as above and that $g(z) = bz^m + \dots$ is a polynomial of degree m such that $J(f) = J(g)$. Then there is a function C in D, analogous to B, such that $C(g) = bC^m$. By the uniqueness of $g(z, \infty)$ for D we have

$$g(z, \infty) = \log|B(z)| + \frac{1}{n-1}\log|a| = \log|C(z)| + \frac{1}{m-1}\log|b|.$$

From the expansion of $g(z, \infty)$ at ∞ we have

$$\frac{1}{n-1}\log|a| = \frac{1}{m-1}\log|b|,$$

whence $ba^m=\delta ab^n$ for some δ such that $|\delta|=1$. Further, since $B(z)\sim C(z)\sim z$ as $|z|\to\infty$ we also have $B(z)\equiv C(z)$ so that

$$B(f) = aB^n, \quad B(g) = bB^m, \tag{4}$$

and

$$B(g\circ f) = \delta B(f\circ g). \tag{5}$$

Equating positive powers of z in the expansions at ∞ of both sides of (4) and using (1) gives

$$g\circ f+b_0=\delta(f\circ g+b_0). \tag{6}$$

Put this in (4) and set $fg=w$. For large $|w|$

$$B(Lw) = \delta B(w), \quad Lw = \delta w+(\delta-1)b_0. \tag{7}$$

Three cases arise.

(i) If $\delta=1$ then by (5) f and g are permutable.

(ii) If δ is not a root of unity it follows from (1) and (7) that $B(z)=z+b_0$, and from (4) that $f(z)=a(z+b_0)^n-b_0$, $g(z)=b(z+b_0)^m-b_0$. In this case J is the circumference given by $|z+b_0|=\varrho$, where $|a|\,\varrho^{n-1}=1$.

(iii) δ is a primitive j-th root of unity for some $j>1$. From (4) and (7) we have $B(fL)=\delta^n B(f)$ and, by equating positive powers at ∞, $f(L)=L^n(f)$. Denoting $M(t)=t+b_0$ and $D(t)=\delta t$ we have $L=M^{-1}DM$ and $F(D)=D^n(F)$, where $F=MfM^{-1}$. Thus $F(t)$ is a polynomial of the form $at^n(1+c_1/t^j+c_2/t^{2j}+\dots)$ and $f(z)$ has the form

$$\begin{aligned} f(z) &= a(z+b_0)^n\{1+c_1(z+b_0)^{-j}+c_2(z+b_0)^{-2j}+\dots\}-b_0 \\ &= (z+b_0)^n f_1\{(z+b_0)^j\}. \end{aligned} \tag{8}$$

Since $f^\nu(L)=L^{n^\nu}(f^\nu)$ we see that D and hence $J(f)$ are invariant under $z\leftrightarrow L(z)$. Conversely a rotational symmetry of $J(f)$ will result via the Green's function in a relation of the form (7) and so one of the cases (ii) or (iii). Let us call the case (iii) a symmetry of order j.

In case (iii) $J(f)=J(g)$ does not necessarily imply that f, g are permutable. For example if δ is a primitive j-th root of unity, $b_0=0$ and $f(z)=p(z^j)$, $g(z)=\delta p(z^j)$ where p is a non-constant polynomial, one has $J(f)=J(g)$, $g\circ f=\delta f\circ g$. However, in case (iii) f and g are related to a pair of permutable polynomials $\hat F$, $\hat G$ as follows. If $F=MfM^{-1}$, $G=MgM^{-1}$ then (6) implies $gf=Lfg$ which leads to $GF=\delta FG$. Since $F(z)=z^n f_1(z^j)$ by (8), with a similar expression $G(z)=z^m g_1(z^j)$, we may set $\hat F(z)=z^n(f_1(z))^j=TFT^{-1}$, where $T(z)=z^j$, and $\hat G(z)=z^m(g_1(z))^j$. The polynomials $\hat F$, $\hat G$ are permutable:

$$\hat G\hat F = TGT^{-1}TFT^{-1} = T\delta FGT^{-1} = TFGT^{-1} = \hat F\hat G.$$

We have now proved Theorem 1 with the following.

Supplement to Theorem 1. *If* $J(f)$ *has a rotational symmetry then* f *has one of the forms given in* (ii) *and in* (iii), (8) *above. In the case of a j-fold symmetry,* $1<j<\infty$, *although* f *and* g *may not be permutable, the related polynomials* $\hat{F}$, $\hat{G}$, *described above, are permutable.*

Proof of Theorem 2. It is at once clear that the case (ii) above has to be excluded in Theorem 2, since $a(z+b_0)^n-b_0$ and $b(z+b_0)^m-b_0$ have the same J if $|ab^n|=|ba^m|$.

Suppose then that f is as in Theorem 1 but does not belong to case (ii) and that g is a polynomial $bz^m+\ldots$ such that $J(f)=J(g)$. We have case (i) or (iii), so that in the above discussion $\delta=1$ or δ is a j-th root of unity for some $j>1$. Since $b^{n-1}\delta=a^{m-1}$ we see that, given f, there is a countable set of choices for δ, m, b. The values of m, b fix g from $B(g)=bB^m$.

3. Proof of Theorem 3. (i) Suppose that f is rational (of degree at least two) and that the critical points of f belong to $N(f)$. D. Sullivan [4] has classified the behaviour of (f^n) in the components of $N(f)$ into five types. Under our assumptions on f it is impossible for two of these types (associated with Siegel discs and Hermann rings) to occur. Examination of the remaining cases shows that a point ξ of $J(f)$ can be a limit point of a sequence $(f^n(c))$, where c is a critical point of f, only if ξ is a fixed point of some f^k, $k\geqq 1$, such that $(f^k)'(\xi)=1$. There is at most a finite set of such values ξ.

(ii) Suppose that α is a cusp of $J(f)$ but that α is not preperiodic. Since all $f^n(\alpha)$ are different we may choose $n_\nu\to\infty$ so that $\alpha_\nu=f^{n_\nu}(\alpha)$ converges, say to $\beta\in J(f)$. By a change of variable it may be assumed that $\beta\neq\infty$. We claim that n_ν may be chosen so that a neighbourhood of β is free of points $f^n(c)$, where c is a critical point of f. If this is not the case, then for every convergent sequence α_ν the limit is one of a finite set of values ξ described in (i). Denote the minimum distance between two such values ξ by δ.

Take a particular choice of $\alpha_\nu=f^{n_\nu}(\alpha)\to\beta$. Then there exist natural numbers j and s such that

$$f^j(z)=z+a_{s+1}(z-\beta)^{s+1}+\ldots,\quad a_{s+1}\neq 0,$$

holds near $z=\beta$. Then (see e.g. [2]) there are s equally spaced cusp domains D_i, $1\leqq i\leqq s$, with cusp at β, such that for some value of r with $0<4r<\delta$ and $B=\{z\colon |z-\beta|<3r\}$ we have

$$J(f)\cap B\subset \textstyle\bigcup_i D_i,\quad |z-\beta|<|f^j(z)|<2|z-\beta|,\quad z\in D_i\cap B.$$

For sufficiently large ν the inequality $|\alpha_\nu-\beta|<r$ holds and so there is a first $k=k_\nu$ such that $f^{kj}(\alpha_\nu)=f^{kj+n_\nu}(\alpha)$ lies in $r<|z-\beta|<2r$. But then there is a limit point of (f^n) which is different from all the ξ of (i). The claim in (ii) is proved.

(iii) We have the cusp α of $J(f)$ (assumed not preperiodic) and a sequence $\alpha_\nu = f^{n_\nu}(\alpha) \to \beta$, where the disc $D(\beta, 3\varrho)$ of centre β, radius 3ϱ, contains no points of the form $f^n(c)$, c critical for f. Denote by $z = g_\nu(w)$ the branch of the inverse of $w = f^{n_\nu}(z)$, chosen so that $w = \alpha_\nu$ corresponds to $z = \alpha$. Then g_ν is analytic and univalent in $D = D(\beta, 3\varrho)$.

Note that by [2, § 31] (i) the g_ν are a normal family in D and (ii) for a domain Δ such that $\bar{\Delta} \subset N(f) \cap D$ and Δ contains no fixed points of f we have $g_\nu(\Delta) \to J(f)$. Since $g_\nu(\alpha_\nu) = \alpha$ it follows that the only limit function of (g_ν) is the constant α, and hence $\lambda_\nu = g'_\nu(\alpha_\nu) \to 0$ as $\nu \to \infty$.

Now consider

$$\varphi_\nu(t) = \{g_\nu(\alpha_\nu + \varrho t) - \alpha\}/(\varrho\lambda_\nu),$$

which belongs to the class S of univalent functions in $|t| < 1$, normalised by $\varphi_\nu(0) = 0$, $\varphi'_\nu(0) = 1$. By replacing φ_ν by a subsequence it may be assumed that $\arg \lambda_\nu \to$ a limit μ and $\varphi_\nu \to \varphi \in S$, locally uniformly in $|t| < 1$. Thus

$$g_\nu(\alpha_\nu + \varrho t) - \alpha = \varrho\lambda_\nu(\varphi(t) + \varepsilon_\nu(t)) \tag{9}$$

where $\varepsilon_\nu(t) \to 0$ locally uniformly in $|t| < 1$.

For any $\beta' \in J(f)$ such that $0 < |\beta' - \beta| < \frac{1}{2}\varrho$, putting $\alpha_\nu + \varrho t_\nu = \beta'$, so that $t_\nu \to t = (\beta' - \beta)/\varrho$, in (9) gives

$$g_\nu(\beta') - \alpha = \varrho\lambda_\nu(\varphi(t) + \varepsilon'_\nu) \tag{10}$$

where $\varphi(t) \neq 0$, $\varepsilon'_\nu \to 0$ as $\nu \to 0$. If the cusp of $J(f)$ at α is in the ϑ-direction, then taking arguments in (10) gives $\vartheta = \mu + \arg \varphi(t)$. We have shown the following result:

All points β' of $J(f)$ near β lie on the analytic arc σ: *through β given by $\varphi((\beta' - \beta)/\varrho) = \tau e^{i(\vartheta - \mu)}$, $\tau > 0$.*

There are now two cases to discuss, in (iv) and (v), according to whether $J(f)$ contains continua or not.

(iv) Suppose that $J(f)$ contains a continuum. Since for any $\eta \in J(f)$ the points $f^{-n}(\eta)$ are dense in $J(f)$, it follows that σ contains a subarc σ' which belongs to $J(f)$. Since fixed points of iterates of f are dense in $J(f)$ we may suppose that the end-points of σ' are fixed points of some iterate f^N and that σ' is interior to an arc of σ which belongs to $J(f)$. By the expanding property of (f^n) on $J(f)$ there is some $k \in \mathbf{N}$ such that $f^{kN}(\sigma') = J(f)$. Now $(f^{kN})' \neq 0$ on J and so at any point, such as $\beta \in J(f)$, we have $\beta = f^{kN}(\beta_1)$ for some $\beta_1 \in \sigma'$ and thus β is an interior point of an arc of $J(f)$. This contradicts what was proved in (iii). Thus the theorem is established in case (iv).

(v) Suppose that $J(f)$ contains no continua, so that $J(f) \cap \sigma$ is a nowhere dense set on σ. Since $\alpha_\nu \in \sigma$ and $(f^{n_\nu})'(\alpha) \neq 0$, while $J(f)$ has a cusp at α, it follows that α_ν is the end-point of a (maximal) arc I_ν of $\sigma \cap N(f)$, while α_ν is a limit point of a sequence $s_{\nu,n} \in J(f) \setminus I_\nu$. We have $\alpha_\nu \to \beta$, $I_\nu \to \beta$ on σ as $\nu \to \infty$.

For large ν and for $\mu>\nu$, $m=n_\mu-n_\nu$ we have $f^m(\alpha_\nu)=\alpha_\mu$, $f^m(\{s_{\nu,n}\})=\{s_{\mu,n}\}$. Thus f^m maps the part of the analytic curve σ near α_ν to the part near α_μ. Now I_ν, I_μ are given by equations of the type $z=h(t)$, $t\in\Delta_\nu=[a_\nu, b_\nu]$ or $t\in[a_\mu, b_\mu]$, respectively, where $h(t)$ is analytic and $h'(t)\neq 0$. Since $\psi(t)=h^{-1}\circ f^m\circ h(t)$ is real analytic at the end, say b_ν, of $[\alpha_\nu, b_\nu]$ which corresponds to $h(t)=\alpha_\nu$, ψ can cease to be analytic (and real) as t traverses Δ_ν only if $t\to t_0<b_\nu$ such that $f^m(t_0)$ is a singularity of h^{-1}. If t_0 is the first such value of t to be reached, then for $t\in[t_0, b_\nu)$ we have $\psi(t)$ real and also $f^m(h(t))\in N(f)$. Thus $f^m(h(t))$ cannot have left I_μ at t_0. Hence no such t_0 exists and thus $f^m(I_\nu)\subset I_\mu$.

Keeping ν fixed, let $\mu\to\infty$. Then for a point z in the interior of I_ν (and hence in $N(f)$), $f^m(z)\to\beta\in J(f)$ as $m=n_\mu-n_\nu\to\infty$. This is possible only if β is a fixed point of some f^p and $(f^p)'(\beta)=1$. However, such a β is a limit of $f^n(c)$ for some critical c, which contradicts the construction of β in (ii). Thus the proof of Theorem 3 is now complete.

4. Proof of Theorem 4. We begin with a simple geometrical observation. If I is any compact subset of **C** and $r>0$, say that $z_0\in I$ is an r-corner if the disc $D(z_0, r)$ has the property that $I\cap D(z_0, r)$ is contained in a sector of $D(z_0, r)$ with the angle $\pi/8$ at the vertex z_0. For a fixed r, I has only a finite set of r-corners. If not there is a sequence z_n of different r-corners such that z_n converges to z'. For large n, $D(z', r)\cap I$ is contained in a square whose diagonal is $z_n z_{n+1}$, and hence $D(z', r)\cap I$ reduces to a single point, which contradicts the construction of z'. Now every cusp of I is an r-corner for some rational r, so the set of cusps of I is at most countably infinite.

Now suppose that $J(f)$ has infinitely many cusps, and hence precisely a countably infinite set of cusps.

If g is a rational function such that $J(f)=J(g)$, g must map cusps of J to cusps. If the degree of g is d then g is determined if its value at $(2d+1)$ points is known. Thus if we take $2d+1$ different cusp points (α_i) and observe that there is at most a countable set of choices for $g(\alpha_i)$ at each such point we see that there is a countable set of g, at most, for each degree d.

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